Fibonacci polynomials and central factorial numbers

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Abstract

I consider some bases of the vector space of polynomials which are defined by Fibonacci and Lucas polynomials and compute the matrices of corresponding basis transformations. It turns out that these matrices are intimately related to Stirling numbers and central factorial numbers and also to Bernoulli numbers, Genocchi numbers and tangent numbers. There is also a close connection with the Akiyama-Tanigawa algorithm. Since such numbers have been extensively studied it would be no surprise if some of these results are already known, but hidden in the literature. I would therefore be very grateful for hints to relevant papers or books or for other comments.

1. Introduction

We consider (a variant of) the Fibonacci polynomials defined by

$$F_n(s) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-k}{k} s^k. \tag{1.1}$$

They satisfy the recursion $F_n(s) = F_{n-1}(s) + sF_{n-2}(s)$ with initial values $F_0(s) = 0$ and $F_1(s) = 1$.

The first terms are

$(0,1,1,1+s,1+2s,1+3s+s^2,1+4s+3s^2,1+5s+6s^2+s^3,1+6s+10s^2+4s^3,\dots).$

The corresponding Lucas polynomials are defined by

$$L_n(s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} s^k \tag{1.2}$$

and satisfy the same recurrence as the Fibonacci polynomials, but with initial values $L_0(s) = 2$ and $L_1(s) = 1$. The first terms of this sequence are

$(2,1,1+2s,1+3s,1+4s+2s^2,1+5s+5s^2,1+6s+9s^2+2s^3,\dots).$

Let

$$\alpha = \frac{1+\sqrt{1+4s}}{2} \tag{1.3}$$

and

$$\beta = \frac{1-\sqrt{1+4s}}{2} \tag{1.4}$$

be the roots of the equation $z^2 - z - s = 0.$

Then for $s \neq -\frac{1}{4}$ the well-known Binet formulae give

$$F_n(s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.5}$$

and

$$L_n(s) = \alpha^n + \beta^n. \tag{1.6}$$

For $s = -\frac{1}{4}$ it is easily verified that

$$F_n\left(-\frac{1}{4}\right) = \frac{n}{2^{n-1}}. \tag{1.7}$$

It is also well known that $L_n(s) = F_{n+1}(s) + sF_{n-1}(s).$ This follows immediately from $s = -\alpha\beta$ and $\alpha^{n+1} - \beta^{n+1} - \alpha\beta\left(\alpha^{n-1} - \beta^{n-1}\right) = \alpha^n\left(\alpha - \beta\right) + \beta^n\left(\alpha - \beta\right).$

Since $\deg F_{2n+1}(s) = \deg F_{2n+2}(s) = \deg L_{2n}(s) = \deg L_{2n+1}(s) = n$ each of the sets $\{F_{2n+1}(s)\}, \{F_{2n+2}(s)\}, \{L_{2n}(s)\}, \{L_{2n+1}(s)\}$ is a basis for the vector space of polynomials in s.

I am interested in the matrices which transform one basis into another. Their entries are related to Bernoulli numbers, Genocchi numbers and tangent numbers. Furthermore there are interesting connections with Stirling numbers and central factorial numbers.

The Genocchi numbers $\left(G_{2n}\right)_{n\geq 1} = \left(1,1,3,17,155,2073,38227,929569,\cdots\right)$ (cf. OEIS A110501) and their relatives $(g_n)_{n\geq 1} = \left(1,-1,0,1,0,-3,0,17,0,-155,\cdots\right)$ (cf. OEIS A036968) are defined by their exponential generating function

$$\frac{2z}{1+e^z} = z + z\frac{1-e^z}{1+e^z} = \sum_{n\geq 1} g_n \frac{z^n}{n!} = z + \sum_{n\geq 1} (-1)^n G_{2n} \frac{z^{2n}}{(2n)!}, \tag{1.8}$$

and the tangent numbers $\left(T_{2n+1}\right)=\left(1,2,16,272,7936,353792,22368256,\cdots\right)$ (cf. OEIS A000182) are defined by

$$\frac{e^{2z}-1}{e^{2z}+1}=\sum_{k\geq 0}(-1)^k T_{2k+1}\frac{z^{2k+1}}{(2k+1)!}. \tag{1.9}$$

Note that by comparing coefficients we get

$$(-1)^{k-1}\frac{g_{2k+2}}{2k+2}=\frac{G_{2k+2}}{2k+2}=\frac{T_{2k+1}}{2^{2k+1}}. \tag{1.10}$$

It is also well-known that $G_{2n}=(-1)^n 2(1-2^{2n})B_{2n}$, where $\left(B_n\right)$ is the sequence of Bernoulli numbers defined by

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k \tag{1.11}$$

for $n>1$ with $B_0=1.$ The list of Bernoulli numbers begins with

$1,-\frac{1}{2},\frac{1}{6},0,-\frac{1}{30},0,\frac{1}{42},0,-\frac{1}{30},0,\frac{5}{66},0,-\frac{691}{2730},0,\frac{7}{6},0,-\frac{3617}{510},\ldots$

For later uses let us also recall the generating function of the Bernoulli numbers

$$\sum_{n\geq 0}B_n\frac{z^n}{n!}=\frac{z}{e^z-1}. \tag{1.12}$$

2. Connection constants

The following theorem gives an explicit computation of some basis transformations.

Theorem 2.1

The bases $\left(F_{2n+2}\right)$ *and* $\left(F_{2n+1}\right)$ *are connected by*

$$F_{2n+2}(s)=\sum_{k=0}^{n}(-1)^{n-k}\frac{G_{2n-2k+2}}{2k+1}\binom{2n+2}{2k}F_{2k+1}, \tag{2.1}$$

and

$$F_{2n+1}(s)=\sum_{k=0}^{n}\binom{2n+1}{2k+1}\frac{B_{2n-2k}}{k+1}F_{2k+2}(s). \tag{2.2}$$

The bases consisting of Lucas polynomials are connected by

$$L_{2n+1}(s)=\sum_{k=0}^{n}(-1)^{n-k}\frac{T_{2n-2k+1}}{2^{2n-2k+1}}\binom{2n+1}{2k}L_{2k}(s)=\sum_{k=0}^{n}(-1)^{n-k}\frac{G_{2n-2k+2}}{2n-2k+2}\binom{2n+1}{2k}L_{2k}(s) \qquad (2.3)$$

and

$$L_{2n}(s)=2\sum_{j=0}^{n}\binom{2n}{2j}\frac{B_{2n-2j}}{2j+1}L_{2j+1}(s). \qquad (2.4)$$

Proof

1) Since $\alpha+\beta=1$ we have

$$e^{\alpha z}-e^{\beta z}=e^{(1-\beta)z}-e^{(1-\alpha)z}=-e^{z}\left(e^{-\alpha z}-e^{-\beta z}\right). \qquad (2.5)$$

This gives

$$\sum_{n\geq 0}\frac{F_n(s)}{n!}z^n=-e^{z}\sum_{n\geq 0}\frac{F_n(s)}{n!}(-z)^n \qquad (2.6)$$

or

$$\frac{1+e^{-z}}{2}\sum_{n}F_n(s)\frac{z^n}{n!}=\sum_{n}F_{2n+1}(s)\frac{z^{2n+1}}{(2n+1)!} \qquad (2.7)$$

This reduces our task to finding the matrix which corresponds to the operator "multiplication by $\frac{1+e^{-z}}{2}$" of the corresponding exponential generating functions.

Comparing coefficients we see that for $x=(x_n)_{n\geq 0}$ and $y=(y_n)_{n\geq 0}$

$$\frac{1+e^{-z}}{2}\sum_{n}x_n\frac{z^n}{n!}=\sum_{n}y_n\frac{z^n}{n!} \qquad (2.8)$$

is equivalent with $y=Mx$

where $M=\left(m(i,j)\right)$ with $m(n,k)=(-1)^{n-k}\frac{1}{2}\binom{n}{k}$ for $k<n$, $m(n,n)=1$ and $m(n,k)=0$ for $k>n$.

For example

$$M_6=\begin{pmatrix}1&0&0&0&0&0\\-\frac{1}{2}&1&0&0&0&0\\\frac{1}{2}&-1&1&0&0&0\\-\frac{1}{2}&\frac{3}{2}&-\frac{3}{2}&1&0&0\\\frac{1}{2}&-2&3&-2&1&0\\-\frac{1}{2}&\frac{5}{2}&-5&5&-\frac{5}{2}&1\end{pmatrix}.$$

The inverse of (2.8) is

$$\sum_n x_n\frac{z^n}{n!}=\frac{2}{1+e^{-z}}\sum_n y_n\frac{z^n}{n!}=\sum_j g_{j+1}\frac{(-z)^j}{(j+1)j!}\sum_k y_k\frac{z^k}{k!}=\sum_n\frac{z^n}{n!}\sum_{k=0}^n(-1)^{n-k}\frac{g_{n-k+1}}{n-k+1}\binom{n}{k}y_k,\text{ thus}$$

$$x_n=\sum_{k=0}^n(-1)^{n-k}\frac{g_{n-k+1}}{n-k+1}\binom{n}{k}y_k. \tag{2.9}$$

For example

$$M_6^{-1}=\begin{pmatrix}1&0&0&0&0&0\\\frac{1}{2}&1&0&0&0&0\\0&1&1&0&0&0\\-\frac{1}{4}&0&\frac{3}{2}&1&0&0\\0&-1&0&2&1&0\\\frac{1}{2}&0&-\frac{5}{2}&0&\frac{5}{2}&1\end{pmatrix}.$$

Thus (2.7) implies

$$F_n(s)=\sum(-1)^{n-1}\frac{g_{n+2-2k}}{n+2-2k}\binom{n}{2k-1}F_{2k-1}$$

and we get as special case

$$F_{2n+2}(s)=-\sum_{k\geq 0}\frac{g_{2n-2k+2}}{2n-2k+2}\binom{2n+2}{2k+1}F_{2k+1}=\sum_{k=0}^n(-1)^{n-k}\frac{G_{2n-2k+2}}{2k+1}\binom{2n+2}{2k}F_{2k+1},$$

i.e. (2.1).

From

$$e^{\alpha z}+e^{\beta z}=e^{(1-\beta)z}+e^{(1-\alpha)z}=e^z\left(e^{-\alpha z}+e^{-\beta z}\right) \tag{2.10}$$

we get in the same way as before

$$\frac{1+e^{-z}}{2}\sum_n L_n(s)\frac{z^n}{n!}=\sum_n L_{2n}(s)\frac{z^{2n}}{(2n)!}. \quad (2.11)$$

In this case (2.9) implies (2.3).

2) Another consequence of (2.5) is

$$\left(1-e^{-z}\right)\sum_{n\geq 0}\frac{F_n(s)}{n!}z^n=2\sum_{n\geq 0}\frac{F_{2n}(s)}{(2n)!}z^{2n}. \quad (2.12)$$

Now

$(1-e^{-z})\sum_k x(k)\frac{z^k}{k!}=\sum_n \frac{z^n}{n!}\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{j}x(j)=\sum_{n\geq 1}y(n-1)\frac{z^n}{n!}$ is equivalent with

$$y(n)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+1}{j}x(j). \quad (2.13)$$

The inverse is

$$\sum_k x(k)\frac{z^k}{k!}=\frac{1}{(1-e^{-z})}\sum_n y(n-1)\frac{z^n}{n!}=\sum_n \frac{b_k z^{k-1}}{k!}\sum_n y(j-1)\frac{z^j}{j!}$$

where $b(n)=B_n$ for $n\neq 1$ and $b(1)=\frac{1}{2}$. This follows from

$$\sum_n b_n\frac{z^n}{n!}=\sum_n B_n\frac{z^n}{n!}+z=\frac{z}{e^z-1}+z=\frac{ze^z}{e^z-1}=\frac{z}{1-e^{-z}}.$$

This implies

$$x(n)=\sum_{j=0}^{n}\binom{n}{j}\frac{b_{n-j}}{j+1}y(j). \quad (2.14)$$

From (2.12) we see that with $x(n)=F_n(s)$ and $y(2n-1)=2F_{2n}(s), y(2n)=0,$ we get

$$F_{2n+1}(s)=\sum_{j=0}^{2n+1}\binom{2n+1}{j}\frac{b_{2n+1-j}}{j+1}y(j)=\sum_{j=0}^{n}\binom{2n+1}{2j+1}\frac{b_{2n-2j}}{2j+2}y(2j+1)=\sum_{j=0}^{n}\binom{2n+1}{2j+1}\frac{b_{2n-2j}}{j+1}F_{2j+2},$$

i.e. (2.2).

(2.10) implies $\left(1-e^{-z}\right)\sum_{n\geq 0}\frac{L_n(s)}{n!}z^n=2\sum_{n\geq 0}\frac{L_{2n+1}(s)}{(2n+1)!}z^{2n+1}.$

Therefore we get from (2.12) with $x(n)=L_n(s)$ and $y(2n)=2L_{2n+1}(s), y(2n+1)=0,$

$$L_{2n}(s)=\sum_{j=0}^{n}\binom{2n}{2j}\frac{b_{2n-2j}}{2j+1}y(2j)=\sum_{j=0}^{n}\binom{2n}{2j}\frac{B_{2n-2j}}{2j+1}2L_{2j+1}(s),$$

i.e. (2.4).

The matrix corresponding to "multiplication by $\left(1-e^{-z}\right)$" of the generating functions is

$$C=\left((-1)^{i-j}[j\le i]\binom{i+1}{j}\right)_{i,j=0}^{\infty}. \tag{2.15}$$

Its inverse is given by

$$C^{-1}=\left([j\le i]\binom{i}{j}\frac{b(i-j)}{j+1}\right)_{i,j=0}^{\infty}. \tag{2.16}$$

For example for $n=5$ the corresponding finite parts C_5 and C_5^{-1} are

$$C_5=\begin{pmatrix}1&0&0&0&0\\-1&2&0&0&0\\1&-3&3&0&0\\-1&4&-6&4&0\\1&-5&10&-10&5\end{pmatrix}$$

and

$$C_5^{-1}=\begin{pmatrix}1&0&0&0&0\\\frac{1}{2}&\frac{1}{2}&0&0&0\\\frac{1}{6}&\frac{1}{2}&\frac{1}{3}&0&0\\0&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&0\\-\frac{1}{30}&0&\frac{1}{3}&\frac{1}{2}&\frac{1}{5}\end{pmatrix}$$

If we apply C^{-1} to the column vector y with entries $y(2n+1)=2F_{n+2}(s), y(2n)=0,$ then the entries $x(n)$ of $x=C^{-1}y$ are $x(n)=F_n(s).$

For even n this is trivial because in $x(2n)=\sum_{j=0}^{n}\binom{2n}{2j+1}\frac{b_{2n-2j-1}}{2j+2}y(2j+1)=\binom{2n}{2n-1}\frac{F_{2n}}{2n}=F_{2n}$

the only non-vanishing term occurs for $j = n-1.$ For odd n we get the identities (2.2).

If we apply C^{-1} to the column vector y with entries $y(2n) = 2L_n(s), y(2n+1) = 0,$ then the entries $x(n)$ of $x = C^{-1}y$ are $x(n) = L_n(s).$ For even n we get (2.4). For odd n the corresponding identity is again trivial.

3. Stirling numbers and central factorial numbers

The matrices (2.15) and (2.16) are intimately connected with Stirling numbers.

In order to make the paper self-contained I shall first state some well-known facts about Stirling numbers and some generalizations (cf. [4]).

Let $w = \left(w(n)\right)_{n\geq 0}$ be an increasing sequence of positive numbers. Define the $w-$Stirling numbers of the second kind $S^w(n,k)$ by

$$S^w(n,k) = S^w(n-1,k-1) + w(k)S^w(n-1,k) \tag{3.1}$$

with initial values $S^w(0,k) = [k=0]$ and $S^w(n,0) = w(0)^n$ and the $w-$Stirling numbers of the first kind $s^w(n,k)$ by

$$s^w(n,k) = s^w(n-1,k-1) - w(n-1)s^w(n-1,k) \tag{3.2}$$

with $s^w(0,k) = [k=0]$ and $s^w(n,0) = (-1)^n \prod_{j=0}^{n-1} w(j).$

This is equivalent with

$$\sum_{k=0}^{n} s^w(n,k)x^k = (x-w(0))(x-w(1))\cdots(x-w(n-1)), \tag{3.3}$$

$$\sum_{k=0}^{n} S^w(n,k)\prod_{j=0}^{k-1}(x-w(j)) = x^n \tag{3.4}$$

and

$$\sum_{n\geq k} S^w(n,k)x^n = \frac{x^k}{(1-w(0)x)(1-w(1)x)\cdots(1-w(k)x)}. \tag{3.5}$$

Let

$$S^w(n) = \left(S^w(i,j)\right)_{i,j=0}^{n-1} \tag{3.6}$$

and

$$s^w(n) = \left(s^w(i,j)\right)_{i,j=0}^{n-1}. \tag{3.7}$$

Then it is clear that

$$s^w(n) = \left(S^w(n)\right)^{-1}. \tag{3.8}$$

The same holds for the infinite matrices $S^w = \left(S^w(i,j)\right)_{i,j\geq 0}$ and $s^w = \left(s^w(i,j)\right)_{i,j\geq 0}$.

For later applications we note the trivial

Proposition 3.1

If an $n\times n-$matrix M_n has different eigenvalues $r(j)$ and if the eigenvector corresponding to $r(j)$ is the column vector $\left(S^w(i,j)\right)_{i=0}^{n-1}$, then $M_n = S^w(n)D(n)s^w(n),$ where $D(n)$ is the diagonal matrix with entries $r(0),\cdots,r(n-1).$

For $w(n)=n$ we get the Stirling numbers $s(n,k)$ of the first kind and $S(n,k)$ of the second kind.

Recall the first values of the Stirling numbers:

$$\left(S(i,j)\right)_{i,j=0}^{6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 \end{pmatrix},$$

$$\left(s(i,j)\right)_{i,j=0}^{6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 \\ 0 & -6 & 11 & -6 & 1 & 0 & 0 \\ 0 & 24 & -50 & 35 & -10 & 1 & 0 \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 \end{pmatrix}$$

Note that $\left(S(i,j)\right)_{i,j\geq 1}$ and $\left(s(i,j)\right)_{i,j\geq 1}$ are the same as $\left(S^{w}(i,j)\right)_{i,j\geq 0}$ and $\left(s^{w}(i,j)\right)_{i,j\geq 0}$ respectively for $w(n)=n+1.$

It is well-known that the generating function of the Stirling numbers of the second kind is $F_k(z)=\frac{(e^z-1)^k}{k!}=\sum_n S(n,k)\frac{z^n}{n!}$.

It can be characterized by

Proposition 3.2

The formal power series $F_k(z)=\sum_n S(n,k)\frac{z^n}{n!}$ *is the uniquely determined solution of the differential equation* $(1-e^{-z})F_k{}'(z)=kF_k(z)$ *with* $S(k,k)=1.$

Proof

It is clear that $F_k(z)=\frac{(e^z-1)^k}{k!}$ satisfies the differential equation.

If on the other hand $F(z)=\sum_{n\geq 0}x(n)\frac{z^n}{n!}$ satisfies the differential equation then

$$\sum_{i\geq 0}(-1)^{i-1}\frac{z^i}{i!}\sum_{j\geq 0}x(j)\frac{z^j}{j!}=k\sum_{n\geq 0}x(n)\frac{z^n}{n!}.$$

Comparing coefficients we see that for $n<k$ all $x(n)$ vanish and that for $n=k$ we get an arbitrary constant which can be chosen to be $x(k)=1.$

Consider the infinite matrix $C=\left(c(i,j)\right)_{i,j\geq 0}$ with $c(i,j)=(-1)^{i-j}\binom{i+1}{j}$ for $j\leq i$ and $c(i,j)=0$ for $j>i,$ which has been defined in (2.15). The eigenvalues are the numbers $1,2,3,\cdots.$

A vector $x_k=(x_k(n))_{n\geq 0}$ is an eigenvector of C with respect to the eigenvalue k if $Cx_k=kx_k$ or equivalently

if $(1-e^{-z})\sum_n x_k(n)\frac{z^n}{n!}=k\sum_n x_k(n-1)\frac{z^n}{n!}$. This means that $F_k(z)=\sum_n x_k(n-1)\frac{z^n}{n!}$ satisfies $(1-e^{-z})F_k{}'(z)=kF_k(z).$

Therefore by Proposition 3.2 $x_k(n)=S(n+1,k)$ is a Stirling number of the second kind.

This leads to

Theorem 3.3

The matrices C *and* C^{-1} *can be factored in the following way:*

$$C=\left(S(i+1,\,j+1)\right)_{i,\,j=0}^{\infty}\left((i+1)([i=j]\right)_{i,\,j=0}^{\infty}\left(s(i+1,\,j+1)\right)_{i,\,j=0}^{\infty} \tag{3.9}$$

and

$$C^{-1}=\left(S(i+1,\,j+1)\right)_{i,j=0}^{\infty}\left(\frac{1}{i+1}([i=j]\right)_{i,\,j=0}^{\infty}\left(s(j+1,k+1)\right)_{j,k=0}^{\infty}. \tag{3.10}$$

This theorem could also be derived from the following simple identities.

The identity

$$C\left([j\le i]\binom{i}{j}\right)_{i,\,j\ge 0}=\left([j\le i]\binom{i+1}{j}\right)_{i,\,j\ge 0} \tag{3.11}$$

is another formulation of the trivial result

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+1}{j}\binom{j}{k}=\binom{n+1}{k}+\sum_{j=0}^{n+1}(-1)^{n-j}\binom{n+1}{j}\binom{j}{k}=\binom{n+1}{k}$$

for $k\le n$.

The identity

$$\left([j\le i]\binom{i}{j}\right)_{i,\,j\ge 0}\left(S(i,\,j)\right)_{i,\,j\ge 0}=\left(S(i+1,\,j+1)\right)_{i,\,j\ge 0} \tag{3.12}$$

is a well-known property of the Stirling numbers and easily proved by induction:

$$\sum_{j=0}^{n}\binom{n}{j}S(j,k)=\sum_{j=0}^{n-1}\binom{n-1}{j}S(j,k)+\sum_{j=0}^{n-1}\binom{n-1}{j-1}S(j,k)=S(n,k+1)+\sum_{j=0}^{n-1}\binom{n-1}{j}S(j+1,k)$$
$$=S(n,k+1)+\sum_{j=0}^{n-1}\binom{n-1}{j}S(j,k-1)+k\sum_{j=0}^{n-1}\binom{n-1}{j}S(j,k)$$
$$=S(n,k+1)+S(n,k)+kS(n,k+1)=S(n+1,k+1).$$

Furthermore we need

$$\left([j\le i]\binom{i+1}{j}\right)_{i,j\ge 0}\left(S(i,j)\right)_{i,j\ge 0}=\left(S(i+1,j+1)(j+1)\right)_{i,j\ge 0}. \tag{3.13}$$

This also is easily proved by induction:

$$\sum_{j=0}^{n}\binom{n+1}{j}S(j,k)=\sum_{j=0}^{n}\binom{n}{j}S(j,k)+\sum_{j=0}^{n}\binom{n}{j-1}S(j,k)=S(n+1,k+1)+\sum_{j=0}^{n-1}\binom{n}{j}S(j+1,k)$$
$$=S(n+1,k+1)+\sum_{j=0}^{n-1}\binom{n}{j}S(j,k-1)+k\sum_{j=0}^{n-1}\binom{n}{j}S(j,k)$$
$$=S(n+1,k+1)+\left(S(n+1,k)-S(n,k-1)\right)+k\left(S(n+1,k+1)-S(n,k)\right)$$
$$=(k+1)S(n+1,k+1)+\left(S(n+1,k)-S(n,k-1)-kS(n,k)\right)=(k+1)S(n+1,k+1).$$

To derive (3.9) observe that (3.11) gives

$$C=\left([j\le i]\binom{i+1}{j}\right)_{i,j\ge 0}\left(\left([j\le i]\binom{i}{j}\right)_{i,j\ge 0}\right)^{-1},$$

(3.13) gives

$$\left([j\le i]\binom{i+1}{j}\right)_{i,j\ge 0}=\left(S(i+1,j+1)(j+1)\right)_{i,j\ge 0}\left(\left(S(i,j)\right)_{i,j\ge 0}\right)^{-1}$$

and (3.12) gives

$$\left(\left([j\le i]\binom{i}{j}\right)_{i,j\ge 0}\right)^{-1}=\left(S(i,j)\right)_{i,j\ge 0}\left(\left(S(i+1,j+1)\right)_{i,j\ge 0}\right)^{-1}.$$

Combining these identities gives (3.9).

Another consequence is

$$\left(\left([j\le i]\binom{i}{j}\right)_{i,j\ge 0}\right)^{-1}\left([j\le i]\binom{i+1}{j}\right)_{i,j\ge 0}=\left(S(i,j)\right)_{i,j\ge 0}\left((i+1)([i=j]\right)_{i,j\ge 0}\left(s(i,j)\right)_{i,j\ge 0}. \tag{3.14}$$

If we choose $w(n)=n^2$ we get the central factorial numbers $t(n,k)=s^w(n,k)$ of the first kind and the central factorial numbers $T(n,k)=S^w(n,k)$ of the second kind respectively.

These numbers have been introduced in [12] with a different notation. Further results can be found in [14], Exercise 5.8.

The following tables show the upper part of the matrices of central factorial numbers. See also OEIS A036969.

$$\left(T(i,j)\right)_{i,j=0}^{6}=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&1&1&0&0&0&0\\0&1&5&1&0&0&0\\0&1&21&14&1&0&0\\0&1&85&147&30&1&0\\0&1&341&1408&627&55&1\end{pmatrix}$$

$$\left(t(i,j)\right)_{i,j=0}^{6}=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&-1&1&0&0&0&0\\0&4&-5&1&0&0&0\\0&-36&49&-14&1&0&0\\0&576&-820&273&-30&1&0\\0&-14400&21076&-7645&1023&-55&1\end{pmatrix}.$$

By (3.5) the generating function of $T(n,k)$ is

$$\sum_n T(n,k)x^n=\frac{x^k}{(1-x)(1-2^2x)\cdots(1-k^2x)}.$$

The following partial fraction expansion is easily verified:

$$\frac{(2k)!\,x^{2k}}{(1-x^2)(1-2^2x^2)\cdots(1-k^2x^2)}=\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\frac{1}{1-(k-j)x}$$

This implies that

$$T(n,k)=\frac{1}{(2k)!}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(j-k)^{2n}. \qquad (3.15)$$

From this we imply the known exponential generating function (cf. [12])

$$T_k(z)=\sum T(n,k)\frac{z^{2n}}{(2n)!}=\frac{e^{-kz}(1-e^z)^{2k}}{(2k)!}=\frac{1}{(2k)!}\left(2\sinh(\frac{z}{2})\right)^{2k}. \qquad (3.16)$$

In the same way as above we get

Proposition 3.4

The formal power series $T_k(z)=\sum_n T(n,k)\frac{z^{2n}}{(2n)!}$ *is the uniquely determined solution of the differential equation* $\frac{e^z-1}{e^z+1}T_k'(z)=kT_k(z)$ *with* $T(k,k)=1.$

For $w(n)=n(n+1)$ we get the Legendre-Stirling numbers $LS(n,k)$ and $ls(n,k)$ studied in [2] and [8] (cf. OEIS A071951 and A129467).

$$\left(LS(i,j)\right)_{i,j=0}^{6}=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&2&1&0&0&0&0\\0&4&8&1&0&0&0\\0&8&52&20&1&0&0\\0&16&320&292&40&1&0\\0&32&1936&3824&1092&70&1\end{pmatrix}$$

$$\left(ls(i,j)\right)_{i,j=0}^{6}=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&-2&1&0&0&0&0\\0&12&-8&1&0&0&0\\0&-144&108&-20&1&0&0\\0&2880&-2304&508&-40&1&0\\0&-86400&72000&-17544&1708&-70&1\end{pmatrix}.$$

There are some interesting relations between central factorial numbers and Legendre-Stirling numbers which are analogous to the corresponding results about Stirling numbers.

Theorem 3.6

$$\sum_{j=0}^{n}\binom{2n-j}{j}LS(j,k)=T(n+1,k+1) \qquad (3.17)$$

and

$$\sum_{j=0}^{n}\binom{2n+1-j}{j}LS(j,k)=(k+1)T(n+1,k+1) \qquad (3.18)$$

Proof

For $n<k+1$ all terms vanish. For $n=k+1$ we have

$$\sum_{j=0}^{k}\binom{2k+1-j}{j}LS(j,k)=\binom{2k+1-k}{k}=(k+1)=(k+1)T(k+1,k+1)$$

and

$$\sum_{j=0}^{k}\binom{2k-j}{j}LS(j,k)=\binom{k}{k}=1=T(k+1,k+1).$$

Now assume that (3.17) and (3.18) are already known up to $n-1.$ Then

$$\sum_{j=0}^{n}\binom{2n-j}{j}LS(j,k)=\sum_{j=0}^{n}\left(\binom{2n-1-j}{j-1}+\binom{2n-1-j}{j}\right)LS(j,k)$$
$$=\sum_{j=0}^{n-1}\binom{2(n-1)-j}{j}LS(j+1,k)+\sum_{j=0}^{n-1}\binom{2(n-1)+1-j}{j}LS(j,k)$$
$$=\sum_{j=0}^{n-1}\binom{2(n-1)-j}{j}LS(j,k-1)+k(k+1)\sum_{j=0}^{n-1}\binom{2(n-1)-j}{j}LS(j,k)+\sum_{j=0}^{n-1}\binom{2(n-1)+1-j}{j}LS(j,k)$$
$$=T(n,k)+k(k+1)T(n,k+1)+(k+1)T(n,k+1)=T(n+1,k+1).$$

In the same way we get

$$\sum_{j=0}^{n}\binom{2n+1-j}{j}LS(j,k)=\sum_{j=0}^{n}\left(\binom{2n-j}{j-1}+\binom{2n-j}{j}\right)LS(j,k)$$
$$=\sum_{j=0}^{n-1}\binom{2(n-1)+1-j}{j}LS(j+1,k)+\sum_{j=0}^{n}\binom{2n-j}{j}LS(j,k)$$
$$=\sum_{j=0}^{n-1}\binom{2(n-1)+1-j}{j}LS(j,k-1)+k(k+1)\sum_{j=0}^{n-1}\binom{2(n-1)+1-j}{j}LS(j,k)+T(n+1,k+1)$$
$$=kT(n,k)+k(k+1)^2T(n,k+1)+T(n+1,k+1)=(k+1)T(n+1,k+1).$$

Corollary 3.7

$$\left([j\le i]\binom{2i-j}{j}\right)\left(LS(i,j)\right)=\left(T(i+1,j+1)\right) \tag{3.19}$$

and

$$\left([j \le i]\binom{2i+1-j}{j}\right)\left(LS(i,j)\right) = \left((j+1)T(i+1,j+1)\right). \qquad (3.20)$$

For example

$$\begin{pmatrix}1&0&0&0&0&0\\1&1&0&0&0&0\\1&3&1&0&0&0\\1&5&6&1&0&0\\1&7&15&10&1&0\\1&9&28&35&15&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&2&1&0&0&0\\0&4&8&1&0&0\\0&8&52&20&1&0\\0&16&320&292&40&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0&0\\1&1&0&0&0&0\\1&5&1&0&0&0\\1&21&14&1&0&0\\1&85&147&30&1&0\\1&341&1408&627&55&1\end{pmatrix}.$$

and

$$\begin{pmatrix}1&0&0&0&0&0\\1&2&0&0&0&0\\1&4&3&0&0&0\\1&6&10&4&0&0\\1&8&21&20&5&0\\1&10&36&56&35&6\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&2&1&0&0&0\\0&4&8&1&0&0\\0&8&52&20&1&0\\0&16&320&292&40&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0&0\\1&2&0&0&0&0\\1&10&3&0&0&0\\1&42&42&4&0&0\\1&170&441&120&5&0\\1&682&4224&2508&275&6\end{pmatrix}.$$

Corollary 3.8

$$\left([j \le i]\binom{2i+1-j}{j}\right)\left(\left([j \le i]\binom{2i-j}{j}\right)\right)^{-1} = \left(T(i+1,j+1)\right)\left([i=j](j+1)\right)\left(\left(T(i+1,j+1)\right)\right)^{-1}. \qquad (3.21)$$

and

$$\left(\left([j \le i]\binom{2i-j}{j}\right)\right)^{-1}\left([j \le i]\binom{2i+1-j}{j}\right) = \left(LS(i,j)\right)\left([i=j](j+1)\right)\left(\left(LS(i,j)\right)\right)^{-1}. \qquad (3.22)$$

Proof

$$\left([j \le i]\binom{2i+1-j}{j}\right)\left(\left([j \le i]\binom{2i-j}{j}\right)\right)^{-1}$$
$$= \left(T(i+1,j+1)\right)\left([i=j](j+1)\right)\left(\left(LS(i,j)\right)\right)^{-1}\left(LS(i,j)\right)\left(\left(T(i+1,j+1)\right)\right)^{-1}$$

In the same way we get (3.22).

Another interesting result is

Theorem 3.9

The following identities hold:

$$\sum_{j=0}^{n}\binom{n+1}{2n-2j}T(j+1,k+1)=LS(n+1,k+1) \tag{3.23}$$

and

$$\sum_{j=0}^{n}\binom{n+1}{2n-2j+1}T(j+1,k+1)=(k+1)LS(n+1,k+1). \tag{3.24}$$

Equivalently this means

$$\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge 0}\left(T(i+1,\,j+1)\right)_{i,j\ge 0}=\left(LS(i+1,\,j+1)\right)_{i,j\ge 0} \tag{3.25}$$

and

$$\begin{aligned}&\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge 0}\left(T(i+1,\,j+1)\right)_{i,j\ge 0}=\left((j+1)LS(i+1,\,j+1)\right)_{i,j\ge 0}\\&=\left(LS(i+1,\,j+1)\right)_{i,j\ge 0}\left([i=j](j+1)\right)_{i,j\ge 0}.\end{aligned} \tag{3.26}$$

Proof

For $n<k$ both sides vanish. For $n=k$ we get 1 and $k+1$ respectively.

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n+1}{2n-2j}T(j+1,k+1)=\sum_{j=0}^{n}\binom{n}{2n-2j}T(j+1,k+1)+\sum_{j=0}^{n}\binom{n}{2n-2j-1}T(j+1,k+1)\\&=\sum_{j=1}^{n}\binom{n}{2(n-1)+2-2j}T(j+1,k+1)+\sum_{j=0}^{n}\binom{n}{2(n-1)+1-2j}T(j+1,k+1)\\&=\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+2,k+1)+(k+1)LS(n,k+1)\\&=\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+1,k)+(k+1)^2\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+1,k+1)+(k+1)LS(n,k+1)\\&=LS(n,k)+((k+1)^2+(k+1))LS(n,k+1)=LS(n+1,k+1).\end{aligned}$$

For the second sum we get

$$\sum_{j=0}^{n}\binom{n+1}{2n-2j+1}T(j+1,k+1)=\sum_{j=0}^{n}\binom{n}{2n-2j+1}T(j+1,k+1)+\sum_{j=0}^{n}\binom{n}{2n-2j}T(j+1,k+1)$$
$$=\sum_{j=1}^{n}\binom{n}{2(n-1)+3-2j}T(j+1,k+1)+\sum_{j=1}^{n}\binom{n}{2(n-1)+2-2j}T(j+1,k+1)$$
$$=\sum_{j=0}^{n-1}\binom{n}{2(n-1)+1-2j}T(j+2,k+1)+\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+2,k+1)$$

$$=\sum_{j=0}^{n-1}\binom{n}{2(n-1)+1-2j}T(j+1,k)+(k+1)^2\sum_{j=0}^{n-1}\binom{n}{2(n-1)+1-2j}T(j+1,k+1)$$
$$+\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+1,k)+(k+1)^2\sum_{j=0}^{n-1}\binom{n}{2(n-1)-2j}T(j+1,k+1)$$
$$=kLS(n,k)+(k+1)^2(k+1)LS(n,k+1)+LS(n,k)+(k+1)^2LS(n,k+1)$$
$$=(k+1)\big(LS(n,k)+(k+1)(k+2)LS(n,k+1)\big)=(k+1)LS(n+1,k+1).$$

Corollary 3.10

$$\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge0}\right)^{-1}\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge0}$$
$$=\big(T(i+1,j+1)\big)_{i,j\ge0}\big([i=j](j+1)\big)_{i,j\ge0}\Big(\big(T(i+1,j+1)\big)_{i,j\ge0}\Big)^{-1} \tag{3.27}$$

and

$$\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge0}\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge0}\right)^{-1}$$
$$=\big(LS(i+1,j+1)\big)_{i,j\ge0}\big([i=j](j+1)\big)_{i,j\ge0}\Big(\big(LS(i+1,j+1)\big)_{i,j\ge0}\Big)^{-1}. \tag{3.28}$$

Proof

The left-hand side of (3.27) equals

$$\big(LS(i+1,j+1)\big)_{i,j\ge0}\big([i=j](j+1)\big)_{i,j\ge0}\Big(\big(T(i+1,j+1)\big)_{i,j\ge0}\Big)^{-1}\big(T(i+1,j+1)\big)_{i,j\ge0}\Big(\big(LS(i+1,j+1)\big)_{i,j\ge0}\Big)^{-1}.$$

In the same way the left-hand side of (3.28) equals

$$\left(T(i+1,j+1)\right)_{i,j\ge0}\left(\left(LS(i+1,j+1)\right)_{i,j\ge0}\right)^{-1}\left(LS(i+1,j+1)\right)_{i,j\ge0}\left([i=j](j+1)\right)_{i,j\ge0}\left(\left(T(i+1,j+1)\right)_{i,j\ge0}\right)^{-1}.$$

Comparing (3.27) with (3.21) we see that

$$\left([j\le i]\binom{2i+1-j}{j}\right)\left(\left([j\le i]\binom{2i-j}{j}\right)\right)^{-1}=\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge0}\right)^{-1}\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge0}. \qquad (3.29)$$

This implies

$$\begin{aligned}&\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge0}\right)\left([j\le i]\binom{2i+1-j}{j}\right)=\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge0}\left(\left([j\le i]\binom{2i-j}{j}\right)\right)\\&=\left(LS(i+1,j+1)\right)_{i,j\ge0}\left([i=j](j+1)\right)_{i,j\ge0}\left(\left(LS(i,j)\right)_{i,j\ge0}\right)^{-1}.\end{aligned} \qquad (3.30)$$

Remark

Michael Schlosser has shown me a simple direct proof of the first identity in (3.30).

It suffices to show that

$$\sum_{j=k}^{n}\binom{n+1}{2n-2j}\binom{2j+1-k}{k}=\sum_{j=k}^{n}\binom{n+1}{2n-2j+1}\binom{2j-k}{k}.$$

This is equivalent with

$$\sum_{j=2k-1}^{2n}(-1)^{j+1}\binom{n+1}{2n-j}\binom{j+1-k}{k}=0.$$

This follows from the Chu-Vandermonde formula:

$$\begin{aligned}&\sum_{j=2k-1}^{2n}(-1)^{j+1}\binom{n+1}{2n-j}\binom{j+1-k}{j+1-2k}\\&=\sum_{i=0}^{2n+1-2k}(-1)^{i}\binom{n+1}{2n+1-2k-i}\binom{i+k}{i}=\sum_{i=0}^{2+1-2k}\binom{n+1}{2n+1-2k-i}\binom{-k-1}{i}=\binom{n-k}{2n-2k+1}=0.\end{aligned}$$

We later need the following result.

Lemma 3.11

Let $w(0)=1$ *and* $\hat{w}(n)=w(n+1).$ *Then*

$$S^{\hat{w}}(n,k)=S^{w}(n+1,k+1)-S^{w}(n,k+1) \tag{3.31}$$

and

$$s^{\hat{w}}(n,k)=-\sum_{j=0}^{k}s^{w}(n+1,j). \tag{3.32}$$

If

$$F_1(n,j)=\sum_{\ell=0}^{n}S^{w}(n,\ell)F\left(\ell\right)s^{w}(\ell,j)=F(0)+\sum_{\ell=1}^{n}S^{w}(n,\ell)F\left(\ell\right)s^{w}(\ell,j) \tag{3.33}$$

for some function $F(\ell)$ *then*

$$\sum_{\ell=0}^{n}S^{\hat{w}}(n,\ell)F\left(\ell+1\right)s^{\hat{w}}(\ell,k)=\sum_{j=0}^{k}\left(F_1(n,j)-F_1(n+1,j)\right). \tag{3.34}$$

Proof

The first assertions follow from

$$\sum_n S^{\hat{w}}(n,k)x^n=\frac{x^k}{(1-w(1)x)(1-w(2)x)\cdots(1-w(k+1)x)}=\frac{(1-x)}{x}\frac{x^{k+1}}{(1-w(0)x)(1-w(1)x)\cdots(1-w(k+1)x)}$$
$$=(1-x)\sum_n S^{w}(n,k+1)x^{n-1}=\sum_n\left(S^{w}(n+1,k+1)-S^{w}(n,k+1)\right)x^n$$

and

$$\sum_{k=0}^{n}s^{w}(n,k)x^k=(x-w(0))\cdots(x-w(n-1))=(x-1)\sum_{k=0}^{n-1}s^{\hat{w}}(n-1,k)x^k$$

or

$$\sum_{k=0}^{n}s^{\hat{w}}(n,k)x^k=-\frac{1}{1-x}\sum_{k=0}^{n+1}s^{w}(n+1,k)x^k.$$

This implies

$$\sum_{\ell=0}^{n} S^{\hat{w}}(n,\ell)F\left(\ell+1\right)s^{\hat{w}}(\ell,k) = -\sum_{\ell=0}^{n}\left(S^{w}(n+1,\ell+1) - S^{w}(n,\ell+1)\right)F\left(\ell+1\right)\sum_{j=0}^{k} s^{w}(\ell+1, j)$$

$$= -\sum_{j=0}^{k}\sum_{\ell=0}^{n} S^{w}(n+1,\ell+1)F\left(\ell+1\right)s^{w}(\ell+1, j) + \sum_{j=0}^{k}\sum_{\ell=0}^{n-1} S^{w}(n,\ell+1)F\left(\ell+1\right)s^{w}(\ell+1, j)$$

$$= -\sum_{j=0}^{k}\sum_{\ell=1}^{n+1} S^{w}(n+1,\ell)F\left(\ell\right)s^{w}(\ell, j) + \sum_{j=0}^{k}\sum_{\ell=1}^{n} S^{w}(n,\ell)F\left(\ell\right)s^{w}(\ell, j)$$

$$= \sum_{j=0}^{k}\left(F_1(n, j) - F_1(n+1, j)\right).$$

4. Some interesting matrices

We know already that

$$\sum_{n\geq 0}\frac{F_n(s)}{n!}z^n = -e^z\sum_{n\geq 0}\frac{F_n(s)}{n!}(-z)^n. \tag{4.1}$$

This can also be written in the form

$$\sum\frac{F_{2n}(s)}{(2n)!}z^{2n} = \frac{e^z-1}{e^z+1}\sum\frac{F_{2n+1}(s)}{(2n+1)!}z^{2n+1}. \tag{4.2}$$

If we define a linear functional λ on the vector space of polynomials in s by

$$\lambda\left(F_{2n+1}(s)\right) = [n=0], \tag{4.3}$$

we get

$$\lambda(F_{2n}(s)) = \left[\frac{z^{2n}}{(2n)!}\right]z\frac{e^z-1}{e^z+1}. \tag{4.4}$$

Using (1.8) this gives (cf. [5],[3])

$$\lambda(F_{2n}(s)) = (-1)^{n-1}G_{2n}. \tag{4.5}$$

By comparing coefficients we get again (cf. [3])

$$F_{2n+2}(s) = \sum_{k=0}^{n} a(n,k)F_{2k+1}(s) \tag{4.6}$$

with

$$a(n,k)=(-1)^{n-k}\frac{1}{2k+1}\binom{2n+2}{2k}G_{2n-2k+2}. \tag{4.7}$$

We call the infinite matrix $A=\left(a(n,k)\right)_{n,k\geq 0}$ and the finite parts $A_n=\left(a(i,j)\right)_{i,j=0}^{n-1}$ Genocchi matrices.

E.g.

$$A_5=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 3 & -5 & 3 & 0 & 0 \\ -17 & 28 & -14 & 4 & 0 \\ 155 & -255 & 126 & -30 & 5 \end{pmatrix}. \tag{4.8}$$

It is clear that A is the matrix version of "multiplication with $\frac{e^z-1}{e^z+1}$" of a certain kind of generating functions. More precisely we have

Proposition 4.1

Let $x=\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ \vdots \end{pmatrix}$, $y=\begin{pmatrix} y_0 \\ y_2 \\ y_2 \\ \vdots \\ \vdots \end{pmatrix}$ *and* $A=\left(a(n,k)\right)_{n,k\geq 0}$.

Then $y=Ax$ *is equivalent with*

$$\sum_{n\geq 0} y_n \frac{z^{2n+2}}{(2n+2)!}=\frac{e^z-1}{e^z+1}\sum_{n\geq 0}\frac{x_n}{(2n+1)!}z^{2n+1}. \tag{4.9}$$

By considering the highest powers of s we see from (1.1) that $a(k,k)=k+1.$ Therefore the eigenvalues of A are $1,2,3,\cdots.$

For later uses we note that

$$\sum_{k=0}^{n} a(n,k)=1. \tag{4.10}$$

This follows from (4.6) for $s=0$.

For special values of s (4.6) gives some interesting identities. Consider for example $s=-1$. Here $(F_n(s))_{n\ge 0}=(0,1,1,0,-1,-1,\cdots)$ is periodic with period 6.

This gives for example $\sum_{k=0}^{n} a(3n+2,3k)=\sum_{k=0}^{n} a(3n+2,3k+2).$

For $s=-\frac{1}{4}$ we get $F_n\left(-\frac{1}{4}\right)=\frac{n}{2^{n-1}}$. Therefore

$$\sum_{k=0}^{n} 4^{n-k}(2k+1)a(n,k)=n+1.$$

By comparing coefficients of s^k we get

$$\sum_{j=k}^{n} a(n,j)\binom{2j-k}{k}=\binom{2n+1-k}{k} \text{ for } n\ge k.$$

This is equivalent with the matrix identity

$$A=\left([j\le i]\binom{2i+1-j}{j}\right)\left([j\le i]\binom{2i-j}{j}\right)^{-1}. \tag{4.11}$$

If we recall that $\left([j\le i]\binom{2i-j}{j}\right)_{i,j\ge 0}\begin{pmatrix}1\\ s\\ s^2\\ s^3\\ \vdots\end{pmatrix}=\begin{pmatrix}F_1\\ F_3\\ F_5\\ F_7\\ \vdots\end{pmatrix}$ and $\left([j\le i]\binom{2i+1-j}{j}\right)_{i,j\ge 0}\begin{pmatrix}1\\ s\\ s^2\\ s^3\\ \vdots\end{pmatrix}=\begin{pmatrix}F_2\\ F_4\\ F_6\\ F_8\\ \vdots\end{pmatrix}$

we see that (4.11) is the same as (4.6).

By (3.21) we get the following representation

$$A=\left(T(i+1,j+1)\right)\left([i=j](j+1)\right)\left(\left(T(i+1,j+1)\right)\right)^{-1}. \tag{4.12}$$

By (3.29) we also have

$$A=\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge 0}\right)^{-1}\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge 0}. \tag{4.13}$$

Thus we get

Theorem 4.2

The Genocchi matrix A *has the following representations*

$$A=\left([j\le i]\binom{2i+1-j}{j}\right)\left([j\le i]\binom{2i-j}{j}\right)^{-1}, \tag{4.14}$$

$$A=\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge 0}\right)^{-1}\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge 0} \tag{4.15}$$

and

$$A=\left(T(i+1,j+1)\right)_{i,j\ge 0}\left([i=j](i+1)\right)_{i,j\ge 0}\left(t(i+1,j+1)\right)_{i,j\ge 0}, \tag{4.16}$$

where $t(i,j)$ *are the central factorial numbers of the first kind and* $T(i,j)$ *the central factorial numbers of the second kind.*

Note the analogy with (3.9) and (3.11).

(4.15) implies $\left(\left([j\le i]\binom{i+1}{2i-2j}\right)_{i,j\ge 0}\right)A=\left([j\le i]\binom{i+1}{2i-2j+1}\right)_{i,j\ge 0}$.

By considering the first column of these matrices we get

$$\sum_{j=0}^{n}\binom{n+1}{2n-2j}(-1)^j G_{2j+2}=[n=0].$$

Slightly reformulated this is

Seidel's identity for Genocchi numbers ([13])

$$\sum_{k=0}^{n}\binom{n}{2k}(-1)^k G_{2n-2k}=[n=1]. \tag{4.17}$$

The matrix $\left([j\le i]\binom{2i-j}{j}\right)^{-1}$ has been evaluated by Dumont and Zeng [6]. We don't need this result. We are only interested in the first column. Let $\left(H_{2n+1}\right)_{n\ge 0}=\left(1,1,2,8,56,608,\cdots\right)$ be the median Genocchi numbers (cf. OEIS A005439).

Lemma 4.3

The elements of the first column of $\left([j \le i]\binom{2i-j}{j}\right)^{-1}$ *are the numbers* $(-1)^n H_{2n+1}$.

The first two columns of

$$\left([j \le i]\binom{2i-j}{j}\right)^{-1}\left([j \le i]\binom{2i+1-j}{j}\right)$$

are $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \\ 0 & 8 \\ \vdots & \vdots \end{pmatrix}$. *The numbers of the second column are*

$r(0) = -H_1 + 1 = 0, r(1) = H_3 + 1 = 2, r(2) = -H_5 = -2, r(3) = H_7 = 8, r(4) = -H_9 = -56, \cdots$ *and in general for* $n \ge 2$ $\quad r(n) = -\lambda(s^n) = (-1)^{n-1} H_{2n+1}$.

The left upper part of the matrix $\left([j \le i]\binom{2i-j}{j}\right)^{-1}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -8 & 13 & -6 & 1 & 0 & 0 \\ 56 & -92 & 45 & -10 & 1 & 0 \\ -608 & 1000 & -493 & 115 & -15 & 1 \end{pmatrix}$$

and of the matrix $\left([j \le i]\binom{2i-j}{j}\right)^{-1}\left([j \le i]\binom{2i+1-j}{j}\right)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -8 & 13 & -6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 1 & 6 & 10 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 8 & -8 & 4 \end{pmatrix}.$$

To prove this lemma we need the fact (cf. [5]) – which can serve as definition of the median Genocchi numbers – that

$$G_{2n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{2n+1-j}{j}H_{2j+1}. \tag{4.18}$$

Since $\sum_{j=0}^{n+1}\binom{2n+1-j}{j}\lambda(s^j)=\lambda\left(F_{2n+2}(s)\right)=(-1)^n G_{2n+2}$ by (4.5) we see that

$$\lambda(s^n)=(-1)^n H_{2n+1}. \tag{4.19}$$

By (4.6) we get

$$\left([j\le i]\binom{2i-j}{j}\right)^{-1}\begin{pmatrix}F_1(s)\\F_3(s)\\F_5(s)\\\vdots\end{pmatrix}=\begin{pmatrix}1\\s\\s^2\\\vdots\end{pmatrix} \tag{4.20}$$

and therefore by applying λ

$$\left([j\le i]\binom{2i-j}{j}\right)^{-1}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\\lambda(s)\\\lambda(s^2)\\\vdots\end{pmatrix}.$$

To prove the second assertion we note that $F_{2n+1}(0)=1$ and therefore (4.20) gives the first column.

To obtain the second column we recall that $\sum_{j=0}^{n}\binom{2n-j}{j}(-1)^j H_{2j+1}=\lambda\left(F_{2n+1}(s)\right)=0$

and therefore $\sum_{j=0}^{n}\binom{2n-j}{j}r(j)=1+(2n-1)=2n.$

(3.22) gives

$$\left(\left([j\le i]\binom{2i-j}{j}\right)\right)^{-1}\left([j\le i]\binom{2i+1-j}{j}\right)=\left(LS(i,j)\right)\left([i=j](j+1)\right)\left(\left(LS(i,j)\right)\right)^{-1}.$$

If we delete the first row and first column we get the matrix

$$\left(LS(i+1,j+1)\right)_{i,j\ge0}\left([i=j](j+2)\right)_{i,j\ge0}\left(\left(\left(LS(i+1,j+1)\right)\right)_{i,j\ge0}\right)^{-1}. \tag{4.21}$$

The left-upper part begins with

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 8 & -8 & 4 & 0 & 0 \\ -56 & 56 & -20 & 5 & 0 \\ 608 & -608 & 216 & -40 & 6 \end{pmatrix}.$$

Theorem 4.2 implies that the eigenvectors of $A=\left(a(i,j)\right)_{i,j\geq 0}$ with respect to the eigenvalue k are

$$x(k)=\begin{pmatrix} T(1,k) \\ T(2,k) \\ T(3,k) \\ T(4,k) \\ \vdots \end{pmatrix}. \tag{4.22}$$

This means that $\sum_{j\geq 0} a(n,j)T(j+1,k)=kT(n+1,k).$

This is equivalent with

$$\frac{e^z-1}{e^z+1}\sum_{n\geq 1}\frac{T(n,k)}{(2n-1)!}z^{2n-1}=k\sum_{n\geq 1}\frac{T(n,k)}{(2n)!}z^{2n}. \tag{4.23}$$

Let

$$T_k(z)=\sum_{n\geq 0}\frac{T(n,k)}{(2n)!}z^{2n}. \tag{4.24}$$

Then (4.23) can be written as

$$\frac{e^z-1}{e^z+1}T_k'(z)=kT_k(z). \tag{4.25}$$

This result can also be obtained from Proposition 3.4.

Corollary 4.4

Define linear functionals φ_k *by* $\varphi_k\left(F_{2n-1}(s)\right) = T(n,k)$ *for* $k \geq 1$. *Then* $\varphi_k\left(F_{2n}(s)\right) = kT(n,k)$.

Proof

This follows from (4.2).

An explicit expression for these functionals gives

Theorem 4.5

The following formulae hold:

$$\varphi_{k+1}(s^n) = LS(n,k), \tag{4.26}$$

$$\varphi_{k+1}\left(F_{2n-1}(s)\right) = \sum_{j=0}^{n-1} \binom{2n-2-j}{j} LS(j,k) = T(n,k+1) \tag{4.27}$$

$$\varphi_{k+1}\left(F_{2n}(s)\right) = \sum_{j=0}^{n-1} \binom{2n-1-j}{j} LS(j,k) = (k+1)T(n,k+1). \tag{4.28}$$

Proof

This is an immediate consequence of Theorem 3.6.

For our next results we need the square of A.

For example

$$A_5^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \\ 17 & -25 & 9 & 0 & 0 \\ -155 & 238 & -98 & 16 & 0 \\ 2073 & -3255 & 1428 & -270 & 25 \end{pmatrix}.$$

Theorem 4.6

The square of A_n *is given by*

$$A_n^2 = \left(aa(i,j)[j \leq i]\right)_{i,j=0}^{n-1} \tag{4.29}$$

with

$$aa(n,k) = (-1)^{n-k} \binom{2n+2}{2k} \frac{n+k+2}{(2k+1)(n+2-k)} G_{2n-2k+4}. \tag{4.30}$$

Proof

Let $f(z) = \sum_{n \geq 0} \frac{x_n}{(2n+1)!} z^{2n+1}.$

Since $y = Ax$ is equivalent with $\sum_{n \geq 0} y_n \frac{z^{2n+2}}{(2n+2)!} = \frac{e^z - 1}{e^z + 1} f(z)$ we see that $z = Ay = A^2 x$ is equivalent with

$$\begin{aligned} &\sum_{n \geq 1} z_{n-1} \frac{z^{2n}}{(2n)!} = \frac{e^z - 1}{e^z + 1} \sum_{n \geq 1} y_{n-1} \frac{z^{2n-1}}{(2n-1)!} = \frac{e^z - 1}{e^z + 1} \left(\frac{e^z - 1}{e^z + 1} f(z) \right)' \\ &= \left(\frac{e^z - 1}{e^z + 1} \right)^2 f'(z) + \frac{e^z - 1}{e^z + 1} \left(\frac{e^z - 1}{e^z + 1} \right)' f(z) \end{aligned} \tag{4.31}$$

Since

$$\left(\frac{e^z - 1}{e^z + 1} \right)^2 = \sum_{n \geq 1} (-1)^{n+1} \frac{G_{2n+2}}{n+1} \frac{z^{2n}}{(2n)!} \tag{4.32}$$

and

$$-\frac{e^z - 1}{e^z + 1} \left(\frac{e^z - 1}{e^z + 1} \right)' = \left(\frac{e^z - 1}{e^z + 1} \right)'' = \sum_{n \geq 1} (-1)^{n-1} \frac{G_{2n}}{2n} \frac{z^{2n-3}}{(2n-3)!}, \tag{4.33}$$

we get

$$\left(\frac{e^z - 1}{e^z + 1} \right)^2 f'(z) = \sum_n \frac{z^{2n}}{(2n)!} \sum_{k=1}^{n} \binom{2n}{2k-2} (-1)^{n-k} \frac{G_{2n-2k+4}}{n-k+2} x(k) \tag{4.34}$$

and

$$-\left(\frac{e^z - 1}{e^z + 1} \right)'' f(z) = \sum_n \frac{z^{2n}}{(2n)!} \sum_{k=0}^{n} \binom{2n}{2k-1} (-1)^{n-k} \frac{G_{2n-2k+4}}{2n-2k+4} x(k). \tag{4.35}$$

From (4.34) and (4.35) and using (4.31) we get immediately (4.30).

It only remains to prove (4.32).

This follows from

$$\left(\frac{e^z-1}{e^z+1}\right)^2=1-2\left(\frac{e^z-1}{e^z+1}\right)'=1-2\sum_{n\ge 0}(-1)^n G_{2n+2}\frac{(2n+1)z^{2n}}{(2n)!(2n+1)(2n+2)}$$
$$=\sum_{n\ge 0}(-1)^{n-1}\frac{G_{2n+2}}{n+1}\frac{z^{2n}}{(2n)!}.$$

Theorem 4.7 (Further properties of A_n^2)

$$aa(n,0)=-a(n+1,0) \tag{4.36}$$

and for $1\le k\le n-1$

$$aa(n,k)=a(n,k-1)-a(n+1,k) \tag{4.37}$$

Proof.

This follows from (4.30).

Intimately connected with the linear functional λ is the linear functional λ^* defined by

$$\lambda^*(F_n(s))=-\lambda(sF_n(s)). \tag{4.38}$$

Since $\lambda(sF_n(s))=\lambda(F_{n+2}(s))-\lambda(F_{n+1}(s))$ we get
$\lambda(sF_{2n}(s))=\lambda(F_{2n+2}(s))-\lambda(F_{2n+1}(s))=\lambda(F_{2n+2}(s))$ for $n>0$ and
$\lambda(sF_{2n+1}(s))=\lambda(F_{2n+3}(s))-\lambda(F_{2n+2}(s))=-\lambda(F_{2n+2}(s))$.

The sequence $\left(\lambda^*\left(F_n(s)\right)\right)$ begins with

$0,1,1,-1,-3,3,17,-17,-155,155,2073,-2073,-38227.$

Therefore

$$\lambda^*(F_{2n}(s)+F_{2n+1}(s))=[n=0] \tag{4.39}$$

and

$$\lambda^*(F_{2n-1}(s)+F_{2n}(s))=(-1)^n\left(G_{2n}+G_{2n+2}\right). \tag{4.40}$$

Thus we are led to consider the basis consisting of the polynomials $F_{2k-2}(s)+F_{2k-1}(s).$

Here we get

Theorem 4.8

Let

$$a_1(n,k)=[k\le n]\sum_{j=0}^{k}a(n,j). \tag{4.41}$$

Then

$$F_{2n+1}(s)=\sum_{k=0}^{n}a_1(n,k)\left(F_{2k}(s)+F_{2k+1}(s)\right). \tag{4.42}$$

The first entries of $\left(a_1(n,k)\right)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0\\ 3 & -2 & 1 & 0 & 0 & 0\\ -17 & 11 & -3 & 1 & 0 & 0\\ 155 & -100 & 26 & -4 & 1 & 0\\ -2073 & 1337 & -346 & 50 & -5 & 1 \end{pmatrix}.$$

Proof

We know that $F_{2n}(s)=\sum_{k=0}^{n-1}a(n-1,k)F_{2k+1}(s).$ Therefore

$F_{2n}(s)+F_{2n+1}(s)=\sum_{k=0}^{n-1}a(n-1,k)F_{2k+1}(s)+F_{2n+1}(s).$ This implies that in (4.42) the matrix $\left(a_1(i,j)\right)$ is the inverse of $I+B$, where I denotes the identity matrix and $b(n,k)=a(n-1,k)$ for $k<n.$ We have to show that $a_1(n,k)$ is given by (4.41).

It suffices to show that $(I+B)\left([j\le i]\sum_{\ell=0}^{j}a(i,\ell)\right)=I.$

This means that for each k

$$\sum_{j=k}^{n-1} a(n-1,j)\left(a(j,0)+a(j,1)+\cdots a(j,k)\right)+a(n,0)+\cdots+a(n,k)=[n=k].$$

Let $k<n.$ By definition $aa(n-1,k)=\sum_{j=0}^{n-1} a(n-1,j)a(j,k).$ Therefore

$$aa(n-1,i)-\sum_{j=k}^{n-1} a(n-1,j)a(j,i)=\sum_{j=0}^{k-1} a(n-1,j)a(j,i).$$

Using (4.37) and (4.10) the left-hand side is equivalent with

$$aa(n-1,0)-\sum_{j=0}^{k-1} a(n-1,j)a(j,0)+aa(n-1,1)-\sum_{j=1}^{k-1} a(n-1,j)a(j,1)+\cdots$$
$$+aa(n-1,k-1)-a(n-1,k-1)a(k-1,k-1)+aa(n-1,k)+a(n,0)+a(n,1)+\cdots+a(n,k)$$
$$=aa(n-1,0)+a(n,0)+aa(n-1,1)+a(n,1)+\cdots+aa(n-1,k)+a(n,k)-\sum_{i=0}^{k-1} a(n-1,i)\sum_{j=0}^{i} a(i,j)$$
$$=a(n-1,0)+a(n-1,1)+\cdots+a(n-1,k-1)-\sum_{j=0}^{k-1} a(n-1,j)=0.$$

For $k=n$ the first sum vanishes and $a(n,0)+\cdots+a(n,n)=1.$

Corollary 4.9

Let

$$a_2(n,k)=[k\le n]\left(a_1(n,k)-a_1(n+1,k)\right). \tag{4.43}$$

Then

$$F_{2n+1}(s)+F_{2n+2}(s)=\sum_{k=0}^{n} a_2(n,k)\left(F_{2k}(s)+F_{2k+1}(s)\right). \tag{4.44}$$

Proof

By Theorem 4.7 we have

$$F_{2n+1}(s)=\sum_{k=0}^{n} a_1(n,k)\left(F_{2k}(s)+F_{2k+1}(s)\right)=\sum_{k=0}^{n-1} a_1(n,k)\left(F_{2k}(s)+F_{2k+1}(s)\right)+F_{2n}(s)+F_{2n+1}(s).$$

Therefore

$$F_{2n+2}(s) = -\sum_{k=0}^{n} a_1(n+1,k)\left(F_{2k}(s)+F_{2k+1}(s)\right)$$

and thus $F_{2n+1}(s)+F_{2n+2}(s) = \sum_{k=0}^{n} a_2(n,k)\left(F_{2k}(s)+F_{2k+1}(s)\right).$

The first terms of $\left(a_2(n,k)\right)$ are

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 & 0 \\ 20 & -13 & 4 & 0 & 0 \\ -172 & 111 & -29 & 5 & 0 \\ 2228 & -1437 & 372 & -54 & 6 \end{pmatrix}.$$

Since $a(n,k) = \sum_{j=0}^{n} T(n+1, j+1)(j+1)t(j+1,k+1)$ we see that $a_2(n,k) = F(n,j)$ if in Lemma 3.11 we choose $w(n) = (n+1)^2$ and $F(\ell) = \ell + 1.$

Therefore $a_2(n,k) = \sum_{\ell=0}^{n} S^{\hat{w}}(n,\ell) F\left(\ell+1\right) s^{\hat{w}}(\ell,k).$

This gives

Theorem 4.10

Let $w(n) = (n+1)^2$ *and thus* $\hat{w}(n) = (n+2)^2$. *Then*

$$\left(a_2(i,j)\right)_{i,j=0}^{n-1} = \left(S^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1} \left([i=j](i+2)\right)_{i,j=0}^{n-1} \left(s^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1}. \qquad (4.45)$$

For example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 16 & 13 & 1 & 0 & 0 \\ 64 & 133 & 29 & 1 & 0 \\ 256 & 1261 & 597 & 54 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 \\ 36 & -13 & 1 & 0 & 0 \\ -576 & 244 & -29 & 1 & 0 \\ 14400 & -6676 & 969 & -54 & 1 \end{pmatrix}.$$

Next we consider the linear functional μ defined by $\mu(F_{2k+2}(s)) = [k=0].$ From (4.2) we see that

$$\sum \frac{\mu(F_{2n+1}(s))}{(2n+1)!} z^{2n+1} = \frac{e^z+1}{e^z-1} \sum \frac{\mu(F_{2n}(s))}{(2n)!} z^{2n} = \frac{z^2}{2}\frac{e^z+1}{e^z-1} = \sum_n (2n+1)B_{2n}\frac{z^{2n+1}}{(2n+1)!}. \quad (4.46)$$

This implies

$$\mu\left(F_{2n+1}(s)\right) = (2n+1)B_{2n} \quad (4.47)$$

and

$$F_{2n+1}(s) = \sum_{j=0}^{n} \binom{2n+1}{2j+1} \frac{B_{2n-2j}}{j+1} F_{2j+2}(s). \quad (4.48)$$

Comparing (4.6) with (4.48) we see that

$$A^{-1} = \left(a(j,k)\right)_{i,j\geq 0}^{-1} = \left(\binom{2j+1}{2k+1}\frac{B_{2j-2k}}{k+1}\right)_{i,j\geq 0}. \quad (4.49)$$

The first terms of the sequence $\left((2n+1)B_{2n}\right)_{n\geq 0}$ are $1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, -\frac{3}{10}, \frac{5}{6}, -\frac{691}{210}, \frac{35}{2}, -\frac{3617}{30}$.

Let $w(j,k) = \binom{2j+1}{2k+1}\frac{B_{2j-2k}}{k+1}[k \leq j]$. Then

$$\left(w(j,k)\right)_{j,k=0}^{5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{7}{12} & \frac{7}{6} & \frac{1}{4} & 0 & 0 \\ -\frac{3}{10} & 1 & -\frac{7}{5} & \frac{3}{2} & \frac{1}{5} & 0 \\ \frac{5}{6} & -\frac{11}{4} & \frac{11}{3} & -\frac{11}{4} & \frac{11}{6} & \frac{1}{6} \end{pmatrix}.$$

By (4.15) we have

$$\left([j\leq i]\binom{i+1}{2i-2j+1}\right)_{i,j\geq 0} A^{-1} = \left([j\leq i]\binom{i+1}{2i-2j}\right)_{i,j\geq 0}.$$

Considering the first column we get

$$\sum_{j=0}^{n}\binom{n+1}{2n-2j+1}(2j+1)B_{2j}=\sum_{j=0}^{n}\binom{n+1}{2j+1}(2n-2j+1)B_{2n-2j}$$
$$=\sum_{j=0}^{n}\binom{n+1}{n-2j}(n+n-2j+1)B_{n+n-2j}=[n=0].$$

Since $B_{2i+1}=0$ for $i>0$ this is equivalent with

$\sum_{i=0}^{n+1}\binom{n+1}{i}(n+i+1)B_{n+i}=0$ for $n>1.$ But it also holds for $n=0$ and $n=1.$

Therefore we get

Kaneko's identity ([7],[10],[13])

$$\sum_{i=0}^{n+1}\binom{n+1}{i}(n+i+1)B_{n+i}=0. \tag{4.50}$$

This identity has first been proved by A.v. Ettingshausen [7] and has been rediscovered by L. Seidel [13],VIII, and by M. Kaneko [10].

(4.49) implies that

$$\left(w(j,k)\right)_{j,k=0}^{n-1}=\left(T(i+1,j+1)\right)_{i,j=0}^{n-1}\left([i=j]\frac{1}{j+1}\right)_{i,j=0}^{n-1}\left(t(i+1,j+1)\right)_{i,j=0}^{n-1}. \tag{4.51}$$

The inverse of $\left(a_2(i,j)\right)_{i,j=0}^{n-1}=\left(S^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1}\left([i=j](i+2)\right)_{i,j=0}^{n-1}\left(s^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1}$ is

$$\left(z(i,j)\right)_{i,j=0}^{n-1}=\left(S^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1}\left([i=j]\frac{1}{i+2}\right)_{i,j=0}^{n-1}\left(s^{\hat{w}}(i,j)\right)_{i,j=0}^{n-1}.$$

By Lemma 3.11 this implies that $z(n,k)=\sum_{j=0}^{k}\left(w(n,j)-w(n+1,j)\right)$ for $k\le n$ and $z(n,k)=0$ else.

As an example consider

$$\left(z(i,j)\right)_{i,j=0}^{5}=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{13}{12} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{7}{15} & -\frac{67}{60} & \frac{29}{20} & \frac{1}{5} & 0 & 0 \\ -\frac{17}{15} & \frac{157}{60} & -\frac{49}{20} & \frac{9}{5} & \frac{1}{6} & 0 \\ \frac{433}{105} & -\frac{3973}{420} & \frac{1191}{140} & -\frac{156}{35} & \frac{15}{7} & \frac{1}{7} \end{pmatrix}.$$

For example we have $z(3,0)=w(3,0)-w(4,0)=\frac{1}{6}+\frac{3}{10}=\frac{7}{15},$

$z(3,1)=w(3,0)-w(4,0)+w(3,1)-w(4,1)=\frac{7}{15}-\frac{7}{12}-1=-\frac{67}{60},\cdots.$

So we have

$$F_{2n}(s)+F_{2n+1}(s)=\sum_{k=0}^{n}z(n,k)\left(F_{2k+1}(s)+F_{2k+2}(s)\right). \tag{4.52}$$

5. Analogous results for Lucas polynomials

For the Lucas polynomials we get

$$\sum\frac{L_{2n+1}(s)}{(2n+1)!}z^{2n+1}=\frac{e^z-1}{e^z+1}\sum\frac{L_{2n}(s)}{(2n)!}z^{2n}. \tag{5.1}$$

This follows from

$$e^{\alpha z}+e^{\beta z}=e^{(1-\beta)z}+e^{(1-\alpha)z}=e^z\left(e^{-\alpha z}+e^{-\beta z}\right). \tag{5.2}$$

We write the series expansion of $\frac{e^z-1}{e^z+1}$ in the form

$$\frac{e^z-1}{e^z+1}=\sum_{k\geq 0}(-1)^k\frac{T_{2k+1}}{2^{2k+1}}\frac{z^{2k+1}}{(2k+1)!}, \tag{5.3}$$

where T_n, $n\geq 1$, are the tangent numbers $1,2,16,272,7936,\cdots.$

Therefore

$$\sum \frac{Y_n}{(2n+1)!} z^{2n+1} = \frac{e^z - 1}{e^z + 1} \sum \frac{X_n}{(2n)!} z^{2n} \tag{5.4}$$

is equivalent with

$$Y_n = \sum_{k=0}^{n} (-1)^k \frac{T_{2k+1}}{2^{2k+1}} \binom{2n+1}{2k+1} X_{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \frac{T_{2n-2k+1}}{2^{2n-2k+1}} \binom{2n+1}{2k} X_k.$$

If we set $b(n,k) = (-1)^{n-k} \frac{T_{2n-2k+1}}{2^{2n-2k+1}} \binom{2n+1}{2k} = (-1)^{n-k} \frac{1}{2} \binom{2n+1}{2k} \frac{G_{2n-2k+2}}{n-k+1},$ then (5.4) is the same as

$$Y_n = \sum_{j=0}^{n} b(n,j) X_j. \tag{5.5}$$

We call the matrices

$$B = \left(b(i,j)\right)_{i,j\geq 0} = \left((-1)^{i-j} \frac{T_{2i-2j+1}}{2^{2i-2j+1}} \binom{2i+1}{2j}\right)_{i,j\geq 0} \tag{5.6}$$

tangent-matrices.

If we compare (5.1) with (5.5) we see that $L_{2n+1}(s) = \sum_{j=0}^{n} b(n,j) L_{2j}(s).$ This is again (2.3).

Let $L_n(s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} l(n,s) s^k.$ Then $L_{2n}(s) = \sum_{k=0}^{n} l(2n,k) s^k$ and $L_{2n+1}(s) = \sum_{k=0}^{n} l(2n+1,k) s^k.$

Therefore we get

$$B = \left(l(2i+1,j)\right)_{i,j\geq 0} \left(\left(l(2i,j)\right)_{i,j\geq 0}\right)^{-1}. \tag{5.7}$$

For example

$$B_5 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{5}{2} & \frac{5}{2} & 0 & 0 \\ -\frac{17}{8} & \frac{21}{2} & -\frac{35}{4} & \frac{7}{2} & 0 \\ \frac{31}{2} & -\frac{153}{2} & 63 & -21 & \frac{9}{2} \end{pmatrix}.$$

Let $w(n)=\left(\frac{2n+1}{2}\right)^2$ and let $U(n,k)=S^w(n,k)$ and $u(n,k)=s^w(n,k).$ respectively.

The first values of the numbers $4^{n-k}U(n,k)$ and $4^{n-k}u(n,k)$ are given by the following tables.

$$\left(4^{i-j}U(i,j)\right)_{i,j=0}^{6}=\begin{pmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&10&1&0&0&0&0\\1&91&35&1&0&0&0\\1&820&966&84&1&0&0\\1&7381&24970&5082&165&1&0\\1&66430&631631&273988&18447&286&1\end{pmatrix}$$

and

$$\left(4^{i-j}u(i,j)\right)_{i,j=0}^{6}=$$
$$\begin{pmatrix}1&0&0&0&0&0&0\\-1&1&0&0&0&0&0\\9&-10&1&0&0&0&0\\-225&259&-35&1&0&0&0\\11025&-12916&1974&-84&1&0&0\\-893025&1057221&-172810&8778&-165&1&0\\108056025&-128816766&21967231&-1234948&28743&-286&1\end{pmatrix}$$

Here we have

$$\sum_{n\geq k}U(n,k)x^{2n}=\frac{x^{2k}}{(1-\left(\frac{1}{2}\right)^2x^2)(1-\left(\frac{3}{2}\right)^2x^2)\cdots(1-\left(\frac{2k+1}{2}\right)^2x^2)}=\frac{x^{2k}}{\prod_{j=-k-1}^{k}(1-\left(\frac{2j+1}{2}\right)x)}.$$

From the partial fraction expansion

$$\frac{x^{2k+1}}{(1-\left(\frac{1}{2}\right)^2x^2)(1-\left(\frac{3}{2}\right)^2x^2)\cdots(1-\left(\frac{2k+1}{2}\right)^2x^2)}=\frac{1}{(2k+1)!}\sum_{j=0}^{2k+1}(-1)^j\binom{2k+1}{j}\frac{1}{1-\frac{2k+1-2j}{2}x}$$

we see that

$$U(n,k)=\frac{1}{(2k+1)!}\sum_{j=0}^{2k+1}(-1)^j\binom{2k+1}{j}\left(\frac{2k+1-2j}{2}\right)^{2n+1}. \qquad (5.8)$$

The exponential generating function is therefore (cf. [12])

$$U_k(z)=\sum_n U(n,k)\frac{z^{2n+1}}{(2n+1)!}=\frac{1}{(2k+1)!}\left(\frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{2}\right)^{2k+1}. \qquad (5.9)$$

Proposition 5.1

The formal power series $U_k(z)=\sum_n U(n,k)\frac{z^{2n+1}}{(2n+1)!}$ *is the uniquely determined solution of the differential equation* $\frac{e^z-1}{e^z+1}U_k{}'(z)=\frac{2k+1}{2}U_k(z)$ *with* $U(k,k)=1.$

Since $b(k,k)=\frac{2k+1}{2}$ the eigenvalues of B are $\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots.$

Theorem 5.2

The tangent matrix B *has the factorization*

$$B=\left(U(i,j)\right)_{i,j\geq 0}\left([i=j]\frac{2j+1}{2}\right)_{i,j\geq 0}\left(u(i,j)\right)_{i,j\geq 0}, \qquad (5.10)$$

where $u(i,j)$ *and* $U(i,j)$ *are the generalized Stirling numbers corresponding to* $w(n)=\left(\frac{2n+1}{2}\right)^2.$

To prove this observe that the eigenvector $U(k)$ corresponding to the eigenvalue $\frac{2k+1}{2}$ of $\left(b(i,j)\right)_{i,j\geq 0}$ satisfies

$$\sum_{j=0}^{n}b(n,j)U_j(k)=\frac{2k+1}{2}U_n(k)$$

which is equivalent with

$$\frac{e^z-1}{e^z+1}\sum\frac{U_n(k)}{(2n)!}z^{2n}=\frac{2k+1}{2}\sum\frac{U_n(k)}{(2n+1)!}z^{2n+1}. \tag{5.11}$$

Let $f_k(z)=\sum\frac{U_n(k)}{(2n+1)!}z^{2n+1}.$ Then (5.11) says that

$\frac{e^z-1}{e^z+1}f_k{}'(z)=\frac{2k+1}{2}f_k(z).$ By Proposition 5.1 we see that $U_n(k)=U(n,k)$ as asserted.

This implies (5.10).

Thus we get again (2.3), i.e.

$$L_{2n+1}(s)=\sum_{k=1}^{n}b(n,k)L_{2k}(s) \tag{5.12}$$

with

$$b(n,k)=(-1)^{n-k}\frac{T_{2n-2k+1}}{2^{2n-2k+1}}\binom{2n+1}{2k}. \tag{5.13}$$

Proposition 5.3

The inverse of B *is*

$$B^{-1}=\left([j\le i]\binom{2i}{2j}\frac{B_{2i-2j}}{2j+1}\right). \tag{5.14}$$

This follows from (5.4) and the well-known series $\frac{z}{2}\frac{e^z+1}{e^z-1}=\sum_{n\ge 0}B_{2n}\frac{z^{2n}}{(2n)!}.$

Then from (5.1) we get again (2.4).

There is also an analogue of Theorem 3.6.

To this end we introduce an analogue of the Legendre-Stirling numbers.

Let $w(n)=\frac{(2n-1)(2n+1)}{4}$ and $V(n,k)=S^w(n,k)$ and $v(n,k)=s^w(n,k).$

Then we get

Theorem 5.4

$$\sum_{j=0}^{n} l(2n, j)V(j,k) = 2U(n,k) \tag{5.15}$$

and

$$\sum_{j=0}^{n} l(2n+1, j)V(j,k) = (2k+1)U(n,k). \tag{5.16}$$

Proof

We use again induction. Let both identities be already proved for $n-1$. Then

$$\sum_{j=0}^{n} l(2n, j)V(j,k) = \sum_{j=0}^{n-1} l(2n-1, j)V(j,k) + \sum_{j=0}^{n-1} l(2n-2, j-1)V(j,k)$$
$$= \sum_{j=0}^{n-1} l(2(n-1)+1, j)V(j,k) + \sum_{j=0}^{n-1} l(2(n-1), j)V(j+1,k)$$
$$= (2k+1)U(n-1,k) + \sum_{j=0}^{n-1} l(2(n-1), j)V(j,k-1) + \frac{(2k-1)(2k+1)}{4}\sum_{j=0}^{n-1} l(2(n-1), j)V(j,k)$$
$$= (2k+1)U(n-1,k) + 2U(n-1,k-1) + \frac{(2k-1)(2k+1)}{4}2U(n-1,k)$$
$$= 2\left(U(n-1,k-1) + \frac{(2k+1)^2}{4}U(n-1,k) \right) = 2U(n,k).$$

And

$$\sum_{j=0}^{n} l(2n+1, j)V(j,k) = \sum_{j=0}^{n} l(2n, j)V(j,k) + \sum_{j=0}^{n-1} l(2n-1, j-1)V(j,k)$$
$$= 2U(n,k) + \sum_{j=0}^{n-1} l(2n-1, j)V(j+1,k)$$
$$= 2U(n,k) + \sum_{j=0}^{n-1} l(2n-1, j)V(j,k-1) + \frac{(2k-1)(2k+1)}{4}\sum_{j=0}^{n-1} l(2(n-1), j)V(j,k)$$
$$= 2U(n,k) + (2k-1)U(n-1,k-1) + \frac{(2k-1)(2k+1)}{4}(2k+1)U(n-1,k)$$
$$= 2\left(U(n,k) + \frac{2k-1}{2}\left(U(n-1,k-1) + \frac{(2k+1)^2}{4}U(n-1,k) \right) \right)$$
$$= 2\left(U(n,k) + \frac{2k-1}{2}U(n,k) \right) = (2k+1)U(n,k).$$

This implies again (5.10).

6. Some interesting identities

For any sequence $(a(n))_{n\ge 0}$ the sequence $c(n)=\sum_{j=0}^{n}(-1)^j\left(\prod_{i=0}^{j-1}w(i)\right)S^w(n,j)a(j)$ is the first column in $\left(S^w(i,j)\right)_{i,j=0}^{\infty}\left(a(i)([i=j]\right)_{i,j=0}^{\infty}\left(s^w(i,j)\right)_{i,j=0}^{\infty}$.

More precisely we have

Theorem 6.1

Let

$$X=\left(x(i,j)\right)_{i,j\ge 0}=\left(S^w(i,j)\right)_{i,j\ge 0}\left(a(i)([i=j]\right)_{i,j\ge 0}\left(s^w(i,j)\right)_{i,j\ge 0}. \qquad (6.1)$$

Then

$$x(n,0)=\sum_{j=0}^{n}(-1)^j S^w(n,j)a(j)\left(\prod_{i=0}^{j-1}w(i)\right) \qquad (6.2)$$

and

$$\sum_{j=0}^{n}s^w(n,j)x(j,0)=(-1)^n a(n)\prod_{j=0}^{n-1}w(j). \qquad (6.3)$$

The sequence $(c(n))_{n\ge 0}$ can be simply computed with the Akiyama-Tanigawa algorithm (cf. [1], [9], [11], [15]):

Theorem 6.2 (Akiyama-Tanigawa algorithm) ([1],[9],[11],[15])

Suppose that $w(n)\neq 0$ *for all* n. *Let* $c(n)=\sum_{j=0}^{n}(-1)^j\left(\prod_{i=0}^{j-1}w(i)\right)S^w(n,j)a(j)$.

Define a matrix $M=\left(m(i,j)\right)_{i,j\ge 0}$ *by*

$m(0, j) = a(j)$ *for* $j \in \mathbb{N}$ *and*

$$m(i, j) = w(j)\big(m(i-1, j) - m(i-1, j+1)\big) \qquad (6.4)$$

Then $m(n,0) = c(n) = \sum_{j=0}^{n} (-1)^j \left(\prod_{i=0}^{j-1} w(i) \right) S^w(n, j) a(j).$

Proof

To prove this we show more generally that

$$m(n,k) = \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i=0}^{k} s(k,i) c(n+i). \qquad (6.5)$$

This holds for $n = 0$ because

$$\frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 0} s^w(k,i) c(i) = \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 0} s^w(k,i) \sum_{j=0}^{i} (-1)^j \left(\prod_{\ell=0}^{j-1} w(\ell) \right) S^w(i, j) a(j)$$

$$= \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{j\geq 0} (-1)^j \left(\prod_{\ell=0}^{j-1} w(\ell) \right) a(j) \sum_{i\geq 0} s^w(k,i) S^w(i, j) = a(k).$$

Now suppose that (6.5) holds for $n-1$. Then

$$w(k)\big(m(n-1,k) - m(n-1,k+1)\big)$$

$$= w(k) \left(\frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 0} s^w(k,i) c(n-1+i) - \frac{(-1)^{k+1}}{\prod_{\ell=0}^{k} w(\ell)} \sum_{i\geq 0} s^w(k+1,i) c(n-1+i) \right)$$

$$= \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 0} \big(w(k) s^w(k,i) + s^w(k+1,i) \big) c(n-1+i)$$

$$= \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 1} s^w(k,i-1) c(n-1+i) = \frac{(-1)^k}{\prod_{\ell=0}^{k-1} w(\ell)} \sum_{i\geq 0} s^w(k,i) c(n+i) = m(n,k).$$

We used the fact that $w(k) s^w(k,0) + s^w(k+1,0) = 0.$

For $w(n) = n+1$ and $a(n) = \frac{1}{n+1}$ this reduces to the original Akiyama-Tanigawa algorithm for $(b(n))$ as shown in [11].

Now we give a list of some formulas.

For $w(n) = n+1$, $a(n) = \frac{1}{n+1}$ we get from (2.16) and (3.10)

$$\sum_{j=0}^{n} S(n+1, j+1)(-1)^j \frac{j!}{j+1} = b(n). \tag{6.6}$$

Another proof of (6.6) can be found in [11], but I suspect that this result must be much older.

Formula (6.3) gives

$$\sum_{j=0}^{n} s(n+1, j+1) b(j) = (-1)^n \frac{n!}{(n+1)}. \tag{6.7}$$

For $w(n) = (n+1)^2$, $a(n) = n+1$ we know from (4.7) and (4.16) that $c(n) = (-1)^n G_{2n+2}$.

This gives

$$(-1)^{n-1} G_{2n} = \sum_{k=1}^{n} (-1)^{k-1} T(n,k) k \left((k-1)!\right)^2 \tag{6.8}$$

and

$$\sum_{k=1}^{n} (-1)^{n-k} t(n,k) G_{2k} = n!(n-1)!. \tag{6.9}$$

The Akiyama-Tanigawa algorithm applied to (6.8) gives another method for computing the Genocchi numbers. Choose $w(n) = (n+1)^2$ and $a(n) = n+1$ in Theorem 7.2.

Then the left upper part of the corresponding matrix $M = (m(i, j))$ is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -1 & -4 & -9 & -16 & -25 & -36 \\ 3 & 20 & 63 & 144 & 275 & 468 \\ -17 & -172 & -729 & -2096 & -4825 & -9612 \\ 155 & 2228 & 12303 & 43664 & 119675 & 276660 \\ -2073 & -40300 & -282249 & -1216176 & -3924625 & -10444428 \end{pmatrix}.$$

In the first column we get $c(n) = (-1)^n G_{2n+2}$.

From (4.30) we deduce the following identity (cf. [14], Exercise 5.8):

$$(-1)^{n-1} G_{2n+2} = \sum_{k=1}^{n} (-1)^{k-1} T(n,k)\left((k)!\right)^2. \tag{6.10}$$

The left upper part of the corresponding Akiyama-Tanigawa matrix is

$$\begin{pmatrix} 1 & 4 & 9 & 16 & 25 & 36 \\ -3 & -20 & -63 & -144 & -275 & -468 \\ 17 & 172 & 729 & 2096 & 4825 & 9612 \\ -155 & -2228 & -12303 & -43664 & -119675 & -276660 \\ 2073 & 40300 & 282249 & 1216176 & 3924625 & 10444428 \\ -38227 & -967796 & -8405343 & -43335184 & -162995075 & -495672372 \end{pmatrix}.$$

(6.3) gives the companion formula

$$\sum_{k=0}^{n} (-1)^{n-k} t(n,k) G_{2k+2} = \left(n!\right)^2. \tag{6.11}$$

For example for $n = 3$ we have $4G_4 + 5G_6 + G_8 = 4 + 15 + 17 = 36 = (3!)^2$.

(4.21) and Lemma 4.3 give

$$\sum_{k=0}^{n} (-1)^{n-k} LS(n+1,k+1)\left((k+1)!\right)^2 = H_{2n+3} \tag{6.12}$$

From (4.45) and (4.44) we deduce for $w(n) = (n+2)^2$ and $a(n) = n+2$

$$\sum_{k=0}^{n} (-1)^{n-k} S^w(n,k)(k+1)!(k+2)! = G_{2n+2} + G_{2n+4} \tag{6.13}$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}s^{w}(n,k)\left(G_{2k+2}+G_{2k+4}\right)=(n+1)!(n+2)!. \tag{6.14}$$

For $w(n)=n+1$ and $a(n)=\dfrac{1}{n+1}$ we get from (4.49)

$$(2n+1)B_{2n}=\sum_{j=0}^{n}(-1)^{j}\frac{(j!)^{2}}{j+1}T(n+1,j+1). \tag{6.15}$$

Here the left upper part of the Akiyama-Tanigawa matrix begins with

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \frac{6}{7} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{9}{20} & -\frac{8}{15} & -\frac{25}{42} & -\frac{9}{14} \\ \frac{1}{6} & \frac{7}{15} & \frac{3}{4} & \frac{104}{105} & \frac{25}{21} & \frac{19}{14} \\ -\frac{3}{10} & -\frac{17}{15} & -\frac{303}{140} & -\frac{16}{5} & -\frac{25}{6} & -\frac{353}{70} \\ \frac{5}{6} & \frac{433}{105} & \frac{261}{28} & \frac{232}{15} & \frac{460}{21} & \frac{4353}{154} \end{pmatrix}.$$

From (5.6) and (5.10) we deduce

$$\sum_{k=0}^{n}(-1)^{n-k}4^{n-k}U(n,k)(2k+1)\left((2k-1)!!\right)^{2}=T_{2n+1}. \tag{6.16}$$

Finally Proposition 5.3 gives

$$\sum_{k=0}^{n}(-1)^{k}U(n,k)\frac{1}{(2k+1)4^{k}}\left((2k-1)!!\right)^{2}=B_{2n}. \tag{6.17}$$

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