A CURIOUS $q$-ANALOGUE OF HERMITE POLYNOMIALS

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Abstract. Two well-known $q$-Hermite polynomials are the continuous and discrete $q$-Hermite polynomials. In this paper we consider a new family of $q$-Hermite polynomials and prove several curious properties about these polynomials. One striking property is the connection with $q$-Fibonacci and $q$-Lucas polynomials. The latter relation yields a generalization of the Touchard-Riordan formula.

1. Introduction

The classical Hermite polynomials have two important properties: (i) they form a family of orthogonal polynomials and (ii) are intimately connected with the commutation properties between the multiplication operator $x$ and the differentiation operator $D$. In contrast to the discrete $q$-Hermite polynomials, which generalize both aspects, the continuous $q$-Hermite polynomials generalize only the first one. The purpose of this paper is to introduce a $q$-analogue which generalizes the second property and establish the missing link with the continuous $q$-Hermite polynomials. It turns out that these new polynomials are in some sense dual to the continuous $q$-Hermite polynomials. Moreover, they provide interesting connections with $q$-Fibonacci and $q$-Lucas polynomials and the Touchard-Riordan formula for the moments of the continuous $q$-Hermite polynomials. In order to provide the reader with the necessary background we first collect some well-known results about the classical Hermite polynomials and their known $q$-analogues.

The normalized Hermite polynomials $H_n(x, s) = s^{n/2}H_n(x/\sqrt{s}, 1)$ ($n \geq 0$) may be defined by the recurrence relation:

$$H_{n+1}(x, s) = xH_n(x, s) - nsH_{n-1}(x, s),$$

(1.1)

with initial values $H_0(x, s) = 1$ and $H_{-1}(x, s) = 0$. By induction, we have

$$H_n(x, s) = (x - sD)^n \cdot 1,$$

(1.2)

where $D = \frac{d}{dx}$ denotes the differentiation operator. It follows that

$$(D H_n(x, s) = nH_{n-1}(x, s).$$

(1.3)

The Hermite polynomials have the explicit formula (see [1, Chapter 6])

$$H_n(x, s) = \sum_{k=0}^{n} \binom{n}{2k} (-s)^k (2k - 1)!! \cdot x^{n-2k}.$$
The first few polynomials are

\[ 1, x, -s + x^2, -3sx + x^3, 3s^2 - 6sx^2 + x^4, 15s^2x - 10sx^3 + x^5. \]

The Hermite polynomials are orthogonal with respect to the linear functional defined by the moments

\[ \mu_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx = \begin{cases} (n-1)!! & \text{if } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \]

In other words, the \( n \)-th moment \( \mu_n \) of the measure of the Hermite polynomials is the number of the complete matchings on \([n] := \{1, \ldots, n\}\), i.e., \( \mu_{2n} = (2n-1)!! \) and \( \mu_{2n+1} = 0 \).

Consider the rescaled Hermite polynomials \( p_n(z, x, s) = H_n(z - x, -s) \), also determined by

\[ p_{n+1}(z, x, s) = (z-x)p_n(z, x, s) + snp_{n-1}(z, x, s) \]  

with initial values \( p_0(z, x, s) = 1 \) and \( p_{-1}(z, x, s) = 0 \). Let \( \mathcal{F} \) be the linear functional on polynomials in \( z \) defined by \( \mathcal{F}(p_n(z, x, s)) = \delta_{n,0} \). Then the moments \( \mathcal{F}(z^n) \) are again the Hermite polynomials

\[ \mathcal{F}(z^n) = (\sqrt{-s})^n \sum_{k=0}^n \binom{n}{k} (x/\sqrt{-s})^{n-k} \mu_k = H_n(x, s). \]

This is equivalent to saying that the generating function of the Hermite polynomials \( H_n(x, s) \) has the following continued fraction expansion:

\[ H(z, x, s) = \sum_{n \geq 0} H_n(x, s)z^n = \frac{1}{1 - xz + \frac{s^2}{1 - xz + \frac{2s^2z^2}{1 - xz + \frac{3s^2z^2}{\ldots}}}}. \]

Two important classes of orthogonal \( q \)-analogues of \( H_n(x, s) \) are the continuous and the discrete \( q \)-Hermite I polynomials, which are both special cases of the Al-Salam–Chihara polynomials. Before we describe these \( q \)-Hermite polynomials, we introduce some standard \( q \)-notations (see [6]). For \( n \geq 1 \) let

\[ [n] := [n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{k=1}^n [k]_q, \quad [2n - 1]_q!! = \prod_{k=1}^n [2k - 1]_q, \]

and \( (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \) with \( (a; q)_0 = 1 \). The \( q \)-binomial coefficient is defined by

\[ \binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \]

for \( 0 \leq k \leq n \) and zero otherwise.

Recall [8] that the Al-Salam–Chihara polynomials \( P_n(x; a, b, c) \) satisfy the three term recurrence:

\[ P_{n+1}(x; a, b, c) = (x - aq^n)P_n(x; a, b, c) - (c + bq^{n-1})[n]_q P_{n-1}(x; a, b, c) \]

with initial values \( P_{-1}(x; a, b, c) = 0 \) and \( P_0(x; a, b, c) = 1 \).
**Definition 1.** Let $\mathcal{F}_{a,b,c}$ be the unique linear functional acting on the polynomials in $z$ that satisfies

$$
\mathcal{F}_{a,b,c}(P_n(z; a, b, c)) = \delta_{n,0}.
$$

Then the continuous $q$-Hermite polynomials are

$$
\tilde{H}_n(x, s|q) = P_n(x; 0, 0, s)
$$
and are also the moments (see [8] and Proposition 16):

$$
\hat{H}_n(x, s|q) = \mathcal{F}_{x,-s,0}(z^n).
$$

The discrete $q$-Hermite polynomials I are

$$
\tilde{h}_n(x, s; q) = P_n(x; 0, (1-q)s, 0),
$$
and the discrete $q$-Hermite polynomials II are

$$
\hat{h}_n(x; q) = (-i)^n \tilde{h}_n(ix, 1; q^{-1}).
$$

It is also convenient to introduce the polynomials

$$
h_n(x, s; q) := P_n(0; -x, 0, s),
$$
which are actually a rescaled version of $\tilde{h}_n(x; q)$ (see Section 4). The main purpose of this paper is to study another $q$-analogue of Hermite polynomials.

**Definition 2.** The $q$-Hermite polynomials $H_n(x, s|q)$ are defined by

$$
H_n(x, s|q) := \mathcal{F}_{x,0,-s}(z^n).
$$

The $q$-Hermite polynomials $\tilde{H}_n(x, s|q)$ have, amongst other facts,

1. orthogonality with an explicit measure,
2. an explicit 3-term recurrence relation,
3. explicit expressions,
4. a combinatorial model using matchings,
5. are moments for other orthogonal polynomials,
6. a closed form expression for Hankel determinants,
7. an explicit Jacobi continued fraction as generating function.

The new $q$-Hermite polynomials $H_n(x, s|q)$ are not orthogonal, i.e., they do not have (1) and (2). Instead they have a nice $q$-analogue of the operator formula (1.2) for the ordinary Hermite polynomials (see Theorem 5), the coefficients of the $H_n(x, s|q)$ appear in the inverse matrix of the coefficients in the continuous $q$-Hermite polynomials (cf. Theorem 6), they have simple connection coefficients with $q$-Lucas and $q$-Fibonacci polynomials (cf. Theorem 12). The discrete $q$-Hermite polynomials $h_n(x, s; q)$ also have (1)–(4), and we will show in Theorem 7 that they are also moments. Moreover, the quotients of two consecutive polynomials $h_n(x, s; q)$ (see Eq.(4.21)) appear as coefficients in the expansion of the $S$-continued fraction of the generating function of the $H_n(x, s|q)$’s, which leads to a second proof of Theorem 5.

This paper is organized as follows: in Section 2, we recall some well-known facts about the general theory of orthogonal polynomials and show how to prove (1.10) by using this theory; we prove the main properties of $H_n(x, s|q)$ and $h_n(x, s; q)$ in Section 3 and Section 4, respectively; in Section 5 we shall establish the connection between our new $q$-Hermite polynomials and the
q-Fibonacci and q-Lucas polynomials. This yields, in particular, a generalization of Touchard-Riordan’s formula for the moments of continuous q-Hermite polynomials (cf. Proposition 15), first obtained by Josuat-Vergès [10].

2. SOME WELL-KNOWN FACTS

In this section we recall some well-known facts about orthogonal polynomials (see [2, 18, 17]). Let \( p_n(x) \) be a sequence of polynomials which satisfies the three term recurrence relation

\[
p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_np_{n-1}(x)
\]

with initial values \( p_0(x) = 1 \) and \( p_{-1}(x) = 0 \).

Define the coefficients \( a(n, k) \) \((0 \leq k \leq n)\) by

\[
\sum_{k=0}^{n} a(n, k)p_k(x) = x^n.
\]

These are characterized by the Stieltjes tableau:

\[
a(0, k) = \delta_{k,0},
\]
\[
a(n, 0) = b_0a(n - 1, 0) + \lambda_1a(n - 1, 1),
\]
\[
a(n, k) = a(n - 1, k - 1) + b_ka(n - 1, k) + \lambda_{k+1}a(n - 1, k + 1).
\]

If \( \mathcal{F} \) is the linear functional such that \( \mathcal{F}(p_n(x)) = \delta_{n,0} \), then

\[
\mathcal{F}(x^n) = a(n, 0).
\]

The generating function of the moments has the continued fraction expansion

\[
\sum_{n \geq 0} \mathcal{F}(x^n)z^n = \frac{1}{1 - b_0z - \lambda_1z^2 - \frac{\lambda_2z^2}{1 - \ldots}}.
\]

The Hankel determinants for the moments are

\[
d(n, 0) = \det(\mathcal{F}(z^{i+j}))_{i,j=0}^{n-1} = \prod_{i=1}^{n-1} \prod_{k=1}^{i} \lambda_k,
\]

and

\[
d(n, 1) = \det(\mathcal{F}(z^{i+j+1}))_{i,j=0}^{n-1} = d(n, 0)(-1)^n p_n(0).
\]

By using the Stieltjes tableau we can give a simple proof of (1.10).

**Proposition 3.** The continuous q-Hermite polynomials \( \tilde{H}_n(x, s|q) \) defined by (1.9), i.e.,

\[
\tilde{H}_{n+1}(x, s|q) = x\tilde{H}_n(x, s|q) - s[n]q\tilde{H}_{n-1}(x, s|q),
\]

are the moments of the measure of the orthogonal polynomials \( p_n(z) := P_n(z; x, -s, 0) \) defined by the recurrence

\[
p_{n+1}(z) = (z - xq^n)p_n(z) + sq^{n-1}[n]qpn_{-1}(z).
\]
Proof. Let \( b_n = q^n x \) and \( \lambda_{n+1} = (-s) q^n [n+1]_q \) for \( n \geq 0 \). It is sufficient to verify that in this case (2.3) is satisfied with

\[
a(n, k) = \binom{n}{k} \tilde{H}_{n-k}(z, s|q).
\] (2.10)

This is clearly equivalent to (2.8).

As a consequence of the previous proposition, and in view of (2.6) and (2.7), we can derive immediately the Hankel determinants

\[
d(n, 0) = (-s)^{\binom{n}{2}} q^{(n-1)} \prod_{j=0}^{n-1} [j]_q!,
\] (2.11)

and

\[
d(n, 1) = d(n, 0) r(n),
\] (2.12)

where \( r(n) = (-1)^n p_n(0; x, -s, 0) \).

Note that the polynomials \( r(n) \) satisfy

\[
r(n) = q^{-n} x r(n-1) + q^{-2} s[n-1]_q r(n-2).
\]

This implies that

\[
r(n) = q^{-\binom{n}{2}} \tilde{H}_n \left( x \sqrt{q}, -s \frac{1}{q} \right).
\] (2.13)

The first few polynomials of the sequence \( \tilde{H}_n(x, s|q) \) are

\[
1, x, -s + x^2, x(-2 + q) x + x^2, (1 + q + q^2) s^2 - (3 + 2q + q^2) s x^2 + x^4,
\]

\[
x(3 + 4q + 4q^2 + 3q^3 + q^4) s^2 - (4 + 3q + 2q^2 + q^3) s x^2 + x^4.
\]

From their recurrence relation we see that

\[
\tilde{H}_{2n}(0, s|q) = (-s)^n [2n - 1]_q!! \quad \text{and} \quad \tilde{H}_{2n+1}(0, s|q) = 0.
\]

3. The \( q \)-Hermite Polynomials \( H_n(x, s|q) \)

By (1.8) the \( q \)-Hermite polynomials \( H_n(x, s|q) \) are the moments of the measure of the orthogonal polynomials \( P_n(z) \) satisfying the recurrence:

\[
P_{n+1}(z) = (z - q^n) P_n(z) + s[n]_q P_{n-1}(z).
\] (3.1)

Recall [13, p.80] that the Al-Salam–Chihara polynomials \( Q_n(x) := Q_n(x; \alpha, \beta) \) satisfy the three term recurrence:

\[
Q_{n+1}(x) = (2x - (\alpha + \beta) q^n) Q_n(x) - (1 - q^n)(1 - \alpha \beta q^{n-1}) Q_{n-1}(x),
\] (3.2)

with \( Q_0(x) = 1 \) and \( Q_1(x) = 0 \). They have the following explicit formulas:

\[
Q_n(x; \alpha, \beta|q) = (\alpha e^{i\theta}; q)_n e^{-i\theta} \varphi_1 \left( \binom{q^{-n}}{\alpha^{-1} q^{-n+1} e^{-i\theta} \beta e^{-i\theta} |q; \alpha^{-1} q e^{i\theta}} \right),
\] (3.3)

where \( x = \cos \theta \).
Comparing (3.1) and (3.2) we have $P_n(z) = \frac{1}{(2\pi i)^n} Q_n(az; \alpha, 0)$ with

$$a = \frac{1}{2} \sqrt{\frac{q-1}{s}} \quad \text{and} \quad \alpha = x \sqrt{\frac{q-1}{s}}. \quad (3.4)$$

Using the known formula for Al-Salam–Chihara polynomials we obtain

$$P_n(z) = \frac{1}{(2\alpha \alpha)^n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} q^k \prod_{i=0}^{k-1} \left(1 + \alpha^2 q^{2i} - 2q^{i\alpha}z\right)$$

$$= \left(\frac{s}{x(q-1)}\right)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \left(-\frac{q}{s}\right)^k \prod_{i=0}^{k-1} \left((q - 1)q^i xz - s - (q - 1)q^{2i} x^2\right). \quad (3.5)$$

The first few polynomials $P_n(z)$ are

- $P_1(z) = z - x,$
- $P_2(z) = z^2 - x(1 + q)z + (s + q_2 x^2),$
- $P_3(z) = z^3 - x[3]q z^2 + (2s + qs + q[3]q x^2)z - (s + qs + q^2 s + q^3 x^2) x.$

A matching $m$ of $\{1, 2, \ldots, n\}$ is a set of pairs $(i, j)$ such that $i < j$ and $i, j \in [n]$. Each pair $(i, j)$ is called an edge of the matching. Let $ed(m)$ be the number of edges of $m$, so $n - 2ed(m)$ is the number of unmatched vertices. Two edges $(i, j)$ and $(k, l)$ have a crossing if $i < k < j < l$ or $k < i < l < j$. Let $cr(m)$ be the number of crossing numbers in the matching $m$. Using the combinatorial theory of Viennot [17], Ismail and Stanton [8, Theorem 6] gave a combinatorial interpretation of the moments of Al-Salam–Chihara polynomials. In particular we derive the following result from [8, Theorem 6].

**Lemma 4.** The moments of the measure of the orthogonal polynomials $\{P_n(x)\}$ are the generating functions for all matchings $m$ of $[n]$:

$$F_{x, 0, -s} (z^n) = \sum_m x^{n - 2ed(m)} (-s)^{ed(m)} q^{c(m) + cr(m)}, \quad (3.6)$$

where $c(m) = \sum_{a - \text{vertices}} |\{\text{edges } i < j : i < a < j\}|$ and the sum extends over all matchings $m$ of $[n]$.

Let $M(n, k)$ be the set of matchings of $\{1, \ldots, n\}$ with $k$ unmatched vertices. Then

$$F_{x, 0, -s} (z^n) = \sum_k c(n, k, q) x^k (-s)^{\frac{n-k}{2}}, \quad (3.7)$$

where

$$c(n, k, q) = \sum_{m \in M(n, k)} q^{c(m) + cr(m)}. \quad (3.8)$$

It is easy to verify that

$$c(n, k, q) = c(n - 1, k - 1, q) + [k + 1] q c(n - 1, k + 1, q) \quad (3.9)$$

with $c(0, k, q) = \delta_{k, 0}$ and $c(n, 0, q) = c(n-1, 1, q)$. Indeed, if $n$ is an unmatched vertex then for the restriction $m_0$ of $m$ to $[n - 1]$ we get $c(m_0) = c(m)$ and $cr(m_1) = cr(m)$. If $n$ is matched with $m(n)$, such that there are $i$ unmatched vertices and $j$ endpoints of edges which cross the
For this is obvious for 

Proof. We first show by induction that 

fractions, see the remark after Theorem 9.

The matrices 

Remark. It should be noted that the method of Varvak [16] (see also [10]) can also be applied to prove Theorem 5. In fact her method proves first that \((x - sD_q)^n \cdot 1\) is a generating function of some rook placements, which is then shown to count involutions with respect to the statistic \(c(m) + cr(m)\) (see [16, Theorem 6.4]). We will give another proof of (3.10) by using continued fractions, see the remark after Theorem 9.

The first few polynomials \(H_n(x, s|q)\) are

\[
1, \ x, \ -s + x^2, \ x(-(2 + q)s + x^2), \ (2 + q)s^2 - (3 + 2q + q^2)sx^2 + x^4, \\
x((5 + 6q + 3q^2 + q^3)s^2 - (4 + 3q + 2q^2 + q^3)sx^2 + x^4), \ldots
\]

Let

\[
\tilde{H}_n(x, s|q) = \sum_k b(n, k, q)x^k(-s)^{n-k}. 
\]

Theorem 6. The matrices \((c(i, j, q))_{i,j=0}^{n-1}\) and \((b(i, j, q))_{i,j=0}^{n-1}(-1)^{\frac{j-i}{2}}\) are mutually inverse.

Proof. We first show by induction that

\[
\tilde{H}_n(x + sD_q, s|q) \cdot 1 = x^n. 
\]

For this is obvious for \(n = 0\). If it is already shown for \(n\) we get

\[
\tilde{H}_{n+1}(x + sD_q, s|q) \cdot 1 = (x + sD_q)\tilde{H}_n(x + sD_q, s|q) \cdot 1 - s[n]_q\tilde{H}_{n-1}(x + sD_q, s|q) \cdot 1 \\
= (x + sD_q)x^n - s[n]_qx^{n-1} = x^{n+1}.
\]
On the other hand we have

\[ \tilde{H}_n(x + sD_q, s|q) \cdot 1 = \sum_{k=0}^{n} b(n, k, q)(-s)^{\frac{n-k}{2}} (x + sD_q)^k \cdot 1 \]

\[ = \sum_{k=0}^{n} b(n, k, q)(-s)^{\frac{n-k}{2}} \sum_{j=0}^{k} c(k, j, q)s^{\frac{k-j}{2}} x^j \]

\[ = \sum_{j=0}^{n} s^{\frac{n-j}{2}} x^j \sum_{k=j}^{n} b(n, k, q)(-1)^{\frac{n-k}{2}} c(k, j, q). \] (3.14)

The result then follows by comparing (3.13) and (3.14).

**Remark.** If we set \( q = 0 \) then (3.9) reduces to the well-known Catalan triangle (see [2, Chap. 7]), which implies

\[ c(2n, 0, 0) = C_n = \frac{1}{n+1} \binom{2n}{n}, \]

\[ c(2n, 2k, 0) = \frac{2k + 1}{n + k + 1} \binom{2n}{n-k} = \binom{2n}{n-k} - \binom{2n}{n-k-1}, \]

\[ c(2n + 1, 2k + 1, 0) = \frac{2k + 2}{n + k + 2} \binom{2n + 1}{n-k} = \binom{2n + 1}{n-k} - \binom{2n + 1}{n-k-1}. \]

The recurrence (3.1) implies that the Hankel determinants of \( H_n(x, s|q) \) are

\[ \det(H_{i+j}(x, s|q))_{i,j}^{n-1} = (-s)^{\binom{n}{2}} \prod_{j=0}^{n-1} [j]_q!, \] (3.15)

and

\[ \det(H_{i+j+1}(x, s|q))_{i,j}^{n-1} = h_n(x, -s; q)(-s)^{\binom{n}{2}} \prod_{j=0}^{n-1} [j]_q!, \] (3.16)

where

\[ h_n(x, -s; q) = (-1)^n P_n(0) = \left( \frac{s}{x(1-q)} \right)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} q^k \prod_{i=0}^{k-1} (1 + x^2(q - 1)q^{2i}/s). \]

4. **The rescaled discrete \( q \)-Hermite polynomials II**

By definition (1.13) and (1.7) we have

\[ h_{n+1}(x, s; q) = q^n x h_n(x, s; q) - [n]_q s h_{n-1}(x, s; q). \] (4.1)
Comparing with the three-term recurrence relation for the discrete $q$-Hermite polynomials II (see (1.12) and (1.7)), we derive

$$h_n(x, s; q) = q^{\left(\frac{n(n-1)}{2}\right)} \sqrt{s^n} h_n \left(\frac{x}{\sqrt{s}}, q\right)$$

(4.2)

$$= \sum_{k=0}^{n} \frac{n!}{2k} [2k]_q \frac{n!}{2k} [2k]_q (\sqrt{n} - 1)! (-s)^k x^{n-2k},$$

(4.3)

where the last expression follows from the known formula for $\tilde{h}_n(x, q)$.

Since $D_q(fg) = D_q(f)g + f(qx)D_q(g)$ and $D_q(x) = 1$, we see that

$$D_q(h_{n+1}(x)) = q^n x D_q(h_n(x)) + q^n h_n(qx) - [n]_q s D_q(h_{n-1}(x)).$$

We find by induction on $n$ that

$$D_q h_n(x, s; q) = [n]_q h_{n-1}(qx, s; q).$$

(4.4)

The first few polynomials $h_n(x, s; q)$ are

$$1, x, qx, q^2 x^2 - s, q^3 x^3 - s[3]_q x, q^4 x^4 - s(q^5 + q^4 + 2q^3 + q^2 + q)x^2 + s^2[3]_q.$$  

The following result shows that the polynomials $h_n(x, s; q)$ are moments of some orthogonal polynomials.

**Theorem 7.** The generating function of $h_n(x, s; q)$ has the continued fraction expansion:

$$\sum_{m \geq 0} h_m(x, s; q) t^m = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{1 - b_2 t - \frac{\lambda_3 t^2}{1 - \cdots}}}},$$

with

$$b_n = q^{n-1} (q^n + q^{n+1} - 1) x \quad \text{and} \quad \lambda_n = -q^{n-1} [n]_q (s + q^{2n-2}(1 - q)x^2).$$

(4.5)

**Proof.** To prove this it suffices to show that the Stieltjes tableau (2.3) is satisfied with

$$a(n, k) = \left[\begin{array}{c} n \\ k \end{array}\right] h_{n-k}(q^k x, s; q).$$

This is easily verified.

Using (2.6) and (2.7), Theorem 7 implies the following Hankel determinant evaluations:

$$\det(h_{i+j}(x, s; q))^{n-1}_{i,j} = (-1)^{\left(\frac{n(n-1)}{2}\right)} \prod_{j=0}^{n-1} [j]_q! (s + q^{2j}(1 - q)x^2)^{n-1-j}$$

(4.6)

and

$$\frac{\det(h_{i+j+1}(x, s; q))^{n-1}_{i,j}}{\det(h_{i+j}(x, s; q))^{n-1}_{i,j}} = w(n),$$

(4.7)
where \( w(n) \) satisfies
\[
w(n + 1) = q^{n-1}(q^n + q^{n+1} - 1)xw(n) + q^{n-1}[n]q(s + q^{2n-2}(1 - q)x^2)w(n - 1).
\]
It is easily verified that
\[
w(n) = \sum_{k=0}^{n} q^{2(n-k)} \left\lfloor \frac{n}{2k} \right\rfloor [2k-1]!! s^k x^{n-2k}
\]
satisfies the same recurrence with the same initial values.

**Lemma 8.** Let \( L_n(x) := h_n(x, (1 - q)s; q) \). Then
\[
sL_n(x) + xL_{n+1}(x) = (x^2 + s)L_n(qx).
\]

**Proof.** First we note that the constant terms of both sides of (4.9) are equal to \( sL_n(0) \). So it suffices to show that the derivatives of the two sides are equal. Applying \( D_q \) to (4.9) and using (4.4) we obtain, after replacing \( x \) by \( x/q \),
\[
s[n]L_{n-1}(x) + xq[n-1]L_n(x) + L_{n+1}(x) = (x^2 + s)q[n]L_{n-1}(qx).
\]
Since \( L_{n+1}(x) = q^n xL_n(x) - (1 - q^n)sL_{n-1}(x) \), we can rewrite the above equation as follows:
\[
sL_{n-1}(x) + xL_n(x) = (x^2 + s)L_{n-1}(qx).
\]
The proof is thus completed by induction on \( n \).

We shall prove the following Jacobi continued fraction expansion for the generating function of \( (x + (1 - q)sD_q)^n \cdot 1 \). This is equivalent to Theorem 5.

**Theorem 9.** Let \( T_n(x, s) = (x + (1 - q)sD_q)^n \cdot 1 \). Then
\[
\sum_{n\geq0} T_n(x, s) t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{1 - \ldots}}},
\]
where the coefficients are
\[
b_n = q^n x, \quad \text{for } n \geq 0; \quad \text{and} \quad \lambda_n = (1 - q^n)s, \quad \text{for } n \geq 1.
\]

**Proof.** Since \( T_n(x, s) = (x + (1 - q)sD_q)T_{n-1}(x, s) \), we have
\[
T_n(x, s) = (x + \frac{s}{x})T_{n-1}(x, s) - \frac{s}{x}T_{n-1}(qx, s).
\]
Equivalently the generating function \( G(x, t) = \sum_{n\geq0} T_n(x, s) t^n \) satisfies the functional equation:
\[
(1 - \frac{x^2 + s}{x} t) G(x, t) = 1 - \frac{s}{x} t G(qx, t).
\]
Suppose that
\[
G(x,t) = \frac{1}{1 - \frac{1}{c_1 t}}, \quad (4.14)
\]
where \(c_n = (g_n - 1)g_{n-1}A\) with \(A := A(x) = -\frac{x^2 + s}{x}\) and \(g_i := g_i(x)\).
Substituting (4.14) in (4.13) and then replacing \(t\) by \(t/A\) we obtain
\[
1 + t - (g_1 - 1)\frac{t}{x^2 + s} = 1 + (g'_1 - 1)\frac{t}{A'}, \quad (4.15)
\]
where \(A' := A(qx)\) and \(g'_i := g_i(qx)\). Comparing this with Wall’s formula (see [12]):
\[
1 + z - (g_1 - 1)\frac{z}{x^2 + s} = 1 + (g'_1 - 1)\frac{z}{A'}, \quad (4.16)
\]
we derive that \(g_0 = 1\) and for \(n \geq 1\),
\[
\begin{align*}
\begin{cases}
g_{2n} = & \frac{A'}{A} g_{2n-1} - 1, \\
g_{2n+1} = & \frac{A'}{A} g_{2n} - 1.
\end{cases}
\end{align*}
\quad (4.17)
\]
For example,
\[
\begin{align*}
g_1 &= \frac{s}{x^2 + s}, \quad g_2 = \frac{A' g'_1 - 1}{A g_1 - 1} = q, \\
g_3 &= \frac{A' g'_2 - 1}{A g_2 - 1} = \frac{s}{x^2 + s} q, \quad g_4 = \frac{A' g'_3 - 1}{A g_3 - 1} = -s + qs + q^3x^2.
\end{align*}
\]
In general we have the following result
\[
\begin{align*}
\begin{cases}
g_{2n} = & \frac{sL_n(x) + xL_{n+1}(x)}{(x^2 + s)L_n(x)}, \\
g_{2n+1} = & \frac{sL_n(x)}{sL_n(x) + xL_{n+1}(x)},
\end{cases}
\end{align*}
\quad (n \geq 0).
\quad (4.18)
This can be verified by induction on \( n \). Suppose that the formula (4.18) is true for \( n \geq 0 \). We prove that the formula holds for \( n + 1 \). By (4.17) we have
\[
g_{2n+2} = A' \frac{g'_{2n+1} - 1}{g'_{2n+1}} g_{2n} = \frac{sL_n(x) + xL_{n+1}(x)}{(x^2 + s)L_{n+1}(x)} \frac{L_{n+1}(qx)}{L_n(qx)}.
\]
It follows from Lemma 1 that
\[
g_{2n+2} = \frac{sL_{n+1}(x) + xL_{n+2}(x)}{(x^2 + s)L_{n+1}(x)}.
\] (4.19)
Since
\[
L_{n+1}(x) - xL_n(x) = (q^n - 1)(xL_n(x) + sL_{n-1}(x)),
\] (4.20)
the verification for \( g_{2n+3} \) is then straightforward. We derive from (4.14) and (4.18) that
\[
\begin{align*}
    &c_{2n} = (g_{2n} - 1)g_{2n-1}A = (1 - q^n)s \frac{L_{n-1}(x)}{L_n(x)}, \quad \text{for } n \geq 1; \\
    &c_{2n+1} = (g_{2n+1} - 1)g_{2n}A = \frac{L_{n+1}(x)}{L_n(x)}, \quad \text{for } n \geq 0.
\end{align*}
\]
(4.21)
Invoking the contraction formula (see [19]), which transforms a \( S \)-continued fraction to a \( J \)-continued fraction,
\[
1 - \frac{c_1 z}{1 - \frac{c_2 z}{1 - \frac{c_3 z}{1 - \frac{c_4 z}{\ddots}}}} = \frac{1}{1 - c_1 z - \frac{c_1 c_2 z^2}{1 - \frac{c_2 c_3 z^3}{1 - \frac{c_3 c_4 z^4}{\ddots}}}},
\] (4.22)
we obtain
\[
\begin{align*}
    b_n &= \frac{h_{n+1}(x,(1 - q)s; q)}{h_n(x,(1 - q)s; q)} + (1 - q^n)s \frac{h_{n-1}(x,(1 - q)s; q)}{h_n(x,(1 - q)s; q)} = q^n x, \\
    \lambda_n &= \frac{h_n(x,(1 - q)s; q)}{h_{n-1}(x,(1 - q)s; q)} \cdot (1 - q^n)s \frac{h_{n-1}(x,(1 - q)s; q)}{h_n(x,(1 - q)s; q)} = (1 - q^n)s.
\end{align*}
\] (4.23)
This completes the proof. \( \square \)

**Remark.** Instead of the contraction formula (4.22), we can also proceed as follows. Define a table \((A(n, k))_{n,k \geq 0}\) by
\[
A(0, 0) = \delta_{k,0}, \\
A(n, 0) = c_1A(n - 1, 1), \\
A(n, k) = A(n - 1, k - 1) + c_{k+1}A(n - 1, k + 1).
\] (4.24)
In this case \( A(2n, 2k + 1) = A(2n + 1, 2k) = 0 \) for all \( n, k \). If we define
\[
a(n, k) = A(2n, 2k),
\]
then it is easily verified that \( a(n, k) \) satisfy (2.3) with
\[
b_0 = c_1, \quad b_n = c_{2n} + c_{2n+1}, \quad \lambda_n = c_{2n}c_{2n-1}.
\] (4.25)
Substituting the values in (4.21) for $c_n$ we obtain (4.23). Therefore
\[ \sum_n A(2n, 0)t^n = \sum_n a(n, 0)t^n = \sum_n T_n(x, s)t^n. \]

As another application of this remark we prove the following result.

**Proposition 10.** Let $w_n(m, q) = \frac{q^{n((2m+1)n+1)}}{2}$. Then
\[ \sum_{m \geq 0} w_n(m, q)t^n = \sum_{m \geq 0} a_n(m, q)t^n = \sum_{n} w_n(m, q)t^n. \]

Proof. Let
\[ A(2n, 2k) = \frac{w_n(m, q)\left(\begin{array}{c} n \\ k \end{array}\right)}{w_k(m, q)\left(\begin{array}{c} k \\ k \end{array}\right)}q^{2m+1}, \quad \text{and} \quad A(2n + 1, 2k + 1) = \frac{w_{n+1}(m, q)\left(\begin{array}{c} n \\ k \end{array}\right)}{w_{k+1}(m, q)\left(\begin{array}{c} k \\ k \end{array}\right)}q^{2m+1}. \]

Then it is easily verified that the table (4.24) holds with $c_{2n} = q^{(2m+1)n-m}(q^{(2m+1)n-1} + q^{(2m+1)(2n+1)-m})$ and $c_{2n+1} = q^{(2m+1)(2n+1)-m}$. Therefore
\[ \sum_n A(2n, 0)t^n = \sum_n a_n(0)t^n = \sum_n w_n(m, q)t^n. \]

5. **Connection with $q$-Fibonacci polynomials and $q$-Lucas polynomials**

In this section we derive some explicit expansion formulae for the $q$-Hermite polynomials $H_n(x, s|q)$ in terms of $q$-Fibonacci polynomials and $q$-Lucas polynomials. We first recall some basic results about the latter polynomials in the $q = 1$ case and then define their $q$-analogue with the ordinary Fibonacci and Lucas polynomials and $q$-operator $D_q$.

The Lucas polynomials are defined by the recurrence
\[ l_n(x, s) = xl_{n-1}(x, s) + sl_{n-2}(x, s) \quad \text{for} \quad n > 2, \]
with initial values $l_1(x, s) = x$ and $l_2(x, s) = x^2 + 2s$. They have the explicit formula
\[ l_n(x, s) = \sum_{2k \leq n} \frac{n}{n-k}\left(\begin{array}{c} n-k \\ k \end{array}\right)s^k x^{n-2k} \quad (n > 0). \]

Furthermore we define $l_0(x, s) = 1$. Note that this definition differs from the usual one in which $l_0(x, s) = 2$.

The Fibonacci polynomials are defined by
\[ f_n(x, s) = xf_{n-1}(x, s) + sf_{n-2}(x, s) \]
with \( f_0(x, s) = 0 \) and \( f_1(x, s) = 1 \). They have the explicit formula

\[
 f_n(x, s) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-k}{k} s^k x^{n-1-2k}. \tag{5.2}
\]

We first establish the following inversion of (5.1) and (5.2), which will be used in the proof of Theorem 12.

**Lemma 11.**

\[
 x^n = \sum_{2k \leq n} \binom{n}{k} s^k l_{n-2k}(x, -s), \tag{5.3}
\]

\[
 x^n = \sum_{2k \leq n+1} \left( \binom{n}{k} - \binom{n}{k-1} \right) s^k f_{n+1-2k}(x, -s). \tag{5.4}
\]

**Proof.** Recall the Tchebyshev inverse relations [15, p. 54-62]:

\[
 b_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} a_{n-2k} \iff a_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k} b_{n-2k}, \tag{5.5}
\]

where \( a_0 = b_0 = 1 \), and

\[
 b_n = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^k \binom{n-k}{k} a_{n-2k} \iff a_n = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( \binom{n}{k} - \binom{n}{k-1} \right) b_{n-2k}. \tag{5.6}
\]

We derive immediately (5.3) from (5.1) and (5.5). Clearly (5.2) is equivalent to the left identity in (5.6) with \( a_n = \left( \frac{x}{\sqrt{s}} \right)^n \) and \( b_n = \frac{f_{n+1}(x, -s)}{(\sqrt{s})^n} \). By inversion we find

\[
 x^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \binom{n}{k} - \binom{n}{k-1} \right) s^k f_{n+1-2k}(x, -s). \tag{5.7}
\]

Now, noticing that

- if \( n \) is odd, then \( \binom{n}{k} = \binom{n}{k-1} \) for \( k = \left\lfloor \frac{n+1}{2} \right\rfloor \),
- if \( n \) is even, then \( \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \),

we see the equivalence of (5.4) and (5.7).

Define the \( q \)-Lucas and \( q \)-Fibonacci polynomials by

\[
 L_n(x, s) = l_n(x + (q-1)sD_q, s) \cdot 1, \tag{5.8}
\]

\[
 F_n(x, s) = f_n(x + (q-1)sD_q, s) \cdot 1. \tag{5.9}
\]
It is known (see [3] and [4]) that they have the explicit formulae

\[
L_n(x, s) = \sum_{k=0}^{[n/2]} q^{k(k+1)} \binom{n-k}{k} s^k x^{n-2k},
\]

(5.10)

\[
F_n(x, s) = \sum_{k=0}^{[n-1/2]} q^{k+1} \binom{n-1-k}{k} s^k x^{n-1-2k},
\]

(5.11)

for \(n > 0\), with \(L_0(x, s) = 1\) and \(F_0(x, s) = 0\).

**Theorem 12.** We have

\[
H_n(x, (q-1)s|q) = \sum_{k=0}^{[n/2]} \binom{n}{k} s^k L_{n-2k}(x, -s)
\]

(5.12)

\[
= \sum_{k=0}^{[n-1/2]} \left( \binom{n}{k} - \binom{n}{k-1} \right) s^k F_{n+1-2k}(x, -s).
\]

(5.13)

**Proof.** Since

\[
L_n(x, -s) = l_n(x - (q-1)s D_q, s) \cdot 1,
\]

\[
F_n(x, -s) = f_n(x - (q-1)s D_q, s) \cdot 1,
\]

the theorem follows by applying the homomorphism \(x \mapsto x - (q-1)s D_q\) to (5.3) and (5.4).

We derive some consequences of the formula (5.13).

**Corollary 13.** We have

\[
H_n(1, q-1|q) = \sum_{k=-\lfloor (n+1)/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k q^{\frac{k(3k+1)}{2}} \binom{n}{\frac{n-3k}{2}}.
\]

(5.14)

**Proof.** Let \(r(j) = \frac{j(3j+1)}{2}\). Then, it follows from [3] that

\[
F_{3n}(1, -1) = \sum_{j=-n}^{n-1} (-1)^j q^{r(j)}, \quad F_{3n+1}(1, -1) = F_{3n+2}(1, -1) = \sum_{j=-n}^{n} (-1)^j q^{r(j)},
\]

or

\[
F_n(1, -1) = \sum_{-n \leq 3j \leq n-1} (-1)^j q^{r(j)}.
\]

Let \(w(n) = \sum_{k \geq 0} \left( \binom{n}{k} - \binom{n}{k-1} \right) F_{n+1-2k}(1, -1)\). Consider a fixed term \((-1)^j q^{r(j)}\). This term occurs in \(F_n(1, -1)\) if \(-\frac{n-1}{3} \leq j \leq \frac{n-1}{3}\). We are looking for all \(k\), such that this term occurs in \(F_{n+1-2k}(1, -1)\). If \(j \geq 0\) then the largest such number is \(k_0 = \left[ \frac{n-3j}{2} \right] \). For \(j \leq \frac{n-2k}{3}\) is
equivalent with \( k \leq k_0 \). Therefore the coefficient of \((-1)^j q^{r(j)}\) in \( w(n) \) is \( \sum_{k=0}^{k_0} \binom{n}{k} - \binom{n}{k-1} = \binom{n}{k_0} \). If \( j < 0 \) then \(-\frac{n+1-2k}{q} \leq j \) is equivalent with \( k \leq \lfloor \frac{n+1+3j}{2} \rfloor \). This gives

\[
H_n(1, q-1|q) = \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^j q^{j(3j+1)/2} \left( \frac{n}{n-3j} \right) + \sum_{j=1}^{\lfloor (n+1)/3 \rfloor} (-1)^j q^{j(3j-1)/2} \left( \frac{n}{n-3j+1} \right).
\]

(5.15)

Now, we have

\[
\left( \frac{n}{n-3j+1} \right) = \left( \frac{n}{n+3j} \right)
\]

because \( \left\lfloor \frac{n-3j+1}{2} \right\rfloor + \left\lfloor \frac{n+3j}{2} \right\rfloor = n \). So (5.15) is equivalent to (5.14).

\[\Box\]

**Corollary 14.** We have

\[
H_{2n}(1, q-1|q) = q^{-n} \sum_{j=-n}^{n} \left( \frac{2n}{n-3j} - \frac{2n}{n-3j-1} \right) q^{2j(3j+1)},
\]

(5.16)

and

\[
H_{2n+1}(1, q-1|q) = q^{-n} \sum_{j=-n}^{n} \left( \frac{2n+1}{n-3j} - \frac{2n+1}{n-3j-1} \right) q^{2j(3j+2)}.
\]

(5.17)

**Proof.** Note that

\[
H_{2n}(1, q-1|q) = \frac{1}{q^n} \sum_{k=0}^{n} \left( \frac{2n}{n-k} - \frac{2n}{n-k-1} \right) q^k F_{2k+1}(1, -\frac{1}{q}),
\]

(5.18)

\[
H_{2n+1}(1, q-1|q) = \frac{1}{q^n} \sum_{k=0}^{n+1} \left( \frac{2n+1}{n+1-k} - \frac{2n+1}{n-k} \right) q^{k-1} F_{2k}(1, -\frac{1}{q}).
\]

(5.19)

Recall (see [3]) that

\[
F_{3n}(1, -\frac{1}{q}) = 0, \quad F_{3n+1}(1, -\frac{1}{q}) = (-1)^n q^{r(n)}, \quad F_{3n+2}(1, -\frac{1}{q}) = (-1)^n q^{r(-n)}.
\]

(5.20)

Hence

- if \( k = 3j \) then \( 2k+1 = 6j+1 \) and \( q^k F_{2k+1}(1, -\frac{1}{q}) = q^{3j} F_{6j+1}(1, -\frac{1}{q}) = q^{2j(3j+1)} \).
- if \( k = 3j+1 \) then \( 2k+1 = 6j+3 \) and \( q^k F_{2k+1}(1, -\frac{1}{q}) = 0 \).
- if \( k = 3j+2 \) then \( 2k+1 = 6j+5 \) and \( q^k F_{2k+1}(1, -\frac{1}{q}) = q^{3j+2} \).
- if \( k = 3j \) then \( 2k = 6j \) and \( q^{k-1} F_{2k}(1, -\frac{1}{q}) = 0 \).
- if \( k = 3j+1 \) then \( 2k = 6j+2 \) and \( q^{k-1} F_{2k}(1, -\frac{1}{q}) = q^{2j(3j+2)} \).
- if \( k = 3j+2 \) then \( 2k = 6j+4 \) and \( q^{k-1} F_{2k}(1, -\frac{1}{q}) = -q^{(3j+1)(2j+2)} \).

Substituting the above values into (5.18) and (5.19) yields (5.16) and (5.17).

\[\Box\]

Finally, from (5.12) and (3.8) we derive two explicit formulae for the coefficient \( c(n, k, q) \).
Proposition 15. If \( k \equiv n \pmod{2} \) then
\[
c(n, k, q) = \sum_{m \in M(n,k)} q^{c(m)+cr(m)}
= (1 - q) \frac{n-k}{2} \sum_{j \geq 0} \left( \frac{n}{2j} \right) (-1)^j q^{j+1} \left[ \frac{k + 2j}{k + j} \right] \left[ \frac{k + j}{k} \right].
\]
(5.21)

We now give a second proof of Proposition 15 using Theorem 6 and the orthogonality of the continuous \( q \)-Hermite polynomials.

Proof. Clearly Theorem 6 is equivalent to
\[
x^n = \sum_{k=n \pmod{2}} c(n, k, q) s^{(n-k)/2} \hat{H}_k(x, s|q).
\]
(5.23)

To compute \( c(n, k, q) \) we can take \( s = 1 \) and let \( \hat{H}_n(x|q) = \hat{H}_n(x, s|q) \). It is known (see [9]) that the continuous \( q \)-Hermite polynomials \( \hat{H}_n(x|q) \) are orthogonal with respect to the linear functional \( \varphi \) defined by
\[
\varphi(x^n) = \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} x^n v(x, q) dx,
\]
(5.24)
where
\[
v(x, q) = \sqrt{(1-q)(q\infty)} \prod_{k=0}^{\infty} \{1 + (2 - (1-q)x^2)q^k + q^{2k}\}.
\]

Since \( \varphi((\hat{H}_k(x|q))^2) = [k]_q! \), it follows from (5.23) that, for \( k \equiv n \pmod{2} \),
\[
c(n, k, q) = \frac{1}{[k]_q!} \varphi(x^n \hat{H}_k(x|q)).
\]
(5.25)

Recall the well-known formula (see [9])
\[
x^{2n} = \sum_{j=-n}^{n} \binom{2n}{n+j} T_{2j}(x/2),
\]
(5.26)
where \( T_n(\cos \theta) = \cos(n \theta) = T_{-n}(\cos \theta) \) is the \( n \)th Chbyeshev polynomial of the first kind. By using the Jacobi triple product formula and the terminating \( q \)-binomial formula, we can prove (see [7, p. 307]) that, for any integer \( j \) and \( a = \sqrt{1-q} \),
\[
\varphi(T_{n-2j}(ax/2) \hat{H}_n(x|q)) = \frac{(-1)^{n+j}}{2^{a^n}} q^{\frac{n-j}{2}} \left\{ (q^{-n+j+1}; q)_n + q^{n-j}(q^{-n+j}; q)_n \right\}.
\]
(5.27)
It follows from (5.25), (5.26) and (5.27) that
\[
c(2n, 2k, q) = \frac{a^{-2n}}{[2k]_q!} \sum_{j=-n}^{n} \left( \frac{2n}{n+j} \right) \varphi(T_{2j}(ax/2)\tilde{H}_{2k}(x|q))
\]
\[
= (1 - q)^{-(n-k)} \frac{(-1)^{k+j}q^{(k+j)}(q^{k}q^{j})(q^{k-j}; q)_{2k}}{(q; q)_{2k}}.
\]
Since \((q^{-k-j+1}; q)_{2k}\) is zero if \(j \neq -n, \ldots, -k\) and \(j \neq k + 1, \ldots, n\), and \((q^{-k-j}; q)_{2k}\) is zero if \(j \neq -n, \ldots, -k - 1\) or \(j \neq k, \ldots, n\), we can split the last summation into the following four summations:
\[
S_1 = \sum_{j=-n}^{-k} \left( \frac{2n}{n+j} \right) \frac{(-1)^{k+j}q^{(k+j)}(q^{k-j+1}; q)_{2k}}{(q; q)_{2k}},
\]
\[
S_2 = \sum_{j=k+1}^{n} \left( \frac{2n}{n+j} \right) \frac{(-1)^{k+j}q^{(k+j)}(q^{k-j+1}; q)_{2k}}{(q; q)_{2k}},
\]
\[
S_3 = \sum_{j=-n}^{-k-1} \left( \frac{2n}{n+j} \right) \frac{(-1)^{k+j}q^{(k+j)}q^{k+j}(q^{k-j}; q)_{2k}}{(q; q)_{2k}},
\]
\[
S_4 = \sum_{j=k}^{n} \left( \frac{2n}{n+j} \right) \frac{(-1)^{k+j}q^{(k+j)}q^{k+j}(q^{k-j}; q)_{2k}}{(q; q)_{2k}}.
\]
It is readily seen, by replacing \(j\) by \(-j\) in \(S_1\) and \(S_3\), that \(S_1 = S_4\) and \(S_2 = S_3\). Therefore,
\[
c(2n, 2k, q) = (1 - q)^{-(n-k)} \frac{(S_2 + S_4)}{(q; q)_{2k}}
\]
\[
= (1 - q)^{-(n-k)} \sum_{j \geq 0} \left( \frac{2n}{n+k+j} \right) (-1)^{j}q^{(j)} \left[ \frac{2k+j}{2k} \right] \left[ \frac{2k+j}{j} \right].
\] (5.28)
This corresponds to (5.21) for even indices. To derive the formula for odd indices we can use (3.9) to get
\[
c(2n + 1, 2k + 1, q) = [2k + 2]_q c(2n, 2k + 2, q) + c(2n, 2k, q),
\]
and then apply (5.28).

Some remarks about the above formula are in order.

(a) Formula (5.22) has been obtained by different means by Josuat-Vergès [10, Proposition 12] and is also used in [5]. It is easy to see that (5.21) and (5.22) are equal by writing
\[
\frac{[k+2j]}{[k+j]} = q^j + \frac{[j]}{[k+j]}.
\]
(b) When \(k = 0\), we recover a formula of Touchard-Riordan (see [2, 9, 14]):
\[
c(2n, 0, q) = \sum_{m \in M(2n, 0)} q^{cr(m)} = \frac{1}{(1-q)^n} \sum_{j=-n}^{n} \left( \frac{2n}{n+j} \right) (-1)^{j}q^{(j)}.
\] (5.29)
(c) Notice that $H_{2n}(0, -1|q) = c(2n, 0, q)$ and $H_{2n+1}(0, -1|q) = c(2n + 1, 0, q) = 0$. Hence
\[
\sum_{n\geq 0} c(n, 0, q)t^n = \frac{1}{1 - \frac{[2]q^2 t^2}{1 - \frac{[3]q^2 t^2}{1 - \ldots}}}. \]

We derive a known result (see [9]): the coefficient $c(n, 0, q)$ coincides with the $n$-th moment of the continuous $q$-Hermite polynomials $\tilde{H}(x, 1|q)$, i.e.,
\[
\mathcal{F}(z^n) = c(n, 0, q),
\]
where $\mathcal{F}$ is the linear functional acting on the polynomials in $z$ defined by $\mathcal{F}(H_n(z, 1|q)) = \delta_{n,0}$.

As in [11] we can derive another double sum expression for $H_n(x, s|q)$. The proof is omitted.

**Proposition 16.** We have
\[
H_n(x, s|q) = \sum_{k=0}^{n} (-1)^k q^{-k} \sum_{i=0}^{k} \left( \frac{s}{x(q-1)} q^{-i} + xq^i \right)^n \times \prod_{j=0, j\neq i}^{k} \frac{1}{q^{-i} - q^{-j} + x^2 \frac{q-1}{s} (q^i - q^j)}. \]

**(5.30)**

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**References**


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