An integrated decay estimate for massless Vlasov fields

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Joint work with
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ArXiv : 1612.09304 [math.AP]
Introduction

- Use of symmetries to study Vlasov fields introduced by FJS15 for the massive/massless Vlasov fields.
- Purpose: Develop a perturbative approach of the stability of Minkowski spacetime, as a solution to the Einstein - massive Vlasov system, similar to the Lindblad-Rodniandki proof.
- But offer also the possibility to extend the study of Vlasov fields on other given backgrounds, for instance black hole backgrounds.
- Existence of steady states for massive particles on Kerr: Sarbach-Rioseco.
- Steady states at "high energy" with BH for massless particles when coupled to gravity: suggested by the work of Andreasson-Fajman-Thaller.
Introduction

- But dispersion is expected for purely massless fields, because of the instability of trapped null geodesics.
- Furthermore, duality Vlasov / wave: massless Vlasov fields offer a good playground to experiment and develop new methods for the wave equation.
- This work: illustrates this duality between wave / massless fields;
- Adapt to massless Vlasov fields the method based on hidden symmetries developed by Andersson-Blue.
- Core of this work: Andersson-Blue paper proving pointwise estimates for the wave equation mixed with symmetry approach by FJS.
- Prove existence of a bounded conserved energy and an integrated decay estimate.
Organization of the talk

Symmetries for Vlasov fields

Defining the energy

Morawetz estimate
Setup

- Purpose here: Compare framework in FJS with a standard Hamiltonian theory of commutators.
- Setup: \((M, g)\) Lorentzian manifold (oriented, time oriented). Consider \(T^*M\).
- Chart \((U, q^\alpha)\) on \(M\); natural extension to \(T^*U\) : \((U, q^\alpha, p_\alpha = \partial q^\alpha)\).
- On \(T^*M\), canonical non-degenerate (symplectic) 2-form:
  \[\epsilon = dq^\alpha \wedge dp_\alpha.\]
- Functions on \(T^*M\) : Hamiltonian.
- Representations of \(dH\) :
  \[dH(V) = \epsilon(X_H, V), \quad X_H = \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha} \frac{\partial}{\partial p_\alpha}.\]
Zero-th order symmetry operator

- Example: Geodesic spray / Liouville vector field:
  \[ H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, \quad \mathcal{X} := X_H = g^{\alpha\gamma} p_\alpha \frac{\partial}{\partial q^\gamma} - \frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial q^\gamma} p_\alpha p_\beta \frac{\partial}{\partial p^\gamma} \]

- If \( X \) is a vector field, \( K_X = X^\alpha p_\alpha \)

- Obvious fact:
  \[ \mathcal{X}(K_X) = p_\alpha p_\beta \left( \pi(X) \right)^{\alpha\beta}. \]

- Hence, zero-th order symmetry operator of \( \mathcal{X} \): if \( X \) Killing,
  \[ \text{If } \mathcal{X}(f) = 0 \text{ then } \mathcal{X}(K_X \cdot f) = 0. \]
First order symmetry operator

- Standard general fact: Taking two Hamiltonians $H, K$:

$$[X_K, X_H] = 0 \text{ iff } \{K, H\} = \frac{\partial H}{\partial p_\alpha} \frac{\partial K}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha} \frac{\partial K}{\partial p_\alpha} = 0$$

- If $X$ is a vector field, $K = X^\alpha p_\alpha$

- Associated Hamiltonian vector field:

$$X_K = X^\alpha \frac{\partial}{\partial q^\alpha} - p_\alpha \frac{\partial X^\alpha}{\partial q^\gamma} \frac{\partial}{\partial p_\gamma}$$

- General formula:

$$[\mathcal{X}, X_K] = -2\pi^{(K)}_{\alpha\gamma} p_\alpha \frac{\partial}{\partial q^\gamma} + \left(\partial_{q^\gamma} \pi^{(K)}\right)^{\alpha\beta} p_\alpha p_\beta \frac{\partial}{\partial p_\gamma}$$

- In particular, if $X$ is Killing

If $\mathcal{X}(f) = 0$ then $\mathcal{X}(X_K \cdot f) = 0$. 
Comparison with FJS

- On $TM$, use the mapping between $T^*M$ and $TM$:
  
  $$x^\alpha = q^\alpha \text{ and } v^\alpha = g^{\alpha\beta} p_\beta.$$ 

- The image of $X_K$ writes:
  
  $$X_K = X^\alpha e_\alpha - v_\alpha \nabla^\gamma X^\alpha \frac{\partial}{\partial v^\gamma}$$

- The image of the commutator:
  
  $$[\mathcal{X}, X_K] = -2\pi (X)^{\alpha\gamma} v_\gamma e_\alpha + \nabla^\gamma \pi^{(X)}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\gamma}$$

- Notion used in FJS: complete lift
  
  $$\tilde{X} = X^\alpha e_\alpha - v^\alpha \nabla_\alpha X^\gamma \frac{\partial}{\partial v^\gamma}$$
Consider \( \phi^t \) be a family of diffeomorphisms of \( M \).

On \( TM \),

\[
\phi_*^t = \left\{ \begin{array}{cc}
TM & \rightarrow & TM \\
(x, v) & \mapsto & (\phi^t(x), d\phi^t_x(v))
\end{array} \right.
\]

\( X \) a vector field on \( M \) arising from \( \phi^t \):

\[
X(x) = \frac{d\phi^t(x)}{dt}.
\]

Complete lift \( \tilde{X} \):

\[
\tilde{X}(x, v) = \frac{d\phi^t_*(x, v)}{dt}.
\]
Comparison with FJS 2

- When $X$ is Killing, $\tilde{X} = X_K$.
- But, perturbatively

$$[\mathcal{X}, X_K] = - 2\pi (X)_{\alpha\gamma} v_{\gamma} e_{\alpha} + \nabla_{\gamma} \pi^{(X)}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\gamma}$$

$$[\mathcal{X}, \tilde{X}] = v^\alpha v^\beta [\nabla_\alpha \nabla_\beta X^\mu - R^\mu_{\beta\alpha\nu} X^\nu] \frac{\partial}{\partial v^\mu}.$$  

- Both coincides because Killing fields are Jacobi fields along geodesics.
- Which one is better perturbatively?
Higher order symmetry operators

• If $Q$ is a symmetric 2-tensor, $K = \frac{1}{2} Q^{\alpha\beta} p_\alpha p_\beta$

• Immediate calculation

$$\mathcal{X} \left( Q^{\alpha\beta} p_\alpha p_\beta \right) = \nabla^{(\alpha} Q^{\beta\gamma)} p_\alpha p_\beta p_\gamma$$

• If $Q$ is Killing,

If $\mathcal{X}(f) = 0$ then $\mathcal{X}(Q^{\alpha\beta} p_\alpha p_\beta \cdot f) = 0$.

• Similarly

$$X_K = Q^{\alpha\beta} p_\alpha \frac{\partial}{\partial q^\alpha} - \frac{1}{2} p_\alpha p_\beta \frac{\partial Q^{\alpha\beta}}{\partial q^\gamma} \frac{\partial}{\partial p_\gamma}$$

• On Minkowski: $Q^{\alpha\beta} = X_1(\alpha X_2^\beta)$, with $X_1, X_2$ Killing.

• Not true on Kerr: Carter constant.
Kerr spacetime

- Slowly rotating black holes ($a \ll M$): stationary, axisymmetric solution to the EE in vacuum. In Boyer-Lindquist coordinates, on $M = \mathbb{R} \times [r_+, +\infty) \times S^2$

$$g = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi$$

$$+ \frac{\Pi \sin^2 \theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

where $r_{\text{outer}} = M + \sqrt{M^2 - a^2}$, and

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta,$$

$$\Pi = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.$$

- Killing vectors: $\partial_t, \partial_\phi$, a non trivial Killing 2-tensor:

$$Q^{ab} = \partial_\theta^a \partial_\theta^b + \cot^2 \theta \partial_\phi^a \partial_\phi^b + a^2 \sin^2 \theta \partial_t^a \partial_t^b$$
Symmetries for Vlasov fields

Defining the energy

Morawetz estimate

Diagram

\[ \partial_t \] is not timelike in the blue region (ergoregion).
Andersson-Blue work on the wave equation

- Difficulty: less symmetries, $\partial_t$ not positive, i.e. no positive energies out of $\partial_t$.
- Work of Andersson-Blue: define a positive energy, reinforced but the use of symmetries; based on the analysis of geodesics, define the appropriate vector fields to perform Morawetz estimates; prove the positive definite energy is approximatively conserved based on this estimates; then commute to get pointwise decay estimates.
- One obstruction: the geodesics analysis is true only at high frequency for the wave equations.
- for massless Vlasov fields, no problem.
The setting

- Mass shell = bundle of future oriented light cone: $\mathcal{C}^+$.
- Massless Vlasov equation: $f: \mathcal{C}^+ \to \mathbb{R}^+$

$$\mathcal{X} f = v^a \left( \frac{\partial}{\partial x^a} - v^b \Gamma^c_{ab} \frac{\partial}{\partial v^c} \right) f.$$ 

- Zero-th order symmetries

$$e = v_a \partial^a_t, \quad l_z = v_a \partial^a_\phi, \quad q = v_a v_b Q^{ab}$$

$$\mathcal{S}_2 = \{ e^2, e l_z, l_z^2, q \} = \{ S_2 \}_{a}.$$ 

- Task: define a positive energy, prove its conservation, establish a morawetz estimates, and prove an integral energy decay.
The volume form

- Usual form: at $x = (x^0, \ldots, x^3)$, $\partial_{x^0}$ timelike
  \[ T_{00} = \rho(f)(x) = \int_{C^+_x} f v_t^2 \sqrt{|g|} \frac{dv^1 dv^2 dv^3}{|v_t|} \]

- Problem: here $\partial_t$ is not uniformly timelike.
- Solution: Use the Gelfand-Leray form:
  \[ C^+_x = \{ S(v) = g_x(v, v); v \text{ future oriented} \} \subset T_x M; \]
  factorize the metric w.r.t. $dv S$:
  \[ \sqrt{|g|} dv^0 \wedge \cdots \wedge dv^3 = dv S \wedge d\mu_{C^+_x} = dv S \wedge \left( \sqrt{|g|} \frac{dv^1 \wedge dv^2 \wedge dv^3}{-v_t} \right) \]

- Orienting $C^+_x$ using the orientation/time orientation of the manifold, one obtains: if $f$ is positive:
  \[ \int_{C^+_x} \left\{ \begin{array}{c} f \sqrt{|g|} \frac{dv^1 \wedge dv^2 \wedge dv^3}{-v_t} \\ \text{oriented form} \end{array} \right\} = \int_{C^+_x} \left\{ \begin{array}{c} f \sqrt{|g|} \frac{dv^1 dv^2 dv^3}{|v_t|} \\ \text{measurable set measure} \end{array} \right\} \geq 0 \]
**Symmetries for Vlasov fields**

**Defining the energy**

**Morawetz estimate**

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**Strengthening the energy**

- Exploit the symmetries of order 2: remind $\mathcal{X} \left( S_{a_1} S_{a_2} f \right) = 0$
  
  $$T_{\alpha\beta} [S_{a_1} S_{a_2} f] = \int_{C_x^+} (S_{a_1} S_{a_2} f) \nu_\alpha \nu_\beta d\mu_{C_x^+}.$$  

- Choose a vector $X^{a_1a_2}$ wisely: depends on the symmetries it applies to!

- Define the energy: on a spacelike Cauchy hypersurface $\Sigma_t$
  
  $$E_X[f](\Sigma_t) = \int_{\Sigma_t} T_{aba_1...a_k} [f] X^{a_1...a_k} d\nu^b_{\Sigma_t}.$$  

- Difference between two slices:
  
  $$E_X[f](\Sigma_{t_2}) - E_X[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_{X,\Omega,q}[f](R) d\mu_g,$$  

  where

  $$\Pi_{X,\Omega,q}[f] = -\frac{1}{2}\Omega^2 T_{aba_1...a_k} [f] \text{Lie} X^{a_1...a_k} (\Omega^{-2} g^{ab})$$

  $$+ T_{aba_1...a_k} g^{ab}[f] q^{a_1...a_k}.$$
Principle

- Principle of the estimates: find $T_\chi$ and $A$, such that

$$E_{T_\chi} \geq 0, \quad \leftarrow \text{choose } T_\chi \text{ timelike and causal}$$

$$\Pi_A \geq 0, \quad \leftarrow \text{choose } q \text{ and } \Omega \text{ wisely and take}$$

$$\Omega^{-2}\Pi_{A,\Omega,q} = \left(-\frac{1}{2}\text{Lie}_A(\Omega^{-2}g^{ab}) - q\Omega^{-2}g^{ab}\right) T_{ab}$$

$$\Pi_{T_\chi} \lesssim \frac{|a|}{M} \Pi_A$$

$$\approx |\text{Lie}_{T_\chi}(g^{ab})|$$

$$E_{T_\chi} \gtrsim |E_A| \text{ provided that } |A| \lesssim |T_\chi|$$
First step: Choosing the vector field:

Definition

\[ T_\perp = \left( \partial_t + \frac{2aMr}{\Pi} \partial_\phi \right)^a = (\partial_t + \omega_\perp \partial_\phi)^a, \]

\[ T^a_\chi = (\partial_t + \chi \omega_\mathcal{H} \partial_\phi)^a, \quad \text{← uniformly timelike} \]

\[ T^{aab}_\chi = T^a_\chi \delta^{ab} \quad \text{← symmetry strengthened VF,} \]

where \( \omega_\mathcal{H} = a/(r_+^2 + a^2) \) is the rotation speed of the horizon,

\( \chi = \chi(r) \) is a function that is 1 for \( r < r_\chi \), smoothly decreasing on \( r \in [r_\chi, r_\chi + M] \), and identically 0 for \( r > r_\chi + M \), and where \( r_\chi \) is chosen sufficiently large. For simplicity, we take \( r_\chi = 10M \).
Lemma

There is a positive constant $\bar{\epsilon}$ such that if $|a| < \bar{\epsilon}M$, $t \in \mathbb{R}$, and $f : C^+ \rightarrow [0, \infty)$ is continuous, then

$$E_{T \perp} [f](\Sigma_t) \simeq \int_{\Sigma_t} \int_{C_x^+} \left( \frac{(r^2 + a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) f \, d\mu_{C_x^+} d\mu_{\Sigma_t},$$

$$E_{T \perp} [f](\Sigma_t) \simeq E_{T \chi} [f](\Sigma_t) \simeq E_{\text{model,3}} [f](t).$$

Proof: Compare wisely polynomials in $v$'s. Exploit some uniformity in the dependency in $a/M$. 
The Morawetz estimate : The starting point

- Key point: Understand how the instability of trapped (orbiting) null geodesics is encoded in the metric coefficients.
- Write

\[ \Sigma g^{ab} = \Delta \partial_r^a \partial_r^b + \frac{1}{\Delta} \mathcal{R}^{ab}, \]

where

\[ \Delta = r^2 - 2Mr + a^2, \]
\[ \Sigma = \Omega^{-2} = r^2 + a^2 \cos^2 \theta, \]
\[ \mathcal{R}^{ab} = -(r^2 + a^2)^2 \partial_t^a \partial_t^b - 4aMr \partial_t^a \partial_\phi^b + (\Delta - a^2) \partial_\phi^a \partial_\phi^b + \Delta Q^{ab}, \]
\[ Q^{ab} = \partial_\theta^a \partial_\theta^b + \cot^2 \theta \partial_\phi^a \partial_\phi^b + a^2 \sin^2 \theta \partial_t^a \partial_t^b. \]
Morawetz estimate: geodesic equation

• Stolen from unpublished notes of Pieter;
• $\gamma$ a null geodesic, with conserved quantities:

$$e = -\dot{\gamma}_t,$$

$$l_z = -\dot{\gamma}_\phi,$$

$$q = \dot{\gamma}_\theta^2 + \frac{\cos^2 \theta}{\sin^2 \theta} \dot{\gamma}_\phi^2 + a^2 \sin^2 \theta \dot{\gamma}_t^2.$$

• Radial component:

$$\Sigma^2 \left( \frac{dr}{d\lambda} \right)^2 = -\mathcal{R}(r; M, a; e, l_z, q),$$

where

$$\mathcal{R}(r; M, a; e, l_z, q) = -(r^2 + a^2)^2 e^2 - 4aMrelz + (\Delta - a^2)l_z^2 + \Delta q.$$  

• Claim: for $a \ll M$, unstable geodesics: around $r \approx 3M$,

$\partial_r \mathcal{R} = 0, \partial_r^2 \mathcal{R} < 0.$
Choosing the Morawetz vector field

Definition

If $z$ and $w$ are smooth functions of $r$ and the parameters $M$ and $a$, the Morawetz vector field and the reduced scalar functions are defined to be

$$A^{ab} = -zwL^{(a} \tilde{R}^{b)} \partial_r,$$

$$q^{ab} = \frac{1}{2} (\partial_r z) w L^{(a} \tilde{R}^{b)},$$

where

$$\mathcal{R} = \mathcal{R}^{ab} \nu_a \nu_b = \mathcal{R}^a S_a = \mathcal{R}^a S^{ab} \nu_a \nu_b$$

$$\tilde{\mathcal{R}}^a = \partial_r \left( \frac{z}{\Delta} \mathcal{R}^a \right),$$

$$\mathcal{L} = \mathcal{L}^a S_a = M^2 e^2 + l_z^2 + q,$$
Choosing the weights $z$ and $w$

- $z$ and $w$ are chosen so that
  - comparability with $\Pi_{T_\chi}$;
  - right combination of $R$ and derivatives;
  - compare with $T_\chi$.

- The same weights as Andersson-Blue.

- Really technical and delicate issue, obtained by carefully comparing expressions of bulk terms.

- Choose $\Omega^{-2} = \Sigma$, 

Flashing the weights $z$ and $w$

**Definition**

Given a positive value for the parameter $\epsilon e^2$, we use the following weights to define the Morawetz vector field,

\[
    z = z_1 z_2, \quad w = w_1 w_2,
\]

\[
    z_1 = \frac{\Delta}{(r^2 + a^2)^2}, \quad w_1 = \frac{(r^2 + a^2)^4}{3r^2 - a^2},
\]

\[
    z_2 = 1 - M^2 \epsilon e^2 \frac{\Delta}{(r^2 + a^2)^2}, \quad w_2 = \frac{1}{2r}.
\]

Same weights as in Andersson-Blue ← real technical difficulty.
Lemma

There are positive constants $\bar{\epsilon}$, $\epsilon_{e^2}$, and $C$ such that if $|a| \leq \bar{\epsilon} M$, $0 < \epsilon_{e^2} \leq \epsilon_{e^2}^-$ and $f : C^+ \to [0, \infty)$ is a solution of the Vlasov equation, then

$$C \Omega^2 \Pi_A \geq M \frac{\Delta^2}{(r^2 + a^2)^2} \nu_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f.$$

and

$$\tilde{\mathcal{R}}' = -2r^{-4}(r - 3M)\mathcal{L}_{\epsilon_{e^2}} + aMO(r^{-4})el_z + a^2 \left(O(r^{-5})q + O(r^{-5})l_z^2\right) + M^2 \epsilon_{e^2} \left(a^2 O(r^{-5})e^2 + O(r^{-5})q + O(r^{-5})l_z^2\right).$$
Closing argument

- Remind

\[ E_{T_x} \geq 0, \, \Pi_A \geq 0, \, \Pi_{T_x} \lesssim \frac{|a|}{M} \Pi_A, \, E_{T_x} \gtrsim |E_A|. \]

- Energy estimates

\[ E_{T_x}[f](\Sigma_{t_2}) - E_{T_x}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_{T_x} \, d\mu_g \]

\[ E_A[f](\Sigma_{t_2}) - E_A[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_A \, d\mu_g \]
Closing argument

- Remind

\[ E_{T_{\chi}} \geq 0, \quad \Pi_{A} \geq 0, \quad \Pi_{T_{\chi}} \lesssim \frac{|a|}{M} \Pi_{A}, \quad E_{T_{\chi}} \gtrsim |E_{A}|. \]

- the energy estimates

\[
E_{T_{\chi}}[f](\Sigma_{t_2}) - E_{T_{\chi}}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_{t}} \Pi_{T_{\chi}} \, d\mu_{g}
\]

\[
E_{A}[f](\Sigma_{t_2}) - E_{A}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_{t}} \Pi_{A} \, d\mu_{g}
\]
Closing argument

- Remind

\[ E_{T\chi} \geq 0, \ \Pi_A \geq 0, \ \Pi_{T\chi} \lesssim \frac{|a|}{M} \Pi_A, \ E_{T\chi} \gtrsim |E_A|. \]

- Energy estimates

\[ E_{T\chi}[f](\Sigma_{t_2}) - E_{T\chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{|a|}{M} \Pi_A \, d\mu_g \]

\[ E_A[f](\Sigma_{t_2}) - E_A[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_A \, d\mu_g \]
Closing argument

- Remind

\[ E_{T\chi} \geq 0, \quad \Pi_A \geq 0, \quad \Pi_{T\chi} \lesssim \frac{|a|}{M} \Pi_A, \quad E_{T\chi} \gtrsim |E_A|. \]

- Energy estimates

\[ E_{T\chi}[f](\Sigma_{t_2}) - E_{T\chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{|a|}{M} \Pi_A d\mu_g \]

\[ E_A[f](\Sigma_{t_2}) - E_A[f](\Sigma_{t_1}) = \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_A d\mu_g \]
Closing argument

- Remind

\[ E_{T \chi} \geq 0, \; \Pi_A \geq 0, \; \Pi_{T \chi} \lesssim \frac{|a|}{M} \Pi_A, \; E_{T \chi} \gtrsim |E_A|. \]

- Energy estimates

\[
E_{T \chi}[f](\Sigma_{t_2}) - E_{T \chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2} \int_{\Sigma_t} \frac{|a|}{M} \Pi_A \, d\mu_g
\]

\[
E_{T \chi}[f](\Sigma_{t_2}) + E_{T \chi}[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_A \, d\mu_g
\]
Closing argument

- Remind

\[ E_T \geq 0, \ \Pi_A \geq 0, \ \Pi_T \chi \lesssim \frac{|a|}{M} \Pi_A, \ E_T \gtrsim |E_A|. \]

- Energy estimates

\[ E_T[f](\Sigma_{t_2}) - E_T[f](\Sigma_{t_1}) \lesssim \frac{|a|}{M} \left( E_T[f](\Sigma_{t_2}) + E_T[f](\Sigma_{t_1}) \right) \]

\[ E_T[f](\Sigma_{t_2}) + E_T[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_A d\mu_g \]
Closing argument

- Remind

\[ E_{T \chi} \geq 0, \quad \Pi A \geq 0, \quad \Pi_{T \chi} \lesssim \frac{|a|}{M} \Pi A, \quad E_{T \chi} \gtrsim |E_A|. \]

- Energy estimates

\[ E_{T \chi}[f](\Sigma_{t_2}) \lesssim \frac{1 + \frac{|a|}{M}}{1 - \frac{|a|}{M}} E_{T \chi}[f](\Sigma_{t_1}) \]

\[ E_{T \chi}[f](\Sigma_{t_2}) + E_{T \chi}[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2} \int_{\Sigma_t} \Pi A d\mu g \]

- Finally use the estimates for \( \Pi A \), and take \( t_1 \to -\infty \) and \( t_2 \to +\infty \).
Theorem 1: Existence of a bounded energy

There are positive constants $C$ and $\bar{\epsilon}$ such that if $M > 0$, $|a| \leq \bar{\epsilon}M$, and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is a smooth solution of the Vlasov equation in the exterior of the Kerr spacetime with parameters $(M, a)$, then, for all $t$ in $\mathbb{R}$,

$$E_{model,3}[f](t) \leq CE_{model,3}[f](0).$$
Theorem 2 : Integrated decay estimates

Theorem (Part 1)

There are positive constants $C$, $\bar{\epsilon}$, and $\bar{r}$ and a function $1_{r \not\in 3M}$ which is identically 1 for $|r - 3M| \geq \bar{r}$ and zero otherwise such that if $M > 0$, $|a| \leq \bar{\epsilon}M$, and $f : C^+ \to [0, \infty)$ is a smooth solution of the Vlasov equation in the exterior of the Kerr spacetime with parameters $(M, a)$, then,

\[
\int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{C_x^+} \left( M \frac{\Delta^2}{(r^2 + a^2)^2} \right) v_r^2 |f|_2 \\
+ 1_{r \not\in 3M} \frac{1}{r} \left( M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) |f|_2 \ d^3 v \ d^4 x \leq CE_{model, 3}[f](0)
\]

where

\[
|f|_2 = \left| M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right|^2 f
\]
Theorem 2: Integrated decay estimates

Theorem (Part 2)

More precisely,

\[
\int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{C_x^+} M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{R}' \tilde{R}' \mathcal{L} f \, d\mu_{C_x^+} \, d\mu_g \\
\leq CE_{model,3}[f](0),
\]