MULTIDIMENSIONAL INTEGRABLE SYSTEMS AND DEFORMATIONS OF LIE ALGEBRA HOMOMORPHISMS

MACIEJ DUNAJSKI, JAMES D.E. GRANT, AND IAN A.B. STRACHAN

Abstract. We use deformations of Lie algebra homomorphisms to construct deformations of dispersionless integrable systems arising as symmetry reductions of anti–self–dual Yang–Mills equations with a gauge group $\text{Diff}(S^1)$.

1. Introduction

A dispersionless limit of PDEs is taken by rescaling the independent variables $X^a \longrightarrow X^a/\varepsilon$, and taking the limit $\varepsilon \longrightarrow 0$. This is a delicate procedure, as the limit of the solutions of a given PDE does not usually correspond to solutions of the limiting dispersionless equation. Moreover, inequivalent PDEs may have the same dispersionless limit, so the problem

- Recover the original PDE from its dispersionless limit

is, of course, ill posed. Some progress can nevertheless be made if the dispersionless equation is integrable, and one insists that its dispersive analogue is also integrable. In the next section we shall explain how dispersionless limits of solitonic PDEs are equivalent to the WKB quasi-classical approximation of the associated linear problems. This suggests that the reconstruction of the dispersive solitonic system should involve a quantisation of some kind.

Such a quantisation procedure has been developed in the seminal work of Kupershmidt [Kup90]. This procedure is based on the Moyal product, and works well if the Lie algebra underlying the dispersionless linear problem is the algebra sdiff(Σ^2) of divergence-free vector fields on a 2-surface Σ^2. This is the case for the dKP and $SU(\infty)$ Toda equations in $2 + 1$ dimensions. Similar progress can also be made in higher dimensions and, indeed, one of us has constructed integrable deformations of Plebanski's first heavenly equation [Str92] by replacing the underlying Poisson bracket with the Moyal bracket.

The idea of deforming integrable systems while retaining the integrability of the resulting equation has now been studied from a number of different points of view:

- Takasaki studied properties of the deformed heavenly equations and described how solutions may be described in terms of a Riemann-Hilbert splitting in a Moyal algebra valued loop group [Tak94a]. Extensions of this led to Moyal-KP hierarchies [Tak94b] and deformations of the self-dual Yang-Mills equations [Tak01]. The deformed Riemann-Hilbert procedure was recently fully developed by Formanski and Przanowski [FP05a, FP05b].
- Nekrasov and Schwarz introduced instantons on noncommutative space-time [NS98]. This led to the development of noncommutative soliton equations. These may be viewed as a deformation of the standard, commutative, soliton equations. Many of these may be studied as reductions of the noncommutative self-dual Yang-Mills equations [Ham06, Lec06].
- Associated to any Frobenius manifold is a hierarchy of integrable equations of hydrodynamic type. Integrable deformations of these equations arise naturally when one studies the genus expansion in the corresponding topological quantum field theories [DZ01].

In the present paper deformations of multidimensional integrable systems are based on the algebra[1] diff(Σ), the Lie algebra of vector fields on Σ, where $\Sigma \cong S^1$ or $\mathbb{R}$. It turns out, however, that this algebra admits no non-trivial deformations [Lic76]. However an alternative method of

---

[1]In the remainder of this paper the superscript, denoting the dimension of the manifold, will be dropped.

deforming these integrable systems may be developed. This method is based on the approach of Ovsienko and Rogers [OR98] where a homomorphism from $\text{diff}(\Sigma)$ to the Poisson algebra on $T^*\Sigma$ can be used to construct non-trivial deformations. We shall use this idea to construct integrable deformations of various equations associated to the algebra $\text{diff}(\Sigma)$.

It should be pointed out that the original, undeformed, equations have a natural interpretation in terms of twistor theory, via the non-linear graviton construction and its variants. It would seem desirable to develop a 'deformed' version of twistor theory that would encode solutions of the deformed equations as some sort of deformed holomorphic conditions. This idea was what was behind the paper [Str97], but the problem remains open (though see [Mer05] for some ideas on how Nijenhuis structures may be deformed).

2. DISPERSIONLESS LIMIT IN $2 + 1$ DIMENSIONS

Certain dispersionless integrable systems can arise from solitonic systems in a following way: Let

$$A\Big(\frac{\partial}{\partial X}\Big) = \frac{\partial^n}{\partial X^n} + a_1(X^a)\frac{\partial^{n-1}}{\partial X^{n-1}} + \cdots + a_n(X^a),$$
$$B\Big(\frac{\partial}{\partial X}\Big) = \frac{\partial^m}{\partial X^m} + b_1(X^a)\frac{\partial^{m-1}}{\partial X^{m-1}} + \cdots + b_m(X^a)$$

be differential operators on $\mathbb{R}$ with coefficients depending on local coordinates $X^a = (X, Y, T)$ on $\mathbb{R}^3$. The overdetermined linear system

$$\Psi_Y = A\Big(\frac{\partial}{\partial X}\Big)\Psi, \qquad \Psi_T = B\Big(\frac{\partial}{\partial X}\Big)\Psi$$

admits a solution $\Psi(X, Y, T)$ on a neighbourhood of initial point (X, Y_0, T_0) for arbitrary initial data $\Psi(X, Y_0, T_0) = f(X)$ if and only if the integrability conditions $\Psi_{YT} = \Psi_{TY}$, or

$$A_T - B_Y + [A, B] = 0 \tag{2.1}$$

are satisfied. The nonlinear system (2.1) for $a_1, \ldots, a_n, b_1, \ldots, b_m$ can be solved by the inverse scattering transform (IST). Integrable systems which admit a Lax representation (2.1) will be referred to as solitonic, or dispersive.

The dispersionless limit [Zak94] is obtained by substituting

$$\frac{\partial}{\partial X^a} = \varepsilon\frac{\partial}{\partial x^a}, \qquad \Psi(X^a) = \exp(\psi(x^a/\varepsilon)),$$

and taking the limit $\varepsilon \longrightarrow 0$. In the limit the commutators of differential operators are replaced by the Poisson brackets of their symbols according to the relation

$$\frac{\partial^k}{\partial X^k}\Psi \longrightarrow (\psi_x)^k\Psi, \qquad [A, B] \longrightarrow \frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial \lambda} = \{A, B\}, \qquad \lambda = \psi_x,$$

where A, B are polynomials in λ, with coefficients depending on $x^a = (x, y, t)$. The dispersionless limit of the system (2.1) is

$$A_t - B_y + \{A, B\} = 0. \tag{2.2}$$

Nonlinear differential equations of the form (2.2) are called dispersionless integrable systems. One motivation for studying of dispersionless integrable systems is their role in constructing partition functions in topological field theories [Kri94].

A natural approach to solving (2.2) would be an attempt to take a quasi-classical limit of the IST which linearises (2.1). This does not yield the expected result, as the quasi-classical limit of the Lax representation for (2.1) is the system of Hamilton–Jacobi equations

$$\psi_y = A(\psi_x, x^a), \qquad \psi_t = B(\psi_x, x^a),$$

with 'two times' t and y, and the initial value problem for (2.2) would require a reconstruction of a potential from the asymptotic form of the Hamiltonians. This classical inverse scattering problem is so far open.

There are alternative methods of solving (2.2) [KMAR01, FK04, TT95, DMT01]. In particular the mini-twistor approach of [DMT01] works as follows: The system (2.2) is equivalent to the integrability $[L, M] = 0$ of a two-dimensional distribution of vector fields

$$L = \partial_t - B_\lambda \partial_x + B_x \partial_\lambda, \qquad M = \partial_y - A_\lambda \partial_x + A_x \partial_\lambda \qquad (2.3)$$

on $\mathbb{R}^3 \times \mathbb{RP}^1$. Assume that L, M are real analytic, and complexify $\mathbb{R}^3$ to $\mathbb{C}^3$. The mini-twistor space Z is the two complex dimensional quotient manifold

$$Z = \mathbb{C}^3 \times \mathbb{CP}^1/(L, M), \qquad \lambda \in \mathbb{CP}^1, x^a \in \mathbb{C}^3.$$

That is to say that the local coordinates on Z lift to functions on $\mathbb{C}^3 \times \mathbb{CP}^1$ constant along L, M.

The mini-twistor space is equipped with a three parameter family of certain rational curves. All solutions to (2.2) can in principle be reconstructed from a complex structure of the mini-twistor space.

In fact the twistor approach outlined above is capable of solving a wider class of equations. We shall therefore generalise the notion of the dispersionless integrable systems by allowing distributions of vector fields more general than (2.3). The derivatives $A_\lambda, A_x, B_\lambda, B_x$ of the symbols (A, B) of operators can be replaced by independent polynomials A_1, A_2, B_1, B_2 in λ with coefficients depending on (x, y, t)

$$L = \partial_t - B_1 \partial_x + B_2 \partial_\lambda, \qquad M = \partial_y - A_1 \partial_x + A_2 \partial_\lambda. \qquad (2.4)$$

If A_1, B_1 are linear in λ and A_2, B_2 are at most cubic in λ then the rational curves in Z have normal bundle $\mathcal{O}(2)$ (the line bundle over $\mathbb{CP}^1$ with transition functions λ^{-2} from the set $\lambda \neq \infty$ to $\lambda \neq 0$ i.e. Chern class 2) and the three–dimensional moduli space of such curves in Z can be parametrised by (x, y, t). Allowing polynomials of higher degrees would lead to hierarchies of dispersionless equations. We take the integrability of this generalised distribution (2.4) as our definition of the dispersionless integrable system. The definition is intrinsic in a sense that it does not refer to an underlying dispersive equation.

3. $\mathrm{Diff}(S^1)$ dispersionless integrable systems

In this section two integrable systems associated with the gauge group $\mathrm{Diff}(S^1)$ will be given. The first has been extensively studied in [Pav03, FK04, Dun04, MAS03, DS05, MS06, OR07], so only a new gauge theoretic description will be given - the reader is referred to these earlier papers for more details. The second system, which arises from a Nahm-type system, is new and this system is discussed in more detail.

3.1. **A $(2+1)$ dimensional dispersionless integrable system.** An example of a dispersionless system which is integrable in the sense of the outlined twistor correspondence is given by the following distribution

$$L = \partial_t - w\partial_x - \lambda\partial_y, \qquad M = \partial_y + u\partial_x - \lambda\partial_x. \qquad (3.1)$$

A linear combination of this distribution leads to a special case of (2.4) with $A_2 = B_2 = 0$. Its integrability leads to the pair of quasi-linear PDEs

$$u_t + w_y + uw_x - wu_x = 0, \qquad u_y + w_x = 0, \qquad (3.2)$$

for two real functions $u = u(x, y, t), w = w(x, y, t)$. This system of equations has recently been studied in [Pav03, FK04, Dun04, MAS03, DS05, MS06, OR07] in connection with Einstein–Weyl geometry, hydrodynamic chains and symmetry reductions of anti–self–dual Yang–Mills equations. From the twistor point of view (3.2) is invariantly characterised [Dun04] by requiring that the mini–twistor space Z fibres holomorphically over $\mathbb{CP}^1$. The second equation can be used to introduce a potential H such that $u = H_x, w = -H_y$. The first equation then gives

$$H_{xt} - H_{yy} + H_y H_{xx} - H_x H_{xy} = 0. \qquad (3.3)$$

The system (3.2) arises as a symmetry reduction of the anti–self–dual Yang Mills equations in signature $(2, 2)$ with the infinite–dimensional gauge group $\mathrm{Diff}(\Sigma)$ and two commuting translational symmetries exactly one of which is null [DS05]. This combined with the embedding of $SU(1, 1) \subset$

$\mathrm{Diff}(\Sigma)$ gives rise to explicit solutions to (3.2) in terms of solutions to the nonlinear Schrödinger equation, and the Korteweg de Vries equation [DS05].

The Lie algebra of the group of diffeomorphisms $\mathrm{Diff}(\Sigma)$, where $\Sigma = S^1$ or $\mathbb{R}$, is isomorphic to the infinite–dimensional Lie algebra of functions on Σ with the Wronskian

$$< f, g > := f g_x - f_x g \tag{3.4}$$

as the Lie bracket, where $f, g \in C^\infty(\Sigma)$, and x is a local coordinate on Σ. An alternative gauge–theoretic interpretation can be given to (3.2): Observe that the first equation in (3.2) can be interpreted as the flatness of a gauge connection on $\mathbb{R}^2$, where the gauge group is $\mathrm{Diff}(\Sigma)$. Indeed, choose local coordinates (t, y) on $\mathbb{R}^2$ and consider $\mathcal{A} \in \Lambda^1(\mathbb{R}^2) \otimes C^\infty(\Sigma)$ of the general form

$$\mathcal{A} = -w\mathrm{d}t + u\mathrm{d}y,$$

where $u, w : \mathbb{R}^2 \to C^\infty(\Sigma)$ depend on (x, y, t).

The flatness of this connection yields

$$\mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (u_t + w_y + < u, w >)\mathrm{d}t \wedge \mathrm{d}y = 0,$$

as claimed. Therefore the connection is a pure gauge, and can be written as $\mathcal{A} = g^{-1}\mathrm{d}g$, where $g = g(x, y, t) \in \mathrm{Map}(\mathbb{R}^2, \mathrm{Diff}(\Sigma))$, and

$$w = -g^{-1}g_t, \qquad u = g^{-1}g_y.$$

The second equation in (3.2) yields the following system

$$(g^{-1}g_y)_y - (g^{-1}g_t)_x = 0, \tag{3.5}$$

where $g = \exp(\mathcal{A})$ is a finite diffeomorphism of Σ, and terms like $g^{-1}g_t$ should be understood as

$$g^{-1}g_t = \mathcal{A}_t - < \mathcal{A}, \mathcal{A}_t > + \frac{1}{2} < \mathcal{A}, < \mathcal{A}, \mathcal{A}_t >> + \dots .$$

3.2. **A $(3+1)$ dimensional dispersionless integrable system.** In this section we shall present another example of an integrable system associated to the Lie algebra of $\mathrm{Diff}(S^1)$. We shall first write it as a Nahm system

$$\dot{\mathbf{e}}_i = \frac{1}{2}\varepsilon_{ijk}[\mathbf{e}_j, \mathbf{e}_k], \qquad i = 1, 2, 3 \tag{3.6}$$

where $\mathbf{e}_i$ are vector fields on an open set in $\mathbb{R}^4$ given by

$$\mathbf{e}_i = \frac{\partial}{\partial y^i} - N_i(x, y^j)\frac{\partial}{\partial x},$$

and (x, y^j) are local coordinates. Rewrite (3.6) as

$$\partial_x N_i + \varepsilon_{ijk}\partial_j N_k - \frac{1}{2}\varepsilon_{ijk} < N_j, N_k > = 0. \tag{3.7}$$

We shall now discuss the origin and possible applications of (3.7)

(1) Any solution to (3.7) defines a hyperHermitian conformal structure represented by the metric

$$g = \mathbf{n}^2 + \delta_{ij}\mathrm{d}y^i\mathrm{d}y^j, \tag{3.8}$$

where

$$\mathbf{n} = \mathrm{d}x + N_i\mathrm{d}y^i.$$

The three complex structures $\mathbf{I}_i, i = 1, 2, 3$ satisfying the algebra of quaternions

$$\mathbf{I}_i\,\mathbf{I}_j = -\delta_{ij}\mathbf{1} + \varepsilon_{ijk}\mathbf{I}_k$$

are given by

$$\mathbf{I}_i(\mathbf{n}) = \mathrm{d}y^i.$$

These formulae together with the algebraic relations satisfied by $\mathbf{I}_j$ determine the complex structures uniquelly, e.g.

$$\mathbf{I}_i(\mathrm{d}y^j) = -\delta_{ij}(\mathbf{n}) + \varepsilon_{ijk}\mathrm{d}y^k.$$

One way to impose integrability of the complex structures is to use the explicit form of the complex structures on the basis $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3, \mathbf{n})$ and demand that the space $\Lambda^{(1,0)}$ is closed under exterior differentiation. We begin by defining a basis of self-dual 2-forms

$$\boldsymbol{\Sigma}^i = \mathbf{n} \wedge \mathrm{d}y^i + \frac{1}{2}\varepsilon^{ijk}\mathrm{d}y^j \wedge \mathrm{d}y^k.$$

The integrability of the complex structures is then equivalent to the anti-self-duality of the two-form $\mathrm{d}\mathbf{n}$:

$$\boldsymbol{\Sigma}^i \wedge \mathrm{d}\mathbf{n} = 0. \tag{3.9}$$

This condition is equivalent to (3.7).

A dual formulation leads to the Lax pair of vector fields, which is a special form of the hyperHermitian Lax pair [Dun99, GS99, Cal]. To see it set $\mathbf{e}_4 = \partial_x$ and define complex vector fields

$$\mathbf{w} = \mathbf{e}_1 - i\mathbf{e}_2, \qquad \mathbf{z} = \mathbf{e}_3 - i\mathbf{e}_4.$$

The system (3.7) is equivalent to the commutativity of the Lax pair

$$[\mathbf{w} - \lambda\overline{\mathbf{z}}, \mathbf{z} + \lambda\overline{\mathbf{w}}] = 0 \tag{3.10}$$

for all values of the parameter λ.

(2) The system (3.7) arises as a symmetry reduction of the anti–self–dual Yang Mills equations on $\mathbb{R}^4$ with the infinite–dimensional gauge group $\mathrm{Diff}(S^1)$ and one translational symmetry. In fact any such symmetry reduction is gauge equivalent to (3.7). To see it consider the flat metric on $\mathbb{R}^4$ which in double null coordinates $w = y^1 + iy^2, z = y^3 + iy^4$ takes the form

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\overline{z} + \mathrm{d}w\mathrm{d}\overline{w},$$

and choose the volume element $\mathrm{d}w \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z}$. Let $A \in T^*\mathbb{R}^4 \otimes \mathfrak{g}$ be a connection one–form, and let F be its curvature two–form. Here $\mathfrak{g}$ is the Lie algebra of some (possibly infinite dimensional) gauge group G. In a local trivialisation $A = A_\mu \mathrm{d}y^\mu$ and $F = (1/2)F_{\mu\nu}\mathrm{d}y^\mu \wedge \mathrm{d}y^\nu$, where $F_{\mu\nu} = [D_\mu, D_\nu]$ takes its values in $\mathfrak{g}$. Here $D_\mu = \partial_\mu + A_\mu$ is the covariant derivative. The connection is defined up to gauge transformations $A \rightarrow b^{-1}Ab - b^{-1}\mathrm{d}b$, where $b \in \mathrm{Map}(\mathbb{R}^4, G)$. The ASDYM equations on A_μ are $F = - * F$, or

$$F_{wz} = 0, \qquad F_{w\overline{w}} + F_{z\overline{z}} = 0, \qquad F_{\overline{wz}} = 0.$$

These equations are equivalent to the commutativity of the Lax pair

$$L = D_w - \lambda D_{\overline{z}}, \qquad M = D_z + \lambda D_{\overline{w}} \tag{3.11}$$

for every value of the parameter λ.

We shall require that the connection possesses a symmetry which in our coordinates is given by $\partial/\partial y^4$. Choose a gauge such that the Higgs field A_4 is a constant in $\mathfrak{g}$. Now choose $G = \mathrm{Diff}(S^1)$, so that the components of the one form A become vector fields on S^1. We can choose a local coordinate x on S^1 such that $A_4 = \partial_x$, and $A_i = -N_i\partial_x$, where $N_i = N_i(x, y^j)$ are smooth functions on $\mathbb{R}^4$. The Lax pair (3.11) is identical to (3.10) and the ASDYM equations reduce to the first order PDEs (3.7).

(3) Example. An ansatz $\mathbf{N}(x, y^j) = f(x)\mathbf{A}(y^j)$, where $\mathbf{N} = (N_1, N_2, N_3)^T$, reduces (3.7) to a pair of linear equations

$$\dot{f} = cf, \qquad c\mathbf{A} + \nabla \wedge \mathbf{A} = 0,$$

where c is a constant. If $c = 0$ then $\mathbf{N}$ may be absorbed into a redefinition of the coordinate x in the metric (3.8). Therefore we assume $c \neq 0$. We set $c = 1$ by rescaling y^j and solve for $f = \exp(x)$ reabsorbing another constant of integration into $\mathbf{A}$. Now define a new coordinate $\rho = \exp(-x)$. Rescaling the metric (3.8) yields

$$\hat{g} = \rho \, \mathrm{d}\mathbf{y}^2 + \rho^{-1}(\mathrm{d}\rho - \mathbf{A} \cdot \mathrm{d}\mathbf{y})^2. \tag{3.12}$$

This metric is hyper–hermitian iff the vector $\mathbf{A}(y^j)$ satisfies the Beltrami equation

$$\mathbf{A} + \nabla \wedge \mathbf{A} = 0. \tag{3.13}$$

This is a slight improvement of the result of [PW02] where it is shown that (3.12) is ASD iff (3.13) holds.

The Beltrami equation implies that $\mathbf{A}$ is divergence–free and satisfies $\triangle \mathbf{A} + \mathbf{A} = 0$, where $\triangle = \nabla^2$ is the scalar Laplacian on $\mathbb{R}^3$ acting on components of $\mathbf{A}$. Existence of solutions of equation (3.13), at least in the analytic case, can be proved by an application of the Cartan-Kähler theorem (c.f. Example 3.7 in Chapter III of [BCG$^+$91]).

(4) The system (3.7) can be put in the hydrodynamic form

$$\partial_x \mathbf{N} = \mathbf{M} \operatorname{curl} \mathbf{N},$$

where

$$\mathbf{M} = - \begin{pmatrix} 1 & N_3 & -N_2 \\ -N_3 & 1 & N_1 \\ N_2 & -N_1 & 1 \end{pmatrix}^{-1}.$$

A different analytic continuation of (3.7) can be obtained at the level of the hyperHermitian geometry. This comes down to looking for conformal structures (3.8) of signature $++--$. To achieve this we regard y_1, y_2, N_1, N_2 as imaginary, and define

$$Y_1 = iy_1, \quad Y_2 = iy_2, \quad Y_3 = y_3, \qquad \mathcal{N}_1 = iN_1, \quad \mathcal{N}_2 = iN_2, \quad \mathcal{N}_3 = N_3.$$

The desired system for $\mathcal{N}_i = \mathcal{N}_i(Y^j, x)$ arises from (3.7).

Clearly there are many further properties of these dispersionless systems that may be studied. We now turn our attention to the construction of non-trivial integrable deformations of equations (3.2) and (3.7).

4. Dispersive deformations

Given a dispersionless integrable system it is natural to ask whether it arises as a limit of some dispersive (or solitonic) system. One would expect the reconstruction of a solitonic system to involve a quantisation of some kind, because taking a dispersionless limit of (2.1) was equivalent to a quasi-classical limit of the wave function $\Psi(X^a)$. This is indeed the case, and the paradigm example is provided by the connection between the Kadomtsev–Petviashvili (KP) equation and its dispersionless limit dKP. One can reconstruct KP from dKP by expressing the latter in the form (2.2), and replacing the Poisson brackets by the Moyal bracket [Kup90, TT95, Str95, Str97, PF99]. The infinite series involved in a Moyal product truncates in this case, because the symbols A and B are polynomials in momentum λ. The deformation parameter can then be set to one, and removed from the construction. It seems however that this beautiful example is rather exceptional, and that the reconstruction of dispersive systems (if at all possible) is in general non-unique and can lead to systems which involve a formal power series.

Generalising the definition of the dispersionless systems to non–Hamiltonian distributions (L, M) like (3.1) makes things even worse, as the Poisson bracket is not present, and the connection with known quantisation procedures of a classical phase–space has been lost. It could be argued that (3.3) should be regarded as its own deformation as it admits a dual (classical, and quantum) description: It is a solitonic system (2.1) with

$$A = H_X \frac{\partial}{\partial X}, \qquad B = H_Y \frac{\partial}{\partial X}$$

or a dispersionless limit (2.2) with $A = \lambda H_x, B = \lambda H_y$.

One attempt to find a dispersive analogue of (3.2) would be to use the centrally extended Virasoro algebra in place of diff(Σ). Recall that such a procedure has been used to produce dispersive systems from dispersionless systems in a different context. Namely [OK87], one can view the periodic Monge equation $u_t = uu_x$ as the equation for affinely parametrised geodesics with respect to the right-invariant metric on Diff(S^1) constructed from the L^2 inner product on the Lie algebra. Going to the central extension, one finds that affinely parametrised geodesics on the Virasoro-Bott group correspond to solutions of the KdV equation. (For a recent review of such

constructions, see [Mic06].) In the current situation, we view a general element of the extended algebra as a pair

$$(f, a) := f(x)\frac{\mathrm{d}}{\mathrm{d}x} - iac$$

where $a \in \mathbb{R}$ does not depend on x, and c is a constant. Assuming that x is a periodic variable the modified commutation relation are

$$[f\partial_x, g\partial_x]_c = <f, g> \partial_x + \frac{ic}{48\pi}\int(f_{xxx}g - fg_{xxx})\mathrm{d}x,$$

and we see that the central term is a function of (t, y) only. Applying this procedure to (3.3) with

$$H(x, y, t) = \sum_k h_k(y, t)L^k, \tag{4.1}$$

where L^k are generators of the centrally extended Virasoro algebra satisfying

$$[L^k, L^m] = (k - m)L^{k+m} + \frac{c}{12}k(k^2 - 1)\delta_{k,-m} \tag{4.2}$$

would modify only one equation in the infinite chain of PDEs for the functions h_k.

In the remaining part of this paper we shall present a construction[2] which leads to non–trivial dispersive analogues of (3.2), or its equivalent form (3.3), and (3.7).

4.1. **Deforming Lie algebra homomorphisms.** To find a non-trivial deformation one would wish to deform the Lie algebra of vector fields on Σ, but this algebra is known not to admit any non-trivial deformations [Lic76].

We shall choose a different route [OR98], and deform the standard homomorphism between diff(Σ) and the Poisson algebra on $T^*\Sigma$, the point being that the homomorphisms between Lie algebras can admit non–trivial deformations even if one of the algebras is rigid. A deformation of (3.3) is achieved in two steps, each introducing a parameter. In the first step we shall deform the embedding of diff(Σ) into the Lie algebra of volume–preserving vector fields on $T^*\Sigma$. This introduces the first parameter μ. The second step is a deformation quantisation of the first one: The Poisson algebra on $T^*\Sigma$ is the quasi-classical limit of the Lie algebra of pseudo–differential operators on Σ, so (working at the level of symbols) one quantises the deformed homomorphism by using the deformed associative product of symbols of pseudo–differential operators rather than a pointwise commutative product of functions. This introduces the second parameter ε. In what follows we shall be interested in the polynomial deformations rather than the formal ones.

The standard embedding $\pi : \mathrm{vect}(\Sigma) \to C^\infty(T^*\Sigma)$ is given by contracting a vector field $X_f = f(x)\partial_x$ with a canonical one–form Θ on $T^*\Sigma$. In our case $T^*\Sigma = \mathbb{R} \times \Sigma$ and the Lie algebras $C^\infty(S^1)$ (with the Wronskian bracket) and $\mathrm{vect}(S^1)$ (with the Lie bracket) are isomorphic so we can regard π as defined on $C^\infty(\Sigma)$.

If λ is a local coordinate on the fibres of $T^*\Sigma$, and $\Theta = \lambda\mathrm{d}x$, the map π is explicitly given by

$$(\pi(f))(\lambda, x) := \lambda f(x).$$

It is a Lie algebra homomorphism as

$$\{\pi(f), \pi(g)\} = \pi(<f, g>).$$

---

[2]An alternative approach which we have not explored would be to consider a quantum deformation of the Virasoro algebra as in [SKAO96] and its free boson realisation

$$[T_m, T_n] = -\sum_{l=1}^{\infty} f_l(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-s}(s^n - s^{-m})\delta_{m+n,0}, \tag{4.3}$$

where $s = qt^{-1}$, and coefficients f_l are given by

$$f(z) = \sum_{l=0}^{\infty} f_l z^l = \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+s^n}z^n\Big).$$

The ordinary Virasoro algebra (4.2) is recovered as $q \to 1$. Applying (4.3) to (4.1) would lead to a q-deformed analog of (3.3).

Given $\mu \in \mathbb{R}$ define [OR98]

$$
\begin{aligned}
(\pi_\mu(f))(\lambda, x) &= (\pi(f))(\lambda, x + \mu/\lambda) = \lambda f(x + \mu/\lambda) \\
&= \lambda \Big(f(x) + f'(x)\frac{\mu}{\lambda} + \frac{1}{2!}f''(x)\Big(\frac{\mu}{\lambda}\Big)^2 + \dots \Big).
\end{aligned}
$$

Note that $\{\pi_\mu(f), \pi_\mu(g)\} = \pi_\mu(< f, g >)$, so that π_μ is also a Lie algebra homomorphism between diff(Σ) and sdiff($T^*\Sigma$).

The next step is motivated by the canonical quantisation $\lambda \to \partial/\partial x$. For any functions $F, G \in C^\infty(T^*\Sigma)$ which are also allowed to depend on a parameter μ define the Kupershmidt-Manin product

$$
F \star G = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial^k F}{\partial \lambda^k} \frac{\partial^k G}{\partial x^k} . \tag{4.4}
$$

(this is equivalent, under an ϵ-valued change of variable to the Moyal product) and set

$$
\{F, G\}_\varepsilon = \frac{1}{\epsilon}(F \star G - G \star F). \tag{4.5}
$$

The Poisson bracket is recovered in the limiting procedure

$$
\lim_{\epsilon \to 0} \{F, G\}_\epsilon = \{F, G\},
$$

but the deformed bracket is equal to the Poisson bracket for all ϵ if F, G are linear in λ. This is why the first deformation parameter μ is needed.

We are now ready to propose the dispersive analog of the dispersionless equation (3.3) and (3.7).

(1) Let $\hat{H}(\lambda, x, y, t; \mu, \epsilon) = \pi_\mu(H)$ take values in an algebra of formal power series in ϵ with an associative product defined by (4.4). The deformed analogue of equations (3.3) is:

$$
\hat{H}_{xt} - \hat{H}_{yy} - \{\hat{H}_x, \hat{H}_y\}_\epsilon = 0. \tag{4.6}
$$

(2) Let $\hat{N}_i(\lambda, x^i; \mu, \epsilon) = \pi_\mu(N_i)$ take values in an algebra of formal power series in ϵ with an associative product defined by (4.4). The deformed analogue of equation (3.7) is:

$$
\partial_x \hat{N}_i + \varepsilon_{ijk} \partial_j \hat{N}_k - \frac{1}{2}\varepsilon_{ijk}\{\hat{N}_j, \hat{N}_k\}_\epsilon = 0. \tag{4.7}
$$

Given a solution $\hat{H}$ of (4.6) such that $(\mu\partial_\mu - \lambda\partial_\lambda)(\lambda^{-1}\hat{H}) = 0$, and $\lambda^{-1}\hat{H}$ is smooth in (μ/λ) we can construct $H(x, y, t)$ satisfying (3.3) by taking any of the two limits $\mu \longrightarrow 0, \epsilon \longrightarrow 0$ and similar remarks hold for equation (4.7). Conversely, formal powers series (in the deformation parameters) solution may be constructed from a solution to the original, undeformed, equation in an analogous manner to the way developed in [Str92]. The extent to which such formal series converge in a suitable space of functions is as yet, however, unclear.

These deformed equations formally retain their integrability; the various manipulations hold at the level of the Lax pair as well as at the level of the equations themselves. However, as remarked in the introduction, a direct twistor theory correspondence for these equations is lacking, though one should be able to adopt the methods developed by Takasaki [Tak94a] and Formanski–Przanowski [FP05a, FP05b] to study the geometry of the corresponding Riemann-Hilbert problem (in some suitable Moyal algebra valued loop group).

References

[BCG⁺91] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991. MR MR1083148 (92h:58007) 6

[Cal] D.M.J. Calderbank, *Integrable background geometries*. 5

[DMT01] Maciej Dunajski, Lionel J. Mason, and Paul Tod, *Einstein-Weyl geometry, the dKP equation and twistor theory*, J. Geom. Phys. **37** (2001), no. 1-2, 63–93. MR MR1807082 (2002c:53081) 3

[DS05] Maciej Dunajski and George Sparling, *A dispersionless integrable system associated to* Diff(S^1) *gauge theory*, Phys. Lett. A **343** (2005), no. 1-3, 129–132. MR MR2149777 (2005m:37157) 3, 4

[Dun99] Maciej Dunajski, *The twisted photon associated to hyper-Hermitian four-manifolds*, J. Geom. Phys. **30** (1999), no. 3, 266–281. MR MR1692233 (2000b:53063) 5

[Dun04] _____, *A class of Einstein-Weyl spaces associated to an integrable system of hydrodynamic type*, J. Geom. Phys. **51** (2004), no. 1, 126–137. MR MR2078688 (2005h:53065) 3

[DZ01] Boris Dubrovin and Youjin Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants*, 2001. 1

[FK04] E. V. Ferapontov and K. R. Khusnutdinova, *The characterization of two-component* (2+1)*-dimensional integrable systems of hydrodynamic type*, J. Phys. A **37** (2004), no. 8, 2949–2963. MR MR2042985 (2004m:37123) 3

[FP05a] Sebastian Formański and Maciej Przanowski, *∗-SDYM fields and heavenly spaces. I. ∗-SDYM equations as an integrable system*, J. Phys. A **38** (2005), no. 20, 4399–4418. MR MR2147629 (2006c:81112) 1, 8

[FP05b] _____, *∗-SDYM fields and heavenly spaces. II. Reductions of the ∗-SDYM system*, J. Phys. A **38** (2005), no. 42, 9371–9385. MR MR2186613 (2006i:81129) 1, 8

[GS99] James D. E. Grant and I. A. B. Strachan, *Hypercomplex integrable systems*, Nonlinearity **12** (1999), no. 5, 1247–1261. MR MR1710101 (2001d:37131) 5

[Ham06] Masashi Hamanaka, *Non-commutative Ward's conjecture and integrable systems*, Nuclear Phys. B **741** (2006), no. 3, 368–389. MR MR2217758 1

[KMAR01] B. Konopelchenko, L. Martínez Alonso, and O. Ragnisco, *The $\overline{\partial}$-approach to the dispersionless KP hierarchy*, J. Phys. A **34** (2001), no. 47, 10209–10217. MR MR1872412 (2003c:37101) 3

[Kri94] Igor Moiseevich Krichever, *The τ-function of the universal Whitham hierarchy, matrix models and topological field theories*, Comm. Pure Appl. Math. **47** (1994), no. 4, 437–475. MR MR1272384 (95g:58106) 2

[Kup90] B. A. Kupershmidt, *Quantizations and integrable systems*, Lett. Math. Phys. **20** (1990), no. 1, 19–31. MR MR1058992 (91i:58051) 1, 6

[Lec06] Olaf Lechtenfeld, *Noncommutative solitons*, 2006. 1

[Lic76] André Lichnerowicz, *Sur l'algèbre de Lie des champs de vecteurs*, Comment. Math. Helv. **51** (1976), no. 3, 343–368. MR MR0431301 (55 #4302) 1, 7

[MAS03] L. Martínez Alonso and A. B. Shabat, *Towards a theory of differential constraints of a hydrodynamic hierarchy*, J. Nonlinear Math. Phys. **10** (2003), no. 2, 229–242. MR MR1976383 (2004b:37145) 3

[Mer05] S. A. Merkulov, *Nijenhuis infinity and contractible differential graded manifolds*, Compos. Math. **141** (2005), no. 5, 1238–1254. MR MR2157137 (2006g:58008) 2

[Mic06] Peter W. Michor, *Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach*, Phase space analysis of partial differential equations, Progr. Nonlinear Differential Equations Appl., vol. 69, Birkhäuser Boston, Boston, MA, 2006, pp. 133–215. MR MR2263211 7

[MS06] S. V. Manakov and P. M. Santini, *A hierarchy of integrable pdes in 2+1 dimensions associated with 2 - dimensional vector fields*, 2006. 3

[NS98] Nikita Nekrasov and Albert Schwarz, *Instantons on noncommutative $\mathbb{R}^4$, and* (2, 0) *superconformal six-dimensional theory*, Comm. Math. Phys. **198** (1998), no. 3, 689–703. MR MR1670037 (2000e:81198) 1

[OK87] V. Yu. Ovsienko and B. A. Khesin, *The super Korteweg-de Vries equation as an Euler equation*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 81–82. MR MR925082 (89g:58095) 6

[OR98] V. Ovsienko and C. Roger, *Deforming the Lie algebra of vector fields on S^1 inside the Poisson algebra on $\dot{T}^*S^1$*, Comm. Math. Phys. **198** (1998), no. 1, 97–110. MR MR1657373 (99k:58202) 2, 7, 8

[OR07] _____, *Looped cotangent Virasoro algebra and non-linear integrable systems in dimension 2+1*, Comm. Math. Phys. **273** (2007), no. 2, 357–378. 3

[Pav03] Maxim V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), no. 9, 4134–4156. MR MR2003950 (2004h:37107) 3

[PF99] Maciej Przanowski and Sebastian Formański, *Searching for a universal integrable system*, Acta Phys. Polon. B **30** (1999), no. 4, 863–879. MR MR1697896 (2000e:37119) 6

[PW02] Radu Pantilie and John C. Wood, *A new construction of Einstein self-dual metrics*, Asian J. Math. **6** (2002), no. 2, 337–348. MR MR1928633 (2003g:53067) 6

[SKAO96] Jun'ichi Shiraishi, Harunobu Kubo, Hidetoshi Awata, and Satoru Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996), no. 1, 33–51. MR MR1401054 (98h:17030) 7

[Str92] I. A. B. Strachan, *The Moyal algebra and integrable deformations of the self-dual Einstein equations*, Phys. Lett. B **283** (1992), no. 1-2, 63–66. MR MR1169656 (93f:58117) 1, 8

[Str95] _____, *The Moyal bracket and the dispersionless limit of the KP hierarchy*, J. Phys. A **28** (1995), no. 7, 1967–1975. MR MR1336506 (96b:58065) 6

[Str97] _____, *A geometry for multidimensional integrable systems*, J. Geom. Phys. **21** (1997), no. 3, 255–278. MR MR1429101 (98b:58066) 2, 6

[Tak94a] Kanehisa Takasaki, *Dressing operator approach to Moyal algebraic deformation of selfdual gravity*, J. Geom. Phys. **14** (1994), no. 2, 111–120. MR MR1288217 (95g:58111) 1, 8

[Tak94b] _____, *Nonabelian KP hierarchy with Moyal algebraic coefficients*, J. Geom. Phys. **14** (1994), no. 4, 332–364. MR MR1303959 (95k:58084) 1

[Tak01] _____, *Anti-self-dual Yang-Mills equations on noncommutative space-time*, J. Geom. Phys. **37** (2001), no. 4, 291–306. MR MR1811147 (2002e:81235) 1

[TT95] Kanehisa Takasaki and Takashi Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. **7** (1995), no. 5, 743–808. MR MR1346289 (97f:58073) 3, 6

[Zak94] V. E. Zakharov, *Dispersionless limit of integrable systems in* $2 + 1$ *dimensions*, Singular limits of dispersive waves (Lyon, 1991), NATO Adv. Sci. Inst. Ser. B Phys., vol. 320, Plenum, New York, 1994, pp. 165–174. MR MR1321203 (96e:35167) 2

DEPARTMENT OF APPLIED MATHEMATICS AND THEORETICAL PHYSICS, UNIVERSITY OF CAMBRIDGE, WILBER-FORCE ROAD, CAMBRIDGE, CB3 0WA, U.K.

E-mail address: M.Dunajski@damtp.cam.ac.uk

URL: http://www.damtp.cam.ac.uk/user/gr/about/members/dunajski.html

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA

E-mail address: james.grant@univie.ac.at

URL: http://www.mat.univie.ac.at/~grant/

DEPARTMENT OF MATHEMATICS, GLASGOW UNIVERSITY, GLASGOW, G12 8QW, SCOTLAND

E-mail address: i.strachan@maths.gla.ac.uk

URL: http://www.maths.gla.ac.uk/~iabs/