

# Pascal's triangle, Hoggatt matrices, and analogous constructions

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## Abstract

We give an overview about some elementary properties of Hoggatt matrices, which are generalizations of Pascal's triangle, and study  $q$ -analogs and Fibonacci analogs and derive a common generalization.

## 1. Introduction

In [4] Daniel C. Fielder and Cecil O. Alford defined generalizations of Pascal's triangle which they called *Hoggatt triangles*. We give an overview about some elementary properties of these triangles and their  $q$ -analogs and give a common generalization with Fibonacci polynomials. I want to thank Christian Krattenthaler for help with some determinants and hypergeometric identities.

Let us first introduce some notations which emphasize the analogy with Pascal's triangle. Let  $d$  be a positive integer. We write

$$\langle n \rangle_d = \binom{n+d-1}{d} = \frac{(d+1)(d+2)\cdots(d+n-1)}{(n-1)!} \quad (1)$$

and

$$\langle n \rangle_d! = \prod_{j=1}^n \langle j \rangle_d \quad (2)$$

and define

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_d = \frac{\langle n \rangle_d}{\langle k \rangle_d} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_d = \prod_{j=0}^{k-1} \frac{\langle n-j \rangle_d}{\langle k-j \rangle_d} = \frac{\langle n \rangle_d!}{\langle k \rangle_d! \langle n-k \rangle_d!} \quad (3)$$

for  $0 \leq k \leq n$  and  $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_d = 0$  for  $k > n$ .

Following [4] we call the matrix

$$H_d = \left( \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_d \right)_{n,k \geq 0}$$

the *Hoggatt matrix* or *Hoggatt triangle* of order  $d$ .

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*Key words and phrases:* Pascal's triangle,  $q$ -analog, Narayana numbers, semistandard Young tableaux, Fibonomial coefficients, Fibonacci polynomials



It is also well-known and easy to verify that

$$\langle n \rangle_k = \det \begin{pmatrix} \binom{n}{k} & \binom{n+1}{k+1} \\ \binom{n+1}{k} & \binom{n+2}{k+1} \end{pmatrix} = \det \begin{pmatrix} \binom{n}{k} & \binom{n}{k-1} \\ \binom{n}{k+1} & \binom{n}{k} \end{pmatrix} \quad (4)$$

These determinants show that all elements  $\langle n \rangle_k$  are integers.

## 2. Main properties

For each  $n$  the entries  $\langle n \rangle_k$  are *palindromic with center of symmetry at  $\frac{n}{2}$*  since

$$\langle n \rangle_k = \langle n \rangle_{n-k} \quad (5)$$

They are also *unimodal* with center of symmetry at  $\frac{n}{2}$ , which means that

$$\langle n \rangle_0 \leq \langle n \rangle_1 \leq \dots \leq \left\langle \left\lfloor \frac{n}{2} \right\rfloor \right\rangle = \left\langle \left\lceil \frac{n+1}{2} \right\rceil \right\rangle \geq \dots \geq \langle n \rangle_{n-1} \geq \langle n \rangle_n.$$

Due to symmetry it suffices to show that we have  $\langle n \rangle_k \leq \langle n \rangle_{k+1}$  or equivalently

$$\langle k+1 \rangle_d \leq \langle n-k \rangle_d \text{ for } k < \left\lfloor \frac{n}{2} \right\rfloor.$$

This is true because for each  $j$  we have  $k+1+j \leq n-k+j$ .

Let us also mention some alternative formulas.

### Proposition 1

$$\langle n \rangle_d! = \prod_{j=0}^{d-1} \frac{(n+j)!}{(d-j)^{n+j}} \quad (6)$$

and

$$\langle n \rangle_k = \prod_{j=0}^{k-1} \frac{\langle n-j \rangle_d}{\langle k-j \rangle_d} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}} = \prod_{j=0}^{d-1} \frac{\binom{n+d-1}{k+j}}{\binom{n+d-1}{j}} \quad (7)$$

**Proof**

Let  $f(n)$  denote the right-hand side of (6). Then  $f(1) = \prod_{j=0}^{d-1} \frac{(1+j)!}{(d-j)^{1+j}} = 1 = \langle 1 \rangle_d!$  and by induction

$$f(n) = \prod_{j=0}^{d-1} \frac{(n+j)!}{(d-j)^{n+j}} = \prod_{j=0}^{d-1} \frac{(n+j-1)! (n+d-1)!}{(d-j)^{n+j-1} (n-1)! d!} = \binom{n+d-1}{d} f(n-1) = \langle n \rangle_d \langle n-1 \rangle_d!$$

Identities (7) follow from

$$\prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}} = \prod_{j=0}^{d-1} \frac{\frac{n+j}{k} \binom{n+j-1}{k-1}}{\frac{k+j}{k} \binom{k+j-1}{k-1}} = \langle n-1 \rangle_d \prod_{j=0}^{d-1} \frac{n+j}{k+j} = \langle n-1 \rangle_d \frac{\binom{n+d-1}{d}}{\binom{k+d-1}{d}} = \langle n \rangle_d$$

and

$$\prod_{j=0}^{d-1} \frac{\binom{n+d-1}{k+j}}{\binom{n+d-1}{j}} = \prod_{j=0}^{d-1} \frac{j!(n+d-1-j)!}{(k+j)!(n-k+d-1-j)!} = \prod_{j=0}^{d-1} \frac{j!(n+j)!}{(k+j)!(n-k+j)!} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}}$$

An analog of (4) is

**Theorem 2**

$$\langle n \rangle_k = \det \left( \binom{n+i+j}{k+j} \right)_{i,j=0}^{d-1} = \det \left( \binom{n}{k+i-j} \right)_{i,j=0}^{d-1}. \quad (8)$$

These determinants show that all  $\langle n \rangle_k$  are integers.

**Proof**

Let us first prove the left-hand side.

$$\binom{n+i+j}{k+j} = \frac{(n+i+j)!}{(k+j)!(n+i-k)!} = \frac{j!(n+i)!}{(k+j)!(n+i-k)!} \frac{(n+i+j)!}{(j)!(n+i)!} = \frac{j!(n+i)!}{(k+j)!(n+i-k)!} \binom{n+i+j}{j}$$

$$\text{implies } \det \left( \binom{n+i+j}{k+j} \right)_{i,j=0}^{d-1} = \prod_{j=0}^{d-1} \frac{j!(n+j)!}{(k+j)!(n+j-k)!} \det \left( \binom{n+i+j}{j} \right)_{i,j=0}^{d-1}$$

$$\text{with } \prod_{j=0}^{d-1} \frac{j!(n+j)!}{(k+j)!(n+j-k)!} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}} = \langle n \rangle_k.$$

It remains to prove that

$$\det \left( \left( \binom{n+i+j}{j} \right)_{i,j=0}^{d-1} \right) = 1. \quad (9)$$

To this end let  $\Delta$  be the difference operator on the polynomials defined by

$$\Delta f(x) = f(x+1) - f(x). \text{ It satisfies } \Delta \binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1} \text{ and therefore}$$

$$\Delta^k \binom{x}{n} = \binom{x}{n-k}. \text{ Writing } \Delta = E - 1 \text{ with } Ef(x) = f(x+1) \text{ we get}$$

$$\Delta^k = (E - 1)^k = \sum_{j=0}^k (-1)^j \binom{k}{j} E^{k-j} \text{ and thus } \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{x+k-j}{n} = \binom{x}{n-k}.$$

Since  $\binom{x}{n-k} = 0$  for  $k > n$  and  $\binom{x}{n-n} = 1$  the matrix  $\left( (-1)^{i-j} \binom{i}{j} \right)_{i,j=0}^{d-1} \left( \binom{n+i+j}{j} \right)_{i,j=0}^{d-1}$  is

upper triangular with all entries 1 in the main diagonal. This implies

$$\det \left( \left( \binom{n+i+j}{j} \right)_{i,j=0}^{d-1} \right) = \det \left( (-1)^{i-j} \binom{i}{j} \right)_{i,j=0}^{d-1} \left( \binom{n+i+j}{j} \right)_{i,j=0}^{d-1} = 1.$$

To compute the determinant  $\det \left( \left( \binom{n}{k+i-j} \right)_{i,j=0}^{d-1} \right) = \det \left( \left( \binom{n}{k+j-i} \right)_{i,j=1}^d \right)$

we use formula (3.12) in [6] for  $q = 1$ :

$$\det \left( \left( \binom{A}{L_i + j} \right)_{i,j=1}^d \right) = \frac{\prod_{1 \leq i < j \leq n} (L_i - L_j) \prod_{i=1}^n (A + i - 1)!}{\prod_{i=1}^n (L_i + n)! \prod_{i=1}^n (A - L_i - 1)!}.$$

Choosing  $A = n$ ,  $L_i = k - i$  and  $n = d$  this gives

$$\det \left( \left( \binom{n}{k+j-i} \right)_{i,j=1}^d \right) = \frac{\prod_{j=0}^{d-1} j! \prod_{j=0}^{d-1} (n+j)!}{\prod_{j=0}^{d-1} (k+j)! \prod_{j=0}^{d-1} (n-k+j)!} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}} = \left\langle \frac{n}{k} \right\rangle_d. \quad (10)$$

Another determinant representation has been given in [7]:

**Corollary 3**

$$\left\langle \frac{n}{k} \right\rangle_d = \det \left( \left( \binom{n+i}{k+j} \right)_{i,j=0}^{d-1} \right). \quad (11)$$

**Proof**

If we subtract row  $i-1$  from row  $i$  in  $\left(\binom{n+i}{k+j}\right)_{i,j=0}^{d-1}$  the new row  $i$  has the entries

$\binom{n+i-1}{k+j-1}$ . If we do this for  $i = d-1, d-2, \dots, 1$  the new matrix has the first row unchanged

and the rest is the matrix  $\left(\binom{n+i-1}{k+j-1}\right)_{i=1}^{d-1}$ . If we iterate this we arrive at  $\left(\binom{n}{k+j-i}\right)_{i,j=0}^{d-1}$ .

**Remark**

In [14] and [7] these ‘‘MacMahon determinants’’ have been proved with the condensation method (cf. [6], Proposition 10). We will use this method in Theorem 8 for the proof of a  $q$ -analog.

There is a nice generalization of the formula

$$\sum_{n \geq 0} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}. \quad (12)$$

In [12] Robert A. Sulanke introduced Narayana numbers  $N(d, n, k)$  of dimension  $d$ . His results imply Theorem 4 which we state without proof.

**Theorem 4**

$$(1-x)^{dk+1} \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_d x^n = \sum_{j=0}^{(d-1)(k-1)} N(d, k, j) x^j. \quad (13)$$

For  $d=3$  the polynomials  $\sum_{j=0}^{2(k-1)} N(3, k, j) x^j$  are  $1, 1+3x+x^2, 1+10x+20x^2+10x^3+x^4, 1+22x+113x^2+119x^3+113x^4+22x^5+x^6, \dots$

For  $d=2$  we get the Narayana numbers  $N(2, n, k) = \frac{1}{k+1} \binom{n}{k} \binom{n-1}{k} = N_{n,k}$  in the usual notation. In our notation  $N_{n,k} = \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_2$ .

Let us give a direct proof for this case.

**Theorem 5**

$$\frac{\sum_{j=0}^{k-1} \left\langle \begin{matrix} k-1 \\ j \end{matrix} \right\rangle_2 x^j}{(1-x)^{2k+1}} = \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_2 x^n. \quad (14)$$

**Proof**

Since  $\binom{k+1}{j+1} = \frac{k+1}{j+1} \binom{k}{j}$  (14) is equivalent with

$$(1-x)^{2k+1} \sum_{n \geq 0} \binom{n+k}{k} \binom{n+k+1}{k} x^n = \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{k+1}{j+1} x^j. \quad (15)$$

If  $D = \frac{d}{dx}$  denotes the differentiation operator we get

$$\begin{aligned} \sum_{n \geq 0} \binom{n+k}{k} \binom{n+k+1}{k} x^{n+1} &= \frac{D^k}{k!} \sum_{n \geq 0} \binom{n+k}{k} x^{n+k+1} = \frac{D^k}{k!} \frac{x^{k+1}}{(1-x)^{k+1}} = \frac{D^k (1-(1-x))^{k+1}}{k! (1-x)^{k+1}} \\ &= \frac{D^k}{k!} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (1-x)^{j-k-1} = \sum_{j=0}^{k+1} (-1)^{j+k} \binom{k+1}{j} \binom{j-k-1}{k} (1-x)^{j-2k-1} \\ &= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \binom{2k-j}{k} (1-x)^{j-2k-1}. \end{aligned}$$

It remains to show that

$$\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \binom{2k-j}{k} (1-x)^j = \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{k+1}{j+1} x^{j+1}. \quad (16)$$

Comparing the coefficient of  $z^k$  in

$$\begin{aligned} \sum_{j,\ell} \binom{k-1}{j} \binom{k+1}{\ell} x^\ell z^{j+k+1-\ell} &= (1+z)^{k-1} (x+z)^{k+1} = (1+z)^{k-1} (x-1+1+z)^{k+1} = \sum \binom{k+1}{j} (x-1)^j (1+z)^{2k-j} \\ &= \sum_{j,\ell} \binom{k+1}{j} (x-1)^j \binom{2k-j}{\ell} z^\ell \end{aligned}$$

gives (16) and thus (15).

### 3. A combinatorial interpretation

There is an interesting combinatorial interpretation which I owe to Qiaochu Yuan [13]:

$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_d$  is the number of semistandard Young tableaux with shape  $d^k$  (a box with  $d$  columns and  $k$  rows) and entries in  $\{1, \dots, n\}$ . This is equivalent with all  $k \times d$ -matrices  $(a_{i,j})$  with entries in  $\{1, \dots, n\}$ , such that  $a_{i,j} \leq a_{i,j+1}$  and  $a_{i,j} < a_{i+1,j}$  for all  $i, j$ .

For  $d = 1$  this is equivalent with choosing  $k$  different numbers from  $\{1, \dots, n\}$ .

In general the number of such matrices is given by the semistandard hook length formula (cf. [11]) which gives

$$\begin{aligned}
& \prod_{i=1}^k \prod_{j=1}^d \frac{n-i+j}{(k-i)+(d-j)+1} = \prod_{i=1}^k \frac{(n-i+1)(n-i+2)\cdots(n-i+d)}{(k-i+1)(k-i+2)\cdots(k-i+d)} \\
& = \prod_{i=0}^{k-1} \frac{(n-i)(n-i+1)\cdots(n-i+d-1)}{(k-i)(k-i+1)\cdots(k-i+d-1)} = \prod_{i=0}^{k-1} \frac{\binom{n-i+d-1}{d}}{\binom{k-i+d-1}{d}} = \prod_{i=0}^{k-1} \frac{\langle n-i \rangle_d}{\langle k-i \rangle_d} = \left\langle \frac{n}{k} \right\rangle_d.
\end{aligned}$$

The Jacobi-Trudi identities (cf. [11]) give

**Theorem 6**

$$\left\langle \frac{n}{k} \right\rangle_d = \det \left( \left( \binom{n+d+j-i-1}{n-1} \right)_{i,j=0}^{k-1} \right). \quad (17)$$

We now give an elementary

**Proof**

We consider more generally  $\det \left( \left( \binom{x_i+j}{n-1} \right)_{i,j=0}^{k-1} \right)$ . This is a polynomial in the indeterminates

$x_0, \dots, x_{k-1}$  of degree  $\leq n-1$  in each variable. It vanishes for  $x_i = x_j$  which gives the factor

$\prod_{0 \leq i < j \leq k-1} (x_j - x_i)$ . Since all entries of row  $i$  have the factor  $x_i(x_i-1)\cdots(x_i-n+k+1)$

the determinant has the factor  $\prod_{0 \leq i < j \leq k-1} (x_j - x_i) \prod_{i=0}^{k-1} x_i(x_i-1)\cdots(x_i-n+k+1)$ .

This also is a polynomial of degree  $n$  in each variable. Therefore, there exists a constant  $c$  such that

$$\det \left( \left( \binom{x_i+j}{n-1} \right)_{i,j=0}^{k-1} \right) = c \prod_{0 \leq i < j \leq k-1} (x_j - x_i) \prod_{i=0}^{k-1} \binom{x_i}{n-k}. \quad (18)$$

To compute  $c$  we choose  $x_i = n-1-i$ . Then  $\left( \left( \binom{x_i+j}{n-1} \right)_{i,j=0}^{k-1} \right)$  is a right triangle matrix with

$$\binom{x_i+i}{n-1} = \binom{n-1}{n-1} = 1 \text{ und therefore } \det \left( \left( \binom{x_i+j}{n-1} \right)_{i,j=0}^{k-1} \right) = 1.$$

On the right-hand side of (18) we get

$$\begin{aligned}
& c \prod_{0 \leq i < j \leq k-1} (x_j - x_i) \prod_{i=0}^{k-1} \binom{x_i}{n-k} = c \prod_{0 \leq i < j \leq k-1} (i-j) \prod_{i=0}^{k-1} \binom{x_i}{n-k} \\
& = c(-1)^{\binom{k}{2}} \prod_{i=0}^{k-1} i! \prod_{i=0}^{k-1} \binom{n-1-i}{k-1-i}
\end{aligned}$$



Setting  $f(k) = \prod_{i=0}^{k-1} i! \binom{n-1-i}{k-1-i}$  we get

$$\frac{f(k)}{f(k-1)} = \frac{\prod_{i=0}^{k-1} i! \binom{n-1-i}{k-1-i}}{\prod_{i=0}^{k-2} i! \binom{n-1-i}{k-2-i}} = (k-1)! \prod_{i=0}^{k-2} \frac{(k-2-i)!(n-k+1)!}{(k-1-i)!(n-k)!} =$$

$$(k-1)! \prod_{i=0}^{k-2} \frac{n-k+1}{k-1-i} = (n-k+1)^k$$

and therefore  $f(k) = \prod_{j=0}^{k-1} (n-j)^j$ .

Thus

$$\det \left( \binom{x_i + j}{n-1} \right)_{i,j=0}^{k-1} = (-1)^{\binom{k}{2}} \frac{1}{\prod_{j=0}^{k-1} (n-j)^j} \prod_{0 \leq i < j \leq k-1} (x_j - x_i) \prod_{i=0}^{k-1} \binom{x_i}{n-k}. \quad (19)$$

To compute  $\det \left( \binom{d-i+n-1+j}{n-1} \right)_{i,j=1}^k$  we choose  $x_i = d-i+n-1$  and get

$$(-1)^{\binom{k}{2}} \frac{1}{\prod_{j=0}^{k-1} (n-j)^j} \prod_{0 \leq i < j \leq k-1} (i-j) \prod_{i=0}^{k-1} \binom{d-i+n-1}{n-k} = \prod_{j=0}^{k-1} \frac{j!}{(n-j)^j} \frac{(d-i+n-1)!}{(n-k)!(d-i+k-1)!}$$

$$= \prod_{i=0}^{k-1} \frac{\binom{n-i+d-1}{d}}{\binom{k-i+d-1}{d}} \prod_{i=0}^{k-1} \frac{(n-i-1)!i!}{(k-i-1)!(n-i)^i(n-k)!} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_d,$$

because

$$\prod_{i=0}^{k-1} \frac{(n-i-1)!i!}{(k-i-1)!(n-i)^i(n-k)!} = \prod_{i=0}^{k-1} \frac{(n-i-1)!}{(n-i)^i(n-k)!}$$

$$= \frac{(n-1)!(n-2)! \cdots (n-k)!}{(n-1)(n-2)^2 \cdots (n-k+1)^{k-1} (n-k)^k (n-k-1)!} = 1.$$

#### 4. q-analogs

The above constructions have straightforward  $q$ -analogs. For a real number  $q$  with  $|q| < 1$

$$\text{let } [n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}, \quad [n]_q! = \prod_{j=1}^n [j]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \prod_{j=0}^{k-1} \frac{1-q^{n-j}}{1-q^{k-j}}.$$

As is well known the  $q$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad (20)$$

and are therefore polynomials in  $q$  with nonnegative integer coefficients.

For later use let us mention the following  $q$ - analog of the binomial theorem (cf. e.g. [1])

$$\prod_{j=0}^{n-1} (1 - q^j x) = \sum_{j=0}^n (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j. \quad (21)$$

We define  $\langle n \rangle_{d,q} = \begin{bmatrix} n+d-1 \\ d \end{bmatrix}_q$ ,  $\langle n \rangle_{d,q}! = \prod_{j=1}^n \langle j \rangle_{d,q}$  and get

$$\langle n \rangle_{d,q} = \frac{\langle n \rangle_{d,q}!}{\langle k \rangle_{d,q}! \langle n-k \rangle_{d,q}!} = \prod_{j=0}^{k-1} \frac{\langle n-j \rangle_{d,q}}{\langle k-j \rangle_{d,q}} = \prod_{j=0}^{d-1} \frac{\begin{bmatrix} n+j \\ k \end{bmatrix}_q}{\begin{bmatrix} k+j \\ k \end{bmatrix}_q} = \prod_{j=0}^{d-1} \frac{\begin{bmatrix} n+d-1 \\ k+j \end{bmatrix}_q}{\begin{bmatrix} n+d-1 \\ j \end{bmatrix}_q}, \quad (22)$$

From Theorem 7 we see that these are also polynomials in  $q$  with integer coefficients.

For  $d = 2$  we get  $\langle n \rangle_{2,q} = \frac{1}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$ .

This gives the triangle

$$\left( \begin{array}{cccccc} 1 & & & & & \\ 1 & & 1 & & & \\ 1 & & 1+q+q^2 & & 1 & \\ 1 & & (1+q^2)(1+q+q^2) & & (1+q^2)(1+q+q^2) & 1 \\ 1 & & (1+q^2)(1+q+q^2+q^3+q^4) & & (1+q^2)^2(1+q+q^2+q^3+q^4) & (1+q^2)(1+q+q^2+q^3+q^4) & 1 \end{array} \right)$$

As analog of (8) we get

### Theorem 7

$$\langle n \rangle_{d,q} = \frac{\det \left( \begin{bmatrix} n+i+j \\ k+j \end{bmatrix}_q \right)_{i,j=0}^{d-1}}{q^{\binom{d}{2} + \sum_{j=0}^{d-1} j^2}} = \det \left( q^{\binom{i-j}{2}} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=0}^{d-1}. \quad (23)$$

**Proof**

From  $\begin{bmatrix} n+i+j \\ k+j \end{bmatrix}_q = \frac{[n+i+j]_q!}{[k+j]_q![n-k+i]_q!} = \frac{[j]_q![n+i]_q!}{[k+j]_q![n-k+i]_q!} \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q$  we get

$$\det \left( \begin{bmatrix} n+i+j \\ k+j \end{bmatrix}_q \right)_{i,j=0}^{d-1} = \prod_{j=0}^{d-1} \frac{[j]_q![n+j]_q!}{[k+j]_q![n-k+j]_q!} \det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1}$$

$$\text{with } \prod_{j=0}^{d-1} \frac{[j]_q![n+j]_q!}{[k+j]_q![n-k+j]_q!} = \prod_{j=0}^{d-1} \frac{\begin{bmatrix} n+j \\ k \end{bmatrix}_q}{\begin{bmatrix} k+j \\ k \end{bmatrix}_q} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}.$$

It remains to compute  $\det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1}$ .

Using the identity

$$\sum_{\ell=0}^i (-1)^{i-\ell} \begin{bmatrix} i \\ \ell \end{bmatrix}_q q^{\binom{i-\ell}{2}} \begin{bmatrix} n+\ell+j \\ j \end{bmatrix}_q = q^{i(n+i)} \begin{bmatrix} n+j \\ j-i \end{bmatrix}_q \quad (24)$$

we see that  $\left( (-1)^{i-\ell} q^{\binom{i-\ell}{2}} \begin{bmatrix} i \\ \ell \end{bmatrix}_q \right)_{i,\ell=0}^{d-1} \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1}$  is an upper triangular matrix with entries  $q^{in+i^2}$  in the main diagonal. This implies

$$\det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1} = q^{n \binom{d}{2} + \sum_{j=0}^{d-1} j^2}. \quad (25)$$

To prove (24) we consider  $\begin{bmatrix} x \\ k \end{bmatrix}_q = \prod_{j=0}^{k-1} \frac{[x-j]_q}{[k-j]_q} = \prod_{j=0}^{k-1} \frac{q^{x-j}-1}{q^{k-j}-1} = \prod_{j=0}^{k-1} \frac{q^x - q^j}{q^k - q^j}$  as a polynomial in  $q^x$  with coefficients in  $Q(q)$ . If we define the operator  $E$  on these polynomials by

$Ef(q^x) = f(q^{x+1})$  then we get  $E \begin{bmatrix} x \\ n \end{bmatrix} = \begin{bmatrix} x+1 \\ n \end{bmatrix}$ . Let now  $\Delta = E - 1$  be the difference operator.

We have  $\Delta \begin{bmatrix} x \\ n \end{bmatrix} = \begin{bmatrix} x+1 \\ n \end{bmatrix} - \begin{bmatrix} x \\ n \end{bmatrix} = q^{x-n+1} \begin{bmatrix} x \\ n-1 \end{bmatrix}$ .

More generally we get by induction

$$(E-1)(E-q) \cdots (E-q^{k-1}) \begin{bmatrix} x \\ n \end{bmatrix}_q = q^{k(x+k-n)} \begin{bmatrix} x \\ n-k \end{bmatrix}_q,$$

because

$$\begin{aligned} (E - q^{k-1})q^{(k-1)(x+k-1-n)} \begin{bmatrix} x \\ n-k+1 \end{bmatrix}_q &= q^{(k-1)(x+k-n)} \begin{bmatrix} x+1 \\ n-k+1 \end{bmatrix}_q - q^{(k-1)(x+k-n)} \begin{bmatrix} x \\ n-k+1 \end{bmatrix}_q \\ &= q^{(k-1)(x+k-n)} q^{x+k-n} \begin{bmatrix} x \\ n-k \end{bmatrix}_q = q^{k(x+k-n)} \begin{bmatrix} x \\ n-k \end{bmatrix}_q. \end{aligned}$$

$$\text{Finally by (21) we have } (E-1)(E-q)\cdots(E-q^{k-1}) = \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} E^{k-j}.$$

$$\text{For the computation of } \det \left( q^{\binom{i-j}{2}} \begin{bmatrix} n \\ k-j+i \end{bmatrix}_q \right)_{i,j=0}^{d-1} = \det \left( q^{\binom{j-i}{2}} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d \text{ we use [6],}$$

formula (3.12):

$$\det \left( q^{jL_i} \begin{bmatrix} A \\ L_i + j \end{bmatrix}_q \right)_{i,j=1}^n = q^{\sum_{i=1}^n iL_i} \frac{\prod_{1 \leq i < j \leq n} [L_i - L_j]_q \prod_{i=1}^n [A + i - 1]_q!}{\prod_{i=1}^n [L_i + n]_q! \prod_{i=1}^n [A - L_i - 1]_q!}.$$

First we write

$$\begin{aligned} \det \left( q^{\binom{j-i}{2}} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d &= \det \left( q^{\binom{i+1}{2} + \binom{j}{2} - ij} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d = q^{\sum_{j=1}^d j^2} \det \left( q^{-ij} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d \\ \det \left( q^{\binom{j-i}{2}} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d &= q^{\sum_{j=1}^d j^2 - k \binom{d+1}{2}} \det \left( q^{(k-i)j} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d. \end{aligned}$$

Then we choose  $A = n$ ,  $L_i = k - i$ ,  $n = d$  and get

$$\begin{aligned} q^{\sum_{j=1}^d j^2 - k \binom{d+1}{2}} \det \left( q^{(k-i)j} \begin{bmatrix} n \\ k-i+j \end{bmatrix}_q \right)_{i,j=1}^d &= q^{\sum_{j=0}^d j^2 - k \binom{d+1}{2}} q^{\sum_{i=1}^d (k-i)i} \frac{\prod_{j=0}^{d-1} [j]_q! \prod_{j=0}^{d-1} [n+j]_q!}{\prod_{j=0}^{d-1} [k+j]_q! \prod_{j=0}^{d-1} [n-k+j]_q!} \\ &= \frac{\prod_{j=0}^{d-1} [j]_q! \prod_{j=0}^{d-1} [n+j]_q!}{\prod_{j=0}^{d-1} [k+j]_q! \prod_{j=0}^{d-1} [n-k+j]_q!} = \prod_{j=0}^{d-1} \frac{\begin{bmatrix} n+j \\ k \end{bmatrix}_q}{\begin{bmatrix} k+j \\ k \end{bmatrix}_q} = \left\langle n \right\rangle_{d,q} \left\langle k \right\rangle_{d,q}. \end{aligned}$$

**Theorem 8**

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q} = q^{-\binom{d}{2}(n-k)} \det \left( \left[ \begin{matrix} n+i \\ k+j \end{matrix} \right]_q \right)_{i,j=0}^{d-1}. \quad (26)$$

**Proof**

We use Dodgson's condensation method (cf. [6], Proposition 10, and [14]). Let

$$X(d, n, k) = \det \left( \left[ \begin{matrix} n+i \\ k+j \end{matrix} \right]_q \right)_{i,j=0}^{d-1}. \quad \text{By condensation we get}$$

$$X(d, n, k) = \frac{X(d-1, n, k)X(d-1, n+1, k+1) - X(d-1, n+1, k)X(d-1, n, k+1)}{X(d-2, n+1, k+1)}.$$

The same identity holds for  $X(d, n, k) = q^{\binom{d}{2}(n-k)} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}$ . For we have

$$\frac{q^{\binom{d-1}{2}(n-k)} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d-1,q} q^{\binom{d-1}{2}(n-k)} \left\langle \begin{matrix} n+1 \\ k+1 \end{matrix} \right\rangle_{d-1,q}}{q^{\binom{d}{2}(n-k)} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q} q^{\binom{d-2}{2}(n-k)} \left\langle \begin{matrix} n+1 \\ k+1 \end{matrix} \right\rangle_{d-2,q}} = \frac{1}{q^{n-k}} \frac{[n-k+d-1]_q}{[d-1]_q}$$

and

$$\frac{q^{\binom{d-1}{2}(n-k)} \left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle_{d-1,q} q^{\binom{d-1}{2}(n-k)} \left\langle \begin{matrix} n \\ k+1 \end{matrix} \right\rangle_{d-1,q}}{q^{\binom{d}{2}(n-k)} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q} q^{\binom{d-2}{2}(n-k)} \left\langle \begin{matrix} n+1 \\ k+1 \end{matrix} \right\rangle_{d-2,q}} = \frac{1}{q^{n-k}} \frac{[n-k]_q}{[d-1]_q},$$

which implies

$$\frac{X(d-1, n, k)X(d-1, n+1, k+1) - X(d-1, n+1, k)X(d-1, n, k+1)}{X(d, n, k)X(d-2, n+1, k+1)} = \frac{[n-k+d-1]_q - [n-k]_q}{q^{n-k}[d-1]_q} = 1.$$

Identity (26) holds for  $d=0$  and  $d=1$  and therefore (26) holds for all  $d$  by induction.

A  $q$ - analog of (17) is

**Theorem 9**

$$\det \left( \left[ \begin{matrix} n+d+j-i-1 \\ n-1 \end{matrix} \right]_q \right)_{i,j=0}^{k-1} = q^{\binom{k}{2}d} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}. \quad (27)$$

**Proof**

Here we use [6], formula (3.11):

$$\det \left( \begin{bmatrix} L_i + A + j \\ L_i + j \end{bmatrix} \right)_{i,j=1}^n = q^{\sum_{i=1}^n (i-1)(L_i+i)} \frac{\prod_{1 \leq i < j \leq n} [L_i - L_j]_q \prod_{i=1}^n [L_i + A + 1]_q!}{\prod_{i=1}^n [L_i + n]_q! \prod_{i=1}^n [A + 1 - i]_q!}.$$

We choose  $L_i = d - i$ ,  $A = n - 1$  and  $n = k$  and get

$$\begin{aligned} \det \left( \begin{bmatrix} n + d + j - i - 1 \\ n - 1 \end{bmatrix} \right)_{i,j=0}^{k-1} &= q^{\binom{k}{2}_d} \prod_{j=1}^k \frac{[j-1]_q! [d+n-j]_q!}{[d+k-j]_q! [n-j]_q!} = q^{\binom{k}{2}_d} \prod_{j=0}^{k-1} \frac{[k-1-j]_q! [d+n-1-j]_q!}{[d+k-1-j]_q! [n-1-j]_q!} \\ &= q^{\binom{k}{2}_d} \prod_{j=0}^{k-1} \frac{\begin{bmatrix} n-j+d-1 \\ d \end{bmatrix}_q}{\begin{bmatrix} k-j+d-1 \\ d \end{bmatrix}_q} = q^{\binom{k}{2}_d} \prod_{j=0}^{d-1} \frac{\begin{bmatrix} n+j \\ k \end{bmatrix}_q}{\begin{bmatrix} k+j \\ k \end{bmatrix}_q} = q^{\binom{k}{2}_d} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}. \end{aligned}$$

**Remark**

Since the determinant (27) is closely related to semistandard Young Tableaux it would make

sense from this point of view to define  $q$ -Hoggatt matrices with entries  $q^{\binom{k}{2}_d} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}$  instead

of  $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q}$ . For  $d = 1$  this means to replace the  $q$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  by their

companion form  $q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$ . This would give the nice generating function

$$\sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (1+x)(1+qx) \cdots (1+q^{n-1}x) \text{ for row } n \text{ of the matrix.}$$

For  $d = 2$  we would get

$$\sum_{k=0}^n q^{k(k+1)} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{2,q} = C_{n+1}(q) = \frac{1}{[n+2]} \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix},$$

which is a nice  $q$ - analog of the fact that the sum of the Narayana numbers  $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_2$  are the

Catalan numbers  $C_{n+1}$ .

As  $q$ - analog of (13) we state

## Conjecture 10

For positive integers  $d, k$  we have

$$(1-x)(1-qx)\cdots(1-q^{dk}x)\sum_{n\geq 0}\left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{d,q} x^n = \sum_{j=0}^{(d-1)(k-1)} N(d,k,j,q)x^j. \quad (28)$$

where the coefficients  $N(d,k,j,q)$  are palindromic polynomials in  $q$  with nonnegative coefficients.

For  $d=2$  this reduces to

$$(1-x)(1-qx)\cdots(1-q^{2k}x)\sum_{n\geq 0}\left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{2,q} x^n = \sum_{j=0}^{k-1} q^{j(j+1)} \left\langle \begin{matrix} k-1 \\ j \end{matrix} \right\rangle_{2,q} x^j. \quad (29)$$

The sums  $\sum_{j=0}^{(d-1)(k-1)} N(d,k,j,q) = C_n^{(d)}(q) = [dn]_q! \prod_{j=0}^{d-1} \frac{[j]_q!}{[n+j]_q!}$  are the  $d$ -dimensional  $q$ -Catalan numbers.

## 5. Fibonacci-Hoggatt triangles

Let  $F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j}$  denote the Fibonacci numbers which satisfy  $F_n = F_{n-1} + F_{n-2}$  with

initial values  $F_0 = 0$  and  $F_1 = 1$  and  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  with  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ .

Let us write  $(n)_F = F_n$ ,  $(n)_F! = F_n F_{n-1} \cdots F_1$  and define the Fibonomial coefficients by

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1} = \frac{(n)_F!}{(k)_F! (n-k)_F!}.$$

The first terms are (cf. OEIS [8], A010048)

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 2 & 2 & 1 & & \\ 1 & 3 & 6 & 3 & 1 & \\ 1 & 5 & 15 & 15 & 5 & 1 \end{pmatrix}.$$

The Fibonacci numbers satisfy

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

$$\text{From } \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-k} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} F_{n-k-1} & F_{n-k} \\ F_{n-k} & F_{n-k+1} \end{pmatrix} \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

we get by comparing the top right elements

$$F_n = F_{k+1}F_{n-k} + F_{n-k-1}F_k.$$

This is equivalent with

$$\binom{n}{k}_F = F_{k+1} \binom{n-1}{k}_F + F_{n-k-1} \binom{n-1}{k-1}_F, \quad (30)$$

which shows that the Fibonomials  $\binom{n}{k}_F$  are nonnegative integers.

L. Carlitz [3] found the analog of (21)

$$h_n(x) = \sum_{j=0}^n (-1)^{\binom{j+1}{2}} \binom{n}{j}_F x^j = \prod_{j=0}^{n-1} (1 - \alpha^{n-j-1} \beta^j x). \quad (31)$$

Let us reproduce his proof. In formula (21) we set  $q = \frac{\beta}{\alpha}$  and get

$$\prod_{j=0}^{n-1} \left( 1 - \frac{\beta^j}{\alpha^j} x \right) = \sum_{j=0}^n (-1)^j \left( \frac{\beta}{\alpha} \right)^{\binom{j}{2}} \left[ \begin{matrix} n \\ j \end{matrix} \right]_{\frac{\beta}{\alpha}} x^j$$

with

$$\left[ \begin{matrix} n \\ j \end{matrix} \right]_q = \frac{(1-q^n) \cdots (1-q^{n-j+1})}{(1-q^j) \cdots (1-q)} = \frac{1}{\alpha^{(n-j)j}} \frac{(\alpha^n - \beta^n) \cdots (\alpha^{n-j+1} - \beta^{n-j+1})}{(\alpha^j - \beta^j) \cdots (\alpha - \beta)} = \alpha^{j^2 - nj} \binom{n}{j}_F.$$

This gives

$$\prod_{j=0}^{n-1} (1 - \alpha^{-j} \beta^j x) = \sum_{j=0}^n (-1)^j \alpha^{-\binom{j}{2}} \beta^{\binom{j}{2}} \alpha^{j^2 - nj} \binom{n}{j}_F x^j.$$

Replacing  $x \rightarrow \alpha^{n-1} x$  we get

$$\prod_{j=0}^{n-1} (1 - \alpha^{n-j-1} \beta^j x) = \sum_{j=0}^n (-1)^j (\alpha\beta)^{\binom{j}{2}} \binom{n}{j}_F x^j = \sum_{j=0}^n (-1)^{\binom{j+1}{2}} \binom{n}{j}_F x^j.$$

Since  $\alpha^n + \beta^n = L_n$  are the Lucas numbers 2, 1, 3, 4, 7, 11, 18,  $\dots$ , we see that

$$\begin{aligned} h_n(x) &= \prod_{j=0}^{n-1} (1 - \alpha^{n-j-1} \beta^j x) = (1 - \alpha^{n-1} x) (1 - \beta^{n-1} x) \prod_{j=1}^{n-2} (1 + \alpha^{n-2-j} \beta^{j-1} x) \\ &= (1 - \alpha^{n-1} x) (1 - \beta^{n-1} x) \prod_{j=0}^{n-3} (1 + \alpha^{n-3-j} \beta^j x) = (1 - L_{n-1} x + (-1)^{n-1} x^2) p_{n-2}(-x). \end{aligned}$$



This gives

$$h_k(x) = \sum_{j=0}^k (-1)^{\binom{j+1}{2}} \binom{k}{j}_F x^j = \prod_{j=0}^{\lfloor \frac{k}{2} \rfloor} u_{k-2j}((-1)^j x) \quad (32)$$

with

$$\begin{aligned} u_k(x) &= 1 - L_{k-1}x + (-1)^{k-1}x^2, \\ u_1(x) &= 1 - x, \quad u_0(x) = 1. \end{aligned} \quad (33)$$

As analog of (12) we get

$$\frac{1}{h_{k+1}(x)} = \frac{1}{\sum_{j=0}^{k+1} (-1)^{\binom{j+1}{2}} \binom{k+1}{j}_F x^j} = \sum_{n \geq 0} \binom{n+k}{k}_F x^n. \quad (34)$$

**Proof**

Since  $h_1(x) = 1 - x$  and  $h_2(x) = 1 - x - x^2$  identity (34) is true for  $k = 0$  and  $k = 1$ .

By (32) identity (34) is equivalent with  $u_{k+1}(x) \sum_{n \geq 0} \binom{n+k}{k}_F x^n = \sum_{n \geq 0} \binom{n+k-2}{k-2}_F (-x)^n$ , i.e.

$$(1 - L_k x + (-1)^k x^2) \sum_{n \geq 0} \frac{F_{n+1} \cdots F_{n+k}}{F_1 \cdots F_k} x^n = \sum_{n \geq 0} \frac{F_{n+1} \cdots F_{n+k-2}}{F_1 \cdots F_{k-2}} (-x)^n.$$

This is equivalent with

$$F_{n+k-1} F_{n+k} - L_k F_n F_{n+k-1} + (-1)^k F_{n-1} F_n = (-1)^n F_{k-1} F_k,$$

which is easily verified.

In [5] the authors studied Fibo-Narayana numbers defined by

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{2,F} = \frac{1}{F_{k+1}} \binom{n}{k}_F \binom{n+1}{k}_F = \frac{1}{F_{n+1}} \binom{n+1}{k}_F \binom{n+1}{k+1}_F.$$

Let us more generally define Fibo-Hoggatt numbers

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F} = \prod_{j=0}^{k-1} \frac{\langle n-j \rangle_{d,F}}{\langle k-j \rangle_{d,F}} \quad (35)$$

with  $\langle n \rangle_{d,F} = \binom{n+d-1}{d}_F$  and consider the corresponding Fibonacci-Hoggatt matrices

$$H_{d,F} = \left( \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F} \right)_{n,k \geq 0}.$$

As in (7) we get

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}_F}{\binom{k+j}{k}_F} = \prod_{j=0}^{d-1} \frac{\binom{n+d-1}{k+j}_F}{\binom{n+d-1}{j}_F}. \quad (36)$$

For example for  $d = 3$  we get

$$\left( \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{3,F} \right)_{i,j=0}^5 = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 15 & 15 & 1 & & \\ 1 & 60 & 300 & 60 & 1 & \\ 1 & 260 & 5200 & 5200 & 260 & 1 \end{pmatrix}.$$

As an analog of the first identity (8) we get

### Theorem 11

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F} = \frac{\det \left( \binom{n+i+j}{k+j}_F \right)_{i,j=0}^{d-1}}{\det \left( \binom{n+i+j}{j}_F \right)_{i,j=0}^{d-1}}. \quad (37)$$

### Proof

This follows from

$$\begin{aligned} \det \left( \binom{n+i+j}{k+j}_F \right)_{i,j=0}^{d-1} &= \det \left( \frac{(j)_F!(n+i)_F!}{(k+j)_F!(n-k+i)_F!} \binom{n+i+j}{j}_F \right)_{i,j=0}^{d-1} \\ &= \prod_{j=0}^{d-1} \frac{(j)_F!(n+j)_F!}{(k+j)_F!(n-k+j)_F!} \det \left( \binom{n+i+j}{j}_F \right)_{i,j=0}^{d-1} \end{aligned}$$

if we observe that

$$\prod_{j=0}^{d-1} \frac{(j)_F! (n+j)_F!}{(k+j)_F! (n-k+j)_F!} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}_F}{\binom{k+j}{k}_F} = \left\langle \frac{n}{k} \right\rangle_{d,F}.$$

If we set  $a(d, n) = \det \left( \binom{n+i+j}{j}_F \right)_{i,j=0}^{d-1}$  then we get  $a(2, n) = F_n$ , but for  $n > 3$  no other interpretation seems to be known. For example for  $d = 3$  we get 1, 5, 7, 53, 187, 853, ...

As analog of (10) we get as special case of Theorem 15

### Theorem 12

$$\det \left( (-1)^{\binom{i-j}{2}} \binom{n}{k-i+j}_F \right)_{i,j=0}^{d-1} = \left\langle \frac{n}{k} \right\rangle_{d,F}. \quad (38)$$

This shows that all  $\left\langle \frac{n}{k} \right\rangle_{d,F}$  are positive integers.

There is also a nice analog of (17):

### Theorem 13

$$\left\langle \frac{n}{k} \right\rangle_{d,F} = (-1)^{d \binom{k}{2}} \det \left( \binom{n+d+j-i-1}{n-1}_F \right)_{i,j=0}^{k-1}. \quad (39)$$

### Proof

By Binet's formula we have  $F_n = \alpha^{n-1} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)}$  with  $\frac{\beta}{\alpha} = -\frac{3-\sqrt{5}}{2} = -\frac{1}{\alpha^2}$ . If we set  $\frac{\beta}{\alpha} = q$ ,

then we get

$$F_n = (-q)^{\frac{1-n}{2}} \frac{1-q^n}{1-q}. \quad (40)$$

This implies

$$\binom{n}{k}_F = (-q)^{\frac{k^2-nk}{2}} \left[ \frac{n}{k} \right]_q. \quad (41)$$

We can now formulate  $\det \left( \left( \begin{matrix} n+d+j-i-1 \\ n-1 \end{matrix} \right)_F \right)_{i,j=0}^{k-1}$  in terms of  $q$ -binomial coefficients as

$$\begin{aligned} \det \left( \left( \begin{matrix} n+d+j-i-1 \\ n-1 \end{matrix} \right)_F \right)_{i,j=0}^{k-1} &= \det \left( (-q)^{\frac{(1-n)(d+j-i)}{2}} \left[ \begin{matrix} n+d+j-i-1 \\ n-1 \end{matrix} \right] \right)_{i,j=0}^{k-1} \\ &= (-q)^{\frac{(1-n)dk}{2}} \det \left( \left[ \begin{matrix} n+d+j-i-1 \\ d+j-i \end{matrix} \right] \right)_{i,j=0}^{k-1}. \end{aligned}$$

Using the above result we get by (41)

$$\begin{aligned} \det \left( \left( \begin{matrix} n+d+j-i-1 \\ n-1 \end{matrix} \right)_F \right)_{i,j=0}^{k-1} &= (-q)^{\frac{(1-n)dk}{2}} q^{\binom{k}{2}_d} \prod_{j=0}^{k-1} \left[ \begin{matrix} n-j+d-1 \\ d \\ k-j+d-1 \\ d \end{matrix} \right]_{-q} \\ &= (-q)^{\frac{(1-n)dk}{2}} q^{\binom{k}{2}_d} (-q)^{\frac{dnk-dk^2}{2}} \prod_{j=0}^{k-1} \left( \frac{\begin{matrix} n-j+d-1 \\ d \end{matrix}}{\begin{matrix} k-j+d-1 \\ d \end{matrix}} \right)_F = (-1)^{\binom{k}{2}_d} \prod_{j=0}^{k-1} \left( \frac{\begin{matrix} n-j+d-1 \\ d \end{matrix}}{\begin{matrix} k-j+d-1 \\ d \end{matrix}} \right)_F = (-1)^{\binom{k}{2}_d} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F} \end{aligned}$$

As analogs of (28) and (29) we state

#### Conjecture 14

$$\left( \sum_{j=0}^{dk+1} (-1)^{\binom{j+1}{2}} \binom{dk+1}{j}_F x^j \right) \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{d,F} x^n = N(d, k, x) \text{ is a polynomial of degree } (d-1)(k-1).$$

For  $d=2$  we get more precisely

$$\left( \sum_{j=0}^{2k+1} (-1)^{\binom{j+1}{2}} \binom{2k+1}{j}_F x^j \right) \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{2,F} x^n = \sum_j \left\langle \begin{matrix} k-1 \\ j \end{matrix} \right\rangle_{2,F} x^j.$$

#### Remark

The polynomials  $N(d, k, x)$  in general are not symmetric or unimodal. For example for  $d=3$  and  $k=5$  Mathematica gives

$$N(3, 5, x) = 1 + 105x + 9450x^2 - 7917x^3 + 166712x^4 + 7917x^5 + 9450x^6 - 105x^7 + x^8.$$

## 6. A common generalization

A generalization which contains all above cases is given by the Fibonacci polynomials

$$F_n(s, t) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} t^j s^{n-2j}. \quad (42)$$

They satisfy  $F_n(s, t) = sF_{n-1}(s, t) + tF_{n-2}(s, t)$  with initial values  $F_0(s, t) = 0$  and  $F_1(s, t) = 1$ .

For  $s = 2$  and  $t = -1$  we have  $F_n(2, -1) = n$  which gives the original Hoggatt matrices, for  $s = 1 + q$  and  $t = -q$  we get  $F_n(1 + q, -q) = [n]_q$  which gives the  $q$ -analogs and for  $s = t = 1$  we get the Fibonacci analogs.

Binet's formulae give  $F_n(s, t) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  with  $\alpha = \frac{s + \sqrt{s^2 + 4t}}{2}$  and  $\beta = \frac{s - \sqrt{s^2 + 4t}}{2}$ .

We also have

$$\begin{pmatrix} 0 & 1 \\ t & s \end{pmatrix}^n = \begin{pmatrix} tF_{n-1}(s, t) & F_n(s, t) \\ tF_n(s, t) & F_{n+1}(s, t) \end{pmatrix},$$

From

$$\begin{pmatrix} tF_{n-1}(s, t) & F_n(s, t) \\ tF_n(s, t) & F_{n+1}(s, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t & s \end{pmatrix}^{n-k} \begin{pmatrix} 0 & 1 \\ t & s \end{pmatrix}^k = \begin{pmatrix} tF_{n-k-1}(s, t) & F_{n-k}(s, t) \\ tF_{n-k}(s, t) & F_{n-k+1}(s, t) \end{pmatrix} \begin{pmatrix} tF_k(s, t) & F_k(s, t) \\ tF_{k+1}(s, t) & F_{k+1}(s, t) \end{pmatrix}$$

we get by comparing the top right elements

$$F_n(s, t) = F_{k+1}(s, t)F_{n-k}(s, t) + tF_{n-k-1}(s, t)F_k(s, t).$$

This is equivalent with

$$\binom{n}{k}_{F(s,t)} = F_{k+1}(s, t) \binom{n-1}{k}_{F(s,t)} + tF_{n-k-1}(s, t) \binom{n-1}{k-1}_{F(s,t)}, \quad (43)$$

which shows that the Fibonomials

$$\binom{n}{k}_{F(s,t)} = \prod_{j=0}^{k-1} \frac{F_{n-j}(s, t)}{F_{k-j}(s, t)} \quad (44)$$

are polynomials in  $s, t$  with nonnegative integer coefficients.

A combinatorial proof of this fact has been given in [2] and an arithmetic one in [10].

The first terms of the Fibonomial triangle are

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & s & 1 \\ 1 & s^2+t & s^2+t & 1 \\ 1 & s(s^2+2t) & (s^2+t)(s^2+2t) & s(s^2+2t) & 1 \\ 1 & s^4+3s^2t+t^2 & (s^2+2t)(s^4+3s^2t+t^2) & (s^2+2t)(s^4+3s^2t+t^2) & s^4+3s^2t+t^2 & 1 \end{pmatrix}$$

As above we get

$$h_k(x, s, t) = \sum_{j=0}^k (-1)^{\binom{j+1}{2}} t^{\binom{j}{2}} \binom{k}{j}_{F(s,t)} x^j = \prod_{j=0}^{\lfloor \frac{k}{2} \rfloor} u_{k-2j}((-t)^j x, s, t) \quad (45)$$

with  $u_n(x, s, t) = 1 - L_{n-1}(s, t)x + (-t)^{n-1}x^2$  for  $n > 1$  and  $u_0(x, s, t) = 1$  and  $u_1(x, s, t) = 1 - x$ ,

where the Lucas polynomials  $L_n(s, t)$  satisfy  $L_n(s, t) = sL_{n-1}(s, t) + tL_{n-2}(s, t)$  with initial values  $L_0(s, t) = 2$  and  $L_1(s, t) = s$ .

This implies as before that

$$\frac{1}{h_{k+1}(x, s, t)} = \frac{1}{\sum_{j=0}^{k+1} (-1)^{\binom{j+1}{2}} t^{\binom{j}{2}} \binom{k+1}{j}_{F(s,t)} x^j} = \sum_{n \geq 0} \binom{n+k}{k}_{F(s,t)} x^n. \quad (46)$$

The Hoggatt coefficients can be defined by

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d, F(s,t)} = \prod_{j=0}^{k-1} \frac{\binom{n-j+d-1}{d}_{F(s,t)}}{\binom{k-j+d-1}{d}_{F(s,t)}}. \quad (47)$$

An extension of a result which in [5] has been obtained for  $d = 2$  is

**Theorem 15**

$$\det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d, F(s,t)}. \quad (48)$$

This implies that all Hoggatt coefficients are polynomials in  $s, t$  with integer coefficients.

**Proof**

By Binet's formula we have  $F_n(s,t) = \alpha^{n-1} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)}$  with  $\frac{\beta}{\alpha} = -\frac{t}{\alpha^2}$ . If we set  $\frac{\beta}{\alpha} = q$ , then

we get

$$F_n = \left(-\frac{q}{t}\right)^{\frac{1-n}{2}} \frac{1-q^n}{1-q}. \quad (49)$$

This implies

$$\binom{n}{k}_{F(s,t)} = \left(-\frac{q}{t}\right)^{\frac{k^2-nk}{2}} \left[ \begin{matrix} n \\ k \end{matrix} \right]_q. \quad (50)$$

We can now formulate  $\det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1}$  in terms of  $q$ -binomial coefficients

as

$$\begin{aligned} \det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} &= \det \left( (-t)^{\binom{i-j}{2}} \left(-\frac{q}{t}\right)^{\frac{(k-j+i)(k-j+i-n)}{2}} \left[ \begin{matrix} n \\ k-j+i \end{matrix} \right]_q \right)_{i,j=0}^{d-1} \\ &= (-1)^{\frac{k(k-n)d}{2} + \frac{k(n-k)d}{2} + \frac{k(k-n)d}{2}} \det \left( q^{\binom{i-j}{2}} \left[ \begin{matrix} n \\ k-j+i \end{matrix} \right]_q \right)_{i,j=0}^{d-1}. \end{aligned}$$

The last determinant has been computed above as

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,q} = \prod_{j=0}^{d-1} \frac{\left[ \begin{matrix} n+j \\ k \end{matrix} \right]_q}{\left[ \begin{matrix} k+j \\ k \end{matrix} \right]_q} = \prod_{j=0}^{d-1} \frac{\left(-\frac{t}{q}\right)^{\frac{k^2-(n+j)k}{2}} \binom{n+j}{k}_{F(s,t)}}{\left(-\frac{t}{q}\right)^{\frac{k^2-(k+j)k}{2}} \binom{n+j}{k}_{F(s,t)}} = \left(-\frac{t}{q}\right)^{\frac{-k(n-k)d}{2}} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,t)}.$$

This gives

$$\det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} = (-1)^{\frac{k(k-n)d}{2} + \frac{k(n-k)d}{2} + \frac{k(k-n)d}{2}} \left(-\frac{t}{q}\right)^{\frac{-k(n-k)d}{2}} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,t)} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,t)}.$$

The same proof as above gives

**Theorem 16**

$$\det \left( \binom{n+d+j-i-1}{n-1} \right)_{F(s,t)}^{k-1}_{i,j=0} = (-t)^{\binom{k}{2}^d} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,t)}. \quad (51)$$

Let us also mention

**Theorem 17**

$$\frac{\det \left( \binom{n+i+j}{k+j} \right)_{F(s,t)}^{d-1}_{i,j=0}}{\det \left( \binom{n+i+j}{j} \right)_{F(s,t)}^{d-1}_{i,j=0}} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,t)} \quad (52)$$

and

$$\frac{\det \left( \binom{n+i+k}{k+j} \right)_{F(s,t)}^{d-1}_{i,j=0}}{\det \left( \binom{n+i}{j} \right)_{F(s,t)}^{d-1}_{i,j=0}} = \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{d,F(s,t)}. \quad (53)$$

The proof follows in the same way as in Theorem 11.

Let us consider two extreme special cases.

Taking limits for  $t \rightarrow 0$  we get  $F_n(s, 0) = s^{n-1}$  for  $n \geq 1$  and  $F_0(s, 0) = 0$ .

For the Fibonomials we get  $\binom{n}{k}_{F(s,0)} = s^{k(n-k)}$  for  $0 \leq k \leq n$ .

This follows by induction from (43). From (47) we get that the entries of the Hoggatt matrices are

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{d,F(s,0)} = s^{dk(n-k)}.$$

Taking limits for  $s \rightarrow 0$  gives more interesting results.

$F_{2n}(0, t) = 0$  and  $F_{2n+1}(0, t) = t^n$  by the definition of the Fibonacci polynomials.

The Lucas polynomials reduce to  $L_{2n}(0, t) = 2t^n$  and  $L_{2n+1}(0, t) = 0$ .



Therefore we get

$$h_{2n}(x, 0, t) = (t^{2n-1}x^2 - 1)^n \quad (54)$$

and

$$h_{2n+1}(x, 0, t) = (1 - t^n x)^{n+1} (1 + t^n x)^n. \quad (55)$$

Comparing with (45) we get

$$\begin{aligned} \binom{2n}{2j}_{F(0,t)} &= \binom{n}{j} t^{2j(n-j)}, & \binom{2n}{2j+1}_{F(0,t)} &= 0, \\ \binom{2n+1}{2j}_{F(0,t)} &= \binom{n}{j} t^{j(2n+1-2j)}, & \binom{2n+1}{2j+1}_{F(0,t)} &= \binom{n}{j} t^{(2n-1-2j)j+n}. \end{aligned} \quad (56)$$

For example

$$\left( \binom{n}{k}_{F(0,t)} \right)_{n,k=0}^7 = \begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 1 & t & t & 1 & & & & \\ 1 & 0 & 2t^2 & 0 & 1 & & & \\ 1 & t^2 & 2t^3 & 2t^3 & t^2 & 1 & & \\ 1 & 0 & 3t^4 & 0 & 3t^4 & 0 & 1 & \end{pmatrix}. \quad (57)$$

Let us also mention the Hoggatt triangle  $H_{2,F(0,t)}$ .

Here we get by (56)

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{2,F(0,t)} = \prod_{j=0}^{k-1} \frac{\binom{n+1-j}{2}_{F(0,t)}}{\binom{k+1-j}{2}_{F(0,t)}} = \prod_{j=0}^{k-1} \frac{\left\lfloor \frac{n+1-j}{2} \right\rfloor t^{n-j}}{\left\lfloor \frac{k+1-j}{2} \right\rfloor t^{k-j}} = t^{k(n-k)} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \left( \left\lfloor \frac{n+1}{2} \right\rfloor \right) \left( \left\lfloor \frac{k}{2} \right\rfloor \right) \left( \left\lfloor \frac{k+1}{2} \right\rfloor \right). \quad (58)$$

for  $0 \leq k \leq n$  and  $= 0$  else.

The first terms are

$$\left( \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{2,F(0,t)} \right)_{n,k=0}^7 = \begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & t & 1 & & & & & \\ 1 & 2t^2 & 2t^2 & 1 & & & & \\ 1 & 2t^3 & 4t^4 & 2t^3 & 1 & & & \\ 1 & 3t^4 & 6t^6 & 6t^6 & 3t^4 & 1 & & \\ 1 & 3t^5 & 9t^8 & 9t^9 & 9t^8 & 3t^5 & 1 & \end{pmatrix}. \quad (59)$$

For  $t = 1$  this is OEIS [8], A088855. For  $t = 1$  the row sums are  $\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$  and for  $t = -1$  the sum

of row  $2n$  is the Catalan number  $C_n$  and the sum of row  $2n-1$  is the central binomial coefficient  $\binom{2n}{n}$ .

A companion to Conjecture 13 is

### Conjecture 18

$$\left( \sum_{j=0}^{dk+1} (-1)^{\binom{j+1}{2}} t^{\binom{j}{2}} \binom{dk+1}{j}_{F(s,t)} x^j \right) \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{d,F(s,t)} x^n = N(d, k, s, t, x) \quad (60)$$

is a polynomial of degree  $(d-1)(k-1)$ .

For  $d = 2$  we get more precisely

$$\sum_{j=0}^{2k+1} (-1)^{\binom{j+1}{2}} \binom{2k+1}{j}_F t^{\binom{j}{2}} x^j \sum_{n \geq 0} \left\langle \begin{matrix} n+k \\ k \end{matrix} \right\rangle_{2,F(s,t)} x^n = \sum_j \left\langle \begin{matrix} k-1 \\ j \end{matrix} \right\rangle_{2,F(s,t)} t^{j^2+j} x^j. \quad (61)$$

For  $s \rightarrow 0$  identity (61) reduces to

$$(1-t^k x)^{k+1} (1+t^k x)^k \sum_{n=0}^{\infty} \left( \begin{matrix} \lfloor \frac{n+k}{2} \rfloor \\ \lfloor \frac{k}{2} \rfloor \end{matrix} \right) \left( \begin{matrix} \lfloor \frac{n+k+1}{2} \rfloor \\ \lfloor \frac{k+1}{2} \rfloor \end{matrix} \right) t^{kn} x^n = \sum_{j=0}^k \left( \begin{matrix} \lfloor \frac{k-1}{2} \rfloor \\ \lfloor \frac{j}{2} \rfloor \end{matrix} \right) \left( \begin{matrix} \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{j}{2} \rfloor \end{matrix} \right) t^{kj} x^j. \quad (62)$$

Let us prove this identity. It suffices to consider  $t = 1$ .

Let  $k = 2\ell$ . Then

$$\sum_{n=0}^{\infty} \left( \begin{matrix} \lfloor \frac{2n+k}{2} \rfloor \\ \lfloor \frac{k}{2} \rfloor \end{matrix} \right) \left( \begin{matrix} \lfloor \frac{2n+k+1}{2} \rfloor \\ \lfloor \frac{k+1}{2} \rfloor \end{matrix} \right) x^{2n} = \sum_{n=0}^{\infty} \binom{n+\ell}{\ell}^2 x^{2n},$$

$$\sum_{n=0}^{\infty} \left( \begin{matrix} \lfloor \frac{2n+1+k}{2} \rfloor \\ \lfloor \frac{k}{2} \rfloor \end{matrix} \right) \left( \begin{matrix} \lfloor \frac{2n+1+k+1}{2} \rfloor \\ \lfloor \frac{k+1}{2} \rfloor \end{matrix} \right) x^{2n+1} = \sum_{n=0}^{\infty} \binom{n+\ell}{\ell} \binom{n+1+\ell}{\ell} x^{2n+1}.$$

Thus the left-hand side is

$$(1-x)(1-x^2)^{2\ell} \left( \sum_{n=0}^{\infty} \binom{n+\ell}{\ell} x^{2n} + \sum_{n=0}^{\infty} \binom{n+\ell}{\ell} \binom{n+1+\ell}{\ell} x^{2n+1} \right)$$

$$= (1-x)(1-x^2)^{2\ell} \left( {}_2F_1 \left[ \begin{matrix} \ell+1, \ell+1 \\ 1 \end{matrix}; x^2 \right] + (\ell+1)x {}_2F_1 \left[ \begin{matrix} \ell+1, \ell+2 \\ 2 \end{matrix}; x^2 \right] \right).$$

The right-hand side reduces in an analogous way to  ${}_2F_1 \left[ \begin{matrix} -\ell, -\ell+1 \\ 1 \end{matrix}; x^2 \right] + \ell x {}_2F_1 \left[ \begin{matrix} -\ell+1, -\ell+1 \\ 2 \end{matrix}; x^2 \right]$ .

Euler's transformation formula (cf. [1], (2.2.7))

$${}_2F_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; z \right] = (1-z)^{c-a-b} {}_2F_1 \left[ \begin{matrix} c-a, c-b \\ c \end{matrix}; z \right] \text{ gives}$$

$${}_2F_1 \left[ \begin{matrix} \ell+1, \ell \\ 1 \end{matrix}; x^2 \right] (1-x^2)^{2\ell} = {}_2F_1 \left[ \begin{matrix} -\ell, 1-\ell \\ 1 \end{matrix}; x^2 \right],$$

$${}_2F_1 \left[ \begin{matrix} \ell+1, \ell+1 \\ 2 \end{matrix}; x^2 \right] (1-x^2)^{2\ell} = {}_2F_1 \left[ \begin{matrix} 1-\ell, 1-\ell \\ 2 \end{matrix}; x^2 \right].$$

By comparing coefficients we get

$${}_2F_1 \left[ \begin{matrix} \ell+1, \ell \\ 1 \end{matrix}; x^2 \right] + \ell x {}_2F_1 \left[ \begin{matrix} \ell+1, \ell+1 \\ 2 \end{matrix}; x^2 \right] = (1-x) {}_2F_1 \left[ \begin{matrix} \ell+1, \ell+1 \\ 1 \end{matrix}; x^2 \right] + (\ell+1)x(1-x) {}_2F_1 \left[ \begin{matrix} \ell+1, \ell+2 \\ 2 \end{matrix}; x^2 \right]$$

which proves (62) for even  $k$ . In a similar way the formula is proved for odd  $k$ .

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