Hertz potentials, peeling, and the Cauchy problem

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Introduction

- **Context**: Study of the nonlinear stability of Kerr black holes.
- **Previous result**: Minkowski nonlinear stability ('91), relies heavily on decay result for higher spin fields ('89) on flat background.
- **Question**: develop alternate methods to study the asymptotics of higher spin fields
- **Exploit symmetries of the spacetime/higher spin fields (Maxwell fields and linearized gravity).**
- **Structure of potentials** (Hertz, Debye, etc) strongly tight to the structure of the space-time.
Introduction : Hertz potentials

- Penrose (63') : representations of massless spin-s fields on flat space-time : local representation by a potential of order 2s, satisfying a wave equation.
- Penrose proved peeling from a decay assumption on $\chi$.
- (Cohen-Kegeles 76) : on Kerr black holes :

$$F = \overline{d\delta} G, \ G \text{ solution of a wave equation}$$

is an uncharged solution of the Maxwell equations.

- Conjecture : Any Maxwell fields can be written as :

$$F = F_{Coulomb} + \overline{d\delta} G$$

- Conjecture : in this situation, $\overline{d\delta} G$ radiates/decays, under suitable assumptions.
Today’s talk’s framework and purpose

- Background: flat space-time.
- Cauchy problem for massless spin-s fields of arbitrary spin but especially Maxwell (spin 1) and linearized gravity (spin 2).
- Construct a potential satisfying a wave equation, whose initial data lie in a Sobolev space insuring good decay properties.
- Deduce decay/peeling properties.
- Important result: Christodoulou-Klainerman ’89 on linear fields.
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Standard decay results for the scalar wave equation

Construction of potentials

Decay of linear fields using potentials
Scalar fields on flat background

- Restrict our attention to fields on Minkowski space:
  \[ (\mathbb{R}^4, dt^2 - dx^2 - dy^2 - dz^2) \]

- Consider the Cauchy problem for the wave equation:
  \[
  \begin{cases}
    \Box \phi = 0 \\
    \phi|_{t=0} = f \in H^k_{\sigma}(\mathbb{R}^3) \\
    \partial_t \phi|_{t=0} = g \in H^{k-1}_{\sigma-1}(\mathbb{R}^3)
  \end{cases}
  \]

- Purpose: How does this field decay?
Decay of solutions of the linear wave equation

- Background: flat space-time.
- Obtained by energy estimates (Klainerman 83-87, $\sigma = -\frac{3}{2}$) or by conformal compactification (Penrose 65, stronger assumptions on the initial data, $\sigma = -2$).
- There exist for arbitrary weights (Asakura ’86, d’Ancona-Georgiev-Kubo ’01, Szpak ’08).
- Obtain decay estimates in two directions:
  - Interior decay: along time directions ($t > 3r$);
  - Exterior decay: along null directions ($\frac{t}{3} < r < 3t$).
Decay of solutions of the linear wave equation

**Theorem (Klainerman)**

Let $s_0 \geq 2$. Let $u$ be a solution of the wave equation with initial date in $H_{-\frac{3}{2}}^{s_0}(\mathbb{R}^3) \times H_{-\frac{5}{2}}^{s_0-1}(\mathbb{R}^3)$. Then

1. for $t > 3r$

   $$|\phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0}}{< t >^{\frac{3}{2}}},$$

2. for $\frac{r}{3} < t < 3r$:

   $$|\phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0}}{< u >^{\frac{1}{2}} < v >^{1}},$$
Decay of solutions of the linear wave equation 2

Theorem (Klainerman)

Let \( s_0 \geq 2, (j, k, l) \in \mathbb{N}^3 \). Let \( u \) be a solution of the wave equation with data in \( H^{s_0+j+k+l}_{-\frac{3}{2}} \). Then

1. for \( t > 3r \)
   \[
   |\nabla^j \phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0+j} < t >^{\frac{3}{2}+j}}{
   }
   
2. for \( \frac{r}{3} < t < 3r \):
   \[
   |\partial_u^i \partial_v^k \nabla_{S^2}^l \phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0+j+k+l} < u >^{\frac{1}{2}+j} < v >^{1+k+l}}{
   }
   
   \( u = t - r \) and \( v = t + r \).
Decay result for arbitrary weights

- For arbitrary $\sigma$?
- $\phi|_{t=0} = f \in H_{\sigma}^{k}$ and $\partial_{t}\phi|_{t=0} = g \in H_{\sigma-1}^{k-1}$, $k \geq 3$:
  \[ f(x) \lesssim < r >^\sigma \| f \|_{2,\sigma} \quad \text{and} \quad g(x) \lesssim < r >^{\sigma-1} \| g \|_{2,\sigma-1} \]

- Integral representation:
  \[ \phi(t, x) = \frac{1}{4\pi} \left( \int_{S^2} t (g(x + t\omega) + \partial_{\omega} f(x + t\omega)) + f(x + t\omega) \mu_{S^2} \right) \]

- Asymptotic behavior is given by:
  \[ J_{\sigma} = \int_{S^2} < |x + t\omega| >^\sigma \mu_{S^2}. \]
Decay result for arbitrary weights

\[ J_\sigma = \begin{cases} 
8\pi \frac{\langle u \rangle^{2+\sigma} - \langle v \rangle^{2+\sigma}}{(2 + \sigma)(\langle u \rangle^2 - \langle v \rangle^2)} & \text{if } \sigma \neq -2 \text{ and } \langle u \rangle \neq \langle v \rangle, \\
8\pi \log \left( \frac{\langle u \rangle}{\langle v \rangle} \right) & \text{if } \sigma = -2 \text{ and } \langle u \rangle \neq \langle v \rangle, \\
4\pi \langle v \rangle^\sigma & \text{if } \langle u \rangle = \langle v \rangle. 
\end{cases} \]

- For the full solution, combine \( J_\sigma \) and \( J_{\sigma-1} \), hence the discussion arises on \( \sigma = -1 \).
- For higher order derivatives, one uses commutations with \( \Box \).
Decay for arbitrary weights

Proposition

If \((f, g)\) in \(H_\sigma^m \times H^m_{\sigma-1}\), \(m \geq j + k + l + 3\), one denotes:

\[
l_\sigma = \|(f, g)\|_{H^{j+k+l+3}_\sigma \times H^{j+k+l+2}_{\sigma-1}}
\]

then:

\[
|\partial_u^k \partial_v^l \nabla^m S_2 \phi| \leq C l_\sigma \begin{cases} 
< u >^{1+\sigma-k} < v >^{-1-l-m} & \text{if } \sigma < k - 1 \\
\log < v > - \log < u > & \text{if } \sigma = k - 1 \\
\frac{< v >^{l+m} (< v > - < u >)}{< v >^{\sigma-l-m-k}} & \text{if } \sigma > k - 1
\end{cases}
\]
Problem

- Give a proper analytic framework to Penrose’s representation of massless fields of spin 2s:

\[ \phi_{A\ldots F} = \nabla_{AA'} \cdots \nabla_{FF'} \xi^{A'\ldots F'}, \text{ where } \Box \xi = 0. \]

- Proper analytic representation: Cauchy problem for the field to a Cauchy problem for the potential + control of the norm of the initial data of the potential.

- Cases of interest: Maxwell and linearized gravity on flat background.

- Methods: elementary elliptic theory.
Maxwell equations

- Geometric background: Minkowski background \((\mathbb{R}^4, \eta)\).
- Consider the Faraday (skew-symmetric) tensor (2-form): \(F\).
- Link with electric and magnetic fields (1-forms on \(\mathbb{R}^3\)): \(T^a = (1, 0, 0, 0)\):

\[
E = F(T, \bullet) \quad \text{and} \quad B = (\ast F)(T, \bullet)
\]

where \(\ast F\) is the Hodge dual.
- Maxwell equations:

\[
\bar{d}F = 0 \quad \text{and} \quad \bar{\delta}F = 0 \\
\nabla_{[a}F_{bc]} = 0 \quad \text{and} \quad \nabla^a F_{ab} = 0
\]
Cauchy problem for the Maxwell equations

- Hyperbolic system of order 1 with 6 real unknowns with geometric constraints on the initial data.
  \[
  \begin{cases}
  (\overline{d} + \overline{\delta})F = 0 \\
  F|_{t=0} \in H^k_\sigma(\mathbb{R}^3, \Lambda^2)
  \end{cases}
  \]

- Geometric constraints on the initial data:
  \[
  D^a E_a = D^a B_a = 0
  \]

- Purpose: construct initial data for $G$ such that $F$ of the form:
  \[
  F = \overline{d}\delta G
  \]
Construction of a potential for the Maxwell field

- Assume \( F = \overline{d\delta G} \).
- Restrict to \( t = 0 \):
  \[
  E = -\delta dH - \delta \star \partial_t K \\
  B = -\delta dK + \delta \star \partial_t H
  \]
- Solve for a given set of initial data with:
  \[
  H = K = 0.
  \]
- Take \( E, B \in H^{k}_\sigma \) in the image of \( \Delta \):
  \[
  E = (d\delta + \delta d) \tilde{H} \\
  B = (d\delta + \delta d) \tilde{K}
  \]
  with \( \tilde{H}, \tilde{K} \in H^{k+2}_{\sigma+2} \).
Construction of a potential

- Use the geometric constraints:
  \[
  \delta E = 0 \implies E = \delta \left( d\tilde{H}(+\text{sth}) \right) \\
  \delta B = 0 \implies B = \delta \left( d\tilde{K}(+\text{sth}) \right)
  \]

- Take as initial data for the potential: \((0, -\star d\tilde{H})\) and \((0, \star d\tilde{K})\).

- Conditions to admit a potential:
  \[
  E, B \in H^k_\sigma(\Lambda^1) \perp \text{Ker}(\Delta) \cap L^2_{-3-\sigma}(\Lambda^1)
  \]
Construction of a potential

Lemma

Let $\sigma < -2$. If $E, B$ in $H^k_\sigma(\Lambda^1)$ satisfy the constraints equation,

$$\delta E = \delta B = 0,$$

there exist $E_1, B_1$ such that

$$E + \delta \ast E_1, B + \delta \ast B_1 \perp_{L^2} \ker(\Delta) \subset L^2_{-3-\sigma}.$$

Proof: Projection + rescaling
Existence of potentials for Maxwell fields

Proposition

Let $\sigma$ in $\mathbb{R} \setminus \mathbb{Z}$ and $s_0 \geq 3$.
Let $E_0, B_0$ be two solutions of the constraints in $H_{\sigma}^{s_0}$.
Then there exist two 2-forms $(G_0, G_1)$ in $H_{\sigma+2}^{s_0+2} \times H_{\sigma+1}^{s_0+1}$ such that:

$$
\|G_0\|_{s_0+2, \sigma+2}^2 + \|G_1\|_{s_0+1, \sigma+1}^2 \leq C \|F_0\|_{s_0, \sigma}^2
$$

and:

$$
F = \overline{d\delta} G
$$

where $G$ is the solution of the wave equation $(\overline{d\delta} + \overline{\delta d}) G = 0$ with initial data $(G_0, G_1)$. 
Linearized gravity – tensor version

- \( W_{abcd} \), a 4-tensor satisfying the symmetries of the Weyl spinor.
- Consider the Cauchy problem:

\[
\begin{aligned}
\nabla^a W_{abcd} &= 0 \\
W_{abcd} &= \psi_{abcd} \in H^k + \text{constraints}
\end{aligned}
\]

- Introduce \( E, B \):

\[
E_{cd} = T^a T^b W_{abcd} \quad \text{and} \quad B_{cd} = T^a T^b (\ast W_{abcd}) \\
D^a E_{ab} &= 0 \\
D^a B_{ab} &= 0
\]

- Hyperbolic system of order 1 of 10 unknowns.
- Unfortunately, no simple tensor notations:

\[
W = \nabla \nabla \underbrace{\nabla \nabla \xi}_{\text{Bergman potential}} \quad \underbrace{\nabla \nabla \xi}_{\text{Lanczos potential}}
\]
Linearized gravity – Spinor version

- $\phi_{ABCD}$, a totally symmetric spinor.
- Consider the Cauchy problem:
  \[
  \begin{cases}
  \nabla^{AA'} \phi_{ABCD} = 0 \\
  \phi_{ABCD} = \psi_{ABCD} \in H_k^\sigma + \text{constraints } D^{AB} \psi_{ABCD} = 0
  \end{cases}
  \]
- Hyperbolic system of order 1 of 5 complex unknowns.
- In spinors, the potential writes:
  \[
  \phi_{ABCD} = \nabla_{AA'} \nabla_{BB'} \nabla_{CC'} \nabla_{DD'} \xi^{A'B'C'D'}.
  \]

\[\text{Bergman potential}\]
\[\text{Lanczos potential} \]
Sketch of the proof

- Principle of the proof is the same except that one has to work with $\Delta^2$.
- Only important change: integrability condition on $E_{ab}$ and $B_{ab}$:

**Proposition**

*If $E_{ab}$ satisfies the constraint,*

$$\nabla^a E_{ab} = 0 \text{ or } \delta_2 E = 0$$

*then there exists a $G_{ab}$ such that*

$$\mathcal{R}(G) = E$$

*where $\mathcal{R}$ is the linearized Cotton-York tensor.*
Integration of 2-tensors

- Conformal rigidity: the deformation of a metric $g_0$, $\{g_t\}_t$, is conformally rigid iff there exist a family of diffeomorphisms $\phi_t$ and functions $u_t$ such that:

$$\phi_t^* g_0 = e^{u_t} g_t$$

with $\phi_0 = Id$ and $u_0 = 0$.

- The conformal Killing equation:

$$L_X g_0 - \frac{1}{3} \text{Tr}(L_X g_0) g_0 = h \text{ or } 2D(ABX_{CD}) = h_{ABCD}$$

can only be integrated provided that:

$$0 = \epsilon_{abcd} \mathcal{R}(h)_{dc} = 2D[a\sigma_b]c \text{ where }$$

$$\sigma_{ab} = D(a D^c h_b)_c - \frac{1}{2} \Delta h_{ab} - \frac{1}{4} g_{ab} D^c D^d h_{cd}.$$  

$\mathcal{R}$ is the linearized Cotton-York tensor.
De Rham and Gasqui-Goldschmidt complexes

- Previous works: Gasqui-Golschmidt '84; Beig '97
- Solving

\[ \delta \ast f = \omega \ (\omega \in \Lambda^1) \text{ or } L_X g_0 - \frac{1}{3} \text{Tr}(L_X g_0)g_0 = h \]

requires that the lhs satisfy constraints.
- Constraints are solved by the differential complexes:

\[
\begin{align*}
C^\infty(M, \mathbb{R}) & \xrightarrow{d} \Lambda^1 & \xrightarrow{\delta^*} & \Lambda^1 & \xrightarrow{\delta} & C^\infty(M, \mathbb{R}) \\
\Lambda^1(M) & \xrightarrow{L} S_0^2(M, g) & \xrightarrow{R} S_0^2(M, g) & \xrightarrow{\delta_2} & \Lambda^1(M)
\end{align*}
\]

\(L\) : conformal Killing operator, \(\delta_2\) : divergence on 2 tensors.
Existence of a potential for spin-2 fields

Proposition

Let $\sigma$ in $\mathbb{R} \setminus \mathbb{Z}$ and $s_0 \geq 3$.

Let $\psi_{ABCD}$ be a solution to the constraints in $H_{\sigma}^{s_0}$.

Then there exists $(\xi_0, \xi_1)$ in $H_{\sigma+4}^{s_0+4} \times H_{\sigma+3}^{s_0+3}$ such that:

$$\|\xi_0\|_{s_0+2,\sigma+4}^2 + \|\xi_1\|_{s_0+1,\sigma+3}^2 \leq C \|\psi\|_{s_0,\sigma}^2$$

and

$$\phi_{ABCD} = \nabla_{AA'} \nabla_{BB'} \nabla_{CC'} \nabla_{DD'} \xi^{A'B'C'D'}.$$ 

Bergman potential

Lanczos potential

where $\xi$ is the solution of the wave equation $\Box \xi = 0$ with initial data $(\xi_0, \xi_1)$. 
Purpose: Study the asymptotic behavior of spin $s$ fields satisfying the Dirac equation on flat background by methods which could be extended to Kerr background.

Work of reference: Christodoulou-Klainerman (’89)

Here: derive the same kind of decay result using representation of fields using potentials by reducing the tensor equation to a scalar wave equation.
Asymptotics for the Maxwell equations

<table>
<thead>
<tr>
<th>Weight – ID</th>
<th>$-\frac{7}{2}$</th>
<th>$-\frac{5}{2}$ (ABJ)</th>
<th>$-\frac{5}{2}$ (CK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight – ID potential</td>
<td>$-\frac{3}{2}$</td>
<td>$-\frac{1}{2}$ (ABJ)</td>
<td>X</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interior decay $t &gt; 3r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^{-\frac{7}{2}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exterior decay $\frac{t}{3} &lt; r &lt; 3t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(\partial_u, e_{S^2})$, $\alpha$, $\phi_{-1}$, $\phi_2$</td>
</tr>
<tr>
<td>$F(\partial_v, \partial_u)$, $\rho$, $F(e_{S^2}, e_{S^2})$, $\sigma$, $\phi_0$, $\phi_1$</td>
</tr>
<tr>
<td>$F(\partial_v, e_{S^2})$, $\alpha$, $\phi_1$, $\phi_0$</td>
</tr>
</tbody>
</table>

One can also derive these components to complete the peeling result. The result is still the same as the one as CK.
Asymptotics for the spin-2 field

- In the interior region, for all weight $\sigma$ : $|\phi_{ABCD}| \lesssim \frac{1}{\langle t \rangle^\sigma}$.
- For the weight $\sigma = -\frac{11}{2}$, the exterior decay result is:
  $$|\phi_i| \lesssim \frac{1}{\langle v \rangle^{1+4-i} \langle u \rangle^{\frac{1}{2}+i}}.$$ 
- for the weight $\sigma = -\frac{7}{2}$:
  - for $i = 2, 3, 4$, 
    $$|\phi_i| \lesssim \frac{1}{\langle v \rangle^{1+4-i} \langle u \rangle^{rac{5}{2}+i}}.$$ 
  - for $i = 0, 1$, 
    $$|\phi_i| \lesssim \langle r \rangle^{-\frac{7}{2}}$$
- Exactly the same result as CK.
Conclusion / Perspectives

- We recover fully Christodoulou-Klainerman results; works for arbitrary spin, arbitrary weight.
- Purpose: extend this to Kerr space time.
- In this context, Maxwell fields of the form $d\delta G$ are not charged.
- Make sense in this context to:

$$ F = F_{\text{Coulomb}} + d\delta G $$
Conclusion / Perspectives

- Hard: Need a proper elliptic theory, results on the wave equation are partially complete.
- Ideally: Hodge-Helmholtz-Kodaira decomposition on non compact manifolds with boundary for weighted Sobolev spaces.
- Also: Reduce the potential to one scalar potential (Debye potential) to use result for the scalar wave equation.
- There exists another process: spin lowering, which can generate both symmetries amongst solutions and potentials, using the existence of the Killing spinor.