Direct and inverse spectral theory of singular left-definite Sturm–Liouville operators

Extended version

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Abstract. We discuss direct and inverse spectral theory of Sturm–Liouville operators in the left-definite setting. In particular, we describe all self-adjoint realizations in a modified Sobolev space and develop Weyl–Titchmarsh theory for these operators (with separated boundary conditions and strongly singular coefficients). Consequently, we apply de Branges’ subspace ordering theorem to obtain inverse uniqueness results for the associated spectral measure. The results can be applied to solve the inverse spectral problem associated with the Camassa–Holm equation.

1. Introduction

Consider the left-definite Sturm–Liouville problem

\[-\frac{d}{dx} \left( p(x) \frac{d}{dx} y(x) \right) + q(x) y(x) = z r(x) y(x)\]

(1.1)
on some interval \((a, b)\). Here, by left-definite we mean that the real-valued function \(r\) is allowed to change sign but \(p\) and \(q\) are assumed to be non-negative. In the case of a regular left endpoint, Bennewitz [5], Brown and Weikard [7] recently developed Weyl–Titchmarsh theory for such problems, analogously to the right-definite case. Moreover, they were also able to prove that the associated spectral measure uniquely determines the left-definite Sturm–Liouville problem up to a so-called Liouville transform.

In the present paper we give an alternative proof of this result, using de Branges’ subspace ordering theorem for Hilbert spaces of entire functions. In fact, this approach allows us to deal with a larger class of problems. For instance, we allow the left endpoint to be quite singular and the weight function \(r\) to be a real-valued Borel measure. However, at a second glance our approach is not too different from the approach taken in [5] and [7]. The authors there prove Paley–Wiener type results to describe the spectral transforms of functions with compact support in order to obtain an appropriate Liouville transform. We will show that these spaces of transforms are actually hyperplanes in some de Branges spaces associated with

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our left-definite Sturm–Liouville problem. This will allow us to apply de Branges’
subspace ordering theorem to obtain a suitable Liouville transform.

As in [5] and [7], the main motivation for this work is the Camassa–Holm equa-
tion, an integrable, non-linear wave equation which models unidirectional propa-
gation of waves on shallow water. Due to its many remarkable properties, this
equation has gotten a lot of attention recently and we only refer to e.g. [8], [9], [10],
[11], [24] for further information. Associated with the Camassa–Holm equation is
the left-definite Sturm–Liouville problem

\[-y''(x) + \frac{1}{4} y(x) = z \omega(x) y(x)\]  (1.2)

on the real line. Direct, and in particular inverse spectral theory of this weighted
Sturm–Liouville problem are of peculiar interest for solving the Camassa–Holm
equation. Provided \(\omega\) is strictly positive and smooth enough, it is possible to
transform this problem into a Sturm–Liouville problem in potential form and some
inverse spectral theory may be drawn from this. However, in order to incorporate
the main interesting phenomena (wave breaking [10] and multi-peakon solutions
[2], [12]) of the dispersionless Camassa–Holm equation, it is necessary to allow \(\omega\)
at least to be an arbitrary finite signed Borel measure on \(\mathbb{R}\). In [5], [7] the authors were
able to prove an inverse uniqueness result under some restrictions on the measure
\(\omega\), which for example prohibits the case of multi-peakon solutions of the Camassa–
Holm equation. Using the results of the present paper we are able to avoid these
restrictions and to cover the case of arbitrary real finite measures \(\omega\); see [17].

Note that this application also requires us to consider our Sturm–Liouville prob-
lem (1.1) with measure coefficients. For further information on measure Sturm–
Liouville equations see e.g. [4] or [18] and the references therein. Moreover, the fact
that we allow the weight measure to vanish on arbitrary sets, makes it necessary
to work with linear relations instead of operators. Regarding the notion of linear
relations, we refer to e.g. [1], [13], [15], [16], [22] or for a brief review, containing all
facts which are needed here [18, Appendix B].

The paper is organized as follows. After some preliminaries about measure
Sturm–Liouville equations, we give a comprehensive treatment of associated linear
relations in a modified Sobolev space. More precisely, this includes a description of
all self-adjoint realizations of left-definite Sturm–Liouville problems in this Hilbert
space as well as their resolvents in Section 4. In the following two sections we
develop Weyl–Titchmarsh theory for such self-adjoint realization with separated
boundary conditions. This part is modeled after the singular Weyl–Titchmarsh
theory, recently introduced in [21] and [26] for Schrödinger operators. In Section 7
we introduce some de Branges spaces associated with a left-definite self-adjoint
Sturm–Liouville problem. Moreover, we provide some crucial properties of these
spaces, which are needed for the proof of our inverse uniqueness result. This proof
is carried out in the last section. In particular, this section provides an inverse
uniqueness result, which applies to the isospectral problem of the Camassa–Holm
equation. Finally, in the appendix we give a brief review of de Branges’ theory of
Hilbert spaces of entire functions as far as it is needed for the proof of our inverse
uniqueness result.

Before we start, let us recall some facts about functions which are absolutely
continuous with respect to some measure. Therefore let \((a, b)\) be an arbitrary in-
terval and \(\mu\) be a complex-valued Borel measure on \((a, b)\). With \(AC_{loc}((a, b); \mu)\) we
denote the set of all left-continuous functions, which are locally absolutely continuous with respect to the measure \( \mu \). These are precisely the functions \( f \) which can be written in the form
\[
f(x) = f(c) + \int_c^x h(s)d\mu(s), \quad x \in (a, b)
\]
for some \( h \in L^1_{\text{loc}}((a, b); \mu) \), where the integral has to be read as
\[
\int_c^x h(s)d\mu(s) = \begin{cases} 
\int_{[c,x]} h(s)d\mu(s), & \text{if } x > c, \\
0, & \text{if } x = c, \\
-\int_{[x,c]} h(s)d\mu(s), & \text{if } x < c.
\end{cases}
\]
The function \( h \) is the Radon–Nikodým derivative of \( f \) with respect to \( \mu \). It is uniquely defined in \( L^1_{\text{loc}}((a, b); \mu) \) and we write \( df/d\mu = h \). Every function \( f \) which is locally absolutely continuous with respect to \( \mu \) is locally of bounded variation and hence also its right-hand limits
\[
f(x+) = \lim_{\varepsilon \to 0} f(x + \varepsilon), \quad x \in (a, b)
\]
exist everywhere and may differ from \( f(x) \) only if \( \mu \) has mass in \( x \). Furthermore, we will repeatedly use the following integration by parts formula for complex-valued Borel measures \( \mu, \nu \) on \((a, b)\) (see e.g. [23, Theorem 21.67])
\[
\int_\alpha^\beta F(x)d\nu(x) = FG|_\alpha^\beta - \int_\alpha^\beta G(x+)d\mu(x), \quad \alpha, \beta \in (a, b),
\]
where \( F, G \) are left-continuous distribution functions of \( \mu, \nu \) respectively.

2. Measure Sturm–Liouville equations

Let \((a, b)\) be an arbitrary interval and \( \varrho, \varsigma \) and \( \chi \) be complex-valued Borel measures on \((a, b)\). We are going to define a linear differential expression \( \tau \) which is informally given by
\[
\tau f = \frac{d}{d\varrho} \left( -\frac{df}{d\varsigma} + \int f d\chi \right).
\]
In the rest of this paper we will always assume that our measures satisfy the following four properties.

**Hypothesis 2.1.**

(i) The measure \( \varrho \) is real-valued.

(ii) The measure \( \varsigma \) is positive and supported on the whole interval.

(iii) The measure \( \chi \) is positive but not identically zero.

(iv) The measure \( \varsigma \) has no point masses in common with \( \chi \) or \( \varrho \), i.e.
\[
\varsigma(\{x\})\chi(\{x\}) = \varsigma(\{x\})\varrho(\{x\}) = 0, \quad x \in (a, b).
\]

The maximal domain \( \mathcal{D}_\tau \) of functions such that the expression \( \tau f \) makes sense consists of all functions \( f \in AC_{\text{loc}}((a, b); \varsigma) \) for which the function
\[
-\frac{df}{d\varsigma}(x) + \int_x^c f d\chi, \quad x \in (a, b)
\]
is locally absolutely continuous with respect to \( \varrho \), i.e. there is some representative of this function lying in \( AC_{\text{loc}}((a, b); \varrho) \). As a consequence of the assumption on the support of \( \varsigma \), this representative is unique. We then set \( \tau f \in L^1_{\text{loc}}((a, b); \varrho) \) to be the
Radon–Nikodým derivative of this function with respect to \( \varrho \). One easily sees that this definition is independent of \( c \in (a, b) \) since the corresponding functions \( f \) as well as their unique representatives only differ by an additive constant. As usual, the Radon–Nikodým derivative with respect to \( \varsigma \) of some \( f \in \mathcal{D}_r \) is denoted with

\[
(2.2) \quad f^{[1]} = \frac{df}{d\varsigma} \in L^1_{\text{loc}}((a, b); \varsigma)
\]

and referred to as the quasi-derivative of \( f \).

It is easily seen that this definition of \( \tau \) is consistent with classical theory. Indeed, if \( \varrho, \varsigma \) and \( \chi \) are locally absolutely continuous with respect to the Lebesgue measure with densities \( r, p^{-1} \) and \( q \) respectively, then \( \tau \) becomes

\[
\tau f(x) = \frac{1}{r(x)} \left( -\frac{d}{dx} \left( p(x) \frac{df}{dx}(x) \right) + q(x)f(x) \right), \quad x \in (a, b).
\]

The following existence and uniqueness theorem for solutions of measure Sturm–Liouville equations may be found in [18, Theorem 3.1].

**Theorem 2.2.** For each \( g \in L^1_{\text{loc}}((a, b); g) \), \( c \in (a, b) \), \( d_1, d_2 \in \mathbb{C} \) and \( z \in \mathbb{C} \) there is a unique solution of the initial value problem

\[
(\tau - z)f = g \quad \text{with} \quad f(c) = d_1 \quad \text{and} \quad f^{[1]}(c) = d_2.
\]

We say that \( \tau \) is regular at an endpoint if the measures \( \varrho, \varsigma \) and \( \chi \) are finite near this endpoint. In this case, if \( g \) is integrable near this endpoint, then each solution of the equation \((\tau - z)f = g\) may be continuously extended to this endpoint. Moreover, the initial point \( c \) in Theorem 2.2 may be chosen as this endpoint (see e.g. [18, Theorem 3.5]).

Associated with our differential expression \( \tau \) is a linear relation \( T_{\text{loc}} \) in the space \( AC_{\text{loc}}((a, b); \varsigma) \) defined by

\[
T_{\text{loc}} = \{(f, f_\tau) \in AC_{\text{loc}}((a, b); \varsigma)^2 \mid f \in \mathcal{D}_r, \tau f = f_\tau \text{ in } L^1_{\text{loc}}((a, b); g) \}.
\]

Regarding notation we will make the following convention. Given some pair \( f \in T_{\text{loc}} \) we will denote its first component also with \( f \) and the second component with \( f_\tau \). Moreover, if \( g \in AC_{\text{loc}}((a, b); \varsigma) \) and \( f \) is a solution of the equation \((\tau - z)f = g\) for some \( z \in \mathbb{C} \) then this solution \( f \) will often be identified with the pair \((f, g +zf) \in T_{\text{loc}}\).

In the right-definite case, a crucial role is played by the Wronskian of two functions and the associated Lagrange identity. The corresponding quantity in the left-definite case is the function

\[
(2.3) \quad V(f, g^*)(x) = f_\tau(x)g^{[1]}(x)^* - f^{[1]}(x)g_\tau(x)^*, \quad x \in (a, b),
\]

defined for all pairs \( f, g \in T_{\text{loc}} \). For this modified Wronskian the following Lagrange identity holds.

**Proposition 2.3.** For every \( f, g \in T_{\text{loc}} \) and \( \alpha, \beta \in (a, b) \) we have

\[
V(f, g^*)(\beta) - V(f, g^*)(\alpha) = \int_{\alpha}^{\beta} f_\tau(x)g(x)^* - f(x)g_\tau(x)^*d\chi(x)
+ \int_{\alpha}^{\beta} f_\tau^{[1]}(x)g^{[1]}(x)^* - f^{[1]}(x)g_\tau^{[1]}(x)^*d\varsigma(x).
\]
Proof. For each \( h \in AC_{\text{loc}}((a,b); \varsigma) \) an integration by parts shows that

\[
\int_\alpha^\beta f h^* d\chi + \int_\alpha^\beta f^{[1]} h^{[1]*} d\varsigma = \int_\alpha^\beta f_\tau h^* d\varsigma + [f^{[1]} h^*]_\alpha^\beta
\]

holds. Hereby note that the right-hand limit appearing in the integration by parts formula (1.3) may be omitted because of assumption (iv) in Hypothesis 2.1. In particular, choosing \( h = g_\tau \) and subtracting the corresponding equation with the roles of \( f \) and \( g \) reversed yields the claim. \( \Box \)

As a consequence of the Lagrange identity one sees that for each \( z \in \mathbb{C} \) the modified Wronskian \( V(u_1, u_2) \) of two solutions \( u_1, u_2 \) of \((\tau - z)u = 0\) is constant. Furthermore, for \( z \neq 0 \) we have

\[
V(u_1, u_2) \neq 0 \iff u_1, u_2 \text{ linearly independent.}
\]

Indeed, the modified Wronskian of two linearly dependent solutions vanishes obviously. Conversely, \( V(u_1, u_2) = 0 \) implies that for each \( c \in (a,b) \) there is a \( K \in \mathbb{C} \) such that

\[
K u_1(c) = u_2(c) \quad \text{and} \quad K u_1^{[1]}(c) = u_2^{[1]}(c),
\]

where, without loss of generality we assumed that \( u_1 \) is a non-zero solution. Now uniqueness of solutions of the initial value problem implies the linear dependence of \( u_1 \) and \( u_2 \). Another useful identity for the modified Wronskian is the following Plücker identity.

Proposition 2.4. For every \( f_1, f_2, f_3, f_4 \in T_{\text{loc}} \) we have

\[
0 = V(f_1, f_2) V(f_3, f_4) + V(f_1, f_3) V(f_4, f_2) + V(f_1, f_4) V(f_2, f_3).
\]

Proof. The right-hand side is equal to the determinant

\[
\begin{vmatrix}
1 & f_{1,\tau} & f_{2,\tau} & f_{3,\tau} & f_{4,\tau} \\
1 & f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \\
2 & f_{1,\tau} & f_{2,\tau} & f_{3,\tau} & f_{4,\tau} \\
1 & f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]}
\end{vmatrix}.
\]

\( \Box \)

3. Left-definite Sturm–Liouville relations

In order to obtain a linear relation in a Hilbert space we introduce a modified Sobolev space \( H^1(a,b) \). It consists of all functions \( f \) on \((a,b)\) which are locally absolutely continuous with respect to \( \varsigma \) such that \( f \) is square integrable with respect to \( \chi \) and the Radon–Nikodým derivative \( df/d\varsigma \) is square integrable with respect to \( \varsigma \). The space \( H^1(a,b) \) is equipped with the inner product

\[
(f, g) = \int_a^b f(x)g(x)^* d\chi(x) + \int_a^b f^{[1]}(x)g^{[1]}(x)^* d\varsigma(x), \quad f, g \in H^1(a,b).
\]

Hereby note that \( f \) and \( g \) are always continuous in points of mass of \( \chi \) in virtue of property (iv) in Hypothesis 2.1.

Proposition 3.1. The space \( H^1(a,b) \) is a reproducing kernel Hilbert space.
Proof. Because of our assumptions on the measures ς and χ it remains to prove completeness in order to see that $H^1(a,b)$ is a Hilbert space. Therefore let $f_n \in H^1(a,b)$, $n \in \mathbb{N}$ be some Cauchy sequence. The quasi-derivatives of this sequence converge in $L^2((a,b);\varsigma)$ to say $f^1$. Moreover, $f_n(c)$ converges to say $F \in \mathbb{C}$ for some point $c \in (a,b)$ since $\chi \neq 0$. Now the function

$$f(x) = F + \int_c^x f^1(t) d\varsigma(t), \quad x \in (a,b)$$

is locally absolutely continuous with respect to $\varsigma$ with quasi-derivative $f^1$. Since $f$ is the pointwise limit of the functions $f_n$, which converge in the space $L^2((a,b);\varsigma)$, one sees that $f$ is also the limit of $f_n$ in $L^2((a,b);\varsigma)$. Hence $f \in H^1(a,b)$ and the sequence $f_n$ converges to $f$ in $H^1(a,b)$. In order to prove that point evaluations are continuous fix some $c \in (a,b)$ and choose $\alpha, \beta \in (a,b)$ with $\alpha < c < \beta$ such that $\chi([\alpha,\beta]) \neq 0$. Clearly, we have for each function $f \in H^1(a,b)$

$$f(c) = f(x) + \int_c^c f^1(t) d\varsigma(t), \quad x \in (a,b).$$

Now, integration with respect to $\chi$ yields

$$|f(c)|\chi([\alpha,\beta]) = \left| \int_\alpha^\beta f(x) d\chi(x) + \int_\alpha^\beta \int_x^c f^1(t) d\varsigma d\chi(x) \right|$$

$$\leq \chi([\alpha,\beta])^{\frac{1}{2}} \|f\| + \chi([\alpha,\beta])\varsigma([\alpha,\beta])^{\frac{1}{2}} \|f\|, \quad \text{where we used the Cauchy–Schwarz inequality twice.}$$

In order to obtain the maximal relation $T_{\text{max}}$ in $H^1(a,b)$ associated with our differential expression $\tau$ we restrict $T_{\text{loc}}$ by

$$T_{\text{max}} = \{(f,f_\tau) \in H^1(a,b) \times H^1(a,b) \mid (f,f_\tau) \in T_{\text{loc}}\}.$$ 

The following characterization of elements of $T_{\text{max}}$ as weak solutions of our differential equation will be quite useful. Henceforth we will denote with $H^1_L(a,b)$ the linear subspace of $H^1(a,b)$ consisting of all functions with compact support.

**Proposition 3.2.** Some pair $(f,f_\tau) \in H^1(a,b) \times H^1(a,b)$ lies in the maximal relation $T_{\text{max}}$ if and only if

$$\int_a^b f(x) g(x)^* d\chi(x) + \int_a^b f^1(x) g^1(x)^* d\varsigma(x) = \int_a^b f_\tau(x) g(x)^* d\rho(x)$$

for each $g \in H^1_L(a,b)$.

**Proof.** If $(f,f_\tau) \in T_{\text{max}}$, then integration by parts as in (2.4) shows that equation (3.1) is valid for each $g \in H^1_L(a,b)$. In order to prove the converse let $c, d \in (a,b)$ with $c < d$, $K \in \mathbb{C}$ and integrate (3.1) by parts to obtain

$$0 = \int_a^b g^1(x)^* \left( f^1(x) \right) + \int_c^d f d\chi + \int_c^d f_\tau d\rho + K \right) d\varsigma(x), \quad g \in H^1_L(a,b).$$

In particular we may choose

$$g(x)^* = \begin{cases} 0, & \text{if } x \in (a,c], \\ \int_c^x f^1(t) - \int_c^t f d\chi + f_\tau d\rho + K d\varsigma(t), & \text{if } x \in (c,d], \\ 0, & \text{if } x \in (d,b), \end{cases}$$
where
\[ K = -\frac{1}{\zeta(c,a)} \int_c^d \left( f^{\lfloor 1 \rfloor}(t) - \int_c^t f d\chi + \int_c^t f_s d\varrho \right) d\varsigma(t). \]

Now from this
\[ \int_c^d \left| f^{\lfloor 1 \rfloor}(x) - \int_c^x f d\chi + \int_c^x f_s d\varrho + K \right|^2 d\varsigma(x) = 0 \]

we infer that the integrand is locally constant. But this shows that \( f \) lies in \( \mathcal{D}_\tau \) and that \( \tau f = f \tau \) almost everywhere with respect to \( |\varrho| \), which yields the claim. \( \square \)

Now Proposition 3.2 allows us to describe the multi-valued part of \( T_{\text{max}} \).

**Corollary 3.3.** Some \( h \in H^1(a, b) \) lies in the multi-valued part of \( T_{\text{max}} \) if and only if \( h = 0 \) almost everywhere with respect to \( |\varrho| \).

**Proof.** If \( h \in \text{mul}(T_{\text{max}}) \), then the definition of \( T_{\text{max}} \) shows that \( h = 0 \) almost everywhere with respect to \( |\varrho| \). The converse immediately follows from Proposition 3.2. \( \square \)

We say some function \( f \in AC_{\text{loc}}((a, b); \varsigma) \) lies in \( H^1(a, b) \) near an endpoint if \( f \) is square integrable with respect to \( \chi \) near this endpoint and its quasi-derivative is square integrable with respect to \( \varsigma \) near this endpoint. Furthermore, we say some pair \( f \in T_{\text{loc}} \) lies in \( T_{\text{max}} \) near an endpoint if both components \( f \) and \( f \tau \) lie in \( H^1(a, b) \) near this endpoint. Clearly, some \( f \in T_{\text{loc}} \) lies in \( T_{\text{max}} \) if and only if it lies in \( T_{\text{max}} \) near \( a \) and near \( b \).

**Lemma 3.4.** If \( f \) and \( g \) lie in \( T_{\text{max}} \) near \( a \), then the limit
\[ V(f, g^*)(a) := \lim_{\alpha \to a} V(f, g^*)(\alpha) \]
exists and is finite. A similar result holds for the endpoint \( b \). If \( f \) and \( g \) even lie in \( T_{\text{max}} \), then
\[ (f, g) - \langle f, g \rangle = V(f, g^*)(b) - V(f, g^*)(a) =: V^h_{\alpha}(f, g^*). \]
Moreover, \( V(\cdot, \cdot)(a) \) and \( V(\cdot, \cdot)(b) \) are continuous bilinear forms on \( T_{\text{max}} \) with respect to the product topology.

**Proof.** Letting \( \alpha \) tend to \( a \) in the Lagrange identity in Proposition 2.3 shows that the limit exists and is finite. Similarly one sees that the limit \( V(f, g^*)(\beta) \) as \( \beta \to b \) exists and is finite if \( f \) and \( g \) lie in \( T_{\text{max}} \) near \( b \). Now if \( f, g \in T_{\text{max}} \), then equation 3.2 follows by letting \( \alpha \to a \) and \( \beta \to b \) in the Lagrange identity. In order to prove continuity fix some \( c \in (a, b) \) and let \( \alpha, \beta \in (a, b) \) with \( \alpha < c < \beta \). Clearly for each \( f \in T_{\text{max}} \) we have
\[ f^{\lfloor 1 \rfloor}(c) = f^{\lfloor 1 \rfloor}(x) - \int_c^x f d\chi + \int_c^x f_s d\varrho, \quad x \in (a, b). \]
Integration with respect to \( \varsigma \) and using the Cauchy–Schwarz inequality twice yields the estimate
\[ |f^{\lfloor 1 \rfloor}(c)| \leq \varsigma((\alpha, \beta))^{-\frac{1}{2}} ||f|| + \chi((\alpha, \beta))^\frac{1}{2} ||f|| + |\varrho|((\alpha, \beta))C_{\alpha,\beta} ||f \varrho||, \]
where the constant \( C_{\alpha,\beta} \in \mathbb{R} \) is chosen such that \( |g(x)| \leq C_{\alpha,\beta} ||g|| \) for all \( x \in [\alpha, \beta] \) and \( g \in H^1(a, b) \). Hence the mapping \( f \mapsto f^{\lfloor 1 \rfloor}(c) \) is continuous on \( T_{\text{max}} \). Since \( f \mapsto f \tau(c) \) is also continuous in virtue of Proposition 3.1 we infer that \( V(\cdot, \cdot)(c) \)
is a continuous bilinear form on $T_{\text{max}}$. Now the claim is easily deduced from the Lagrange identity, using the Cauchy–Schwarz inequality. \hfill \Box

If $\tau$ is regular at an endpoint, say $a$, then for each $f$ which lies in $T_{\text{max}}$ near $a$ the limits
\[ f(a) = \lim_{\alpha \to a} f(\alpha), \quad f^{[1]}(a) = \lim_{\alpha \to a} f^{[1]}(\alpha) \quad \text{and} \quad f_\tau(a) = \lim_{\alpha \to a} f_\tau(\alpha) \]
exist and are finite. Indeed, for the first and the last limit this follows from the fact that in this case $f^{[1]}$ and $f^{[1]}_\tau$ are integrable near $a$ with respect to $\varsigma$. Moreover, the existence of the second limit may be seen from
\[ -f^{[1]}(\beta) + f^{[1]}(\alpha) + \int_\alpha^\beta f d\chi = \int_\alpha^\beta f_\tau d\varrho, \quad \alpha, \beta \in (a, b), \quad \alpha < \beta. \]

Now the modified Wronskian near a regular endpoint $a$ is given by
\[ V(f, g)(a) = f_\tau(a)g^{[1]}(a) - f^{[1]}(a)g_\tau(a), \]
if $f$ and $g$ lie in $T_{\text{max}}$ near $a$. Of course similar results hold if the right endpoint $b$ is regular.

In the remaining part of this section we will collect some more properties of the modified Sobolev space $H^1(a, b)$ and the maximal relation $T_{\text{max}}$. Therefore we need the following result on solutions of our differential equation which has been proven in [6, Theorem 3].

Theorem 3.5. For each $z \in \mathbb{C}$ all solutions of $(\tau - z)u = 0$ lie in $H^1(a, b)$ near $a$ if and only if $\varsigma$, $\chi$ are finite near $a$ and the function
\[ z \int_x^c d\varrho, \quad x \in (a, b) \]
is square integrable with respect to $\varsigma$ near $a$ for some $c \in (a, b)$. A similar result holds for the endpoint $b$.

In the following $H^1_0(a, b)$ will denote the closure of $H^1_c(a, b)$ in $H^1(a, b)$.

Proposition 3.6. We have $H^1(a, b) = H^1_0(a, b) \oplus \ker(T_{\text{max}})$, with
\[ \dim \ker(T_{\text{max}}) = \begin{cases} 
0, & \text{if } \varsigma + \chi \text{ is infinite near both endpoints}, \\
1, & \text{if } \varsigma + \chi \text{ is finite near precisely one endpoint}, \\
2, & \text{if } \varsigma + \chi \text{ is finite}.
\end{cases} \]

Proof. Proposition 3.2 shows that $\ker(T_{\text{max}}) = H^1_c(a, b)^\perp$. Now suppose that $\varsigma + \chi$ is infinite near both endpoints and take some $u \in \ker(T_{\text{max}})$. An integration by parts as in (2.4) shows that for each $\alpha, \beta \in (a, b)$ with $\alpha < \beta$
\[ \int_\alpha^\beta |u|^2 d\chi + \int_\alpha^\beta |u^{[1]}|^2 d\varsigma = \text{Re}(u^{[1]}(\beta)u(\beta)^*) - \text{Re}((u^{[1]}(\alpha)u(\alpha)^*). \tag{3.3} \]
From this we see that the function $\text{Re}(u^{[1]}u^*)$ is increasing on $(a, b)$. Further integration by parts yield
\[ |u(\beta)|^2 - |u(\alpha)|^2 = 2 \int_\alpha^\beta \text{Re}(u^{[1]}(x)u(x)^*) d\varsigma(x) \tag{3.4} \]
as well as
\begin{equation}
|u^{[1]}(\beta)|^2 - |u^{[1]}(\alpha)|^2 = 2 \int_{\alpha}^{\beta} \text{Re}(u^{[1]}(x)u(x)^*)d\chi(x)
\end{equation}
for each \( \alpha, \beta \in (a, b) \) with \( \alpha < \beta \). Now if there was some \( c \in (a, b) \) such that \( \text{Re}(u^{[1]}(c)u(c)^*) > 0 \), then the equations (3.4), (3.5) and the monotonicity of the function \( \text{Re}(u^{[1]}u^*) \) would show that the modulus of \( u \) and its quasi-derivative are non-decreasing to the right of \( c \). Moreover, if there was some \( c \in (a, b) \) with \( \text{Re}(u^{[1]}(c)u(c)^*) < 0 \) one would similarly show that the modulus of \( u \) and its quasi-derivative are non-increasing to the left of \( c \). Hence both cases would imply that \( u \) does not lie in \( H^1(a, b) \) since for each endpoint at least one, \( \varsigma \) or \( \chi \) is infinite. But from this we conclude that \( u \) vanishes identically.

Now suppose \( \varsigma + \chi \) is finite near exactly one endpoint, say \( a \). Then it is easily seen that for each \( f \in H^1(a, b) \) the limit of \( f(x) \) as \( x \to a \) exists and is finite. Moreover, point evaluation in \( a \) is continuous on \( H^1(a, b) \). Indeed, this may be shown literally in the proof of Proposition 3.1. Since there are functions in \( H^1(a, b) \) which do not vanish in \( a \), the dimension of the kernel of \( T_{\text{max}} \) is not zero. Hence the claim follows, since Theorem 3.5 shows that the dimension is at most one. The remaining case immediately follows from Theorem 3.5.

**Corollary 3.7.** There is a (up to scalar multiples) unique non-trivial real solution \( w_a \) of \( \tau u = 0 \) which lies in \( H^1(a, b) \) near \( a \) and satisfies
\begin{equation}
\lim_{\alpha \to a} g(\alpha)w_a^{[1]}(\alpha) = 0, \quad g \in H^1(a, b).
\end{equation}

Similarly, there is a (up to scalar multiples) unique non-trivial real solution \( w_b \) of \( \tau u = 0 \) with corresponding properties near the endpoint \( b \). The solutions \( w_a \) and \( w_b \) are linearly independent.

**Proof.** If \( \varsigma + \chi \) is finite near \( a \), the solution \( w_a \) of \( \tau u = 0 \) with initial conditions
\[ w_a(a) = 1 \quad \text{and} \quad w_a^{[1]}(a) = 0 \]
has the claimed properties. Any other such solution has to vanish at \( a \) in view of (3.6) and thus is a scalar multiple of \( w_a \). Now if \( \varsigma + \chi \) is not finite near \( a \), Proposition 3.6 shows that there is a (up to scalar multiples) unique solution of \( \tau u = 0 \) which lies in \( H^1(a, b) \) near \( a \). Moreover, we may suppose that \( w_a \) is real since otherwise one could take the real or imaginary part of \( w_a \). Next we show that the mapping
\[ g \mapsto \lim_{\alpha \to a} g(\alpha)w_a^{[1]}(\alpha) \]
is continuous on \( H^1(a, b) \). Indeed, integration by parts as in (2.4) shows
\[ \int_a^c gw_a d\varsigma + \int_a^c g^{[1]}w_a^{[1]}d\chi = g(c)w_a^{[1]}(c) - g(\alpha)w_a^{[1]}(\alpha), \quad g \in H^1(a, b), \]
hence this limit exists. Moreover, it may be estimated by
\[ |\lim_{\alpha \to a} g(\alpha)w_a^{[1]}(\alpha)| \leq |w_a^{[1]}(c)||g(c)| + ||g||\sqrt{\int_a^c w_a^2 d\varsigma + \int_a^c w_a^{[1]}^2 d\varsigma} \]
for each \( g \in H^1(a, b) \) and thus is continuous. Since obviously (3.6) is valid for \( g \in H^1(a, b) \) we infer that it is also valid for \( g \in H^1_c(a, b) \). If \( \ker(T_{\text{max}}) \neq \{0\} \), i.e. when \( \varsigma + \chi \) is finite near \( b \), then \( \ker(T_{\text{max}}) = \text{span}\{w_a\} \). Hence we have to
prove that (3.6) holds for \( g = w_a \) in this case. Moreover, note that the proof of Proposition 3.6 shows that the function

\[ x \mapsto w_a(x)w_a^{[1]}(x) \]

is strictly positive and increasing on \((a, b)\). Now if the limit as \( x \to a \) was strictly positive, one would obtain a contradiction from (3.4) or from (3.5). But this guarantees (3.6) for all \( g \in H^1(a, b) \) because of the decomposition \( H^1(a, b) = \ker(T_{\text{max}}) \oplus H^1_0(a, b) \). Finally, if \( w_a \) and \( w_b \) were linearly dependent, then (3.3) would show that their norm vanishes. \( \square \)

Note that the functions

\[ x \mapsto w_a(x)w_a^{[1]}(x) \quad \text{and} \quad x \mapsto w_b(x)w_b^{[1]}(x) \]

are increasing on \((a, b)\) in view of (3.3) and strictly positive (respectively negative). Now for each fixed \( c \in (a, b) \) we define the function

\[
\delta_c(x) = \frac{1}{W(w_b, w_a)} \begin{cases} 
  w_a(x)w_a(c), & \text{if } x \in (a, c], \\
  w_a(c)w_b(x), & \text{if } x \in (c, b),
\end{cases}
\]

with

\[
W(w_b, w_a) = w_b(x)w_a^{[1]}(x) - w_a^{[1]}(x)w_b(x),
\]

where the right-hand side is independent of \( x \in (a, b) \) and non-zero since \( w_a \) and \( w_b \) are linearly independent solutions of \( \tau u = 0 \). With this definition the point evaluation in \( c \) is given by

\[
f(c) = \langle f, \delta_c \rangle, \quad f \in H^1(a, b).
\]

More precisely, this follows from splitting the integrals on the right-hand side, integrating by parts as in (2.4) twice and using the properties from Corollary 3.7. Furthermore, if the measures \( \varsigma \) and \( \chi \) are finite near an endpoint, say \( a \), then \( f(x) \) has a finite limit as \( x \to a \) for each \( f \in H^1(a, b) \) and

\[
f(a) = \lim_{\alpha \to a} f(\alpha) = \langle f, \delta_a \rangle, \quad f \in H^1(a, b),
\]

where the function \( \delta_a \) is given by

\[
\delta_a(x) = -\frac{w_b(x)}{w_b^{[1]}(a)}, \quad x \in (a, b).
\]

In fact, this follows from a simple integration by parts using the properties of Corollary 3.7. Of course, a similar result holds for the right endpoint \( b \) if \( \varsigma \) and \( \chi \) are finite near \( b \). As a consequence of this we see that some function \( f \in H^1(a, b) \) lies in \( H^1_0(a, b) \) if and only if \( f \) vanishes in each endpoint near which \( \varsigma \) and \( \chi \) are finite.

We are now turning back to our maximal relation \( T_{\text{max}} \). Since we are interested in self-adjoint restrictions of \( T_{\text{max}} \), we will first compute its adjoint relation.

**Theorem 3.8.** The maximal relation \( T_{\text{max}} \) is closed with adjoint given by

\[
T_{\text{max}}^* = \{ f \in T_{\text{max}} \mid \forall g \in T_{\text{max}} : V(f, g)(a) = V(f, g)(b) = 0 \}.
\]
Proof. Let $T_0 \subseteq T_{\max}$ consist of all $f \in T_{\max}$ such that $f_r \in H^1_c(a,b)$, $f$ is a scalar multiple of $w_a$ near $a$ and a scalar multiple of $w_b$ near $b$. Then the range of $T_0$ is actually equal to $H^1_c(a,b)$. Indeed, if $g \in H^1_c(a,b)$ is given, then the function

$$f(x) = W(w_b, w_a)^{-1}\left( w_b(x) \int_a^x w_a g d\varrho + w_a(x) \int_x^b w_b g d\varrho \right), \quad x \in (a,b)$$

is a solution of $\tau f = g$ (see [13], Proposition 3.3]) which is a scalar multiple of $w_a$, $w_b$ near the respective endpoints and hence $g \in \text{ran}(T_0)$. Moreover, for each $f \in T_0$, $g \in T_{\max}$ the limits of $V(f,g)(x)$ as $x \to a$ and as $x \to b$ vanish in view of Corollary 3.7. Hence Lemma 3.4 shows that $T_{\max} \subseteq T_0^*$. Conversely, if $(f_1, f_2) \in T_0^*$, then integration by parts as in (2.4) and using Corollary 3.7 shows that

$$\langle f_1, g_r \rangle = \langle f_2, g \rangle = \int_a^b f_2(x) g_r(x)^* d\varrho(x)$$

for each $g \in T_0$. Now since $\text{ran}(T_0) = H^1_c(a,b)$ we infer that $(f_1, f_2) \in T_{\max}$ in view of Proposition 3.2. Thus $T_{\max}$ is the adjoint of $T_0$ and hence closed. Finally we obtain

$$T_{\max}^* = T_0 \subseteq \{ f \in T_{\max} : \forall g \in T_{\max} : V(f,g)(a) = V(f,g)(b) = 0 \} \subseteq T_{\max}^*,$$

where we used Lemma 3.4 and the fact that $V(\cdot, \cdot)(a)$ and $V(\cdot, \cdot)(b)$ are continuous on $T_{\max} \times T_{\max}$.

The adjoint of $T_{\max}$ is referred to as the minimal relation $T_{\min}$. This linear relation is obviously symmetric with adjoint $T_{\max}$. Moreover, the following corollary shows that its deficiency indices are equal. Hence there are always self-adjoint restrictions of $T_{\max}$.

**Corollary 3.9.** The deficiency indices of the minimal relation $T_{\min}$ are equal and at most two, i.e.

$$n(T_{\min}) := \dim\text{ran}(T_{\min} - i)^\perp = \dim\text{ran}(T_{\min} + i)^\perp \leq 2.$$

**Proof.** The dimensions are less than two because of

$$\text{ran}(T_{\min} \pm i)^\perp = \ker(T_{\max} \mp i) \subseteq \ker(T_{\loc} \mp i),$$

and since there are at most two linearly independent solutions of $(\tau \pm i)u = 0$. Moreover, note that the solutions of the equation $(\tau - i)u = 0$ are precisely the complex conjugates of the solutions of $(\tau + i)u = 0$. From this it is easily seen that the complex conjugation is a conjugate-linear isometry from the kernel of the relation $T_{\max} + i$ onto the kernel of $T_{\max} - i$ and hence their dimensions are equal. \qed

4. Self-adjoint restrictions

In this section we will determine the self-adjoint restrictions of $T_{\max}$. Therefore we will distinguish two different cases at each endpoint. We say $\tau$ is in the limit-circle (l.c.) case at $a$, if for each $z \in \mathbb{C}^\times$ (the cross indicates that zero is removed from this set) all solutions of $(\tau - z)u = 0$ lie in $H^1_c(a,b)$ near $a$. Furthermore, we say $\tau$ is in the limit-point (l.p.) case at $a$ if for each $z \in \mathbb{C}^\times$ there is some solution of $(\tau - z)u = 0$ which does not lie in $H^1_c(a,b)$ near $a$. Similarly one defines the l.c. and l.p. cases for the right endpoint $b$. It is obvious that $\tau$ is only either in the l.c.
or in the l.p. case at each endpoint. The next lemma shows that \( \tau \) indeed is in one of these cases at each endpoint.

**Lemma 4.1.** If there is a \( z_0 \in \mathbb{C}^\times \) such that all solutions of \( (\tau - z_0)u = 0 \) lie in \( H^1(a,b) \) near \( a \), then for each \( z \in \mathbb{C}^\times \) all solutions of \( (\tau - z)u = 0 \) lie in \( H^1(a,b) \) near \( a \). A similar result holds at the endpoint \( b \).

This lemma has been proven by Bennewitz and Brown in [6, Lemma 4]. In contrast to the right-definite theory, Theorem 3.5 shows that there are precise criteria for the l.c. case to prevail. In particular, if \( \tau \) is in the l.c. case at some endpoint, then all solutions of \( \tau u = 0 \) lie in \( H^1(a,b) \) near this endpoint. However, note that it might happen that \( \tau \) is in the l.p. case at an endpoint although all solutions of \( \tau u = 0 \) lie in \( H^1(a,b) \) near this endpoint.

The next aim is to determine the deficiency index of \( T_{\min} \), depending on whether or not the endpoints are in the l.c. or l.p. case. We denote the set of all points of regular type of \( T \) or not the endpoints are in the l.c. or l.p. case. We denote the set of all points of regular type of \( T_{\min} \) with \( r(T_{\min}) \).

**Lemma 4.2.** Given \( z \in r(T_{\min}) \) there is a non-trivial solution of \( (\tau - z)u = 0 \) which lies in \( H^1(a,b) \) near \( a \). A similar result holds for the endpoint \( b \).

Proof. We may assume that the measures \( \varrho, \varsigma \) and \( \chi \) are finite near the endpoint \( b \). If there were no solutions of \( (\tau - z)u = 0 \) which lie in \( H^1(a,b) \) near \( a \), we would have \( \ker(T_{\max} - z) = \{0\} \). Since the mapping

\[
 z \mapsto \dim \text{ran}(T_{\min} - z)^\perp
\]

is constant on each connected component of \( r(T_{\min}) \) (see [16, Theorem 2.4] or [15, Theorem 6.1]) and thus

\[
 0 = \dim \ker(T_{\max} - z^*) = \dim \text{ran}(T_{\min} - z)^\perp = n(T_{\min}), \quad z \in r(T_{\min}),
\]

we would infer that \( T_{\min} = T_{\max} \). Let \( g \in H^1(a,b) \) such that \( g \) vanishes near \( a \) and is equal to \( w_a \) near \( b \). Now the solution \( h \) of \( \tau f = g \) which is equal to \( w_a \) near \( a \) lies in \( H^1(a,b) \) since \( h \) and its quasi-derivative are bounded near the endpoint \( b \). From this we get

\[
 V(h, w_a)(b) = g(b)w_a^1(b) = w_a(b)w_a^1(b) > 0,
\]

contradicting Theorem 3.8. \[\square\]

**Corollary 4.3.** If \( z \in r(T_{\min})^\perp \) and \( \tau \) is in the l.p. case at \( a \), then there is a (up to scalar multiples) unique non-trivial solution of \( (\tau - z)u = 0 \), which lies in \( H^1(a,b) \) near \( a \). A similar result holds for the endpoint \( b \).

Proof. If there were two linearly independent solutions which lie in \( H^1(a,b) \) near \( a \), \( \tau \) would be in the l.c. case at \( a \). \[\square\]

The following result shows that one might tell from the modified Wronskian whether \( \tau \) is in the l.c. or in the l.p. case.

**Proposition 4.4.** \( \tau \) is in the l.p. case at \( a \) if and only if

\[
 (4.1) \quad V(f, g)(a) = 0, \quad f, g \in T_{\max}
\]

\( \tau \) is in the l.c. case at \( a \) if and only if there is a \( v \in T_{\max} \) such that

\[
 (4.2) \quad V(v, v^*)(a) = 0 \quad \text{and} \quad V(f, v^*)(a) \neq 0 \quad \text{for some} \quad f \in T_{\max}.
\]

Similar results hold at the endpoint \( b \).
Proof. Assume that \( \tau \) is in the l.p. case at \( a \) and that the measures \( \varrho \), \( \varsigma \) and \( \chi \) are finite near \( b \). Then \( T_{\text{max}} \) is a two-dimensional extension of \( T_{\text{min}} \), since \( \dim \ker(T_{\text{max}} - i) = 1 \) by Corollary 4.3. Now let \( h \in T_{\text{max}} \) be defined as in the proof of Lemma 4.2. Then \( w_a \) and \( h \) are linearly independent modulo \( T_{\text{min}} \). Indeed, if \( c_1 w_a + c_2 h \in T_{\text{min}} \) for some \( c_1, c_2 \in \mathbb{C} \) then

\[
c_2 w_a(b)g[1](b) - c_2 h[1](b)g_r(b) - c_1 w_a[1](b)g_r(b) = 0, \quad g \in T_{\text{max}}.
\]

Now choosing \( g = w_a, g_r = 0 \) yields \( c_2 = 0 \) and from \( g = f, g_r = f_r \) we obtain \( c_1 = 0 \). Hence

\[
T_{\text{max}} = T_{\text{min}} + \text{span}\{w_a, h\},
\]

and a simple calculation shows that \( V(f, g)(a) = 0 \) for all \( f, g \in T_{\text{max}} \). If the measures \( \varrho, \varsigma \) and \( \chi \) are not finite near \( b \), then pick some \( c \in (a, b) \). For each \( f, g \in T_{\text{max}} \) the restrictions \( f|_{(a, c)} \) and \( g|_{(a, c)} \) lie in the maximal relation induced by our differential expression restricted to the interval \( (a, c) \). Since this restriction does not change the l.p. classification at \( a \) as well as the modified Wronskian near \( a \), the claim follows from the case which we just proved.

Now suppose \( \tau \) is in the l.c. case at \( a \) and in the l.p. case at \( b \). According to Corollary 4.3, \( T_{\text{max}} \) is a two-dimensional extension of \( T_{\text{min}} \). In particular, there is some \( v \in T_{\text{max}} \) which does not lie in \( T_{\text{min}} \). We may assume that \( V(v, v^*)(a) = 0 \), since otherwise replace \( v \) by \( v + v^* \) or \( v - v^* \). Now from Theorem 3.8 and the first part of the proof we infer that

\[
T_{\text{min}} = \{ f \in T_{\text{max}} | \forall g \in T_{\text{max}} : V(f, g)(a) = 0 \}.
\]

Since \( v \) does not lie in \( T_{\text{min}} \) there is some \( f \in T_{\text{max}} \) such that \( V(f, v^*)(a) \neq 0 \). Finally consider the case when both endpoints are in the l.c. case. Then we have \( u_1, u_2 \in T_{\text{max}} \), where \( u_1, u_2 \) are two linearly independent real solutions of \( \tau u = u \). Now if one chooses \( v = u_1 \), then the function \( f = u_2 \) has the claimed property. \( \square \)

The results obtained so far allow us to determine the deficiency index of the minimal relation.

Theorem 4.5. The deficiency index of \( T_{\text{min}} \) is given by

\[
n(T_{\text{min}}) = \begin{cases} 0, & \text{if } \tau \text{ is in the l.c. case at no endpoint,} \\ 1, & \text{if } \tau \text{ is in the l.c. case at precisely one endpoint,} \\ 2, & \text{if } \tau \text{ is in the l.c. case at both endpoints.} \end{cases}
\]

Proof. If \( \tau \) is in the l.c. case at both endpoints, all solutions of \( (\tau - i)u = 0 \) lie in \( H^1(a, b) \) and hence in \( T_{\text{max}} \). Therefore \( n(T_{\text{min}}) = \dim \ker(T_{\text{max}} - i) = 2 \). Provided \( \tau \) is in the l.c. case at exactly one endpoint, there is (up to scalar multiples) exactly one non-trivial solution of \( (\tau - i)u = 0 \) in \( H^1(a, b) \), by Corollary 4.3. Now suppose \( \tau \) is in the l.p. case at both endpoints and \( u \in H^1(a, b) \) is a solution of \( (\tau - i)u = 0 \). Then \( u^* \) is a solution of \( (\tau + i)u = 0 \) and both \( u \) and \( u^* \) lie in \( T_{\text{max}} \). Now the Lagrange identity yields

\[
V(u, u^*)(\beta) - V(u, u^*)(\alpha) = 2i \left( \int_{\alpha}^{\beta} |u|^2 d\chi + \int_{\alpha}^{\beta} |u[1]|^2 d\varsigma \right), \quad \alpha, \beta \in (a, b).
\]

As \( \alpha \to a \) and \( \beta \to b \), the left-hand side converges to zero by Proposition 4.4 and the right-hand side converges to \( 2i\|u\|^2 \). Hence \( u \) vanishes identically and \( n(T_{\text{min}}) = 0 \). \( \square \)
We will now characterize the self-adjoint restrictions of $T_{\text{max}}$. Therefore recall that these are precisely the $n(T_{\text{min}})$-dimensional symmetric extensions of $T_{\text{min}}$ (see e.g. [18, Corollary B.6]). Clearly, the simplest case is when both endpoints are in the l.p. case. The following result is an immediate consequence of Theorem 4.5.

**Theorem 4.6.** If $\tau$ is in the l.p. case at both endpoints then $T_{\text{min}} = T_{\text{max}}$ is a self-adjoint relation.

Next we will turn to the case when precisely one endpoint is in the l.p. case. But before, we need some more properties of the modified Wronskian.

**Lemma 4.7.** Suppose $\tau$ is in the l.c. case at $a$ and let $v \in T_{\text{max}}$ such that (4.2) holds. Then for each $f, g \in T_{\text{max}}$ we have

$$V(f, v^*)(a) = 0 \iff V(f^*, v)(a) = 0$$

and

$$V(f, v^*)(a) = V(g, v^*)(a) = 0 \Rightarrow V(f, g)(a) = 0.$$  

Similar results hold at the endpoint $b$.

**Proof.** Pick some $h \in T_{\text{max}}$ such that $V(h, v^*)(a) \neq 0$. Choosing $f_1 = v, f_2 = v^*, f_3 = h$ and $f_4 = h^*$ in the Plücker identity in Proposition 2.4, we see that also $V(h, v)(a) \neq 0$. Now choosing $f_1 = f, f_2 = v, f_3 = v^*$ and $f_4 = h$ in the Plücker identity yields (4.3), whereas the choice $f_1 = f, f_2 = g, f_3 = v^*$ and $f_4 = h$ yields (4.4). □

**Theorem 4.8.** Suppose $\tau$ is in the l.c. case at $a$ and in the l.p. case at $b$. Then some relation $S$ is a self-adjoint restriction of $T_{\text{max}}$ if and only if there is a $v \in T_{\text{max}}$ with (4.2) such that

$$S = \{ f \in T_{\text{max}} \mid V(f, v^*)(a) = 0 \}.$$  

A similar result holds if $\tau$ is in the l.c. case at $b$ and in the l.p. case at $a$.

**Proof.** Since $n(T_{\text{min}}) = 1$, the self-adjoint restrictions of $T_{\text{max}}$ are precisely the one-dimensional, symmetric extensions of $T_{\text{min}}$. Hence some relation $S$ is a self-adjoint restriction of $T_{\text{max}}$ if and only if there is a $v \in T_{\text{max}} \setminus T_{\text{min}}$ with $V(v, v^*)(a) = 0$ such that

$$S = T_{\text{min}} + \text{span}\{v\}.$$  

Thus we have to prove that

$$T_{\text{min}} + \text{span}\{v\} = \{ f \in T_{\text{max}} \mid V(f, v^*)(a) = 0 \}.$$  

The subspace on the left-hand side is included in the right one because of Theorem 3.8 and $V(v, v^*)(a) = 0$. If the subspace on the right-hand side were larger, it were equal to $T_{\text{max}}$ and hence would imply $v \in T_{\text{min}}$ in view of Theorem 3.8 □

It remains to consider the case when both endpoints are in the l.c. case. To this end recall that we introduced the convenient short hand notation

$$V_a^b(f, g^*) = V(f, g^*)(b) - V(f, g^*)(a), \quad f, g \in T_{\text{max}}.$$
Theorem 4.9. Suppose $\tau$ is in the l.c. case at both endpoints. Then some relation $S$ is a self-adjoint restriction of $T_{\max}$ if and only if there are some $v, w \in T_{\max}$, linearly independent modulo $T_{\min}$, with

\begin{equation}
V_a^b(v, v^*) = V_a^b(w, w^*) = V_a^b(v, w^*) = 0, \tag{4.6}
\end{equation}

such that

\begin{equation}
S = \{ f \in T_{\max} | V_a^b(f, v^*) = V_a^b(f, w^*) = 0 \}. \tag{4.7}
\end{equation}

Proof. Because of $n(T_{\min}) = 2$, the self-adjoint restrictions of $T_{\max}$ are precisely the two-dimensional, symmetric extensions of $T_{\min}$. Hence a relation $S$ is a self-adjoint restriction of $T_{\max}$ if and only if there are $v, w \in T_{\max}$, linearly independent modulo $T_{\min}$, with [4.6] such that

\[ S = T_{\min} + \text{span}\{v, w\}. \]

Therefore, we have to prove that

\[ T_{\min} + \text{span}\{v, w\} = \{ f \in T_{\max} | V_a^b(f, v^*) = V_a^b(f, w^*) = 0 \} = T, \]

where we denote the subspace on the right-hand side by $T$. In fact, the subspace on the left-hand side is contained in $T$ by Theorem 3.8 and (4.6). In order to prove that it is not larger indeed, consider the linear functionals $F_v, F_w$ on $T_{\max}$ defined by

\[ F_v(f) = V_a^b(f, v^*) \quad \text{and} \quad F_w(f) = V_a^b(f, w^*) \quad \text{for} \quad f \in T_{\max}. \]

The intersection of the kernels of these functionals is precisely $T$. Furthermore, these functionals are linearly independent. In fact, if $c_1, c_2 \in \mathbb{C}$ such that $c_1 F_v + c_2 F_w = 0$, then for all $f \in T_{\max}$ we had

\[ 0 = c_1 F_v(f) + c_2 F_w(f) = c_1 V_a^b(f, v^*) + c_2 V_a^b(f, w^*) = V_a^b(f, c_1 v^* + c_2 w^*). \]

But this shows that $c_1 v^* + c_2 w^* \in T_{\min} = T^*_{\max}$ in view of Lemma 3.4 and since $v, w$ are linearly independent modulo $T_{\min}$ we infer $c_1 = c_2 = 0$. Now the linear independence of our functionals guarantees that

\[ \text{ker} F_v \not\subseteq \text{ker} F_w \quad \text{and} \quad \text{ker} F_w \not\subseteq \text{ker} F_v. \]

Hence there are some $f_v, f_w \in T_{\max}$ such that $V_a^b(f_v, v^*) = V_a^b(f_w, w^*) = 0$ but $V_a^b(f_v, w^*) \neq 0$ and $V_a^b(f_w, v^*) \neq 0$. Since both functions, $f_v$ and $f_w$, do not lie in $T$ and are linearly independent, $T$ is at most a two-dimensional extension of $T_{\min}$. \( \square \)

In the rest of this paper we will only be interested in self-adjoint restrictions of $T_{\max}$ with separated boundary conditions. These are precisely the relations of the form

\begin{equation}
S = \{ f \in T_{\max} | V(f, v^*)(a) = V(f, w^*)(b) = 0 \}, \tag{4.8}
\end{equation}

where $v, w \in T_{\max}$ such that

\begin{equation}
V(v, v^*)(a) = 0 \quad \text{and} \quad V(f, v^*)(a) \neq 0 \quad \text{for some} \quad f \in T_{\max}, \tag{4.9a}
\end{equation}

if $\tau$ is in the l.c. case at $a$ and

\begin{equation}
V(w, w^*)(b) = 0 \quad \text{and} \quad V(f, w^*)(b) \neq 0 \quad \text{for some} \quad f \in T_{\max}, \tag{4.9b}
\end{equation}

if $\tau$ is in the l.c. case at $b$. Hence some $f \in T_{\max}$ lies in $S$ if and only if it satisfies the boundary condition $V(f, v^*)(a) = 0$ at $a$ and the boundary condition $V(f, w^*)(b) = 0$ at $b$. Note that boundary conditions at l.p. endpoints are actually superfluous, since each $f \in T_{\max}$ satisfies the boundary condition there in view
of Proposition 4.4 Finally, let us mention that each relation of the form (4.8) is self-adjoint indeed. If \( \tau \) is not at both endpoints in the l.c. case, this is evident from Theorem 4.6 and Theorem 4.8. Actually, in these cases all self-adjoint restrictions of \( T_{\text{max}} \) have separated boundary conditions. For the case when \( \tau \) is in the l.c. case at both endpoints, note that \( S \) is symmetric in view of Lemma 4.7 and Lemma 3.4 and that \( S \) is at most a two-dimensional restriction of \( T_{\text{max}} \).

When \( \tau \) is regular at an endpoint, say \( a \), then the boundary condition at this endpoint may be given in a simpler form. In fact, if \( v \in T_{\text{max}} \) such that (4.9a) holds then it may be shown that there is some \( \alpha \in [0, \pi) \) such that for each \( f \in T_{\text{max}} \)

\[
(4.10) \quad V(f, u^*)(a) = 0 \iff f_\alpha(a) \cos \alpha - f^{[1]}(a) \sin \alpha = 0.
\]

Conversely, if some \( \phi_a \in [0, \pi) \) is given, then there is a \( v \in T_{\text{max}} \) with (4.9a) such that (4.10) holds for each \( f \in T_{\text{max}} \). In this case, the boundary conditions corresponding to \( \phi_a = 0 \) are called Dirichlet boundary conditions, whereas the ones corresponding to \( \phi_a = \pi/2 \) are called Neumann boundary conditions. Moreover, note that if \( z \in \mathbb{C} \) and \( u \) is a solution of \( (\tau - z)u = 0 \), the boundary condition at the regular endpoint \( a \) takes the form

\[
zu(a) \cos \phi_a - u^{[1]}(a) \sin \phi_a = 0.
\]

**Proposition 4.10.** If \( S \) is a self-adjoint restriction of \( T_{\text{max}} \) with separated boundary conditions, then all non-zero eigenvalues of \( S \) are simple.

**Proof.** Suppose \( \lambda \in \mathbb{R} \) is an eigenvalue and \( u_j, j = 1, 2 \) are solutions of \( (\tau - \lambda)u = 0 \). If \( \tau \) is in the l.p. case at \( a \), then clearly \( V(u_1, u_2) \) vanishes. Otherwise, the same is true because of (4.4) in Lemma 4.7 and hence \( u_1, u_2 \) are linearly dependent. \( \square \)

It might happen that zero is a double eigenvalue indeed. This is due to the fact that there are cases where each solution of \( \tau u = 0 \) lies in \( H^1(a, b) \) near both endpoints and furthermore, satisfies the boundary condition at each endpoint which is in the l.c. case. For example, this happens for Dirichlet boundary conditions at regular endpoints or if \( \xi \) and \( \chi \) are finite near an endpoint which is in the l.p. case.

Next we will determine the resolvent of self-adjoint restrictions of \( T_{\text{max}} \) with separated boundary conditions.

**Theorem 4.11.** Suppose \( S \) is a self-adjoint restriction of \( T_{\text{max}} \) with separated boundary conditions and let \( z \in \rho(S)^\times \). Furthermore, let \( u_a \) and \( u_b \) be non-trivial solutions of \( (\tau - z)u = 0 \) such that

\[
u_a \begin{cases} 
\text{satisfies the boundary condition at } a \text{ if } \tau \text{ is in the l.c. case at } a, \\
\text{lies in } H^1(a, b) \text{ near } a \text{ if } \tau \text{ is in the l.p. case at } a,
\end{cases}
\]

and

\[
u_b \begin{cases} 
\text{satisfies the boundary condition at } b \text{ if } \tau \text{ is in the l.c. case at } b, \\
\text{lies in } H^1(a, b) \text{ near } b \text{ if } \tau \text{ is in the l.p. case at } b.
\end{cases}
\]

Then the resolvent \( R_z \) is given by

\[
(4.11) \quad R_z g(x) = \langle g, G_z(x, \cdot) \rangle, \quad x \in (a, b), \quad g \in H^1(a, b),
\]

where

\[
(4.12) \quad G_z(x, y) + \frac{\delta_z(y)}{z} = \frac{1}{V(u_b, u_a)} \begin{cases} 
u_a(y)u_b(x), & \text{if } y \leq x, \\
u_a(x)u_b(y), & \text{if } y > x.
\end{cases}
\]
Proof. First of all note, that the solutions \( u_a, u_b \) are linearly independent, since otherwise \( z \) were an eigenvalue of \( S \). Now if \( g \in H^1(a, b) \), then the function \( f_g \) given by

\[
f_g(x) = \frac{z}{V(u_b, u_a)} \left( u_b(x) \int_a^x u_a g \, d\rho + u_a(x) \int_x^b u_b g \, d\rho \right), \quad x \in (a, b)
\]
is a solution of \((\tau - z)f = g\) by [13, Proposition 3.3]. Moreover, \( f_g \) is a scalar multiple of \( u_a \) near \( a \) and a scalar multiple of \( u_b \) near \( b \). As a consequence \( f_g \in T_{\text{max}} \) satisfies the boundary conditions of \( S \) and therefore \( R_z g = f_g \). Now an integration by parts as in [24] shows that \( R_z g \) is given by (4.11). Furthermore, by continuity this holds for all \( g \in H^1(a, b) \). Hence it remains to consider \( R_z w \) when \( w \in \ker(T_{\text{max}}) \). In this case integration by parts yields

\[
\langle w, G_z(x, \cdot)^* \rangle = \frac{1}{z} \frac{V(u_b, w)(b) u_a(x)}{V(u_b, u_a)(b)} + \frac{1}{z} \frac{V(w, u_a)(a) u_b(x)}{V(u_b, u_a)(a)} - \frac{w(x)}{z}
\]

for each \( x \in (a, b) \). Obviously, this function is a solution of \((\tau - z)f = w\), since \( w \) is a solution of \( \tau u = 0 \). Moreover, if \( \tau \) is in the l.p. case at \( a \), then the second term vanishes in view of Proposition [4.4]. For the same reason the first term vanishes if \( \tau \) is in the l.p. case at \( b \) and hence this function even lies in \( H^1(a, b) \). Finally, using the Plücker identity it is easily verified that this function also satisfies the boundary conditions at each endpoint which is in the l.c. case.

If \( z \in \rho(S)^c \) and \( \tau \) is in the l.p. case at some endpoint, then Corollary 4.3 shows that there is always a \((u \text{ to scalar multiples})\) unique non-trivial solution of \((\tau - z)u = 0\), lying in \( H^1(a, b) \) near this endpoint. Moreover, if \( \tau \) is in the l.c. case at some endpoint, then there exists a \((u \text{ to scalar multiples})\) unique non-trivial solution of \((\tau - z)u = 0\), satisfying the boundary condition at this endpoint. Hence functions \( u_a \) and \( u_b \), as in Theorem 4.11 always exist.

5. Singular Weyl–Titchmarsh function

In the rest of this paper \( S \) will always denote some self-adjoint restriction of \( T_{\text{max}} \) with separated boundary conditions as in (4.8). Our aim in this section is to construct a singular Weyl–Titchmarsh function as it has been introduced recently in [21, 26] and [13] in the right-definite case. To this end we first need a non-trivial real analytic solution \( \phi_z, z \in \mathbb{C}^\times \) of \((\tau - \tau)u = 0\) such that \( \phi_z \) lies in \( S \) near \( a \), i.e. \( \phi_z \) lies in \( H^1(a, b) \) near \( a \) and satisfies the boundary condition at \( a \) if \( \tau \) is in the l.c. case there.

**Hypothesis 5.1.** For every \( z \in \mathbb{C}^\times \) there is a non-trivial solution \( \phi_z \) of \((\tau - \tau)u = 0\) such that \( \phi_z \) lies in \( S \) near \( a \) and the functions

\[
z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{[1]}(c)
\]

are real analytic in \( \mathbb{C}^\times \) with at most poles at zero for each \( c \in (a, b) \).

In order to introduce a singular Weyl–Titchmarsh function we furthermore need a second real analytic solution \( \theta_z, z \in \mathbb{C}^\times \) of \((\tau - \tau)u = 0\) such that \( V(\theta_z, \phi_z) = 1 \).

**Lemma 5.2.** If Hypothesis 5.1 holds, then for each \( z \in \mathbb{C}^\times \) there is a solution \( \theta_z \) of \((\tau - \tau)u = 0\) such that \( V(\theta_z, \phi_z) = 1 \) and the functions

\[
z \mapsto \theta_z(c) \quad \text{and} \quad z \mapsto \theta_z^{[1]}(c)
\]
are real analytic in $\mathbb{C}^\times$ with at most poles at zero for each $c \in (a, b)$.

Proof. Following literally the proof of [26, Lemma 2.4] there is a real analytic solution $u_z, z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that the usual Wronskian satisfies
\[
W(u_z, \phi_z) = u_z(x)\phi_z^{[1]}(x) - u_z^{[1]}(x)\phi_z(x) = 1, \quad x \in (a, b), \quad z \in \mathbb{C}^\times.
\]

Now the solutions $\theta_z = z^{-1}u_z, z \in \mathbb{C}^\times$ have the claimed properties.

Given a real analytic fundamental system $\theta_z, \phi_z, z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ as in Hypothesis [5.1] and Lemma [5.2] we may define a complex valued function $M$ on $\rho(S)^\times$ by requiring that the solutions
\[
(5.3) \quad \psi_z = \theta_z + M(z)\phi_z, \quad z \in \rho(S)^\times
\]
lie in $S$ near $b$, i.e. they lie in $H^1(a, b)$ near $b$ and satisfy the boundary condition at $b$ if $\tau$ is in the l.c. case there. By Corollary [13] and the fact that there is up to scalar multiples precisely one solution of $(\tau - z)u = 0$ satisfying the boundary condition at $b$ if $\tau$ is in the l.c. case there, $M$ is well-defined. This function $M$ is referred to as the singular Weyl–Titchmarsh function of $S$, associated with the fundamental system $\theta_z, \phi_z, z \in \mathbb{C}^\times$. The solutions $\psi_z, z \in \rho(S)^\times$ are called the Weyl solutions.

**Theorem 5.3.** The singular Weyl–Titchmarsh function $M$ is analytic with
\[
(5.4) \quad M(z) = M(z^*)^*, \quad z \in \rho(S)^\times.
\]

Proof. From Theorem [4.11] we have for each $c \in (a, b)$
\[
R_z\delta_c(x) = \langle \delta_c, G_z(x, \cdot)^* \rangle = G_z(x, c), \quad x \in (a, b), \quad z \in \rho(S)^\times
\]
and hence
\[
\langle R_z\delta_c, \delta_c \rangle = G_z(c, c), \quad z \in \rho(S)^\times.
\]
Moreover, because of $V(\psi_z, \phi_z) = 1, z \in \rho(S)^\times$ the diagonal of the kernel $G_z$ is given by
\[
(5.5) \quad \langle R_z\delta_c, \delta_c \rangle = G_z(c, c) = \psi_z(c)\phi_z(c) - \frac{\delta_z(c)}{z} = M(z)\phi_z(c)^2 + \theta_z(c)\phi_z(c) - \frac{w_a(c)w_b(c)}{W(w_b, w_a)}, \quad z \in \rho(S)^\times.
\]
But this shows that $M$ is analytic since for each $z \in \mathbb{C}^\times$ there is some $c \in (a, b)$ such that $\phi_z(c) \neq 0$. Moreover, equation [5.4] is valid because the solutions
\[
\theta_z + M(z^*)^*\phi_z = (\theta_z + M(z)\phi_z)^*, \quad z \in \rho(S)^\times
\]
lie in $S$ near $b$ in view of Lemma [4.7].

**Remark 5.4.** Note that solutions as in Hypothesis [5.1] and Lemma [5.2] are not unique. In fact, any other such solutions are given by
\[
\tilde{\theta}_z = e^{-g(z)}z^{-k}\theta_z - f(z)\phi_z \quad \text{and} \quad \tilde{\phi}_z = e^{g(z)}z^k\phi_z, \quad z \in \mathbb{C}^\times
\]
for some $k \in \mathbb{Z}$, some real entire function $e^g$ without zeros and some function $f$ which is real analytic in $\mathbb{C}^\times$ and has at most a pole at zero. The corresponding singular Weyl–Titchmarsh functions are related via
\[
\tilde{M}(z) = e^{-2g(z)}z^{-2k}M(z) + e^{-g(z)}z^{-k}f(z), \quad z \in \rho(S)^\times.
\]
In particular, the maximal domain of holomorphy in $\mathbb{C}^\times$ or the structure of poles and singularities in $\mathbb{C}^\times$ do not change.
Next we will construct a real analytic fundamental system \( \theta_z, \phi_z, z \in \mathbb{C}^\times \) of \((\tau - z)u = 0 \) as in Hypothesis 5.1 and Lemma 5.2 provided that \( \tau \) is in the l.c. case at the endpoint \( a \).

**Theorem 5.5.** Suppose \( \tau \) is in the l.c. case at \( a \). Then there is a real analytic fundamental system \( \theta_z, \phi_z, z \in \mathbb{C}^\times \) of \((\tau - z)u = 0 \) as in Hypothesis 5.1 and Lemma 5.2 such that for all \( z_1, z_2 \in \mathbb{C}^\times \) we have additionally

\[
V(\theta_{z_1}, \phi_{z_2})(a) = 1 \quad \text{and} \quad V(\theta_{z_1}, \theta_{z_2})(a) = V(\phi_{z_1}, \phi_{z_2})(a) = 0.
\]

**Proof.** Let \( \theta, \phi \) be a real fundamental system of \( \tau u = u \) with \( V(\theta, \phi) = 1 \) such that \( \phi \) lies in \( S \) near \( a \). Fix some \( c \in (a, b) \) and let for each \( z \in \mathbb{C}^\times \), \( u_{z,1}, u_{z,2} \) be the real fundamental system of \((\tau - z)u = 0 \) with initial conditions

\[
z u_{z,1}(c) = u_{z,2}^{[1]}(c) = 1 \quad \text{and} \quad u_{z,1}^{[1]}(c) = z u_{z,2}(c) = 0.
\]

We introduce the solutions

\[
\phi_z(x) = V(u_{z,1}, \phi)(a) u_{z,2}(x) - V(u_{z,2}, \phi)(a) u_{z,1}(x), \quad x \in (a, b), \; z \in \mathbb{C}^\times
\]

and the second solutions

\[
\theta_z(x) = V(u_{z,1}, \theta)(a) u_{z,2}(x) - V(u_{z,2}, \theta)(a) u_{z,1}(x), \quad x \in (a, b), \; z \in \mathbb{C}^\times.
\]

From this definition it is obvious that \( V(\phi_z, \phi)(a) = V(\theta_z, \theta)(a) = 0 \) for each \( z \in \mathbb{C}^\times \) and hence also \( V(\phi_{z_1}, \phi_{z_2})(a) = V(\theta_{z_1}, \theta_{z_2})(a) = 0 \) for all \( z_1, z_2 \in \mathbb{C}^\times \) in view of Lemma 4.7. Moreover, using the Plücker identity one obtains \( V(\theta_z, \phi_z)(a) = V(\theta, \phi_z)(a) = 1 \) for each \( z \in \mathbb{C}^\times \) and thus also the remaining equality. Furthermore, a direct calculation shows that \( \theta_z = \theta_z^* \) and \( \phi_z = \phi_z^* \) for \( z \in \mathbb{C}^\times \). Hence it remains to prove that the functions \( V(u_{z,1}, \theta)(a), V(u_{z,2}, \theta)(a), V(u_{z,1}, \phi)(a) \) and \( V(u_{z,2}, \phi)(a) \) are analytic in \( z \in \mathbb{C}^\times \). Indeed, from the Lagrange identity we obtain for each \( z \in \mathbb{C}^\times \)

\[
V(u_{z,1}, \theta)(a) = V(u_{z,1}, \theta)(c) - (z - 1) \lim_{\alpha \to a} \int_{\alpha}^{c} \theta u_{z,1} \, d\chi + \int_{\alpha}^{c} \theta^{[1]} u_{z,1}^{[1]} \, d\kappa.
\]

The integrals on the right-hand side are analytic in \( \mathbb{C}^\times \) since the integrands are bounded locally uniformly in \( \mathbb{C}^\times \) (see [13, Theorem 3.6]). Hence in order to prove that the limit is also analytic we need to show that the integrals are bounded as \( \alpha \to a \), locally uniformly in \( z \in \mathbb{C}^\times \). But this holds since

\[
\int_{a}^{c} |u_{z,1}(x)|^2 \, d\chi(x) + \int_{a}^{c} |u_{z,1}^{[1]}(x)|^2 \, d\kappa(x), \quad z \in \mathbb{C}^\times
\]

is locally uniformly bounded as an inspection of the proof of [6, Lemma 4] shows. Analyticity of the remaining functions may be proved similarly.

If \( \tau \) is regular at \( a \) and the boundary condition there is given by

\[
f \varphi(a) \cos \varphi_{\alpha} - f^{[1]}(a) \sin \varphi_{\alpha} = 0, \quad f \in S
\]

for some \( \varphi_{\alpha} \in [0, \pi) \), then a real analytic fundamental system \( \theta_z, \phi_z, z \in \mathbb{C}^\times \) of \((\tau - z)u = 0 \) is given for example by the initial conditions

\[
z \phi_z(a) = -\theta_z^{[1]}(a) = \sin \varphi_{\alpha} \quad \text{and} \quad \phi_z^{[1]}(a) = z \theta_z(a) = \cos \varphi_{\alpha}, \quad z \in \mathbb{C}^\times.
\]

It is easily verified that this fundamental system satisfies the properties claimed in Theorem 5.5.
Corollary 5.6. If $\tau$ is in the l.c. case at $a$ and $\theta_z$, $\phi_z$, $z \in \mathbb{C^x}$ is a real analytic fundamental system of $(\tau - z)u = 0$ as in Theorem 5.5 then the singular Weyl–Titchmarsh function $M$ is a Herglotz–Nevanlinna function.

Proof. In order to prove the claim we will show that

\begin{equation}
0 < \langle \psi_z, \psi_z \rangle^2 = \frac{\text{Im}(M(z))}{\text{Im}(z)}, \quad z \in \mathbb{C \setminus \mathbb{R}}.
\end{equation}

Indeed, if $z_1$, $z_2 \in \mathbb{C \setminus \mathbb{R}}$, then

\begin{align*}
V(\psi_{z_1}, \psi_{z_2})(a) &= V(\theta_{z_1}, \theta_{z_2})(a) + M(z_2)V(\theta_{z_1}, \phi_{z_2})(a) \\
&\quad + M(z_1)V(\phi_{z_1}, \theta_{z_2})(a) + M(z_1)M(z_2)V(\phi_{z_1}, \phi_{z_2})(a) \\
&= M(z_2) - M(z_1).
\end{align*}

If $\tau$ is in the l.p. case at $b$, then furthermore we have $V(\psi_{z_1}, \psi_{z_2})(b) = 0$, since clearly $\psi_{z_1}$, $\psi_{z_2} \in T_{\max}$. But this also holds if $\tau$ is in the l.c. case at $b$, since then $\psi_{z_1}$ and $\psi_{z_2}$ satisfy the same boundary condition at $b$. Now using the Lagrange identity from Proposition 2.3 yields

\begin{equation}
(z_1 - z_2)^2 \langle \psi_{z_1}, \psi_{z_2} \rangle = V(\psi_{z_1}, \psi_{z_2}^*)(b) - V(\psi_{z_1}, \psi_{z_2}^*)(a) = M(z_1) - M(z_2^*).
\end{equation}

In particular for $z \in \mathbb{C \setminus \mathbb{R}}$, using $M(z^*) = M(z)^*$ as well as

$\psi_{z^*} = \theta_{z^*} + M(z^*)\phi_{z^*} = \psi_{z^*}$,

we get

\begin{equation}
0 < \langle \psi_z, \psi_z \rangle^2 = \langle \psi_z, \psi_z \rangle = \frac{M(z) - M(z^*)}{z - z^*} = \frac{\text{Im}(M(z))}{\text{Im}(z)}.
\end{equation}

The relation between two fundamental systems in Remark 5.4 can be strengthened if they satisfy the additional properties of Theorem 5.5. In fact, it is not hard to see that in this case $f$, $e^q$ are constant and $k = 0$. Moreover, from the construction in the proof of Theorem 5.5 it follows that for such fundamental systems the functions in (5.1) and (5.2) have at most simple poles at zero.

As in the right-definite case (see [21, Lemma 3.2], [26, Lemma 2.2], [18, Theorem 9.6]) we may give a necessary and sufficient condition for Hypothesis 5.1 to hold. Therefore fix some $c \in (a, b)$ such that $\chi((a, c)) \neq 0$ and consider the maximal relation in $H^1(a, c)$ associated with our differential expression restricted to $(a, c)$. With $S_c$ we denote the self-adjoint restriction of this relation with Dirichlet boundary conditions at $c$ and the same boundary conditions as $S$ near $a$.

**Theorem 5.7.** Hypothesis 5.1 holds if and only if the self-adjoint relation $S_c$ has purely discrete spectrum.

Proof. Consider the real analytic fundamental system $\theta_{c,z}$, $\phi_{c,z}$, $z \in \mathbb{C^x}$ of $(\tau - z)u = 0$ with initial conditions

$z\phi_{c,z}(c) = \theta_{c,z}[1](c) = 0$ and $\phi_{c,z}[1](c) = z\theta_{c,z}(c) = 1$, $\quad z \in \mathbb{C^x}$.

Now define the function $M_c$ on $\mathbb{C \setminus \mathbb{R}}$ by requiring that the solutions

$\psi_{c,z} = \theta_{c,z} - M_c(z)\phi_{c,z}$, $\quad z \in \mathbb{C \setminus \mathbb{R}}$

lie in $S_c$ near $a$. Of course $M_c$ is the restriction of a Weyl–Titchmarsh function of $S_c$ with the roles of the left and the right endpoint reversed. Now if we suppose that $S_c$
has purely discrete spectrum then the function $M_c$ turns out to be a meromorphic Herglotz–Nevanlinna function. By the Weierstraß product theorem there is a real entire function $G$ which has simple zeros at precisely all poles of $M_c$ which are not removable. Thus the real analytic solution $\phi_z, z \in \mathbb{C}$, given by

$$\phi_z(x) = G(z)\psi_{c,z}(x) - G(z)M_c(z)\theta_{c,z}(x), \quad x \in (a,b),$$

has the properties claimed in Hypothesis 5.1. Conversely, if Hypothesis 5.1 holds then $\psi_{c,z}$ and $\phi_z$ are linearly dependent for each $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore

$$0 = V(\psi_{c,z}, \phi_z) = V(\theta_{c,z}, \phi_z) - M_c(z) V(\psi_{c,z}, \phi_z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and hence $M_c$ is actually meromorphic in $\mathbb{C}$. From (5.5) we infer that for each $f$ in a dense subspace of $H^1(a,c)$ the functions

$$\langle (S_c - z)^{-1} f, f \rangle_{H^1(a,c)}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

have an analytic continuation to $\mathbb{C} \setminus \Omega$, where $\Omega$ is the set of poles of $M_c$. But from this we infer that the operator-valued spectral measure $E_c$ of $S_c$ (see [18, Lemma B.3 and Lemma B.4]) satisfies $E_c(\mathbb{R} \setminus \Omega) = 0$, i.e. the spectrum of $S_c$ is purely discrete. □

If Hypothesis 5.1 holds at both endpoints then it is easily seen that all associated singular Weyl–Titchmarsh functions are meromorphic in $\mathbb{C}$. Now a similar argument as in the proof of Theorem 5.7 shows that in this case the spectrum of $S$ is purely discrete. In particular, $S$ has purely discrete spectrum provided that $\tau$ is in the l.c. case at both endpoints.

6. Spectral transformation

In this section let $S$ again be a self-adjoint restriction of $T_{\text{max}}$ with separated boundary conditions such that Hypothesis 5.1 holds. Moreover, we will assume that zero is not an eigenvalue of $S$. For example this excludes the case where $\xi$ and $\chi$ are finite near an endpoint which is in the l.p. case. Another case excluded hereby are Dirichlet boundary conditions at regular endpoints.

For all functions $f, g \in H^1(a,b)$ there is a unique complex Borel measure $E_{f,g}$ on $\mathbb{R}$ such that

$$\langle R_z f, g \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S).$$

(6.1)

In fact, these measures are obtained by applying a variant of the spectral theorem to the operator part

$$S_{\mathbb{D}} = S \cap (\mathbb{D} \times \mathbb{D}), \quad \mathbb{D} = \text{dom}(S) = \text{mul}(S)^{\perp},$$

of $S$ (see e.g. [18, Lemma B.4]). In this section we will introduce a spectral transformation for our self-adjoint linear relation $S$. Therefore we first need to construct our spectral measure.

**Lemma 6.1.** There is a unique Borel measure $\mu$ on $\mathbb{R}$ such that

$$E_{\delta_{\alpha}, \delta_{\beta}}(B) = \int_{\mathbb{R}} \phi_{\lambda}(\alpha)\phi_{\lambda}(\beta)d\mu(\lambda)$$

(6.2)

for all $\alpha, \beta \in (a,b)$ and each Borel set $B \subseteq \mathbb{R}$. In particular,

$$\langle R_z \delta_{\alpha}, \delta_{\beta} \rangle = \int_{\mathbb{R}} \frac{\phi_{\lambda}(\alpha)\phi_{\lambda}(\beta)}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S).$$

(6.3)
As in the proof of Theorem 5.3 one sees that for \( \alpha, \beta \in (a, b) \)

\[
\langle R_z, \delta_\alpha, \delta_\beta \rangle = M(z)\varphi_z(\alpha)\varphi_z(\beta) + H_{\alpha, \beta}(z), \quad z \in \rho(S)^\times,
\]

where \( H_{\alpha, \beta} \) is a real analytic function on \( \mathbb{C}^\times \). In particular, this shows that

\[
\langle R_z, \delta_\alpha, \delta_\beta \rangle = \langle R_z, \delta_\alpha, \delta_\beta \rangle^*, \quad z \in \rho(S)^\times
\]

and hence \( E_{\delta_\alpha, \delta_\beta} \) is a real-valued measure, in view of equation (6.1) and the Stieltjes inversion formula. Moreover, the Stieltjes inversion formula shows that for each continuous function \( F \) on \( \mathbb{R} \)

\[
(*) \quad \int_{\lambda_1}^{\lambda_2} F(\lambda) dE_{\delta_\alpha, \delta_\beta}(\lambda) = \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 - \delta}^{\lambda_2 - \delta} F(\lambda) \text{Im}(R_{\lambda + i\varepsilon} \delta_\alpha, \delta_\beta) d\lambda,
\]

where \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 < \lambda_2 \). Now if \( [\lambda_1, \lambda_2] \subseteq \mathbb{R}^\times \), then we define

\[
(6.5) \quad \mu([\lambda_1, \lambda_2]) = \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 - \delta}^{\lambda_2 - \delta} \text{Im}(M_{\lambda + i\varepsilon}) d\lambda.
\]

Given \( \lambda_0 \in \mathbb{R}^\times \), pick \( \alpha, \beta \in (a, b) \) such that \( \varphi_{\lambda_0}(\alpha) \neq \varphi_{\lambda_0}(\beta) \) and set

\[
F(\lambda) = \begin{cases} 
\varphi_{\lambda_0}^{-1}(\alpha)\varphi_{\lambda_0}(\beta)^{-1}, & \text{if } \lambda \in (-\infty, \lambda_-), \\
\varphi_{\lambda_0}^{-1}(\alpha)\varphi_{\lambda_0}(\beta)^{-1}, & \text{if } \lambda \in [\lambda_-, \lambda_+], \\
\varphi_{\lambda_0}^{-1}(\alpha)\varphi_{\lambda_0}(\beta)^{-1}, & \text{if } \lambda \in (\lambda_+, \infty),
\end{cases}
\]

where \( [\lambda_-, \lambda_+] \subseteq \mathbb{R}^\times \) is such that \( \lambda_0 \in (\lambda_-, \lambda_+) \) and \( \varphi_{\lambda_0}(\alpha) \neq \varphi_{\lambda_0}(\beta) \) for all \( \lambda \in [\lambda_-, \lambda_+] \). With this definition, from (6.4) and (*) we get

\[
(6.6) \quad \mu([\lambda_1, \lambda_2]) = \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 - \delta}^{\lambda_2 - \delta} \varphi_{\lambda_0}^{-1}(\alpha)\varphi_{\lambda_0}(\beta)^{-1} \text{Im}(R_{\lambda + i\varepsilon} \delta_\alpha, \delta_\beta) d\lambda
\]

for all \( \lambda_1, \lambda_2 \in (\lambda_-, \lambda_+) \) with \( \lambda_1 < \lambda_2 \). Hereby we used that

\[
\lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 - \delta}^{\lambda_2 - \delta} \varepsilon \text{Im}(\langle R_{\lambda + i\varepsilon} \delta_\alpha, \delta_\beta \rangle) d\lambda = 0,
\]

which holds since the function \( z \mapsto \langle R_z, \delta_\alpha, \delta_\beta \rangle \) actually is the difference of two Herglotz–Nevanlinna functions (see e.g. [20, Theorem 2.3]). In particular, equation (6.6) shows that the limit in (6.5) actually exists. Moreover, choosing \( \alpha = \beta \) it follows that the distribution function \( \lambda \mapsto \mu([\lambda, \lambda]) \) is non-decreasing on \( (\lambda_-, \lambda_+) \), hence \( \mu \) uniquely extends to a positive Borel measure on \( \mathbb{R}^\times \). Equation (6.6) furthermore shows that (6.2) holds for all Borel sets \( B \subseteq \mathbb{R}^\times \) not containing zeros of the analytic functions \( z \mapsto \varphi_z(\alpha) \) and \( z \mapsto \varphi_z(\beta) \). Moreover, if \( \lambda_0 \in \mathbb{R}^\times \) was such that \( \varphi_{\lambda_0}(\alpha)\varphi_{\lambda_0}(\beta) = 0 \) and \( E(\{\lambda_0\}) \neq 0 \), then we would obtain the contradiction

\[
E_{\delta_\alpha, \delta_\beta}(\{\lambda_0\}) = \langle E(\{\lambda_0\}), \delta_\alpha, \delta_\beta \rangle = \langle \delta_\alpha, \varphi_{\lambda_0}, \delta_\beta \rangle = \varphi_{\lambda_0}(\alpha)\varphi_{\lambda_0}(\beta) = 0,
\]

since \( \varphi_{\lambda_0} \) would be an eigenfunction in that case. Hence we see that (6.2) holds for all Borel sets \( B \subseteq \mathbb{R} \) (also note that \( E_{\delta_\alpha, \delta_\beta}(\{0\}) = 0 \) since zero is no eigenvalue of \( S \)).

We are now able to define our spectral transform. If \( F \) is a Borel measurable function on \( \mathbb{R}^\times \), then we denote with \( M_F \) the maximally defined operator of multiplication with \( F \) in \( L^2(\mathbb{R}^\times; \mu) \).
Lemma 6.2. There is a unique bounded linear operator
\[ \mathcal{F} : H^1(a, b) \to L^2(\mathbb{R}^\times; \mu) \]
such that for each \( c \in (a, b) \) we have \( \mathcal{F}\delta_c(\lambda) = \phi_\lambda(c) \) for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \). The operator \( \mathcal{F} \) is a surjective partial isometry with initial subspace \( \mathcal{D} \). Its adjoint is given by
\[ \mathcal{F}^* g(x) = \int_{\mathbb{R}^\times} \phi_\lambda(x) g(\lambda) d\mu(\lambda), \quad x \in (a, b), \ g \in L^2(\mathbb{R}^\times; \mu), \]
its (in general multi-valued) inverse is given by
\[ \mathcal{F}^{-1} = \{(g, f) \in L^2(\mathbb{R}^\times; \mu) \times H^1(a, b) | \mathcal{F}^* g - f \in \text{mul}(S)\}. \]

Proof. First of all note that the functions \( \delta_c, c \in (a, b) \) are linearly independent since otherwise there were distinct points\( c, c_1, \ldots, c_N \in (a, b) \) and \( a_n \in \mathbb{C}, \ n = 1, \ldots, N \) such that
\[ g(c) = \sum_{n=1}^{N} a_n g(c_n), \quad g \in H^1(a, b). \]
But this gives a contradiction since there are functions in \( H^1(a, b) \), vanishing in the points \( c_1, \ldots, c_N \) but not in \( c \). Hence there is a unique linear operator \( \mathcal{F} \) on the linear span of all functions \( \delta_c, c \in (a, b) \) such that for all \( c \in (a, b) \) one has \( \mathcal{F}\delta_c(\lambda) = \phi_\lambda(c) \) for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \). Thereby, note that the functions \( z \mapsto \phi_\lambda(c), c \in (a, b) \) are square integrable with respect to \( \mu \) in view of Lemma 6.1. Furthermore, from Lemma 6.1 one sees that for each \( c_1, c_2 \in (a, b) \)
\[ \langle P\delta_{c_1}, \delta_{c_2} \rangle = E_{\delta_{c_1}, \delta_{c_2}}(\mathbb{R}) = \int_{\mathbb{R}^\times} \phi_\lambda(c_1) \phi_\lambda(c_2) d\mu(\lambda) = \langle \mathcal{F}\delta_{c_1}, \mathcal{F}\delta_{c_2} \rangle \mu, \]
where \( P \) is the orthogonal projection from \( H^1(a, b) \) onto \( \mathcal{D} \). Moreover, from linearity we also have
\[ (*) \quad \langle \mathcal{F}f_1, \mathcal{F}f_2 \rangle _\mu = \langle Pf_1, f_2 \rangle, \quad f_1, f_2 \in \text{span}\{\delta_c | c \in (a, b)\}. \]
In particular, \( \mathcal{F} \) is bounded on the linear span of all functions \( \delta_c, c \in (a, b) \) and since this span is dense in \( H^1(a, b) \), \( \mathcal{F} \) uniquely extends to a bounded linear operator on \( H^1(a, b) \). Moreover, equation (\ref{equation6.7}) still holds for all functions \( f_1, f_2 \in H^1(a, b) \) and hence the operator \( \mathcal{F} \) is a partial isometry with initial subspace \( \mathcal{D} \). Also note that (6.2) from Lemma 6.1 now extends to all of \( H^1(a, b) \) by continuity, i.e.
\[ (6.7) \quad E_{f,g}(B) = \int_{B} \mathcal{F} f(\lambda) \mathcal{F} g(\lambda)^* d\mu(\lambda), \quad f, g \in H^1(a, b) \]
for each Borel set \( B \subseteq \mathbb{R} \). In order to prove that \( \mathcal{F} \) is onto, let \( c \in (a, b) \) and \( F, G \) be some bounded measurable functions on \( \mathbb{R}^\times \). Since \( E \) is the spectral measure of the operator part \( S_D \) of \( S \) (see e.g. [18] Lemma B.4) we get
\[ GF(S_D)P\delta_c(x) = \langle GF(S_D)P\delta_c, \delta_x \rangle = \int_{\mathbb{R}^\times} G(\lambda) F(\lambda) dE_{\delta_c, \delta_x}(\lambda) \]
\[ = \int_{\mathbb{R}^\times} G(\lambda) F(\lambda) \phi_\lambda(c) \phi_\lambda(x) d\mu(\lambda) \]
for each \( x \in (a, b) \). On the other side a similar calculation yields
\[
GF(S_D)P\delta_c(x) = G(S_D)F(S_D)P\delta_c(x)
\]
\[
= \int_{\mathbb{R}^\times} G(\lambda)\mathcal{F}F(S_D)P\delta_c(\lambda)\phi_\lambda(x)d\mu(\lambda).
\]
Since \( G \) was arbitrary, we infer that \( F(\lambda)\phi_\lambda(c)\phi_\lambda(x) = \mathcal{F}F(S_D)P\delta_c(\lambda)\phi_\lambda(x) \) for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \). Moreover, since for each given \( \lambda_0 \in \mathbb{R}^\times \) there is some point \( x \in (a, b) \) such that \( \phi_{\lambda_0}(x) \neq 0 \) we even get that \( F(\lambda)\phi_\lambda(c) = \mathcal{F}F(S_D)P\delta_c(\lambda) \) for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \). But this shows that the range of \( \mathcal{F} \) contains all characteristic functions of intervals in \( \mathbb{R}^\times \). More precisely, let \( \lambda_0 \in \mathbb{R}^\times \) and choose some \( c \in (a, b) \) such that \( \phi_{\lambda_0}(c) \neq 0 \). Then for each small enough interval \( J \subseteq \mathbb{R}^\times \) around \( \lambda_0 \) the function
\[
F(\lambda) = \begin{cases} \phi_\lambda(c)^{-1}, & \text{if } \lambda \in J, \\ 0, & \text{if } \lambda \in \mathbb{R}^\times \setminus J, \end{cases}
\]
is bounded and \( \mathcal{F}F(S_D)P\delta_c = M_F\mathcal{F}\delta_c = 1_J \) lies in the range of \( \mathcal{F} \), which proves surjectivity. Now a simple calculation shows that the adjoint of \( \mathcal{F} \) is given by
\[
\mathcal{F}^*g(x) = \langle \mathcal{F}^*g, \delta_x \rangle = \langle g, \mathcal{F}\delta_x \rangle_{\mu} = \int_{\mathbb{R}^\times} \phi_\lambda(x)g(\lambda)d\mu(\lambda), \quad x \in (a, b)
\]
for each \( g \in L^2(\mathbb{R}^\times; \mu) \). Furthermore, the inverse is given as in the claim because of the equivalence
\[
\mathcal{F}^*g - f \in \text{mul}(S) \iff \forall h \in L^2(\mathbb{R}^\times; \mu) : \langle f, \mathcal{F}^*h \rangle = \langle \mathcal{F}\mathcal{F}^*g, h \rangle_{\mu} \iff (f, g) \in \mathcal{F} = \mathcal{F}^{**},
\]
which holds for every \( f \in H^1(a, b) \) and \( g \in L^2(\mathbb{R}^\times; \mu) \). Here we used that \( \mathcal{F}\mathcal{F}^* \) is the identity since \( \mathcal{F} \) is a surjective partial isometry. \( \square \)

There is also another way of defining the linear operator \( \mathcal{F} \) as some integral transform, which may be more appealing. The approach taken here simplifies the proofs but it has the main disadvantage that the transforms are not given very explicitly. Hence we will show next that these two ways of introducing the spectral transform are actually the same. Indeed, provided that \( \tau \) is in the l.c. case at \( b \), we obviously have
\[
\mathcal{F}\delta_c(\lambda) = \phi_\lambda(c) = \langle \phi_\lambda, \delta_c \rangle, \quad \lambda \in \mathbb{R}^\times
\]
and hence by continuity also
\[
\mathcal{F}f(\lambda) = \langle \phi_\lambda, f^* \rangle = \int_a^b \phi_\lambda(x)f(x)d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x)f^{[1]}(x)d\varsigma(x), \quad \lambda \in \mathbb{R}^\times
\]
for each \( f \in H^1(a, b) \). Otherwise, if \( \tau \) is in the l.p. case at \( b \), then this is not possible since \( \phi_\lambda \) does not lie in \( H^1(a, b) \) unless \( \lambda \) is an eigenvalue. However, we still have the following general result.

**Proposition 6.3.** If \( f \in H^1(a, b) \) vanishes near \( b \), then
\[
\mathcal{F}f(\lambda) = \int_a^b \phi_\lambda(x)f(x)d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x)f^{[1]}(x)d\varsigma(x)
\]
for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \).
Proof. First of all note that for \( \lambda \in \mathbb{R}^\times \) and \( c \in (a, b) \), integration by parts shows that

\[
(*) \quad \int_a^b \phi_\lambda \delta_c \, d\lambda + \int_a^b \phi_\lambda^{[1]} \delta_c^{[1]} \, d\xi = \phi_\lambda(c) + \frac{w_a(c)w_b^{[1]}(x)}{W(w_b, w_a)} \phi_\lambda(x), \quad x \in (c, b).
\]

Now pick some \( \beta \in (a, b) \) such that \( f \) vanishes on \([\beta, b]\) and consider the space \( H_\beta \) of functions in \( H^1(a, b) \) which are equal to a scalar multiple of \( w_b \) on \([\beta, b]\). It is not hard to see that this space is closed and that it contains all functions \( \delta_c \), \( c \leq \beta \). Moreover, the linear span of these functions is even dense in \( H_\beta \), i.e. \( f \) lies in the closure of \( \text{span}\{\delta_c \mid c \leq \beta\} \). Now for each \( n \in \mathbb{N} \) let \( N(k) \in \mathbb{N} \) and \( a^k_n \in \mathbb{C} \), \( c^k_n \in (a, \beta) \) for \( n = 1, \ldots, N(k) \) such that the functions

\[
f_k(x) = \sum_{n=1}^{N(k)} a^k_n \delta_{c^k_n}(x), \quad x \in (a, b), \quad k \in \mathbb{N}
\]

converge to \( f \) in \( H^1(a, b) \) as \( k \to \infty \). Using equation \((*)\) we may estimate for each \( \lambda \in \mathbb{R}^\times \)

\[
\left| \int_a^b \phi_\lambda(x)f(x) \, d\lambda (x) + \int_a^b \phi_\lambda^{[1]}(x)f^{[1]}(x) \, d\xi (x) - \sum_{k=1}^{N(k)} a^k_n \phi_\lambda(c^k_n) \right| \\
\leq \left| \int_a^\beta \phi_\lambda(f - f_k) \, d\lambda + \int_a^\beta \phi_\lambda^{[1]}(f^{[1]} - f_k^{[1]}) \, d\xi \right| + \left| \phi_\lambda(\beta) \frac{w_k^{[1]}(\beta)}{w_b(\beta)} f_k(\beta) \right|.
\]

The first term converges to zero since \( f_k \) converges to \( f \) in \( H^1(a, b) \) as \( k \to \infty \). Moreover, the second term converges to zero since \( f_k(\beta) \) converges to zero as \( k \to \infty \). But this proves the claim since \( \mathcal{F}f_k(\lambda) \) converges to \( \mathcal{F}f(\lambda) \) for almost all \( \lambda \in \mathbb{R}^\times \) with respect to \( \mu \).

We are now ready to prove the main theorem of this section.

**Theorem 6.4.** The self-adjoint relation \( S \) is given by \( S = \mathcal{F}^{-1} M_{id} \mathcal{F}|_{\mathcal{D}} \).

**Proof.** First of all note that for each \( f \in \mathcal{D} \) we have (see e.g. [18, Lemma B.4])

\[
f \in \text{dom}(S) \iff \int_{\mathbb{R}} |\lambda|^2 dE_{f,f}(\lambda) < \infty \iff \int_{\mathbb{R}^\times} |\lambda|^2 |\mathcal{F}f(\lambda)|^2 d\mu(\lambda) < \infty \iff \mathcal{F}f \in \text{dom}(M_{id}) \iff f \in \text{dom}(\mathcal{F}^{-1} M_{id} \mathcal{F}).
\]

Furthermore, if \( (f, f_\tau) \in S \), then from Lemma 6.1 and [18, Lemma B.4] we infer that for each \( x \in (a, b) \)

\[
P f_\tau(x) = \langle P f_\tau, \delta_x \rangle = \int_{\mathbb{R}} \lambda dE_{f_\tau, \delta_x}(\lambda) = \int_{\mathbb{R}^\times} \lambda \mathcal{F}f(\lambda) \phi_\lambda(x) d\mu(\lambda) = \int_{\mathbb{R}^\times} M_{id} \mathcal{F}f(\lambda) \phi_\lambda(x) d\mu(\lambda) = \mathcal{F}^* M_{id} \mathcal{F}f(x).
\]

This and Lemma 6.2 show that \( (M_{id} \mathcal{F}f, f_\tau) \in \mathcal{F}^{-1} \), which is equivalent to \( (f, f_\tau) \in \mathcal{F}^{-1} M_{id} \mathcal{F} \). Conversely, if we assume that \( (g, g_\tau) \in \mathcal{F}^{-1} M_{id} \mathcal{F}|_{\mathcal{D}} \), then \( (M_{id} \mathcal{F}g, g_\tau) \in \mathcal{F}^{-1} \) (also note that \( g \) lies in the domain of \( S \)). From this we infer that \( \mathcal{F}^* M_{id} \mathcal{F}g - g_\tau \in \text{mul}(S) \) and because of \( (g, \mathcal{F}^* M_{id} \mathcal{F}g) \in S \), we furthermore get \( (g, g_\tau) \in S \).
Note that all of the multi-valuedness of $S$ is only contained in the inverse of our spectral transform. Moreover, the self-adjoint operator part $S_D$ of $S$ is unitarily equivalent to the operator of multiplication $M_{id}$ in $L^2(\mathbb{R}^\times; \mu)$. In fact, $F$ is unitary as an operator from $\mathcal{D}$ onto $L^2(\mathbb{R}^\times; \mu)$ and maps the operator part of $S$ onto multiplication with the independent variable. Now the spectrum of $S$ can be read off from the boundary behavior of the singular Weyl–Titchmarsh function $M$ in the usual way.

**Corollary 6.5.** The spectrum of $S$ is given by

$$\sigma(S)^\times = \text{supp}(\mu) = \{\lambda \in \mathbb{R}^\times \mid 0 < \limsup_{\varepsilon \downarrow 0} \text{Im} M(\lambda + i\varepsilon) \}.\$$

**Proof.** Since the operator part $S_D$ of $S$ is unitarily equivalent to $M_{id}$ we infer that (see e.g. [18, Lemma B.3]) $\sigma(S)^\times = \sigma(M_{id})^\times = \text{supp}(\mu)$. Now let $[\lambda_1, \lambda_2] \subseteq \mathbb{R}^\times$ and $c \in (a, b)$ such that $\phi_\lambda(c) \neq 0$ for $\lambda \in [\lambda_1, \lambda_2]$. Then equation (6.4) shows that

$$\{\lambda \in \mathbb{R}^\times \mid 0 < \limsup_{\varepsilon \downarrow 0} \text{Im} M(\lambda + i\varepsilon) \} \cap [\lambda_1, \lambda_2]$$

is equivalent to the operator of multiplication $M$.

Moreover, the measures $\mu$ and $E_{\delta_c, \delta_c}$ restricted to $[\lambda_1, \lambda_2]$ are mutually absolutely continuous, hence $\text{supp}(\mu) \cap [\lambda_1, \lambda_2] = \text{supp}(E_{\delta_c, \delta_c}) \cap [\lambda_1, \lambda_2]$. Now the claim follows from standard results (see e.g. [33, Lemma 3.14]).

With similar arguments as in the proof of the previous corollary one may show that the set of eigenvalues of $S$ is given by

$$\sigma_\mu(S) = \{\lambda \in \mathbb{R}^\times \mid \lim_{\varepsilon \downarrow 0} \text{Im} M(\lambda + i\varepsilon) > 0\}.\$$

The value of $\mu$ at such an eigenvalue may be computed explicitly.

**Proposition 6.6.** If $\lambda \in \sigma(S)$ is an eigenvalue, then

$$\mu(\{\lambda\}) = \|\phi_\lambda\|^2.\$$

**Proof.** Under this assumptions $\phi_\lambda$ is an eigenfunction, i.e. $(\phi_\lambda, \lambda \phi_\lambda) \in S$ and $Ff(\lambda) = (f, \phi_\lambda), f \in H^1(a, b)$. Now from equation (6.7) we get

$$\|\phi_\lambda\|^2 = E_{\delta_c, \phi_\lambda}(\{\lambda\}) = \text{F}\phi_\lambda(\lambda) \text{F}\phi_\lambda(\lambda)^* \mu(\{\lambda\}) = \|\phi_\lambda\|^4 \mu(\{\lambda\}),$$

since $E(\{\lambda\})$ is the orthogonal projection onto the space spanned by $\phi_\lambda$.

Note that the measure $\mu$ is uniquely determined by the property that the mapping $\delta_c \mapsto \phi_\lambda(c), c \in (a, b)$ uniquely extends to a partial isometry onto $L^2(\mathbb{R}^\times; \mu)$, which maps $S$ onto multiplication with the independent variable. Because of this, the measure $\mu$ is referred to as the spectral measure of $S$ associated with the real analytic solutions $\phi_z$, $z \in \mathbb{C}^\times$.

**Remark 6.7.** Given another real analytic solution as in Remark 5.4 the corresponding spectral measures are related by

$$\tilde{\mu}(B) = \int_B e^{-2g(\lambda)}\lambda^{-2k}d\mu(\lambda)$$

for each Borel set $B \subseteq \mathbb{R}^\times$, where $k \in \mathbb{Z}$ and $e^g$ is the real entire function from Remark 5.4. In particular, the measures are mutually absolutely continuous and the associated spectral transforms just differ by a simple rescaling with a positive function.
7. Associated de Branges spaces

As in the previous sections let $S$ be some self-adjoint restriction of $T_{\text{max}}$ (with separated boundary conditions) which does not have zero as an eigenvalue. The aim of the present section is to describe the spaces of transforms of functions in $H^1(a,b)$ with compact support. It will turn out that these spaces are hyperplanes in some de Branges spaces associated with our left-definite Sturm–Liouville problem, at least if we somewhat strengthen Hypothesis $2.1$. In fact, in this section we will assume that for each $z \in \mathbb{C}$ there is a non-trivial solution $\phi_z$ of $(\tau - z)u = 0$ such that $\phi_z$ lies in $S$ near $a$ and the functions

$$z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{-1}(c)$$

are real entire for each $c \in (a,b)$. In particular, note that the solution $\phi_0$ is always a scalar multiple of the solution $w_a$ (due to the assumption that zero is not an eigenvalue of $S$). For example, if $\tau$ is regular at $a$ and the boundary condition at $a$ is given by (4.10) for some $\varphi_{\alpha} \in (0, \pi)$, then such a real entire solution $\phi_z$, $z \in \mathbb{C}$ of $(\tau - z)u = 0$ is given by the initial conditions

$$\phi_z(a) = \sin \varphi_{\alpha} \quad \text{and} \quad \phi_z^{-1}(a) = \cos \varphi_{\alpha}, \quad z \in \mathbb{C}.$$

Furthermore, we will assume that the measure $\zeta$ is absolutely continuous with respect to the Lebesgue measure. This will guarantee that our chain of de Branges spaces is continuous in some sense, which simplifies the discussion to some extent. However, we do not have to impose additional assumptions on the measures $\chi$ and $\theta$.

First of all we will introduce the de Branges spaces associated with $S$ and our real entire solution $\phi_z$, $z \in \mathbb{C}$. For a brief review of de Branges’ theory of Hilbert spaces of entire functions see Appendix $A$ whereas for a detailed account we refer to de Branges’ book [14]. Now fix some $c \in (a,b)$ and consider the entire function

$$(7.1) \quad E(z,c) = z \phi_z(c) + i \phi_z^{-1}(c), \quad z \in \mathbb{C}.$$ 

Then this function is a de Branges function, i.e. it satisfies

$$\left| E(z,c) \right| > \left| E(z^*,c) \right|, \quad z \in \mathbb{C}^+,$$

where $\mathbb{C}^+$ is the upper open complex half-plane. Indeed, a simple calculation, using the Lagrange identity from Proposition 2.1 shows that

$$\frac{E(z,c)E^\#(\zeta^*,c) - E(\zeta^*,c)E^\#(z,c)}{2i(\zeta^* - z)} = \frac{\zeta^* \phi_z(c) \phi_z^{-1}(c) - z \phi_z(c) \phi_z^{-1}(c)^*}{\zeta^* - z}$$

$$= \int_a^c \phi_z^* \phi_z \, d\chi + \int_a^c \phi_z^{-1*} \phi_z^{-1} \, d\kappa$$

for each $\zeta, z \in \mathbb{C}^+$. In particular, choosing $\zeta = z$ this equality shows that our function $E(\cdot,c)$ is a de Branges function. Hence it gives rise to a de Branges space $B(c)$ equipped with the inner product

$$[F, G]_{B(c)} = \frac{1}{\pi} \int_\mathbb{R} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda,c)|^2} \, d\lambda, \quad F, G \in B(c).$$

Moreover, note that $E(\cdot,c)$ does not have any real zeros $\lambda$. Indeed, if $\lambda \neq 0$ this would mean that both, $\phi_\lambda$ and its quasi-derivative vanish in $c$ and if $\lambda = 0$ this would contradict the fact that $\phi_0$ is a scalar multiple of $w_a$. 

The reproducing kernel $K(\zeta, z, c)$ of the de Branges space $B(c)$ is given as in equation (A.1). A similar calculation as above, using the Lagrange identity shows that it may as well be written as

$$(7.2) \quad K(\zeta, z, c) = \int_a^c \phi_\zeta(x)^* \phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^* \phi_z^{[1]}(x) d\xi(x), \quad \zeta, z \in \mathbb{C}.$$ 

In the following, the function $K(0, \cdot, c)$ will be of particular interest. An integration by parts as in (2.4) shows that this function may as well be written as

$$(7.3) \quad K(0, z, c) = \phi_0^{[1]}(c) \phi_z(c), \quad z \in \mathbb{C},$$

where the boundary term at $a$ vanishes since $\phi_0$ is a scalar multiple of $w_a$.

We want to link the de Branges space $B(c)$ to our generalized Fourier transform $\mathcal{F}$, using Proposition 6.3. Therefore consider the modified Sobolev space $H^1(a, c)$ and define the transform of a function $f \in H^1(a, c)$ as

$$(7.4) \quad \hat{f}(z) = \int_a^c \phi_z(x) f(x) d\chi(x) + \int_a^c \phi_z^{[1]}(x)^* f^{[1]}(x) d\xi(x), \quad z \in \mathbb{C}.$$ 

We will now identify the de Branges space $B(c)$ with the space of transforms of functions from the subspace

$$D(c) = \text{span}\{\phi_z | z \in \mathbb{C}\}$$

of $H^1(a, c)$, equipped with the norm inherited from $H^1(a, c)$.

**Theorem 7.1.** The transformation $f \mapsto \hat{f}$ is a partial isometry from the modified Sobolev space $H^1(a, c)$ onto $B(c)$ with initial subspace $D(c)$.

**Proof.** For all $\zeta \in \mathbb{C}$, the transform of the function $f_\zeta = \phi_\zeta^* |_{(a, c)}$ is given by

$$\hat{f}_\zeta(z) = \int_a^c \phi_\zeta(x)^* \phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^* \phi_z^{[1]}(x) d\xi(x) = K(\zeta, z, c)$$

for each $z \in \mathbb{C}$ and hence lies in the de Branges space $B(c)$. Moreover, for some given $\zeta_1, \zeta_2 \in \mathbb{C}$ we have

$$\langle f_{\zeta_1}, f_{\zeta_2} \rangle_{H^1(a, c)} = \int_a^c \phi_{\zeta_1}(x)^* \phi_{\zeta_2}(x) d\chi(x) + \int_a^c \phi_{\zeta_1}^{[1]}(x)^* \phi_{\zeta_2}^{[1]}(x) d\xi(x) = K(\zeta_1, \zeta_2, c) = [K(\zeta_1, \cdot, c), K(\zeta_2, \cdot, c)]_{B(c)} = [\hat{f}_{\zeta_1}, \hat{f}_{\zeta_2}]_{B(c)}.$$ 

Now, since the linear span of the functions $K(\zeta, \cdot, c), \zeta \in \mathbb{C}$ is dense in $B(c)$, our transformation uniquely extends to a unitary linear map $V$ from $D(c)$ onto $B(c)$. Moreover, because the functionals $f \mapsto \hat{f}(z)$ and $f \mapsto Vf(z)$ are continuous on $D(c)$ for each fixed $z \in \mathbb{C}$, we conclude that $V$ is nothing but our transform restricted to $D(c)$. Finally, it is easily seen that transforms of functions which are orthogonal to $D(c)$ vanish identically. \quad \Box

In the following, the closed linear subspace

$$B^0(c) = \{ F \in B(c) | F(0) = 0 \}$$

of functions in $B(c)$ which vanish at zero will be of particular interest. This subspace consists precisely of all transforms of functions in $H^1(a, c)$ which vanish in $c$. In fact, an integration by parts shows that

$$\hat{f}(0) = \phi_0^{[1]}(c) f(c), \quad f \in H^1(a, c),$$
where the boundary term at \( a \) vanishes since \( \phi_0 \) is a scalar multiple of \( w_a \). Moreover, the orthogonal complement of \( B^2(c) \) consists of all scalar multiples of the function \( K(0, \cdot, c) \). Hence it corresponds to the one-dimensional subspace of \( D(c) \) spanned by the function \( \phi_0,(a,c) \).

The crucial properties of the de Branges spaces \( B(c) \), \( c \in (a, b) \) only hold if \( c \) lies in the support
\[
\text{supp}(\varrho) = \{ x \in (a, b) | \forall \varepsilon > 0 : |\varrho|((x - \varepsilon, x + \varepsilon)) > 0 \}
\]
of \( \varrho \). However, for the proof of our inverse uniqueness result a modified set \( \Sigma \) instead of \( \text{supp}(\varrho) \) will be more convenient. This set \( \Sigma \subseteq \text{supp}(\varrho) \cup \{ a, b \} \) is defined as follows. Take \( \text{supp}(\varrho) \) and add \( a \) if \( \tau \) is regular at \( a \), there are no Neumann boundary conditions at \( a \) and \( |\varrho| \) has no mass near \( a \). Under similar conditions one adds the endpoint \( b \). Moreover, if \( a \) has not been added, then remove the point \( a_\varepsilon = \inf \text{supp}(\varrho) \) unless \( |\varrho|((a_\varepsilon, c)) = 0 \) for some \( c \in (a_\varepsilon, b) \). Similarly, if \( b \) has not been added, then remove the point \( b_\varepsilon = \sup \text{supp}(\varrho) \) unless \( |\varrho|((c, b_\varepsilon)) = 0 \) for some \( c \in (a, b_\varepsilon) \). The following lemma gives a hint why this definition might be useful.

**Lemma 7.2.** The closure of the domain of \( S \) is given by
\[
\mathfrak{D} = \overline{\text{span}\{ \delta_c \mid c \in \Sigma \}}.
\]

**Proof.** The multi-valued part of \( S \) is given by
\[
\mu_\ast (S) = \{ h \in \mu_\ast (T_{\text{max}}) \mid V((0, h), v^\ast)(a) = V((0, h), w^\ast)(b) = 0 \}.
\]
Now if \( c \in \Sigma \cap (a, b) \), then \( \delta_c \perp \mu_\ast (S) \) since each \( h \in \mu_\ast (T_{\text{max}}) \) vanishes almost everywhere with respect to \( |\varrho| \). Moreover, if \( a \in \Sigma \) then \( \tau \) is regular at \( a \) and there are no Neumann boundary conditions at \( a \). Thus, each \( h \in \mu_\ast (S) \) vanishes in \( a \) in view of \( \mu_\ast \) and hence \( \delta_a \perp \mu_\ast (S) \). Similarly one shows that \( \delta_a \perp \mu_\ast (S) \) provided that \( b \in \Sigma \). Hence the closure of the linear span of all functions \( \delta_c \), \( c \in \Sigma \) is orthogonal to \( \mu_\ast (S) \) and hence contained in \( \mathfrak{D} \). In order to prove the converse let
\[
h \in \overline{\text{span}\{ \delta_c \mid c \in \Sigma \}}^\perp.
\]
Since \( h \) is continuous this implies that \( h \) vanishes on \( \text{supp}(\varrho) \), hence \( h \) lies in \( \mu_\ast (T_{\text{max}}) \). Now suppose that
\[
(V((0, h), v^\ast)(a) = \lim_{\alpha \to a} h(\alpha)v^\ast)(\alpha) \neq 0,
\]
then \( \tau \) is necessarily in the l.c. case at \( a \). If \( \varrho \) had mass near \( a \), we would infer that \( h(\alpha) = 0 \) since \( h \) vanishes on \( \text{supp}(\varrho) \). Hence \( \tau \) is even regular at \( a \) and \( (***) \) implies that there are no Neumann boundary conditions at \( a \). Therefore \( a \) lies in \( \Sigma \) and hence \( h(\alpha) = \langle h, \delta_\alpha \rangle = 0 \), contradicting \( (***) \). A similar argument for the right endpoint \( b \) shows that \( h \) lies in \( \mu_\ast (S) \), which finishes the proof.

Also note that functions in \( \mathfrak{D} \) are uniquely determined by their values on \( \Sigma \). In fact, if \( f_1, f_2 \in \mathfrak{D} \) such that \( f_1(c) = f_2(c), c \in \Sigma \), then \( f_1 - f_2 \) lies in the orthogonal complement of \( \mathfrak{D} \) in view of **Lemma 7.2** and hence \( f_1 = f_2 \).

Now before we state our main embedding theorem, it remains to introduce the de Branges spaces \( B(a) \) if \( a \in \Sigma \) and \( B(b) \) if \( b \in \Sigma \). First of all if \( a \in \Sigma \), then let \( B(a) \) be the one-dimensional space spanned by the entire function \( z \mapsto \phi_\ast(a) \). It does not matter which inner product this space is equipped with; each one turns \( B(a) \) into a de Branges space as is easily seen from [14, Theorem 23]. In particular,
note that $B^0(a) = \{0\}$. Finally if $b \in \Sigma$, then let $B(b)$ be the de Branges space associated with the de Branges function

$$E(z, b) = z \phi_z(b) + i \phi_z^1(b), \quad z \in \mathbb{C}.$$  

The space $B(b)$ has the same properties as the other de Branges spaces $B(c)$, $c \in (a, b)$. For example the reproducing kernel is given as in (7.2) and Theorem 7.1 holds with $c$ replaced by $b$.

The following result is basically a consequence of Theorem 7.1 and Proposition 6.3 linking our transformation with the generalized Fourier transform $F$. In the following, $\mu$ will denote the spectral measure associated with the real analytic solutions $\phi_z$, $z \in \mathbb{C}^\times$ as constructed in the previous section. However, note that in the present case we may extend $\mu$ to a Borel measure on $\mathbb{R}$ by setting $\mu(\{0\}) = 0$.

**Theorem 7.3.** For each $c \in \Sigma$ the de Branges space $B(c)$ is a closed subspace of $L^2(\mathbb{R}; \mu)$ with

$$(7.5) \quad \langle F, G \rangle_\mu = [P^0 F, P^0 G]_{B(c)} + \frac{F(0) G(0)^*}{|\phi_0(c)|^2} \|c_\mu\|^2_{H^1(a, b)}, \quad F, G \in B(c),$$

where $P^0$ is the orthogonal projection from $B(c)$ onto $B^0(c)$.

**Proof.** First of all note that for $z \in \mathbb{C}$ and $h \in \text{mul}(S) \subseteq \text{mul}(T_{\text{max}})$ we have

$$(*) \quad \int_a^c \phi_z(x) h(x)^* d\lambda(x) + \int_a^c \phi_z^1(x) h^1(x)^* d\kappa(x) = \lim_{x \to a} \phi_z^1(x) h(x)^*,$$

since $h$ vanishes almost everywhere with respect to $|\alpha|$ (in particular note that $h(c) = 0$). Moreover, the limit on the right-hand side is zero since

$$\lim_{x \to a} \phi_z^1(x) h(x)^* = V((0, h^*), \phi_z)(a) = 0$$

and both, $(0, h^*)$ and $(\phi_z, z \phi_z)$ lie in $S$ near $a$. Now, given some arbitrary functions $f, g \in \text{span}\{\phi_z|_{(a, c)} | z \in \mathbb{C}\}$, let

$$f_0(x) = \frac{f(c)}{\phi_0(c)} \phi_0(x) \quad \text{and} \quad f_1(x) = f(x) - f_0(x), \quad x \in (a, c)$$

and similarly for the function $g$. The extensions $\tilde{f}_1, \tilde{g}_1$ of $f_1, g_1$, defined by

$$\tilde{f}_1(x) = \begin{cases} f_1(x), & \text{if } x \in (a, c], \\ 0, & \text{if } x \in (c, b), \end{cases}$$

and similarly for $\tilde{g}_1$, lie in $H^1(a, b)$ since $f_1(c) = g_1(c) = 0$. Moreover, these extensions even lie in $\mathcal{D}$, because Lemma 6.2 and Theorem 7.1 shows that they are orthogonal to $\text{mul}(S)$. Now we get the identity

$$(7.6) \quad \langle \tilde{f}_1, \tilde{g}_1 \rangle_\mu = \langle F \tilde{f}_1, F \tilde{g}_1 \rangle_\mu = \langle \tilde{f}_1, \tilde{g}_1 \rangle_{H^1(a, b)} = \langle f_1, g_1 \rangle_{H^1(a, c)}$$

where we used Proposition 6.3 Lemma 6.2 and Theorem 7.1. Moreover, from (7.3) (also note that $\delta_{\epsilon} \in \mathcal{D}$) we get

$$\langle \tilde{f}_0, \tilde{g}_0 \rangle_\mu = f_0(c) g_0(c)^* \left| \frac{\phi_z^1(c)}{\phi_0(c)} \right|^2 \quad \int_{\mathbb{R}} |\phi_{\lambda}(c)|^2 d\mu(\lambda) = \frac{f_0(0) g_0(0)^*}{|\phi_0(c)|^2} \|c_\mu\|_{H^1(a, b)}^2.$$
Furthermore,

\[ \langle \hat{f}_1, \hat{g}_0 \rangle_\mu = g_0(c)^* \frac{\phi_0^{[2]}(c)}{\phi_0(c)} \int_\mathbb{R} \phi_0(c) \hat{f}_1(\lambda) d\mu(\lambda) = g_0(c)^* \frac{\phi_0^{[2]}(c)}{\phi_0(c)} f_1(c) = 0, \]

i.e. the function \( \hat{g}_0 \) is orthogonal to \( \hat{f}_1 \) not only in \( B(c) \) but also in \( L^2(\mathbb{R}; \mu) \). Using these properties, we finally obtain

\[ \langle \hat{f}, \hat{g} \rangle_\mu = \langle \hat{f}_1, \hat{g}_1 \rangle_\mu + \langle \hat{f}_0, \hat{g}_0 \rangle_\mu = [P^* \hat{f}; P^* \hat{g}]_{B(c)} + \frac{\hat{f}(0) \hat{g}(0)^*}{|\phi_0(c)|^2} \| \delta_c \|^2_{H^1(a,b)}, \]

Hence (7.5) holds for all \( F, G \) in a dense subspace of \( B(c) \). Now it is quite easy to see that \( B(c) \) is actually continuously embedded in \( L^2(\mathbb{R}; \mu) \) and that (7.5) holds for all \( F, G \in B(c) \). Moreover, \( B(c) \) is a closed subspace of \( L^2(\mathbb{R}; \mu) \) since the norms \( \| \cdot \|_{B(c)} \) and \( \| \cdot \|_{\mu} \) are equivalent on \( B(c) \).

In particular, note that under the assumption of Theorem 7.3, the subspace \( B^0(c) \) is isometrically embedded in \( L^2(\mathbb{R}; \mu) \). Moreover, the embedding \( B(c) \to L^2(\mathbb{R}; \mu) \) preserves orthogonality and a simple calculation shows that for functions \( F \) in the orthogonal complement of \( B^0(c) \) we have

\[ \| F \|^2_{B(c)} = \frac{|F(0)|^2}{\phi_0^{[2]}(c) \phi_0(c)} = \left( 1 - \frac{w_2^{[2]}(c)}{w_2(c) w_1^{[2]}(c)} \right) \| F \|^2_{\mu}, \]

at least if \( c \neq a \). This difference between \( B^0(c) \) and its orthogonal complement stems from the fact that the functions in \( H^1(a, c) \) corresponding to \( B^0(c) \) are isometrically embedded in \( H^1(a, b) \), whereas the functions corresponding to its orthogonal complement are not.

The following results contain further properties of our de Branges spaces which are needed for the inverse uniqueness theorem in the next section. First of all, we will show that they are totally ordered and strictly increasing.

**Proposition 7.4.** If \( c_1, c_2 \in \Sigma \) with \( c_1 < c_2 \), then

\[ B(c_1) \subset B(c_2). \]

Moreover, if \( |\rho|((c_1, c_2)) = 0 \) then \( B(c_1) \) has codimension one in \( B(c_2) \).

**Proof.** If \( \delta \in H^1(a, c_2) \) is such that

\[ \langle f, \delta \rangle_{H^1(a, c_2)} = f(c_1), \quad f \in H^1(a, c_2), \]

then the modified Sobolev space \( H^1(a, c_2) \) may be decomposed into

\[ H^1(a, c_2) = H^1_\perp(a, c_2) \oplus \text{span}\{\delta\} \oplus H^1_\perp(a, c_2). \]

Here \( H^1_\perp(a, c_2) \) is the subspace of functions in \( H^1(a, c_2) \) vanishing on \( (c_1, c_2) \) and \( H^1_\perp(a, c_2) \) is the subspace of functions in \( H^1(a, c_2) \) vanishing on \( (a, c_1) \). Now the transforms of functions in \( H^1_\perp(a, c_2) \) are precisely the transforms of functions in \( H^1(a, c_1) \) which vanish in \( c_1 \), i.e. \( B^0(c_1) \). The transform of the subspace \( \text{span}\{\delta\} \) is precisely the orthogonal complement of \( B^0(c_1) \). Hence one sees that \( B(c_1) \) is contained in \( B(c_2) \). In order to prove that \( B(c_2) \) is larger indeed, suppose that the function \( z \mapsto \phi_z(c_2) \) belongs to \( B(c_1) \). Since this function is orthogonal to \( B^0(c_2) \) it is also orthogonal to \( B^0(c_1) \) by Theorem 7.3. Thus we infer that the functions \( z \mapsto \phi_z(c_1) \) and \( z \mapsto \phi_z(c_2) \) are linearly dependent. Now from Lemma 6.2
(hereby also note that \(\delta_{c_1}\) and \(\delta_{c_2}\) lie in \(\mathcal{D}\)) one sees that \(\delta_{c_1}\) and \(\delta_{c_2}\) are also linearly dependent, which gives a contradiction.

It remains to prove that the space of transforms of functions in \(H^1_b(a, c_2)\) is at most one-dimensional provided that \(|g((c_1, c_2))| = 0\). Indeed, for each function \(f \in H^1_b(a, c_2)\) an integration by parts shows that

\[
\hat{f}(z) = \phi_2^{(1)}(c_2)f(c_2) - \phi_2^{(1)}(c_1)f(c_1) + z \int_{c_1}^{c_2} \phi_z f \, dg = \phi_2^{(1)}(c_2)f(c_2), \quad z \in \mathbb{C},
\]

since \(f\) vanishes on \((a, c_1)\) and \(|g((c_1, c_2))| = 0\).

The following result shows that our de Branges spaces are continuous in some sense. This is due to the assumption that the measure \(\varsigma\) is absolutely continuous with respect to the Lebesgue measure. Otherwise, there would be jumps of dimension one in points where \(\varsigma\) has mass.

**Proposition 7.5.** If \(c, \alpha_n, \beta_n \in \text{supp}(\varrho)\), \(n \in \mathbb{N}\) are such that \(\alpha_n \uparrow c\) and \(\beta_n \downarrow c\) as \(n \to \infty\), then

\[
(7.8) \quad \bigcup_{n \in \mathbb{N}} B(\alpha_n) = B(c) = \bigcap_{n \in \mathbb{N}} B(\beta_n),
\]

where the closure is taken in \(L^2(\mathbb{R}; \mu)\).

**Proof.** From Proposition 7.4 it is clear that

\[
\bigcap_{n \in \mathbb{N}} B(\alpha_n) \subseteq B(c) \subseteq \bigcup_{n \in \mathbb{N}} B(\beta_n).
\]

If \(F \in B^0(c)\), then there is an \(f \in H^1(a, c)\) with \(f(c) = 0\) such that \(\hat{f} = F\). Now choose a sequence \(f_k \in H^1(a, c), k \in \mathbb{N}\) of functions which vanish near \(c\), such that \(f_k \to f\) as \(k \to \infty\). By our assumptions the transform of each of these functions lies in \(B(\alpha_n)\), provided that \(n \in \mathbb{N}\) is large enough, i.e.

\[
\hat{f}_k \in \bigcup_{n \in \mathbb{N}} B(\alpha_n), \quad k \in \mathbb{N}.
\]

Consequently the transform of \(f\) lies in the closure of this union. Moreover, for each \(n \in \mathbb{N}\) the entire function \(z \mapsto \phi_z(\alpha_n)\) lies in \(B(\alpha_n)\). Now since \(\delta_{\alpha_n} \to \delta_{c}\) in \(H^1(a, b)\), Lemma 6.2 shows that the entire function \(z \mapsto \phi_z(c)\) lies in the closure of our union which proves the first equality in (7.8).

Next, if \(F \in B(\beta_n)\) for each \(n \in \mathbb{N}\), then there are \(f_n \in D(\beta_n)\) such that

\[
F(z) = \int_a^{\beta_n} \phi_z(x)f_n(x) d\chi(x) + \int_a^{\beta_n} \phi_z^{(1)}(x)f_n^{(1)}(x) d\varsigma(x), \quad z \in \mathbb{C}, n \in \mathbb{N}.
\]

Moreover, from Theorem 7.1 and Theorem 7.3 we infer

\[
\|f_n\|_{H^1(a, \beta_n)}^2 = \|F\|_{B(\beta_n)}^2 \leq \left(1 + \frac{w_k^{(1)}(\beta_n)}{w_k(\beta_n)} \frac{w_n(\beta_n)}{w_n^{(1)}(\beta_n)}\right) \|F\|_\mu^2, \quad n \in \mathbb{N},
\]

where the coefficient on the right-hand side is bounded uniformly for all \(n \in \mathbb{N}\) by the properties of the solutions \(w_{\alpha}\) and \(w_{\beta}\). Hence there is some subsequence of \(f_n, n \in \mathbb{N}\) converging weakly in \(H^1(a, c)\) to say \(f\). Now this yields for all \(z \in \mathbb{C}\)

\[
F(z) = \hat{f}(z) + \chi(\{c\})\phi_z(c)f(c) + \lim_{n \to \infty} \int_{(c, \alpha_n)} \phi_z f_n d\chi + \int_{(c, \beta_n)} \phi_z^{(1)} f_n^{(1)} d\varsigma,
\]
where the limit is actually zero. In fact, for each $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have
\[
\left| \int_{(c,\beta_n)} \phi_z f_n d\chi + \int_{(c,\beta_n)} \phi_z^{[1]} f_n^{[1]} d\zeta \right| \leq C_z \| f_n \|_{H^1(a,\beta_n)} \sqrt{\chi((c,\beta_n))} + \zeta((c,\beta_n)),
\]
where $C_z \in \mathbb{R}$ is such that the moduli of $\phi_z$ and $\phi_z^{[1]}$ on $(c, \beta_1)$ are bounded by $C_z$. But this shows that $F$ actually is the transform of a function in $H^1(a, b)$ and hence lies in $B(c)$ which finishes the proof. \qed

Finally we will prove that our de Branges spaces decrease to zero near $a$ and fill the whole space $L^2(\mathbb{R}; \mu)$ near $b$.

**Proposition 7.6.** The de Branges spaces $B(c)$, $c \in \Sigma$ satisfy
\[
(7.9) \quad \bigcap_{c \in \Sigma} B^2(c) = \{0\} \quad \text{and} \quad \bigcup_{c \in \Sigma} B(c) = L^2(\mathbb{R}; \mu).
\]

**Proof.** First suppose that $\text{supp}(\varrho) \cap (a, c) \neq \emptyset$ for each $c \in (a, b)$ and pick some $F \in \bigcap_{c \in \Sigma} B^2(c)$. Then for each $\zeta \in \mathbb{C}$ we have
\[
|F(\zeta)|^2 = |[F, K(\zeta, \cdot, c)]_{B(c)}|^2 \leq \| F \|^2_{B(c)} |K(\zeta, \cdot, c), K(\zeta, \cdot, c)|_{B(c)}
\]
\[
\leq \| F \|^2_{B(c)} K(\zeta, \zeta, c)
\]
for each $c \in \text{supp}(\varrho)$. Now from (7.2) we infer that $K(\zeta, \zeta, c) \to 0$ as $c \to a$ and hence that $F = 0$. Otherwise, if $\alpha_\varrho = \inf \text{supp}(\varrho) > a$, then the subspace $D^2(\alpha_\varrho) = \{ f \in D(\alpha_\varrho) \mid f(\alpha_\varrho) = 0 \}$, corresponding to $B^2(\alpha_\varrho)$, is at most one-dimensional. In fact, this is because each function $\phi_z|_{(a, \alpha_\varrho)}$, $z \in \mathbb{C}$ is a solution of $\tau u = 0$ on $(a, \alpha_\varrho)$ in this case. Consequently, the functions in $D^2(\alpha_\varrho)$ are also solutions of $\tau u = 0$ on $(a, \alpha_\varrho)$. Moreover, if $\zeta + \chi$ is infinite near $a$, then each $f \in D^2(\alpha_\varrho)$ is a scalar multiple of $w_a$ on $(a, \alpha_\varrho)$ with $f(\alpha_\varrho) = 0$ and hence vanishes identically. Also if $\zeta + \chi$ is finite near $a$ and there are Neumann boundary conditions at $a$, one sees that $f$ is a scalar multiple of $w_a$ and hence identically zero. We conclude that the first equality in the claim holds in these cases. Finally, if $\zeta + \chi$ is finite near $a$ and there are no Neumann boundary conditions at $a$, then $a \in \Sigma$ and hence clearly $B^2(a) = \{0\}$. For the second equality note that the linear span of functions $z \mapsto \phi_z(c)$, $c \in \Sigma$ is dense in $L^2(\mathbb{R}; \mu)$ in view of Lemma 7.2 and Lemma 6.2. \qed

### 8. Inverse uniqueness results

The present section is devoted to our inverse uniqueness result. We will prove that the spectral measure determines a left-definite Sturm–Liouville operator up to some Liouville transformation (see e.g. [7] or [3] for the right-definite case). Therefore let $S_1$ and $S_2$ be two self-adjoint left-definite Sturm–Liouville relations (with separated boundary conditions), both satisfying the assumptions made in the previous section, i.e. zero is not an eigenvalue of $S_1$ and $S_2$ and there are real entire solutions satisfying the boundary condition at the left endpoint. Moreover, again we assume that the measures $\zeta_1$ and $\zeta_2$ are absolutely continuous with respect to the Lebesgue measure. All remaining quantities corresponding to $S_1$ respectively $S_2$ are denoted with an additional subscript.

We will first state a part of the proof of our inverse uniqueness result as a separate lemma. Note that the equality in the claim of this lemma has to be read as sets of
entire functions and not as de Branges spaces. In general the norms of these spaces will differ from each other.

**Lemma 8.1.** Suppose that the function

\[
E_1(z, x_1) - E_2(z, x_2), \quad z \in \mathbb{C}^+
\]

is of bounded type for some \(x_1 \in \Sigma_1\) and \(x_2 \in \Sigma_2\). If \(\mu_1 = \mu_2\), then there is an increasing continuous bijection \(\eta\) from \(\Sigma_1\) onto \(\Sigma_2\) such that

\[
B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.
\]

**Proof.** First of all note that by the definition of de Branges spaces and Proposition 7.4 the function in (8.1) is of bounded type for all \(x_1 \in \Sigma_1\) and \(x_2 \in \Sigma_2\). We will first consider the case when \(\Sigma_1\) consists of finitely many (strictly increasing) points \(x_{1,n}, n = 1, \ldots, N\) separately. In this case \(\mu_1 = \mu_2\) is supported on \(N\) points, since \(\mathcal{F}_1\) is a unitary map from \(\mathcal{D}_1\) onto \(L^2(\mathbb{R}; \mu_1)\). Hence, \(\Sigma_2\) also consists of finitely many (strictly increasing) points \(x_{2,n}, n = 1, \ldots, N\). Now let \(\eta\) be the unique strictly increasing bijection from \(\Sigma_1\) onto \(\Sigma_2\), i.e. \(\eta(x_{1,n}) = x_{2,n}, n = 1, \ldots, N\). Using the properties of our de Branges spaces it is quite simple to see that

\[
\dim B_1(x_{1,n}) = \dim B_2(x_{2,n}) = n, \quad n = 1, \ldots, N,
\]

and therefore the claim follows from Theorem 7.3 and Theorem A.1.

Now suppose that \(\Sigma_1\) consists of infinitely many points and fix some arbitrary \(x_1 \in \Sigma_1 \setminus \{\inf \Sigma_1, \sup \Sigma_1\}\). Then from Theorem 7.3 and Theorem A.1 we infer that for each \(x_2 \in \Sigma_2\) either \(B_1(x_1) \subset B_2(x_2)\) or \(B_1(x_1) \supset B_2(x_2)\) and hence also \(B_1^2(x_1) \subset B_2^2(x_2)\) or \(B_1^2(x_1) \supset B_2^2(x_2)\). In order to define \(\eta(x_1) \in (a_2, b_2)\) we are first going to show that both of the sets

- \(J_- = \{x_2 \in \Sigma_2 \mid B_2(x_2) \subset B_1(x_1)\}\),
- \(J_+ = \{x_2 \in \Sigma_2 \mid B_2(x_1) \supset B_2(x_2)\}\),

are non-empty. Indeed, if \(J_-\) was empty, then \(B_1^2(x_1) \subset B_2^2(x_2)\) for each \(x_2 \in \Sigma_2\) and hence

\[
B_1^2(x_1) \subseteq \bigcap_{x_2 \in \Sigma_2} B_2^2(x_2) = \{0\},
\]

in view of Proposition 7.6. Thus we obtained the contradiction \(x_1 = \inf \Sigma_1\), since otherwise there would be some \(\tilde{x}_1 \in \Sigma_1\) with \(\tilde{x}_1 < x_1\) such that \(B_1(\tilde{x}_1) \not\subset B_1(x_1)\). Furthermore, if \(J_+\) was empty, then \(B_2(x_2) \subset B_1(x_1)\) for each \(x_2 \in \Sigma_2\) and hence

\[
L^2(\mathbb{R}; \mu_1) = \bigcup_{x_2 \in \Sigma_2} B_2(x_2) \subseteq B_1(x_1) \subseteq L^2(\mathbb{R}; \mu_1).
\]

But from this we infer the contradiction \(x_1 = \sup \Sigma_1\), since otherwise there would be an \(\tilde{x}_1 \in \Sigma_1\) with \(\tilde{x}_1 > x_1\) such that \(B_1(\tilde{x}_1) \not\subset B_1(x_1) \subseteq L^2(\mathbb{R}; \mu_1)\). Hence we showed that \(J_-\) and \(J_+\) are non-empty. Now, if \(J_- = \{a_2\}\) then the space \(B_2(\alpha_{e_2})\) is two-dimensional and \(\alpha_{e_2}\) does not lie in \(J_+\) since otherwise

\[
B_2(\alpha_{e_2}) \subset B_1(x_1) \subset B_2(\alpha_{e_2}).
\]

Thus in this case we may set \(\eta(x_1) = \alpha_{e_2}\) and obtain \(B_1(x_1) = B_2(\eta(x_1))\). Furthermore, if \(J_+ = \{b_2\}\) then the space \(B_2(\beta_{e_2})\) has codimension one in \(L^2(\mathbb{R}; \mu)\) and
$\beta_{\varphi_2}$ does not lie in $J_-$ since otherwise
\[
B_2(\beta_{\varphi_2}) \subseteq B_1(x_1) \subseteq B_2(b_2).
\]
Again, we may define $\eta(x_1) = \beta_{\varphi_2}$ and get $B_1(x_1) = B_2(\eta(x_1))$. Now in the remaining cases $J_-$ is bounded from above in $(a_2, b_2)$ with supremum
\[
\eta_-(x_1) = \sup J_-(a_2, b_2),
\]
and $J_+$ is bounded from below in $(a_2, b_2)$ with infimum
\[
\eta_+(x_1) = \inf J_+ (a_2, b_2).
\]
Moreover, we have $\eta_\pm(x_1) \in \text{supp}(\varphi_2)$ since $J_\pm \backslash \{a_2, b_2\}$ is contained in $\text{supp}(\varphi_2)$. Now Proposition 7.5 shows that
\[
B_2(\eta_-(x_1)) \subseteq B_1(x_1) \subseteq B_2(\eta_+(x_1)).
\]
If $B_1(x_1) = B_2(\eta_-(x_1))$, set $\eta(x_1) = \eta_-(x_1)$ and if $B_1(x_1) = B_2(\eta_+(x_1))$, set $\eta(x_1) = \eta_+(x_1)$ to obtain $B_1(x_1) = B_2(\eta(x_1))$. Otherwise we have
\[
B_2(\eta_-(x_1)) \subseteq B_1(x_1) \subseteq B_2(\eta_+(x_1)),
\]
and hence $\text{supp}(\varphi_2) \cap (\eta_-(x_1), \eta_+(x_1)) \neq \emptyset$ in view of Proposition 7.4. Now we may choose $\eta(x_1)$ in this intersection and get $B_1(x_1) = B_2(\eta(x_1))$ since $\eta(x_1)$ neither lies in $J_-$ nor in $J_+$.

Up to now we constructed a function $\eta : \Sigma_1 \backslash \{\inf \Sigma_1, \sup \Sigma_1\} \rightarrow \Sigma_2$ such that $B_1(x_1) = B_2(\eta(x_1))$ for each $x_1 \in \Sigma_1 \backslash \{\inf \Sigma_1, \sup \Sigma_1\}$. Now if $\inf \Sigma_1$ lies in $\Sigma_1$ and we set $x_1 = \inf \Sigma_1 \setminus \{\inf \Sigma_1\}$, then $B_1(x_1) = B_2(\eta(x_1))$ is two-dimensional and from Proposition 7.6 we infer that there is an $x_2 \in \Sigma_2$ with
\[
\{0\} \subseteq B_2(x_2) \subseteq B_2(\eta(x_1)) = B_1(x_1).
\]
Hence we may set $\eta(\inf \Sigma_1) = x_2$ and obtain $B_1(\inf \Sigma_1) = B_2(\eta(\inf \Sigma_1))$. Similarly, if $\sup \Sigma_1$ lies in $\Sigma_1$ and we set $x_1 = \sup \Sigma_1 \setminus \{\sup \Sigma_1\}$, then the space $B_1(x_1) = B_2(\eta(x_1))$ has codimension one in $B_1(\sup \Sigma_1) = L^2(\mathbb{R}; \mu_1)$. But because of Proposition 7.6 there is an $x_2 \in \Sigma_2$ such that
\[
B_2(\eta(\beta_{\varphi_1})) \subseteq B_2(x_2) \subseteq L^2(\mathbb{R}; \mu_1).
\]
Again, we may define $\eta(\sup \Sigma_1) = x_2$ and get $B_1(\sup \Sigma_1) = B_2(\eta(\sup \Sigma_1))$. Thus, we extended our function $\eta$ to all of $\Sigma_1$ and are left to prove the remaining claimed properties.

The fact that $\eta$ is increasing is a simple consequence of Proposition 7.4. Now if $x_2 \in \Sigma_2$, then the first part of the proof with the roles of $\Sigma_1$ and $\Sigma_2$ reversed shows that there is an $x_1 \in \Sigma_1$ with $B_1(x_1) = B_2(x_2) = B_1(\eta(x_1))$. In view of Proposition 7.4 this yields $\eta(x_1) = x_2$ and hence $\eta$ is a bijection. Finally, continuity follows from Proposition 7.5. Indeed, if $c, c_n \in \Sigma_1, n \in \mathbb{N}$ such that $c_n \uparrow c$ as $n \to \infty$, then
\[
B_2 \left( \lim_{n \to \infty} \eta(c_n) = \bigcup_{n \in \mathbb{N}} B_2(\eta(c_n)) = \bigcup_{n \in \mathbb{N}} B_1(c_n)) = B_2(\eta(c))
\]
and hence $\eta(c_n) \to \eta(c)$ as $n \to \infty$. Similarly, if $c_n \downarrow c$ as $n \to \infty$, then
\[
B_2 \left( \lim_{n \to \infty} \eta(c_n) = \bigcap_{n \in \mathbb{N}} B_2(\eta(c_n)) = \bigcap_{n \in \mathbb{N}} B_1(c_n)) = B_2(\eta(c))
\]
and hence again $\eta(c_n) \to \eta(c)$ as $n \to \infty$. \qed
Note that the condition that the function in (8.1) is of bounded type is actually equivalent to the function
\[
\frac{\phi_{1,z}(x_1)}{\phi_{2,z}(x_2)} \quad z \in \mathbb{C}^+
\]
being of bounded type for some \(x_1 \in \Sigma_1\) and \(x_2 \in \Sigma_2\). Unfortunately, these conditions are somewhat inconvenient in view of applications. However, note that this assumption is for example fulfilled if for some \(x_1 \in \Sigma_1\) and \(x_2 \in \Sigma_2\) the entire functions \(z \mapsto \phi_{1,z}(x_1)\) and \(z \mapsto \phi_{2,z}(x_2)\) are of finite exponential type such that the logarithmic integrals
\[
\int_{\mathbb{R}} \frac{\ln^+ |\phi_{j,z}(x_j)|}{1 + \lambda^2} d\lambda < \infty, \quad j = 1, 2
\]
are finite. Here \(\ln^+\) is the positive part of the natural logarithm. Indeed, a theorem of Krein [28, Theorem 6.17], [32, Section 16.1] states that in this case the functions \(z \mapsto \phi_{j,z}(x_j)\), \(j = 1, 2\) (and hence also their quotient) are of bounded type in the upper and in the lower complex half-plane. Moreover, note that the conclusion of Lemma 6.1 is also true if for some (and hence all) \(x_1 \in (a_1, b_1)\) and \(x_2 \in (a_2, b_2)\) the functions \(E_1(\cdot, x_1)\), \(E_2(\cdot, x_2)\) are of exponential type zero, i.e.
\[
\ln^+ |E_j(z, x_j)| = o(|z|), \quad j = 1, 2
\]
as \(|z| \to \infty\) in \(\mathbb{C}\). The proof therefore is literally the same, except that one has to apply Theorem A.2 instead of Theorem A.1.

With all the work done in Lemma 8.1 it is now quite simple to show that the spectral measure determines our self-adjoint Sturm–Liouville relation up to a Liouville transform. Here, a Liouville transform \(\mathcal{L}\) is a unitary map from \(\mathcal{D}_2\) onto \(\mathcal{D}_1\) given by
\[
\mathcal{L}f_2(x_1) = \kappa(x_1)f_2(\eta(x_1)), \quad x_1 \in \Sigma_1, \; f_2 \in \mathcal{D}_2,
\]
where \(\eta\) is an increasing continuous bijection from \(\Sigma_1\) onto \(\Sigma_2\) and \(\kappa\) is a non-vanishing real function on \(\Sigma_1\). We say that the Liouville transform \(\mathcal{L}\) maps \(S_1\) onto \(S_2\) if
\[
S_2 = \mathcal{L}^* S_1 \mathcal{L},
\]
where \(\mathcal{L}^*\) is the adjoint of \(\mathcal{L}\) regarded as a linear relation in \(H^1(a_2, b_2) \times H^1(a_1, b_1)\).

Note that in this case the operator parts \(S_{1,\mathcal{D}}, S_{2,\mathcal{D}}\) of \(S_1, S_2\) are unitarily equivalent. In fact, the Liouville transforms \(\mathcal{L}\) maps \(S_{2,\mathcal{D}}\) onto \(S_{1,\mathcal{D}}\).

**Theorem 8.2.** Suppose that the function
\[
\frac{E_1(z, x_1)}{E_2(z, x_2)} \quad z \in \mathbb{C}^+
\]
is of bounded type for some \(x_1 \in \Sigma_1\) and \(x_2 \in \Sigma_2\). If \(\mu_1 = \mu_2\), then there is a Liouville transform \(\mathcal{L}\) mapping \(S_1\) onto \(S_2\).

**Proof.** By Lemma 8.1 there is an increasing continuous bijection \(\eta\) from \(\Sigma_1\) onto \(\Sigma_2\) such that \(B_1(x_1) = B_2(\eta(x_1))\) and hence also \(B_1^*(x_1) = B_2^*(\eta(x_1))\) for each \(x_1 \in \Sigma_1\). According to Theorem 7.3 for each fixed \(x_1 \in \Sigma_1\) the entire functions
\[
z \mapsto \phi_{1,z}(x_1) \quad \text{and} \quad z \mapsto \phi_{2,z}(\eta(x_1))
\]
are orthogonal to \( B_1^2(x_1) = B_2^2(\eta(x_1)) \) in \( L^2(\mathbb{R}; \mu_1) \). From this we infer that
\[
\phi_{1,z}(x_1) = \kappa(x_1) \phi_{2,z}(\eta(x_1)), \quad z \in \mathbb{C}
\]
for some \( \kappa(x_1) \in \mathbb{R}^\times \) and hence also
\[
(\ast) \quad \mathcal{F}_1 \delta_{1,x_1} = \kappa(x_1) \mathcal{F}_2 \delta_{2,\eta(x_1)}.
\]
Now the linear relation
\[
\mathcal{L} = \mathcal{F}_1^* \mathcal{F}_2|_{\mathcal{D}_2}
\]
is a unitary mapping from \( \mathcal{D}_2 \) onto \( \mathcal{D}_1 \) by Lemma 6.2 and moreover, equation \((\ast)\) shows that
\[
(\delta_{1,x_1}, \kappa(x_1) \delta_{2,\eta(x_1)}) \in \mathcal{L}^* = \mathcal{F}_2^{-1} \mathcal{F}_1, \quad x_1 \in \Sigma_1.
\]
From this one sees that the transform of some function \( f_2 \in \mathcal{D}_2 \) is given by
\[
\mathcal{L} f_2(x_1) = \langle \mathcal{L} f_2, \delta_{1,x_1} \rangle \mathcal{H}^{1(a_1,b_1)} = \kappa(x_1) \langle f_2, \delta_{2,\eta(x_1)} \rangle \mathcal{H}^{1(a_2,b_2)}
\]
at each point \( x_1 \in \Sigma_1 \). Finally, we conclude that
\[
S_2 = \mathcal{F}_2^{-1} \text{Id} \mathcal{F}_2|_{\mathcal{D}_2} = \mathcal{F}_2^{-1} \mathcal{F}_1 \mathcal{F}_1^{-1} \text{Id} \mathcal{F}_1 \mathcal{F}_2|_{\mathcal{D}_2} = \mathcal{L}^* \mathcal{F}_1^{-1} \text{Id} \mathcal{F}_1 \mathcal{L}
\]
from Theorem 6.4.

We will now show to which extent the spectral measure determines the coefficients. For the proof we need a result on the high energy asymptotics of solutions of our differential equation (see e.g. [4, Section 6]). Henceforth we will denote with \( r_j, j = 1,2 \) the densities of the absolute continuous parts of \( \varrho_j \) with respect to the Lebesgue measure and with \( p_j^{-1}, j = 1,2 \) the densities of \( \varsigma_j \) with respect to the Lebesgue measure.

**Lemma 8.3.** For each \( j = 1,2 \) and all points \( x_j, \tilde{x}_j \in (a_j, b_j) \) we have the asymptotics
\[
\sqrt{\frac{y}{2}} \ln \left| \frac{\phi_{j,ip}(x_j)}{\phi_{j,ip}(\tilde{x}_j)} \right| \to \int_{\tilde{x}_j}^{x_j} \sqrt{\frac{|r_j(x)|}{p_j(x)}} \, dx,
\]
as \( y \to \infty \) in \( \mathbb{R}^+ \).

**Proof.** By our assumptions, the Lebesgue decomposition of the measure \( \varrho_j \) with respect to \( \varsigma_j \) is given by
\[
\varrho_j = r_j p_j \varsigma_j + \varrho_{j,s},
\]
where \( \varrho_{j,s} \) is the singular part of \( \varrho_j \) with respect to the Lebesgue measure. Now the results in [4, Section 6] show that (the square root is the principal one with branch cut along the negative real axis)
\[
\ln \left| \frac{\phi_{j,ip}(x_j)}{\phi_{j,ip}(\tilde{x}_j)} \right| = \Re \left( \int_{\tilde{x}_j}^{x_j} \sqrt{-iy r_j(x)} p_j(x) d\varsigma_j(x) + o(\sqrt{y}) \right)
\]
\[
= \sqrt{\frac{y}{2}} \int_{\tilde{x}_j}^{x_j} \sqrt{\frac{|r_j(x)|}{p_j(x)}} \, dx + o(\sqrt{y}),
\]
as \( y \to \infty \) in \( \mathbb{R}^+ \), which yields the claim. \( \square \)
We are now able to establish a relation between the measure coefficients. However, this is only possible on sets where the support of the weight measure has enough density. Otherwise there would be too much freedom for the remaining coefficients.

**Corollary 8.4.** Let \( \alpha_1, \beta_1 \in (a_1,b_1) \) with \( \alpha_1 < \beta_1 \) such that \( r_1 \neq 0 \) almost everywhere on \( (\alpha_1, \beta_1) \) and \( r_2 \neq 0 \) almost everywhere on \( (\eta(x_1), \eta(x)) \) with respect to the Lebesgue measure. If the function

\[
\frac{E_1(z,x_1)}{E_2(z,x_2)}, \quad z \in \mathbb{C}^+
\]

is of bounded type for some \( x_1 \in (a_1, b_1) \), \( x_2 \in (a_2, b_2) \) and \( \mu_1 = \mu_2 \), then the functions \( \eta \) and \( \kappa \) from the Liouville transform of Theorem 8.2 satisfy

\[
\kappa' \eta' = \sqrt{\frac{p_2 \circ \eta'}{p_1} \frac{|r_1|}{|r_2 \circ \eta|}} \quad \text{and} \quad \kappa^2 = \sqrt{\frac{p_2 \circ \eta'}{p_1} \frac{|r_2 \circ \eta|}{|r_1|}}
\]

almost everywhere on \( (\alpha_1, \beta_1) \) with respect to the Lebesgue measure and for the measure coefficients we have

\[
\kappa_2 \circ \eta = \kappa^2 \kappa_1, \quad \phi_2 \circ \eta = \kappa^2 \phi_1 \quad \text{and} \quad \chi_2 \circ \eta = \kappa^2 \chi_1 - \kappa \kappa_1',
\]

as measures on \( (\alpha_1, \beta_1) \).

**Proof.** From equation (8.3) and the asymptotics in Lemma 8.3 we infer that

\[
\int_{x_1}^{x_2} \sqrt{\frac{|r_1(x)|}{p_1(x)}} \, dx = \int_{\eta(x_1)}^{\eta(x_2)} \sqrt{\frac{|r_2(x)|}{p_2(x)}} \, dx, \quad x_1, x_2 \in (a_1, b_1).
\]

In view of the Banach–Zarecki theorem (see e.g. [29, Chapter IX; Theorem 3.4], [23, Theorem 18.25]) this shows that \( \eta \) is locally absolutely continuous on \( (\alpha_1, \beta_1) \) with derivative given as in the claim. More precisely, this follows from an application of [29, Chapter IX; Exercise 13] and [29, Chapter IX; Theorem 3.5]. Furthermore, since \( \phi_{1,0}, \phi_{2,0} \) are scalar multiples of \( w_{1,a}, w_{2,a} \) respectively, we also have

\[
(*): \quad w_{1,a}(x_1) = C_a \kappa(x_1)w_{2,a}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)
\]

for some constant \( C_a \in \mathbb{R}^\kappa \). In particular, this shows that \( \kappa \) is locally absolutely continuous on \((\alpha_1, \beta_1)\). In fact, the substitution rule for Lebesgue–Stieltjes integrals (see e.g. [19]) shows that

\[
w_{2,b}(\eta(x_1)) - w_{2,b}(\eta(x)) = \int_{\eta(x)}^{\eta(x_1)} w_{2,b} \, d\kappa_2
\]

\[
= \int_{\eta(x)}^{\eta(x_1)} w_{2,b} \circ \eta \, d\kappa_2 \circ \eta, \quad x_1, x \in (\alpha_1, \beta_1)
\]

and hence the function \( x_1 \mapsto w_{2,b}(\eta(x_1)) \) is locally of bounded variation on \((\alpha_1, \beta_1)\). Therefore, from [29, Chapter IX; Theorem 3.5] we infer that this function is even locally absolutely continuous on \((\alpha_1, \beta_1)\) and hence so is \( \kappa \). Moreover, in view of Lemma 6.2, equation (8.3) yields

\[
\kappa(x_1)^2 = \frac{\left\| \delta_{1,1} \right\|_{H^1(\alpha_1, \beta_1)}}{\left\| \delta_{2,0}(x_1) \right\|_{H^1(\alpha_2, \beta_2)}} = \frac{W(w_{2,b}, w_{2,a})}{W(w_{1,b}, w_{1,a})} \frac{w_{1,a}(x_1)w_{1,b}(x_1)}{w_{2,a}(\eta(x_1))w_{2,b}(\eta(x_1))}
\]
for each \( x_1 \in (\alpha_1, \beta_1) \). Inserting \([4]\) we get from this equation
\[
 w_{1,b}(x_1) = C_{\alpha,1} W^{(w_{1,b}, w_{1,a})}_{(w_{2,b}, w_{2,a})} \kappa(x_1) w_{2,b}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1).
\]
Plugging this expression and equation \([4]\) into the definition of the Wronskian \( W(w_{1,b}, w_{1,a}) \) one obtains
\[
 1 = \frac{\kappa(x_1)^2 \eta'(x_1) p_1(x_1)}{p_2(\eta(x_1))}, \quad x_1 \in (\alpha_1, \beta_1),
\]
which shows that \( \kappa \) is given as in the claim. Next, differentiating equation \([8.3]\) yields
\[
 \kappa(x_1) \phi_{1,z}^{[1]}(x_1) = \kappa^{[1]}(x_1) \phi_{1,z}(x_1) + \phi_{2,z}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)
\]
for each \( z \in \mathbb{C} \). From this we get for all \( \alpha, \beta \in (\alpha_1, \beta_1) \)
\[
 \int_{\alpha}^{\beta} \phi_{1,z} \kappa d\chi_1 - z \int_{\alpha}^{\beta} \phi_{1,z} \kappa d\eta_1 = \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{[1]} + \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\chi_2 \circ \eta - z \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\eta_2 \circ \eta,
\]
where we used the integration by parts formula \([1.3]\), the differential equation and the substitution rule. In particular, choosing \( z = 0 \) this shows that the coefficients \( \chi_1 \) and \( \chi_2 \) are related as in the claim (note that \( \phi_{1,0} \) does not have any zeros). Using this relation, one sees from the previous equation that for each \( z \in \mathbb{C}^\times \) and \( \alpha, \beta \in (\alpha_1, \beta_1) \) we actually have
\[
 \int_{\alpha}^{\beta} \phi_{1,z} \kappa d\eta_1 = \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\eta_2 \circ \eta.
\]
Now since for each \( x_1 \in (\alpha_1, \beta_1) \) there is some \( z \in \mathbb{C}^\times \) such that \( \phi_{1,z}(x_1) \neq 0 \), this shows that the coefficients \( \eta_1 \) and \( \eta_2 \) are related as in the claim. \( \square \)

In particular, note that these relations among our measures show that under the assumptions of Corollary \([8.4]\) for every \( z \in \mathbb{C} \) and each solution \( u_2 \) of \((\tau_2 - z)u = 0\), the function
\[
 u_1(x_1) = \kappa(x_1) u_2(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)
\]
(a solution of \((\tau_1 - z)u = 0\) on \((\alpha_1, \beta_1)\). Moreover, linear independence is preserved under this transformation.

In the remaining part of this section we will prove one more inverse uniqueness result, tailor-made to fit the requirements of the isospectral problem of the Camassa–Holm equation. There, we do not want the measures \( \eta_1 \) and \( \eta_2 \) to necessarily have dense support; hence we can not apply Corollary \([8.4]\). However, we will assume that the intervals and the coefficients on the left-hand side of the differential equation are fixed, i.e.
\[
 a := a_1 = a_2, \quad b := b_1 = b_2, \quad \varsigma := \varsigma_1 = \varsigma_2 \quad \text{and} \quad \chi := \chi_1 = \chi_2,
\]
and that \( \tau_1 \) and \( \tau_2 \) are in the l.p. case at both endpoints. Another crucial additional assumption we will make for this inverse uniqueness result is that the norms of point evaluations (note that the modified Sobolev spaces are the same for both relations) \( ||\delta_c||_{H^1(a,b)} \) are independent of \( c \in (a, b) \). For example this is the case when \( \varsigma \) and \( \chi \) are scalar multiples of the Lebesgue measure, as it is the case for the isospectral
problem of the Camassa–Holm equation. Moreover, we suppose that our real entire solutions \( \phi_{1,z} \) and \( \phi_{2,z} \) coincide at \( z = 0 \), i.e.

\[
(8.4) \quad \phi_{1,0}(x) = \phi_{2,0}(x), \quad x \in (a, b).
\]

As a consequence of these assumptions, the coefficient of the second term on the right-hand side of (7.5) in Theorem 7.3 is the same for both problems. Now the weight measure on the right-hand side of our differential equation is uniquely determined by the spectral measure. In view of application to the isospectral problem of the Camassa–Holm equation we state this result with the assumption that our de Branges functions are of exponential type zero. Of course the same result holds if their quotient is of bounded type in the upper complex half-plane.

**Theorem 8.5.** Suppose that \( E_1(\cdot, c) \) and \( E_2(\cdot, c) \) are of exponential type zero for some \( c \in (a, b) \). If \( \mu_1 = \mu_2 \), then we have \( \varrho_1 = \varrho_2 \) and \( S_1 = S_2 \).

**Proof.** The (remark after the) proof of Lemma 8.1 shows that there is an increasing continuous bijection \( \eta \) from \( \Sigma_1 \) onto \( \Sigma_2 \) such that

\[
B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.
\]

Moreover, the proof of Theorem 8.2 (see equation (8.3)) shows that

\[
\phi_{1, z}(x_1) = \kappa(x_1) \phi_{2, z}(\eta(x_1)), \quad z \in \mathbb{C}, \ x_1 \in \Sigma_1
\]

for some non-zero real function \( \kappa \) on \( \Sigma_1 \). In particular, from Lemma 6.2 we infer for each \( x_1 \in \Sigma_1 \)

\[
\| \delta_{x_1} \|_{H^1(a,b)}^2 = \| F_1 \delta_{x_1} \|_{\mu_1}^2 = \| \kappa(x_1) F_2 \delta_{\eta(x_1)} \|_{\mu_1}^2 = \kappa(x_1)^2 \| \delta_{\eta(x_1)} \|_{H^1(a,b)}^2
\]

and hence \( \kappa(x_1)^2 = 1 \) in view of our additional assumptions. Moreover, Theorem 7.3 shows that \( B_1(x_1) \) and \( B_2(\eta(x_1)) \) actually have the same norm and hence

\[
\phi_{1,0}^{[1]}(x_1) \phi_{1,0}(x_1) = K_1(0, 0, x_1) = K_2(0, 0, \eta(x_1)) = \phi_{1,0}^{[1]}(\eta(x_1)) \phi_{1,0}(\eta(\eta(x_1))).
\]

Now since the function \( \phi_{1,0}^{[1]} \) is strictly increasing on \( (a, b) \) we infer that \( \eta(x_1) = x_1 \), \( x_1 \in \Sigma_1 \) and in particular \( \Sigma_1 = \Sigma_2 \). Hence we even have (note that (8.4) prohibits \( \kappa(x_1) = -1 \) for some \( x_1 \in \Sigma_1 \))

\[(*) \quad \phi_{1,z}(x_1) = \phi_{2,z}(x_1), \quad x_1 \in \Sigma_1, \quad z \in \mathbb{C}.
\]

Moreover, if \((\alpha, \beta)\) is a gap of \( \Sigma_1 \), i.e. \( \alpha, \beta \in \Sigma_1 \) but \((\alpha, \beta) \cap \Sigma_1 = \emptyset \), then both of these functions are solutions to the same differential equation which coincide on the boundary of the gap. Since their difference is a solution of \( \tau_1 u = 0 \) which vanishes on the boundary of the gap, we infer that (*) holds for all \( x_1 \) in the convex hull of \( \Sigma_1 \) in view of (3.3). Now if \( \underline{a} = \inf \Sigma_1 > a \), then \( \zeta + \chi \) is infinite near \( a \) and for each \( z \in \mathbb{C} \) the solutions \( \phi_{1,z} \) and \( \phi_{2,z} \) are scalar multiples of \( u_a \) on \((a, \underline{a})\). Since they are equal in the point \( \underline{a} \) we infer that (*) also holds for all \( x_1 \) below \( \underline{a} \). Similarly, if \( \overline{b} = \sup \Sigma_1 < b \), then the spectrum of \( S_1 \) (and hence also of \( S_2 \)) is purely discrete. Indeed, the solutions \( \psi_{1,b,z} \), \( z \in \mathbb{C} \) of \((\tau_1 - z) u = 0 \) which are equal to \( u_b \) near \( b \) are real entire and lie in \( S_1 \) near \( b \). Now for each eigenvalue \( \lambda \in \mathbb{R}^k \) the solutions \( \phi_{1,\lambda} \) and \( \phi_{2,\lambda} \) are scalar multiples of \( w_\theta \) on \((\overline{b}, b)\). As before we infer that (*) holds for \( z = \lambda \) and all \( x_1 \in (a, b) \). Finally, from the differential equation we get for each \( \alpha \),
\[ \beta \in (a, b) \text{ with } \alpha < \beta \]

\[
\lambda \int_\alpha^\beta \phi_{1,\lambda} d\varrho_1 = -\phi_{1,\lambda}^{[1]}(\beta) + \phi_{1,\lambda}^{[1]}(\alpha) + \int_\alpha^\beta \phi_{1,\lambda} d\chi = \lambda \int_\alpha^\beta \phi_{2,\lambda} d\varrho_2
\]

\[
= \lambda \int_\alpha^\beta \phi_{1,\lambda} d\varrho_2
\]

for each \( \lambda \in \sigma(S_1) \). But this shows \( \varrho_1 = \varrho_2 \) and hence also \( S_1 = S_2 \). Hereby note that for each \( x \in (a, b) \) there is an eigenvalue \( \lambda \in \mathbb{R} \) such that \( \phi_{1,\lambda}(x) \neq 0 \). Indeed, otherwise we had \( f(x) = 0 \) for each \( f \in \mathcal{D}_1 \), which is not possible unless \( \Sigma_1 = \emptyset \). \( \square \)

Note that the condition that the differential expressions are in the l.p. case may be relaxed. For example it is sufficient to assume that \( \tau_j, j = 1, 2 \) are in the l.p. case at \( a \) unless \( \inf \Sigma_j = a \) and in the l.p. case at \( b \) unless \( \sup \Sigma_j = b \). The proof therefore is essentially the same.

### Appendix A. Hilbert spaces of entire functions

In this appendix we will briefly summarize some results of de Branges’ theory of Hilbert spaces of entire functions as far as it is needed for the proof of our inverse uniqueness theorem. For a detailed discussion we refer to de Branges’ book [14].

First of all recall that an analytic function \( N \) in the upper open complex half-plane \( \mathbb{C}^+ \) is said to be of bounded type if it can be written as the quotient of two bounded analytic functions. For such a function the number

\[
\limsup_{y \to \infty} \frac{\ln |N(iy)|}{y} \in [-\infty, \infty)
\]

is referred to as the mean type of \( N \).

A de Branges function is an entire function \( E \), which satisfies the estimate

\[ |E(z)| > |E(z^*)|, \quad z \in \mathbb{C}^+. \]

Associated with such a function is a de Branges space \( B \). It consists of all entire functions \( F \) such that

\[
\int_{\mathbb{R}} \frac{|F(\lambda)|^2}{|E(\lambda)|^2} d\lambda < \infty
\]

and such that \( F/E \) and \( F^# / E \) are of bounded type in \( \mathbb{C}^+ \) with non-positive mean type. Here \( F^# \) is the entire function given by

\[ F^#(z) = F(z^*), \quad z \in \mathbb{C}. \]

Equipped with the inner product

\[ [F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda^*)}{|E(\lambda)|^2} d\lambda, \quad F, G \in B, \]

the vector space \( B \) turns into a reproducing kernel Hilbert space (see [14, Theorem 21]). For each \( \zeta \in \mathbb{C} \), the point evaluation in \( \zeta \) can be written as

\[ F(\zeta) = [F, K(\zeta, \cdot)], \quad F \in B, \]

where the reproducing kernel \( K \) is given by (see [14, Theorem 19])

\[ K(\zeta, z) = \frac{E(z)E^#(\zeta^*) - E(\zeta^*)E^#(z)}{2i(\zeta^* - z)}, \quad \zeta, z \in \mathbb{C}. \]

(A.1)
Note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including norms), the reproducing kernel $K$ is independent of the actual de Branges function.

One of the main results in de Branges’ theory is the subspace ordering theorem: [14, Theorem 35]. For our application we need to slightly weaken the assumptions of this theorem. In order to state it let $E_1, E_2$ be two de Branges functions with no real zeros and $B_1, B_2$ be the associated de Branges spaces.

**Theorem A.1.** Suppose $B_1, B_2$ are homeomorphically embedded in $L^2(\mathbb{R}; \mu)$ for some Borel measure $\mu$ on $\mathbb{R}$. If $E_1/E_2$ is of bounded type in the open upper complex half-plane, then $B_1$ contains $B_2$ or $B_2$ contains $B_1$.

**Proof.** If a de Branges space $B$ is homeomorphically embedded in $L^2(\mathbb{R}; \mu)$, then $B$ equipped with the inner product inherited from $L^2(\mathbb{R}; \mu)$ is a de Branges space itself. In fact, this is easily verified using the characterization of de Branges spaces in [14, Theorem 23]. Hence, without loss of generality we may assume that $B_1, B_2$ are isometrically embedded in $L^2(\mathbb{R}; \mu)$ and thus apply [14, Theorem 35]. Therefore, also note that $F_1/F_2$ is of bounded type in the upper complex half-plane for all $F_1 \in B_1, F_2 \in B_2$ and hence so is the quotient of any corresponding de Branges functions. □

Note that the isometric embedding in [14, Theorem 35] is only needed to deduce that the smaller space is actually a de Branges subspace of the larger one. The inclusion part is valid under much more general assumptions; see [30, Theorem 5] or [31, Theorem 3.5].

Adapting the proof of [14, Theorem 35], one gets a version of de Branges’ ordering theorem, where the bounded type condition is replaced by the assumption that the functions $E_1, E_2$ are of exponential type zero. Actually this has been done in [27] with the spaces $B_1, B_2$ being isometrically embedded in some $L^2(\mathbb{R}; \mu)$. Again this latter assumption can be weakened.

**Theorem A.2.** Suppose $B_1, B_2$ are homeomorphically embedded in $L^2(\mathbb{R}; \mu)$ for some Borel measure $\mu$ on $\mathbb{R}$. If $E_1, E_2$ are of exponential type zero, then $B_1$ contains $B_2$ or $B_2$ contains $B_1$.

**Proof.** As in the proof of Theorem A.1 the claim can be reduced to the case where the de Branges spaces are isometrically embedded in $L^2(\mathbb{R}; \mu)$. Again this last assumption can be weakened.

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