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# Seven short stories on blowups

*Herwig Hauser*

*To Raoul Bott – with great respect.*

*"At that time, blowups were the poor man's tool to resolve singularities."* This phrase of the late 21st century mathematician J.H.Φ. Leicht could become correct. In our days, however, blowups are still the main device for resolution purposes (cf. fig. 1).

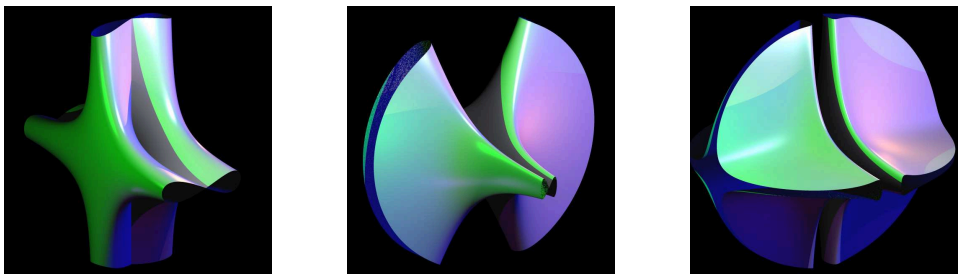

*Figure* 1: Resolution of the surface Helix: $x^2 - x^4 = y^2z^2$ by two blowups.

These notes shall give an informal introduction to the subject. They are complemented by the discussion of many special and less known features of blowups.

The lectures adress to students and geometers who are not experts in the field, but who need to use blowups occasionally or who just want to have a good comprehension of them. References are scattered in the literature and mostly concentrate on only part of the story. This text is neither complete, but hints at least at the variety of properties, results and techniques which are related to blowups and which make them so attractive. Actually, it may serve as the starting point to write a comprehensive treatise on blowups (which should in particular include the solutions to all exercises). The obvious objection from algebraic geometers to such a project will be that blowups are too simple to deserve a separate treatment. The many open and intricate questions listed in these notes may serve as a reply to this reproach.

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The material stems from lectures held by the author at the Mathematical Sciences Research Institute (MSRI) at Berkeley in April and May 2004 and during the Conference on Geometry and Topology at Gökova, Turkey, in June 2005. Both classes were taught ex tempore (i.e., without preparation) in order to incorporate better the doubts and questions of the audience. And the nicest compliment was the remark of a solid algebraic geometer saying "I thought to know already all about blowups, but I am no longer sure about this".

We are very grateful to Frank Sottile and Selman Akbulut for the kind invitations to Berkeley and Gökova, and all the people from MSRI for their warm hospitality. The students of the classes interfered many times with interesting remarks and inquiries – this helped a lot to shape the material. As a pop star would formulate it: *"You are the best audience I ever had".*

Here is a list of reading literature. Eisenbud-Harris' book on the geometry of schemes is probably the source which gives the most inspiration on the topic [EiH]. Several phenomena were taken from there. Another excellent reading is the beginning of Hironaka's Annals article [Hi 1]. There are almost no proofs, but many of the main properties appear with precise details and in all generality. Abhyankar's papers can be very stimulating if one really succeeds to read the information hidden between the lines. In Hartshorne's book [Hs], blowups have a prominent place, but the discussion lacks the concrete calculations and the various aspects which can arise in examples. Let us also mention the articles of Encinas-Villamayor [Vi 1, Vi 2, EV 1, EV 2] and Bierstone-Milman [BM 1 to BM 6], the survey of Lipman [Lp 3], the notes of Cutkosky, Matsuki and Kollár [Cu 7, Ma, Ko]. Above all, we recommend reading Zariski, especially his paper on the resolution of threefolds [Za 4] and therein the section on resolution of surfaces.

The proof of resolution in characteristic zero which we develop in big lines towards the end of these notes stem's from the author's collaboration with Santiago Encinas [EH]. The enclosed pictures were produced by Sebastian Gann with the ray-tracing program POV-Ray.

## 1. Introduction and examples

By a *resolution* of an algebraic set (variety, scheme) $X$ with singularities we understand a map $\pi : X' \to X$ which represents $X$ as the image of a manifold $X'$, i.e., which *parametrizes* $X$. It is by no means clear how to find such a parametrization.

*Example* 1: The first example of a resolution of a surface is the contraction of the cylinder $X'$ given by $x^2 + y^2 = 1$ in $\mathbb{A}^3$ to the (double) cone $X : x^2 + y^2 - z^2 = 0$. The map is induced by $\pi : \mathbb{A}^3 \to \mathbb{A}^3$, $(x, y, z) \to (xz, yz, z)$. It collapses the $xy$-plane $z = 0$ to the origin of $\mathbb{A}^3$ so that the cone appears as the image of the cylinder contracted along a circle (cf. fig. 2). Despite its simplicity, the example is quite instructive, because it leads

to the problem of reconstructing by a natural and general procedure the cylinder $X'$ and the map $\pi$ from the knowledge of $X$.

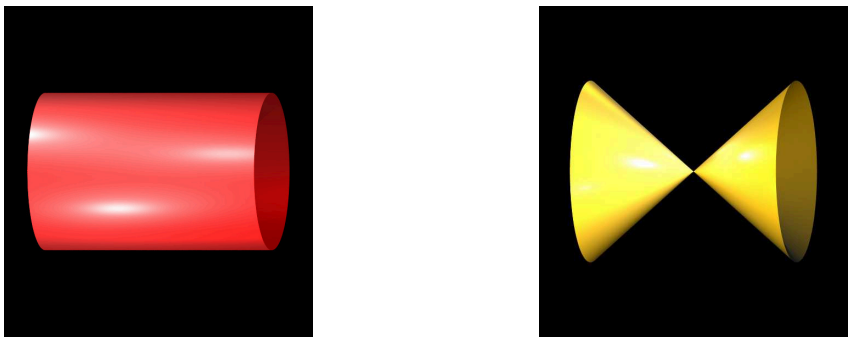

*Figure* 2: Cylinder and cone.

*Example* $1^a$: To answer this question, we consider a yet simpler example, the crossing $X$ of the two diagonals in $\mathbb{A}^2$, given by $x^2 - y^2 = 0$. The manifold $X'$ should of course consist of two separate lines. Therefore, to separate the two branches of $X$, we lift the two diagonals to $\mathbb{A}^2 \times \mathbb{P}^1$ by associating to each point $(x, y)$ the height $z$ given by the slope of the line through $(x, y)$ and $(0, 0)$, more precisely,

$$\sigma : \mathbb{A}^2 \setminus 0 \to \mathbb{A}^2 \times \mathbb{P}^1,$$

$$(x, y) \to ((x, y), (x : y)).$$

This map is defined outside the origin of $\mathbb{A}^2$, not only on $X$. Geometrically it is clear that the image of $X \setminus 0$ under $\sigma$ consists of two "parallel" lines (each deprived of one point). The Zariski-closure $X'$ of $\sigma(X \setminus 0)$ in $\mathbb{A}^2 \times \mathbb{P}^1$ consists of the entire lines and is thus a submanifold of $\mathbb{A}^2 \times \mathbb{P}^1$. The map $X' \to X$ is the restriction of the projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$ onto the first two components. We leave it to the reader to compute the affine or projective equations of $X'$ in $\mathbb{A}^2 \times \mathbb{P}^1$ as well as the chart expression of $\pi : X' \to X$.

The construction works fine because all points on one of the two lines of $X$ have the same slope. What happens if this is not the case?

*Example* $1^b$: Consider the variation of the preceding example, taking $X$ defined in $\mathbb{A}^2$ by $(y - x^2)(y - x - x^3) = 0$, a parabola and a cubic, intersecting over the reals only at the origin. The two components meet transversally at 0. There are now two options how to separate the branches.

(i) Either we associate to points $(x, y) \in \mathbb{A}^2$ again the triple $((x, y), (x : y)) \in \mathbb{A}^2 \times \mathbb{P}^1$. Notice that for points $(x, y)$ on the curve $X$, the projective point $(x : y)$ is the *secant line* through $(x, y)$ and $(0, 0)$.

A computation shows that the Zariski-closure $X'$ of the image of $X\backslash 0$ under the above map $\sigma$ consists of two disjoint regular curves, and it comes with a surjective map $\pi : X' \to X$. This construction is called the *blowup of* $\mathbb{A}^2$ *with center the origin* (or the ideal $(x, y)$ of $K[x, y]$), and $X'$ is the *strict* or *proper transform* of $X$. It's not a bad exercise to carry out this computation.

(ii) The second option is to associate to a point $(x, y)$ on the curve $X$ the triple $(x, y, t) \in \mathbb{A}^2 \times \mathbb{P}^1$, where $t$ is now the *tangent line* to $X$ in $(x, y)$. This construction, of course, works for the moment only at (the regular) points of the curve, and not on whole $\mathbb{A}^2$. The resulting transformation of $X$ is known as *Nash modification* (it corresponds to blowing up the jacobian ideal of the curve given by the partial derivatives of the defining equation, and is thus defined a posteriori again on whole $\mathbb{A}^2$). Though geometrically attractive, it has not been used and exploited systematically beyond curve and surface singularities.

*Nota bene*: Blowups whose centers are points or regular subvarieties of the ambient scheme represent the main modification of varieties to resolve their singularities. Their impact on the singularities, however, is very modest. They only succeed to improve them "picolissimo". Actually, it is very hard in general situations to show that an improvement actually occurs.

Therefore it would be nice to find substitutes which are more powerful and more refined, and Nash modifications are one possible option. You should always keep in mind to look out for possible new modifications to replace classical blowups.

After all, it can be shown by Hironaka's theorem that there does exist (at least in characteristic 0) a non reduced ideal structure on the singular locus of an arbitrarily difficult variety (i.e., an ideal with support the singular locus) such that blowing up the respective ideal resolves the singularities of $X$ ad hoc in one capital stroke, and moreover preserves the regularity of the ambient scheme. Nobody has the slightest idea how this non-reduced structure should look like. Aside of non-significant examples nothing seems to be known. *End of N.b.*

*Example* $1^c$: Consider the curve $X$ defined in $\mathbb{A}^2$ by $(y - x^2)(y - x^k)$ with $k \geq 3$. The two components meet now tangentially at the origin. Taking the blowup with center $(0, 0)$, i.e., corresponding to secant lines, yields (in the relevant chart) the equation $(y - x)(y - x^{k-1})$. The two components now meet less tangentially, but this, obviously, has to be made precise and captured by an intrinsic measure of tangency.

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Here is a first (and somewhat) preliminary definition of the blowup of affine space $\mathbb{A}^n$ with center an ideal $P$ of $K[x_1, \ldots, x_n]$ (said differently, with center $Z$ the subvariety $V(P)$ of $\mathbb{A}^n$): Choose generators $g_1, \ldots, g_k$ of $P$ and consider the map

$$\mathbb{A}^n \setminus Z \to \mathbb{A}^n \times \mathbb{P}^{k-1},$$
$$x \to (x, g_1(x) : \ldots : g_k(x)).$$

This is a well defined injective morphism. The image is the graph $\Gamma(g)$ of $g : \mathbb{A}^n \setminus Z \to \mathbb{P}^{k-1}$, $x \to (g_1(x) : \ldots : g_k(x))$. The Zariski-closure $\widetilde{\mathbb{A}}^n$ of $\Gamma(g)$ in $\mathbb{A}^n \times \mathbb{P}^{k-1}$ is the *blowup of* $\mathbb{A}^n$ *with center* $Z$. It does not depend on the choice of the generators $g_i$ up to isomorphism (show this). The restriction of the projection on the first factor $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ is the associated *blowup map*. It is a birational morphism which is an isomorphism over the complement of $Z$, say $\pi : \widetilde{\mathbb{A}}^n \setminus \pi^{-1}(Z) \cong \mathbb{A}^n \setminus Z$. The subvariety $\pi^{-1}(Z)$ of $\widetilde{\mathbb{A}}^n$ which is contracted under $\pi$ to $Z$ is a hypersurface (show also this) and called the *exceptional divisor* $E$ of the blowup.

From the definition, the algebraic equations of $\widetilde{\mathbb{A}}^n$ in $\mathbb{A}^n \times \mathbb{P}^{k-1}$ as well as the affine chart expressions of $\pi$ can be directly deduced. The projective equations for $\widetilde{\mathbb{A}}^n$ in $\mathbb{A}^n \times \mathbb{P}^{k-1}$ are $u_i g_j(x) - u_j g_i(x) = 0$ for $1 \leq i, j \leq k$ and coordinates $u_1, \ldots, u_k$ on $\mathbb{P}^{k-1}$.

For $X$ a subvariety of $\mathbb{A}^n$ containing $Z$ we obtain in a similar fashion the Zariski-closure $\widetilde{X}$ of the image of

$$\begin{aligned} &X \setminus Z \to X \times \mathbb{P}^{k-1}, \\ &x \to (x, g_1(x) : \ldots : g_k(x)), \end{aligned}$$

called the blowup of $X$ with center $Z$. Again, $\pi : \widetilde{X} \to X$ is the associated blowup map.

It is not difficult to show that $\widetilde{\mathbb{A}}^n$ is regular if $P$ is the reduced ideal of a regular subvariety $Z$ of $\mathbb{A}^n$. For $Z = 0$ the origin in the real plane $\mathbb{R}^2$, we obtain (cum grano salis) for $\widetilde{\mathbb{A}}^2$ the Möbiusband in $\mathbb{R}^3$ (cf. fig. 3).

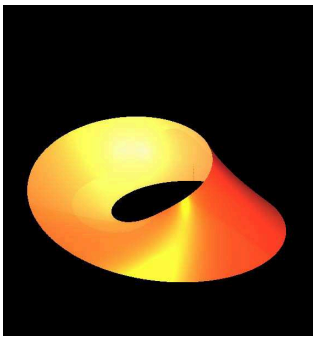

*Figure* 3: The Möbiusband, blowup of $\mathbb{A}^2$ with center a point.

**Exercises.** Writing down the above introductory examples with all details is almost mandatory for being able to enjoy the remaining material. There are also more complicated examples where everybody can try out its resolution instinct.

*Exercise* 2: To warm up, we start with some computations of blowups of affine space.

*Exercise* $2^a$: Determine the affine chart expressions for the blowup map $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ for $n = 2$ and $Z = 0$, $n = 3$ and $Z = 0$, $n = 3$ and $Z = V(x, y) = z$-axis. Show that in the

last case, the blowup map $\pi$ is the cartesian product of the blowup map in the first case with the identity map on the $z$-axis.

*Exercise* $2^b$: Take various curves in $\mathbb{A}^2$ and $\mathbb{A}^3$ (e.g., $y^2 = x^3 - x$, $y^2 = x^2 - x^3$, $y^2 = x^3$ or the parametrized curves $(t^2, t^3, t^4)$ or $(t^3, t^4, t^5)$) and compute their pullbacks under the previous blowups, both projectively and in the affine charts. Distinguish the locations of the intersections of the "pullbacks" of $xy = 0$, $x^2 - y^2 = 0$, $xy(x^2 - y^2) = 0$ with the exceptional divisor $E$ (cf. fig. 4).

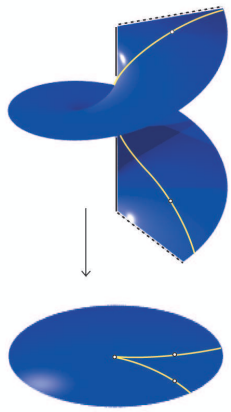

*Figure* 4: Blowup of $\mathbb{A}^2$ with center a point and transform of cuspidal curve.

*Exercise* $2^c$: Show that the ideals $(x, y)$, $(x, y)^2$ and more generally $(x, y)^k$ define the same blowup of $\mathbb{A}^2$ when taken as center.

*Exercise* $2^d$: If the center is a hypersurface, the resulting blowup map is an isomorphism.

*Exercise* $2^e$: Compute the point blowup of the ideal $(x, y^2)$ in $\mathbb{A}^2$. Show that $\widetilde{\mathbb{A}}^2$ is singular. Resolve its singularities by a further blowup.

*Exercise* $2^f$: Compute the blowup of the ideal $(x, y^2)(x, y)$ in $\mathbb{A}^2$. Show that $\widetilde{\mathbb{A}}^2$ is regular.

*Exercise* $2^g$: Let $\widetilde{\mathbb{A}}^2$ be the blowup of $\mathbb{A}^2$ with center the origin. Let $a \in \widetilde{\mathbb{A}}^2$ be the point corresponding to the direction of the line $y = 0$ in $\mathbb{A}^2$ (make this precise). Blow it up. You get a composition of two blowups, say $\tau : \widehat{\mathbb{A}}^2 \to \mathbb{A}^2$. Show that it is the blowup of $\mathbb{A}^2$ with center an ideal (not reduced) supported by the origin. Determine this ideal. Show that it is not unique.

*Exercise* $2^h$: Compute the blowup of $\mathbb{A}^3$ with center the circle $x^2 + y^2 = 1$, $z = 0$. Make sure to compute all affine charts. Show that $\tilde{\mathbb{A}}^3$ is regular.

*Exercise* $2^i$: Show that the exceptional divisor $E$ of a blowup is indeed a hypersurface. Compute its equation in various circumstances.

*Exercise* $2^j$: Blow up the "arithmetic line" $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x])$ in the ideal $(x, p)$, $p \in \mathbb{Z}$ a prime. What could be the "affine" equations of $\widetilde{\mathbb{A}}^1_{\mathbb{Z}}$?

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*Exercise* 3: We next try to resolve some singularities: The double-cone $x^2+y^2 = z^2$ in $\mathbb{A}^3$ should not pose to many problems, as well as the Dingdong surface $x^2 + y^2 = z^2(1 - z)$ (figure out first how this creature looks like?). More interesting is the Whitney-umbrella $x^2 - y^2z = 0$ in $\mathbb{A}^3$. The singular locus is the $z$-axis. Again, it is worth to try first a simpler example, namely

*Exercise* $3^a$: The cylinder over the cusp $x^2 - y^3$ in $\mathbb{A}^3$. The equation is $x^2 - y^3 = 0$, but considered in three variables. What is the correct blowup to be applied? The question applies to any varieties which are cartesian products (cf. fig. 5).

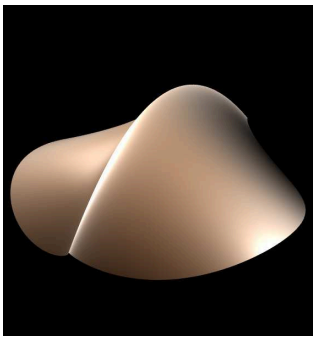

*Figure* 5: Plop, the cartesian product of a cusp with the parabola.

*Exercise* $3^b$: A somewhat more delicate example is the Limão $x^2 = y^3z^3$ in $\mathbb{A}^3$ (cf. fig. 6), which appears as a modification of the equation of the Whitney-umbrella $x^2 = y^2z$ (cf. fig. 7). The singular locus consists of the $y$- and the $z$-axis; it has therefore a singular point at the origin. What is the correct choice of center of blowup? Can you preserve by your blowups the symmetry between $y$ and $z$.

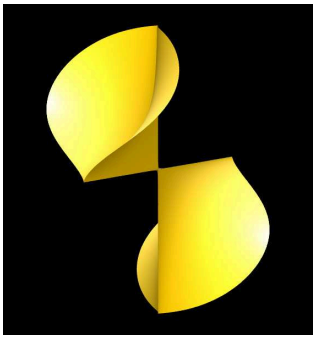

*Figure* 6: Limão, of equation $x^2 - y^3z^3 = 0$.

*Exercise* 4: Let $f$ be any polynomial in three variables, and denote by $o$ its order of vanishing at 0, i.e., the order of the Taylor expansion at 0. Blow up the origin in $\mathbb{A}^3$ and consider the strict transform $f'$ of $f$ at a point $a'$ of $\widetilde{\mathbb{A}}^3$ above $a$ (i.e., take the pullback $f^*$ of $f$ and factor from it the maximal power of the equation defining the exceptional divisor). Determine the cases where the order $o'$ of $f'$ at $a'$ has remained constant.

## 2. Choosing the centers of blowup

We now turn to the problem of selecting for each singular variety a suitable center along which the ambient space and the variety shall be blown up. Recall that the blowup map is an isomorphism over the complement of the center, whereas it contracts the exceptional divisor, which is a hypersurface, to the center (which could be rather small). Therefore, our variety will be modified only along the center. And indeed, at its regular points we have no reason to change it, they cannot be improved any further. So will agree to choose our centers *inside the singular locus* of the variety.

This choice, however, is not a must, and there are several papers where the centers are chosen to stick out of the singular locus and even out of the variety. The defect is due to a lack of suitable stratifications of the variety, namely ones whose smallest stratum is regular and could thus be chosen as center. And in fact, allowing singular centers is quite a delicate job, see [Ha 2].

For the moment we will accept solely centers inside the singular locus of the variety. This requirement is known as the *economy of the resolution process*: Only points are modified where it is really necessary, because they are singular. Of course, in the resolution process, each singular point has to belong once in a while to a center of blowup, otherwise it would remain singular forever.

Let us then observe whether our singularities improve under the chosen blowups.

We shall concentrate on surfaces, because they represent a nice playground to develop some of the most prominent ideas. In contrast, the reader should know that dimension three is *the* critical dimension for bad phenomena to happen, especially but not exclusively in positive characteristic. Quasi all "counterexamples" to properties one could hope for are formulated for three-folds.

Let us start with isolated singularities. In this case there is no ambiguity how to choose the center. It should coincide with the finitely many singular points, so it suffices to consider one of these.

*Example* 4: The double-cone $X$ given by $x^2 + y^2 - z^2 = 0$ in $\mathbb{A}^3$. The only singularity sits in the origin. Blowing it up gives three charts, with a symmetry between the $x$- and $y$-chart. In the $x$-chart we obtain $x^2(1 + y^2 - z^2) = 0$, in the $z$-chart $z^2(x^2 + y^2 - 1) = 0$. Again, we will omit the exceptional monomial factors $x^2$ and $z^2$, so that the interesting equations are $1 + y^2 - z^2 = 0$ and $x^2 + y^2 - 1 = 0$. All three charts of the blowup $X'$ are regular.

The exceptional divisor $E$ of $\pi : \tilde{\mathbb{A}}^3 \to \mathbb{A}3$ is the two-dimensional projective space $\mathbb{P}^2$, and $X'$ meets $E$ entirely in the $z$-chart (so we can discard the other charts). The intersection is the circle $x^2 + y^2 - 1 = 0$, which shows that $X'$ is indeed the cylinder (because outside 0 we did not alter $X$). We recover our example from the beginning of the first lecture. Check that the blowup map $\pi : X' \to X$ is precisely the contraction described there.

*Example* $4^a$: Let us now change coordinates. As a first variation of the preceding example, "rotate" the double-cone by taking for $X$ the equation $x^2 - yz = 0$, with a symmetry between $y$ and $z$. The equation is obtained from the previous example by substituting there $y - z$ by $y$ and $y + z$ by $-z$. The equations for $X'$ are now in the $x$-chart $1 - yz = 0$ and in the $y$-chart $x^2 - z = 0$. Therefore, the intersection of $X'$ with $E$ is no longer contained in one of the three affine charts of $E$. This becomes relevant when computing local invariants of $X'$ at points of $E$. The fewer the charts are that have to be considered, the shorter are the computations.

As a general observation, the choice of suitable coordinates (suitable with respect to different intentions as e.g. achieving small complexity or defining coordinate independent data) is a very subtle point and marks the entire literature on the subject.

*Example* $4^b$: What happens if we do not take the origin of the cone as center, but a whole line through the origin? For simplicity we take $X$ given by $x^2 - yz = 0$. There are two types of lines through 0, those contained in $X$ and those which are transversal to $X$ (in an heuristic sense, meaning simply that they are not contained in $X$). Let first the center $Z \subseteq X$ be the $z$-axis of ideal $(x, y)$. The transform $X'$ of $X$ has equation (replace $(x, y)$ by $(x, yx)$ respectively $(xy, y)$ and leave $z$ untouched) $x - yz = 0$, respectively $x^2y - z = 0$. It is hence non-singular, but the global geometry seems to be somewhat twisted. We leave it to the ambitious (or curious) reader to figure out whether $X'$ is again the cylinder.

(Glueing affine charts may be tedious. We will encounter this task at many more occasions, and a certain routine will be helpful.)

Let us now take as center $Z \not\subseteq X$ the "transversal line" of ideal $(y, z)$, i.e., the $x$-axis. By symmetry, we only have to consider the $y$-chart. The transform $X'$ is given there by $x^2 - y^2z = 0$, which is the Whitney-umbrella mentioned at the end of the first lecture. It has a whole line as singular locus, and the origin is by all means the worst singularity on this surface. It would be difficult to qualify $X'$ as being less singular than the double-cone: Our blowup did not improve the singularities – it made them *worse*.

This comes a bit as a surprise, but cannot be avoided: If the center leaves the variety, or if it remains inside but is too small, the singularities may become more involved (further examples still to come). Moreover, in our example, the singularities are no longer isolated. We will see later that for instance the normality of surfaces need not persist under point blowups. This has a simple meta-mathematic interpretation: Normal surface singularities are isolated, but the variety has really to squeeze around these points in an intricate way in order not to create singular curves. Now, when considering blowups of normal surfaces, we think of all secants going through regular points and the (isolated) singular point. And

there may appear in the limit – when the regular points come close to the singular one – very complicated configurations of secants.

In particular, as a side-product, we see that the blowup of $X$ in a center $Z$ not contained in $X$ does not coincide with its blowup with center the intersection $Z \cap X$ (neither for the set-theoretic nor for the scheme-theoretic intersection $Z \cap X$). There is no way to expect commutativity of blowups with respect to restricting the center to $X$.

*Exercise* 1: Prove this accurately. What happens if you take as center the scheme-theoretic intersection $Z \cap X$?

Fortunately, many other and very practical commutativity properties do hold, we will list and prove them in detail. The first one appears in the next example.

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*Example* 5: Let us take for $X$ two transversal planes in $\mathbb{A}^3$, say of equation $x^2 - y^2 = 0$, considered in three variables. So $X$ is the cartesian product of (or the cylinder over) the union $C$ of two transversal lines in $\mathbb{A}^2$ with regular factor the $z$-axis, say $X = C \times \mathbb{A}^1$. We have two options:

(i) Blow up the origin in $\mathbb{A}^2$, get $\pi : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$ and then take $\pi \times \mathrm{id}_{\mathbb{A}^1} : \widetilde{\mathbb{A}}^2 \times \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$. This is certainly an honest attempt, because $X$ is a cartesian product, a fact which should be reflected by the transformation we choose.

(ii) The second option is to blow up $\mathbb{A}^3$ along the singular locus of $X$, i.e., with center the $z$-axis. The ideal is $(x, y)$.

It is easily checked (especially, if you have done some of the earlier exercises), that both options are valid and resolve $X$, yielding for $X'$ two separate (parallel) planes in $\mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{A}^1$, respectively $\mathbb{A}^3 \times \mathbb{P}^1$. And as you probably suspected by what was said before, the two options coincide (not hard to be verified). Perfect! We state explicitly our first functorial property:

*Blowups commute with taking cartesian products with regular factors.*

More explicitly: The blowup of $\mathbb{A}^n$ along a coordinate subspace $Z$ equals the cartesian product of the point-blowup in a transversal subspace $V$ of $\mathbb{A}^n$ (of complementary dimension) with the identity on $Z$.

*Exercise* 2: Prove this in all generality without referring to affine charts but by recalling that the blowup is the Zariski-closure of a graph.

*Exercise* $3^a$: This sounds nice, so will immediately test it in a concrete situation. Compute the blowup of $\mathbb{A}^3$ with center the circle $Z$ of ideal $(x^2 + y^2 - 1, z)$. You should at least remember here that this coincides with an earlier exercise. If you don't remember, you can almost be sure that you did not do this exercise. What are the affine chart expressions of the blowup map? Next, define a surface whose singular locus is precisely this circle. What is its transform under the blowup?

*Exercise* $3^b$: Blow up the plain circle $Z$ from before inside the surface $Z \times \mathbb{A}^2$.

*Exercise* 4: Let us take for $X$ the cylinder over the cusp, given by the equation $x^2 - y^3 = 0$ in $\mathbb{A}^3$. You should have no problems in defining the center and the induced resolution of $X$ by one blowup. But how does $X'$ intersect the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{A}^1$? Are you pleased how they meet? If yes, go back to the blowup of the curve $x^2 - y^3$ in $\mathbb{A}^2$ and contemplate your satisfaction. If not, what do you suggest to remedy your discomfort?

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To relax briefly and to prepare the field for new exercises, another definition of the blowup of affine space $\mathbb{A}^n$ in an ideal $P = I_Z = (g_1, \ldots, g_k)$ of $K[x_1, \ldots, x_n] = K[x]$ will be given. This time, we indicate the affine coordinate rings of $\widetilde{\mathbb{A}}^n$ as subrings of the quotient field $K(x)$. The $i$-th one is given as

$$K[x_1, \ldots, x_n, g_1/g_i, \ldots, g_k/g_i].$$

*Exercise* 5: Determine how these rings have to be glued in order to define $\widetilde{\mathbb{A}}^n$.

*Exercise* 6: Computations are fun, so show with this definition that $P = (x, y^2)$ in $\mathbb{A}^2$ and $P = (x, yz)$ in $\mathbb{A}^3$ give singular blowups $\widetilde{\mathbb{A}}^2$ and $\widetilde{\mathbb{A}}^3$, whereas $P = (x, y^2)(x, y)$ in $\mathbb{A}^2$ and $P = (x, yz)(x, y)(x, z)$ in $\mathbb{A}^3$ give regular ones. You may consult [Ha 2].

*Exercise* 7: This is a nice little exercise$^+$ in discrete or toric geometry. Determine the monomial ideals in $K[x_1, \ldots, x_n]$ which produce a regular blowup of affine space.

*Exercise* 8: This is a possibly nasty exercise$^{++}$ in discrete geometry. Determine for monomial ideals in $K[x_1, \ldots, x_n]$ a natural and simple multiplication procedure by other monomial ideals (with smaller exponents) such that the entire product (which may have many many generators) produces a regular blowup of affine space (not too hard for $n = 2$ or $n = 3$).

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We are back to singularities.

*Exercise* 9: Here comes the first serious example, the Whitney-umbrella $x^2 - y^2 z = 0$ in $\mathbb{A}^3$. Though easy, it shows many of the decisive features, so we will study it with patience. You are invited to carry out all computations in detail.

This surface has non-isolated singularities along the $z$-axis. Locally in the euclidean topology, at points on the $z$-axis but off the origin, it looks like two transversal planes (this is not the case not in the Zariski-toplogy), which reminds us example 5. At the origin, the surface bends around, making 0 strikingly more singular than the other points on the $z$-axis.

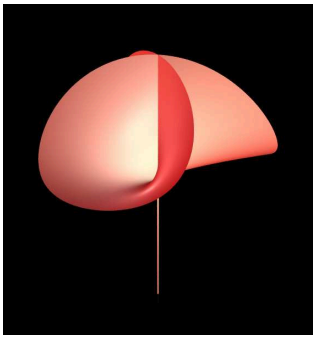

*Figure* 7: The Whitney-umbrella $x^2 = y^2 z$.

According to the philosophy that the worst singularities should be taken care of first, we blow up the origin with reduced ideal $(x, y, z)$ in $\mathbb{A}^3$. We get three affine charts and a new surface $X'$ of respective equations:

$x$-chart: $1 - xy^2 z = 0$, no intersection with exceptional divisor $E$.

$y$-chart: $x^2 - yz = 0$, our old acquaintance the double-cone, which, for any point of view, should be simpler than the Whitney-umbrella. Note that the intersection of $X'$ with $E$ is the $z$-axis (with multiplicity 2).

$z$-chart: $x^2 - y^2 z = 0$, and – surprise – the same singularity we had before blowing up pops up again at the origin of this chart. The singularity has survived our attack without damage (cf. fig. 8). The intersection of $X'$ with $E$ is now the $y$-axis (again with multiplicity 2).

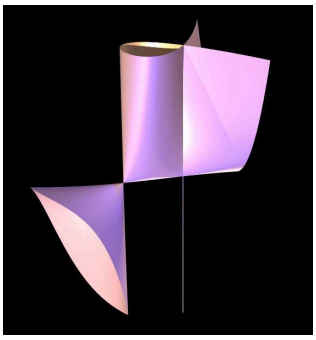

*Figure* 8: The Pirat, point blowup of the Whitney-umbrella.

*Tautology* : If the singularity remains the same, there is no way to declare it less singular than the original one.

*Conclusion* : Our choice of center was not appropriate (if we believe that resolutions exist). We should have probably better taken a larger center.

*Parenthesis* : We have cheated here a little bit. Even though the singularity was preserved in the $z$-chart, the situation has improved in a certain (and not yet revealed) sense. And, actually, all existing resolution algorithms and their implementations choose for the Whitney-umbrella the origin as the first center of blowup. The reason for this "misbehaviour" of the algorithms will be explained later.

*Exercise* 10: In the $y$-chart we have an isolated singularity at the origin (the one of the double-cone), in the $z$-chart we have a non-isolated singularity (the one of the Whitney-umbrella). Explain this strange occurence. Think again of the configuration of secant lines along the $z$-axis of $X$. How do the three charts patch?

*Exercise* 11: (mandatory) Show that blowing up the $z$-axis in $\mathbb{A}^3$ resolves the Whitney-umbrella in one step. Could this be expected geometrically before doing the calculation?

*Exercise* $12^a$: Resolve Himmel & Hölle $x^2 - y^2z^2 = 0$ (cf. fig. 9).

*Exercise* $12^b$: Resolve Limão $x^2 - y^3z^3 = 0$.

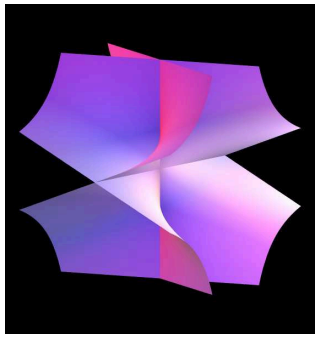

*Figure* 9: Himmel & Hölle, of equation $x^2 - y^2z^2 = 0$.

## 3. Transforms of ideals

Now that we have got an approximate idea how blowups are defined and how they may affect singularities, let us see more systematically how various types of objects transform under blowup. These objects can be polynomials or hypersurfaces, ideals or subvarieties, parametrizations of curves and surfaces, jacobian ideals of polynomials or singular loci of varieties, sum and product decompositions of ideals, coefficient ideals, coordinate systems, automorphisms, group actions, local flags, local invariants, vector fields, differential forms, etc.

Given is a blowup $\pi : W' \to W$ with center $Z \subset W$ of ideal $P$ in $\mathcal{O}_W$ and exceptional divisor $E = \pi^{-1}Z$. We shall mostly assume that $W$ is equal to affine space $\mathbb{A}^n$, so that $P$ is an ideal in $K[x_1, \ldots, x_n]$, with $Z = V(P)$ regular. We wish to *lift* a given object $\mathcal{B}$ in $W$ to an object $\mathcal{B}'$ in $W'$ in a natural and significant way.

There are two options: Either we take the usual pullback $\mathcal{B}^*$ of $\mathcal{B}$ under $\pi$. Or, more subtly, we restrict $\mathcal{B}$ to $W \setminus Z$, get an object $\mathcal{C}$ on $W \setminus Z$, apply to the restriction $\mathcal{C}$ the inverse $\varphi^{-1}$ of the isomorphism $\varphi : W' \setminus E \to W \setminus Z$ induced by $\pi$ and get an object $\mathcal{C}'$ on $W' \setminus E$, which we can then try to extend to an object $\mathcal{B}'$ defined on whole $W'$. If the original object is defined locally on $W$ at a point of $Z$, the lifted object may only exist locally on $W'$ at certain points of $E$.

When realizing this procedure algebraically, we shall fix for convenience (local or affine) coordinates $x_1, \ldots, x_n$ on $W$, and assume that $Z$ is defined by the ideal $P = (x_1, \ldots, x_k)$ for some $k \leq n$. Moreover, we will place ourselves w.l.o.g. in the $k$-th affine chart of $W'$ so that the blowup map $W' \to W$ has chart expression $\pi : \mathbb{A}^n \to \mathbb{A}^n$ sending $x_i$ to $x_i$ if $i \geq k$ and to $x_i x_k$ if $i < k$. The exceptional divisor $E$ has equation $x_k = 0$ in this chart. Of course, our setting specifies a point $a \in Z$ and a point $a'$ above $a$ in $E$ by considering the respective origins of the charts. Note here that the location of $a'$ on $E$ depends on the choice of the coordinates $x_1, \ldots, x_n$ at $a$.

*Hint*: It is much simpler to treat the chart expression of the blowup map as a map of affine space onto *itself* with the same coordinates on source and image rather than to consider two different affine spaces with different coordinate systems.

In any case, we have not specified yet what we mean by coordinates. For the moment it suffices to take any regular system of parameters of $K[x_1, \ldots, x_n]$, later on we shall mostly work in the local context with regular parameters of the formal power series ring $K[[x_1, \ldots, x_n]]$. In the second case, coordinate changes (i.e., $K$-algebra-automorphisms of $K[[x]]$) are easier to handle.

Our study of transforms begins geometrically. Take a subvariety $X$ of $W$ (no component of $X$ should be contained in the center $Z$). How does it lift to $W'$? We follow the recipe described above: Consider $U = X \setminus Z$ with preimage $U' = \pi^{-1}(U) = \pi^{-1}(X \setminus Z)$ in $W' \setminus E$. Clearly, $\pi : U' \to U$ is an isomorphism. We are interested in extending $U'$ over the exceptional divisor $E$. Notice here that the way how $U'$ approaches $E$ may be substantially different from the way how $U$ approaches $Z$. By our assumption, $U$ is Zariski-dense in $X$, i.e., the points of $X$ inside $Z$ are "limit" or "boundary" points of $U$.

It is then more than natural to define $X'$ as the Zariski-closure of $U'$ in $W'$, i.e., to add to $U'$ those points of $E$ to which the points of $U'$ tend in the limit when approaching the exceptional divisor. We call $X'$ for the moment the *geometric transform* of $X$. It is contained in the *total transform* $X^* = \pi^{-1}(X)$ of $X$, which is just the whole inverse image of $X$ under $\pi$ and therefore contains the entire exceptional divisor $E$. So the irreducible components of $X'$ will be precisely the components of $X^*$ which are different from $E$. Clearly, $X'$ is the geometrically interesting object, whereas $X^*$ is easier to work with algebraically.

It will be useful to provide an algebraic description of $X'$ in terms of the ideal defining $X$ in $W$ (the ideal of $X'$ will be the *strict* or *proper transform* of the ideal of $X$, and $X'$ will inherit this name).

So let $I$ be the ideal defining $X$ in $W$. Assume in a first instance that $I$ is principal, i.e., that $X$ is a hypersurface given by a polynomial equation $f = 0$. Let $f^*$ be the pullback of $f$ to $W'$. In the $k$-th chart, $E$ is defined by $x_k = 0$, and $f^*$ is again a polynomial. As $E$ is an irreducible component of $X^*$ (we shall always assume that $Z$ is irreducible), $f^*$ must factor into a power of $x_k$ and another polynomial, say $f'$, not divisible by $x_k$,

$$f^* = x_k^r \cdot f'.$$

So $r$ is the maximal power to which $x_k$ can be factored from $f^*$.

*Exercise* 1: Show that the exponent $r$ of $x_k$ does not depend on the chart. It is called the exceptional multiplicity of $f^*$ or $X^*$. Determine its precise value in terms of $f$.

*Exercise* 2: Verify your formula for $r$ in case $Z = V(x, y, z, u)$ in $\mathbb{A}^6$ and

$$f = x^{33} + y^{21}z^{13} + u^{35}v^{11}w^{11} + x^8y^8z^8u^8v^8w^8 \in K[x, y, z, u, v, w].$$

We can thus write globally $f^* = m_E^r \cdot f'$ where $m_E = 0$ is the reduced equation of $E$ in $W'$. As we just stated, $m_E$ is locally in the affine charts a monomial in one of the variables. Now it is (almost) evident that the polynomial $f'$ defines $X'$ in $W'$. Said differently, the geometric transform $X'$ of $X$ has equation $f' = m_E^{-r} \cdot f^* = 0$ in $W'$.

*Exercise* 3: Prove this with rigor over any ground field, including finite ones.

We call $f'$ the *strict transform* of $f$. If $X$ is not a hypersurface in $W$, things become more complicated. Let again $I$ be the ideal of $X$ in $W$.

*Fact*: The geometric transform $X'$ of $X$ under blowup of $W$ with center the ideal $P$ is defined by the ideal

$$I' = \bigcup_{s \in \mathbb{N}} m_E^{-s} \cdot (I \cap P^s)^*.$$

This formula breaks with the leisurely going style we have gotten used to. So let's have a closer look at it: The intersection $I \cap P^s$ simply means that you have to take all total transforms (i.e., pullbacks) $f^*$ of elements $f$ of $I$ which vanish at least with order $s$ along $Z$; then factor from each of them the $s$-th power of (the equation of) the exceptional divisor (you know that this is possible, because you have done exercise 1 above). Finally take the union over $s$. The resulting ideal defines $X'$. Not too difficult, isn't it.

*Exercise* 4: Prove the fact.

Everybody will agree that the above description of $I'$ is not very handy for practical purposes. The curious reader may ask: "Does it suffice to consider here a generator system of $I$?" – and the answer will be an exercise.

*Exercise* 5: Let $I = (y^2 - xz, yz - x^3, z^2 - x^2y) = (f, g, h) \subset K[x, y, z]$ define the monomial curve $X$ in $\mathbb{A}^3$ of parametrization $(t^3, t^4, t^5)$. Blow up the origin and prove that $I'$ is generated by the strict transforms $f'$, $g'$ and $h'$ of $f$, $g$ and $h$. What are the respective exceptional multiplicities?

*Exercise* $5^a$: By the way, does this blowup resolve $X$? If not, resolve it by further blowups. Is the resolved curve transversal to the exceptional divisor? If not, achieve transversality by still further blowups.

*Exercise* $5^b$, $5^c$, $5^d$: Compute the Nash modification of $X$, then the toric resolution and finally the normalization.

*Exercise* $5^e$: Define directly the transform of a parametrized curve $\mathbb{A}^1 \to W$. Then compute it in case of $(t^3, t^4, t^5)$.

*Exercise* 6: In exercise 5, we saw that the strict transforms of the three generators of the defining ideal suffice to describe the geometric transform of the curve. Try to prove this in all generality for the geometric transform of subvarieties $X$ in $\mathbb{A}^n$ defined by ideals of $K[x_1, \ldots, x_n]$.

*Exercise* $7^a$: Verify your proof in the case of the plane curve $X$ in $\mathbb{A}^3$ defined by $x = 0$ and $y^2 - z^2 = 0$ (caution: the curve is embedded in three-space). What are the respective multiplicities?

*Exercise* $7^b$: Verify your proof in the case of the curve $X$ in $\mathbb{A}^3$ which is the intersection of the two surfaces $S_1 : x^2 - y^3 = 0$ and $S_2 : xy - z^3 = 0$. Be sure to understand completely how the geometric transform $X'$ of $X$ looks like, in particular, what are its irreducible components. Then do the same for the intersection of the geometric transforms of the two surfaces $S_1$ and $S_2$. How do $S_1'$ and $S_2'$ meet the exceptional divisor $E$? Finally, compute in all cases the respective exceptional multiplicities.

*Exercise* 8: Show now that your proof from exercise 6 was erroneous. Then try to fix it by modifying the original assertion on how to compute the strict transform of an ideal in terms of generators.

Here is the answer: Let $I$ be an ideal in $K[x_1, \ldots, x_n]$, and let $f_1, \ldots, f_q$ be a (local) *Macaulay basis* of $I$, i.e., a generator system whose initial homogeneous forms (i.e., of minimal degree) generate the ideal of *all* initial forms of elements of $I$. Then the strict transform $I'$ of $I$ under the blowup of $\mathbb{A}^n$ in a regular center is generated by the strict transforms of $f_1, \ldots, f_q$:

$$\langle f_1, \ldots, f_q \rangle' = \langle f_1', \ldots, f_q' \rangle$$

with $f_i' = m_E^{-\mathrm{ord}_P f_i} \cdot f_i^*$. You may want to check this on your own or look it up in chapter III of Hironaka's Annals paper from 1964. There, the statement is proven in the local context for formal power series. Such generator system were originally called by Hironaka *standard bases*. Only later, the notion of nowadays standard bases (with respect to initial monomials selected by a monomial order on $\mathbb{N}^n$) was introduced.

Standard bases served Hironaka to compute the strict transform of ideals, but more decisively he used them to define his local resolution invariant $\nu_a(I)$ as the vector of orders of the Taylor expansions at the point $a$ of a minimal standard basis. To compare the invariant before and after blowup, it was necessary to dispose also of a standard

basis of the transformed ideal (which is not automatic) and to relate it to the one below. To ensure this, he introduced the concept of *reduced* standard basis (whose existence proof anticipated, at least implicitly, his famous division theorem). Nice thing: Reduced standard bases are persistent under blowup.

The definition of the invariant suggests an immediate refinement: Instead of the vector of the orders of the Taylor expansions of a reduced standard basis, take directly the vector given by the initial monomials of the initial ideal of $I$ with respect to a chosen monomial order. The monomials will be listed increasingly. This vector can be made coordinate independent by taking either generic or most specific (formal) coordinates [Ha 5]. It represents a substitute for $\nu_a(I)$ and also for the more popular Hilbert-Samuel function. Advantage. It is much easier to work with. And, by choosing different monomial orders, it offers more flexibility and information.

*Note*: It seems to be an open problem how to describe the strict transform of a $\mathcal{D}$-module in terms of generators (where the strict transform is defined again by restricting first to $W \setminus Z$ and taking then an extension from $W' \setminus E$ to $W'$).

## 4. Properties of blowups

Blowups can be introduced axiomatically by a universal property. This goes as follows. Let $X$ be any scheme, and let $E$ be a subscheme of $X$. Then $E$ is a *Cartier divisor* in $X$ if, locally on $X$, $E$ is defined by a non-zero divisor $f$ in the structure sheaf of $X$: For any $a \in X$ there is an affine neighborhood $U = \operatorname{Spec} R$ of $a$ in $X$ such that $E \cap U$ equals the subscheme $V(f)$ of $U$ defined by a non-zero divisor $f \in R$.

If $X$ is regular, a Cartier divisor is just a (closed) hypersurface of $X$. If $X$ is singular, the notion is more subtle.

*Exercise* 1: Let $E$ be one of the axes of the cross $X = V(xy)$ in $\mathbb{A}^2$. Is it a Cartier divisor? Same question for $E$ the origin.

*Exercise* 2: Let $E$ be the reduced origin of $X = V(xy, x^2)$ in $\mathbb{A}^2$. Show that $E$ is not a Cartier divisor.

*Exercise* 3: Any line through 0 inside the cone $X = V(x^2 + y^2 - z^2)$ in $\mathbb{A}^3$ is not a Cartier divisor of $X$ (though, locally at points different from the origin, it is).

*Exercise* 4: Are $E = V(x^2)$ in $\mathbb{A}^1$ and $E = V(x^2y)$ in $\mathbb{A}^2$ Cartier divisors?

It can be shown that a hypersurface $E$ of $X$ is a Cartier divisor of $X$ if and only if it is rare in $X$, i.e., if and only if $X \setminus E$ is (scheme-theoretically) dense in $X$. The examples of the exercises show that this property depends in part on the location of $E$ with respect to the (embedded) components of $X$.

Here is the universal property of blowups. The blowup of $X$ in a closed subscheme $Z$ is a morphism $\pi : X' \to X$ such that $E = \pi^{-1}(Z)$ is a Cartier divisor of $X'$ and such that

for any morphism $\tau : Y \to X$ with $\tau^{-1}(Z)$ a Cartier divisor in $Y$ there exists a unique morphism $\rho : Y \to X'$ with $\tau = \pi \circ \rho$.

If it exists, the blowup is unique (up to unique isomorphism). The existence can be proven by different methods, constructing suitably the scheme $X'$ and the morphism $\pi$. For affine schemes, our method with the closure of the graph of the map defined by the equations of $Z$ does the job. There is a more general way to construct the blowup of $X$ as the Proj of the Rees-Algebra $\oplus_{k=0}^{\infty} P^k$ where $P$ denotes the ideal sheaf defining $Z$ in $X$.

The blowup of a regular scheme $X$ in a closed hypersurface $Z$ is an isomorphism, by the universal property. This is no longer true if $X$ is singular and/or $Z$ is not a Cartier divisor.

*Exercise* 5: Show that the blowup of the cone $X = V(x^2 - yz)$ in $\mathbb{A}^3$ along the line $Z = V(x, y)$ is an isomorphism locally at all points outside 0, but not globally on $X$.

*Exercise* 6: Blow up the fat point $X = V(x^2)$ in $\mathbb{A}^2$ in the (reduced) origin $Z = 0$. What do you get?

*Exercise* 7: Blow up the subscheme $X = V(x^2, xy)$ of $\mathbb{A}^2$ in the reduced origin.

*Exercise* 8: Blow up the subscheme $X = V(xz, yz)$ of $\mathbb{A}^3$ first in the origin, then in the $x$-axis. At which points are the resulting morphisms local isomorphisms?

The universal property, as usual, is helpful when proving statements about blowups. The most important is the commutativity of blowups with base change. Many special features of blowups follow from this property.

*Base change.* Let $\pi : X' \to X$ be the blowup of $X$ with center $Z \subset X$, let $\varphi : Y \to X$ be a morphism (the base change) and let $V = \varphi^{-1}(Z)$ be the preimage of $Z$ in $Y$. Consider the fibre product $\widetilde{Y} = Y \times_X X'$ of $Y$ and $X'$ over $X$ with induced diagram

$$\begin{array}{ccc} \widetilde{Y} & \to & X' \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

where $\widetilde{Y} \to Y$ and $\widetilde{Y} \to X'$ denote the canonical projections. Let $Y'$ be the Zariski-closure of the preimage of $Y \setminus V$ in $\widetilde{Y}$. Then the induced morphism $\tau : Y' \to Y$ is the blowup of $Y$ along $V$.

We prove the base change property after a few remarks and exercises.

*Special cases*: Taking in the above situation as base change the inclusion $Y \subset X$ of an open subscheme, we find that blowups can be constructed from local data by gluing blowups of open subsets: Given an open covering $\{U_i\}$ of $X$ the blowups of $U_i$ with center $Z_i$ glue to a global blowup of $X$ if the $Z_i$ patch on the overlaps $U_i \cap U_j$ thus defining a global center $Z$ in $X$.

Also, it follows that blowups are compatible with taking localizations at a point (or at a closed subscheme) and completions. For $a \in Z$ and $a'$ a point of $E$ above $a$, say $\pi(a') = a$, we call the induced local morphism $(X', a') \to (X, a)$ corresponding to the inclusion of the local rings $\mathcal{O}_{X,a} \to \mathcal{O}_{X',a'}$ or their completions the local blowup of $X$ at $a$ and $a'$ with center $Z$.

Taking $Y \subset X$ a closed subscheme of a regular scheme $X$ and a center $Z$ inside $Y$, we recover the earlier notion of strict transform of $Y$: The strict transform of $Y$ under the blowup of $X$ in $Z \subset Y$ coincides with the blowup of $Y$ in $Z$.

If $Z$ is regular but not contained in $Y$, the intersection $V = Z \cap Y$ may be singular and non-reduced. It then yields the blowup of $Y$ in a center which is no longer regular.

*Exercise* 9: Take the cone $Y : x^2 - yz = 0$ in $X = \mathbb{A}^3$ and as center the line $Z : y = z = 0$. The intersection $V = Z \cap Y$ is the origin with ideal $P = (\overline{y}, \overline{z})$ in $\mathcal{O}_Y$, the bars denoting residues. Compute the blowup of $Y$ in $V$ (compare with exercise $4^b$ of chapter II).

*Exercise* 10: Explain, why $\mathcal{O}_{X,a} \to \mathcal{O}_{X',a'}$ is an inclusion of rings. Then take $X$ and $Z$ regular, $a \in Z$, and express the local blowup with respect to (suitably chosen) regular systems of parameters of $\mathcal{O}_{X,a}$ and $\mathcal{O}_{X',a'}$.

We may also use the base change property of blowups for field extensions.

*Exercise* 11: Let $Y$ be the real points of the complex surface $X : x^4 - y^4 - yz^2 = 0$ in $\mathbb{A}^3_{\mathbb{C}}$. Let $Z \subseteq X$ be the curve $x^2 + y^2 = z = 0$. Compare the blowups of $X$ in $Z$ and of $Y$ in $Y \cap Z$. Then do the same for the center defined by the ideal $(x^2+y^2, z)(x^2+y^2, xz, yz, z^2)$.

*Exercise* 12: Consider $L = \mathbb{F}_2$ as a subfield of $K = \mathbb{F}_{32}$, and let $X$ and $Y$ be the surfaces defined by $x^2 + y^3 + z^4 = 0$ in $\mathbb{A}^3_K$, respectively $\mathbb{A}^3_L$. How many points have the blowups of $X$ and $Y$ with center the origin? Do these resolve the surfaces in one blowup?

*Exercise* 13: Let $K = \mathbb{C}((t))$ be the ring of formal Laurent series in one variable. Let $X : tx^2 - y^3 = 0$ be the "cusp" in $\mathbb{A}^2_K$. Blow up the origin and compute $X'$.

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*Proof.* The proof of the base change property goes by diagram chasing. So let $\varphi : Y \to X$ be given and let $\tau : U \to Y$ be a morphism for which $\tau^{-1}(V)$ is a Cartier divisor in $U$. As $(\varphi\tau)^{-1}(Z) = \tau^{-1}(V)$, the universal property of the blowup $X' \to X$ implies that $\varphi\tau : U \to X$ factors through $X' \to X$ via some morphism $\rho : U \to X'$. By the universal property of fibre products, we get from $\tau$ and $\rho$ a morphism $\sigma : U \to \widetilde{Y} = Y \times_X X'$. Its composition with the projection to $Y$ equals $\tau$. We have to show that $\sigma$ maps $U$ into $Y'$. By construction, $\sigma$ maps $\tau^{-1}(Y \setminus V)$ into $Y'$. As $\tau^{-1}(Y \setminus V)$ is dense in $U$ and $Y'$ is closed in $\widetilde{Y}$, we conclude that $\sigma$ maps $U$ into $Y'$, thus getting the required factorization of $\tau$.

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We constructed earlier, via graphs, the blowup $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ of affine space $\mathbb{A}^n$ in a closed subscheme $Z$. It was done by choosing generators $g_1, \ldots, g_k$ of the ideal $P$ defining $Z$ in

$\mathbb{A}^n$, taking the morphism $g : \mathbb{A}^n \setminus Z \to \mathbb{P}^{k-1}$ with projective components $(g_1 : \ldots : g_k)$ and the Zariski-closure $\widetilde{\mathbb{A}}^n$ of its graph in $\mathbb{A}^n \times \mathbb{P}^{k-1}$. So let us establish the universal property for $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$. It will in particular show that, up to isomorphism, the construction does not depend on the choice of the generators $g_1, \ldots, g_k$.

We have already seen that $E = \pi^{-1}(Z)$ is a Cartier divisor, its affine equations were $g_i = 0$ (check this again accurately). Take a morphism $\tau : Y \to \mathbb{A}^n$ with $\tau^{-1}(Z)$ a Cartier divisor in $Y$. We wish to show that $\tau$ factors through $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ via a morphism $\rho : Y \to \widetilde{\mathbb{A}}^n$. The morphism $\tau g : Y \setminus \tau^{-1}(Z) \to \mathbb{P}^{k-1}$ lifts to a morphism $\widetilde{\rho} : Y \setminus \tau^{-1}(Z) \to G$ into its graph $G \subset Y \times \mathbb{P}^{k-1}$, which, by construction, is contained in the graph of $g : \mathbb{A}^n \setminus Z \to \mathbb{P}^{k-1}$. Taking closures on both sides and using that $Y \setminus \tau^{-1}(Z)$ is dense in $Y$, we get the desired morphism $\rho : Y \to \widetilde{\mathbb{A}}^n$.

This proves the existence of the blowup of $\mathbb{A}^n$ in a closed center $Z$. From this, by the base change property for subschemes, we get the existence of blowups of arbitrary affine schemes with centers closed subschemes. The gluing property of blowups then implies the existence of blowups for arbitrary schemes.

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Let us summarize the advantages and drawbacks of our five equivalent definitions of blowups.

*Universal property*: Very handy for proofs and general properties of blowups.

*Rees-algebra and* Proj: Most general explicit construction. Allows to read off global and geometric properties of the blowup as regularity, normality, Cohen-Macaulayness, etc. by applying methods of ring theory and commutative algebra. Very efficient if center is monomial ideal. Somewhat unpleasant for local computations.

*Closure of graph*: Works for affine schemes. Geometrically intuitive. Allows explicit computations for regular centers. Obsolete if the ideal of center has large number of (complicated) generators.

*Affine charts expression of morphism*: Best suited for local computations if center is regular. Allows often to choose privileged coordinates for which the local blowup is given by a monomial substitution of the variables.

*Equations of blowup*: Works for affine schemes $X \subseteq \mathbb{A}^n$ and their blowup $X' \subset \mathbb{A}^n \times \mathbb{P}^{k-1}$. Both, affine or projective coordinates and equations may be taken.

---

We now come to more geometric properties of schemes and how they behave under blowup.

*Exercise* 14: Let $X$ be a regular scheme, and $Z$ a regular closed subscheme. Show that the blowup $X'$ of $X$ along $Z$ is again regular.

Blowups preserve also normal crossings singularities, provided the center is sufficiently transversal. We make this more precise. Two subschemes $X$ and $Z$ of a regular ambient scheme $W$ are said to be *transversal* if the scheme defined by the product of the ideals

of $X$ and $Z$ in $W$ is a normal crossings scheme (we do not take the intersection of the ideals).

*Exercise* 15: Show that a regular subscheme $Z$ of a regular scheme $X$ meets $X$ transversally.

*Exercise* 16: Let $X$ be a surface in $\mathbb{A}^3$ consisting of two regular components $X_1$ and $X_2$ meeting transversally. Let $Z$ be a regular surface in $\mathbb{A}^3$ which is transversal to $X_1$ and $X_2$. Show that $Z$ need not be transversal to $X$.

*Exercise* 17: Let $X$ be a normal crossings subscheme of $\mathbb{A}^n$, and let $Z$ be a regular closed subscheme which is transversal to $X$. Show that the blowup $X'$ of $X$ along $Z$ is again a normal crossings scheme (cf. fig. 10).

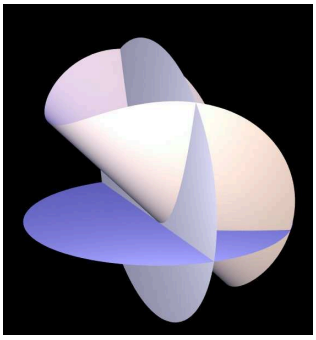

*Figure* 10: Tülle, non-normal crossings surface.

*Exercise* 18*: Let $X$ be a *mikado scheme* in $\mathbb{A}^n$, i.e., a scheme which is a union of regular components all whose intersections are regular (scheme theoretically). Example: A finite union of *linear* subspaces $V_i$ in $\mathbb{A}^n$ so that $V_i \setminus \bigcup_{j \neq i} V_j$ is dense in $V_i$ for all $i$. Show that there is a sequence of blowups over $\mathbb{A}^n$ which transforms $X$ in a normal crossings scheme (taking always total transforms). Write a program for this and determine its complexity. Compare with the papers of De Concini, Procesi [DCP] and Feichtner, Koslov [FK].

*Exercise* 19: Let $X' \to X$ be the blowup of any $X$ in a chosen ideal $P$. Find three other ideals which yield the same blowup.

*Exercise* 20*: Let $\pi : (W', a') \to (W, a)$ be a local blowup of a regular scheme $W$ along a regular center $Z$ (you may choose first $Z$ to be the point $a$). It should be clear what is meant by a local flag $\mathcal{F}$ of regular subschemes in $W$ at $a$. Assume that $Z$ is transversal to $\mathcal{F}$ (in the obvious sense). Show that $\mathcal{F}$ lifts in a natural way to a flag $\mathcal{F}'$ in $W'$ at $a'$. (Hint: Do first the point blowups of $\mathbb{A}^2$ and $\mathbb{A}^3$ in the case where the flags are given by coordinate subspaces. You may wish to consult [Ha 5] for the general case.)

*Exercise* 21: Now choose local coordinates on $W$ at $a$ for which the local blowup from before is given by a monomial substitution of the variables (you may work in the completions of the local rings). Show that the coordinates can be chosen so that the flag in $W$ is given by coordinate subspaces. Then determine the equations of the induced flag in $W'$.

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There are two other important existential properties of blowups: The composition of blowups is again a blowup (in a more complicated center), and any birational projective morphism is a blowup. We refer to Bodnár's paper [Bo] for the first and to Hartshorne [Hs, p. 166] for the second.

*Exercise* 22: Blow up $\mathbb{A}^3$ in the normal crossings center $Z$ of ideal $P = (xy, z)$, getting a scheme $\widetilde{\mathbb{A}}^3$ with an isolated singularity (locally, it is a three-dimensional cone). Show that the blowup of this point in $\widetilde{\mathbb{A}}^3$ yields a regular scheme. Then find an ideal in $\mathbb{A}^3$ which, when taken as center, defines the composition of the two blowups. (Hint: you may find an appropriate candidate for the ideal in one of the exercises of the other chapters.)

*Exercise* 23: Same as before with the irreducible center $P = (x^2 - y^2 - y^3, z)$.

*Exercise* 24**: Let $P$ be an ideal in $X = \mathbb{A}^n$, inducing the blowup $\pi : X' \to X$. Determine explicitly an ideal $R$ in $X$ such that the blowup $\tau : X'' \to X$ of $X$ in the product ideal $P \cdot R$ yields a regular scheme $X''$ and such that $\tau$ factors through $\pi$, i.e., $\tau = \pi\rho$ with $\rho : X'' \to X'$. In particular, the morphism $\rho$ will define a resolution of the singularities of $X'$. (A decent solution to exercise 24 in non-trivial cases is probably worth a publication in an equally decent journal.)

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## 5. Improving singularities by blowups

One of the roles of blowups in algebraic geometry is to eliminate successively the singularities of a given variety. As this cannot be done by just one blowup (actually, it could be done, it is just that we don't know how to choose a suitable center), a whole series of different blowups is necessary. Some invariant shall document that the singularities become better with each blowup. This relies on a numerical measure of the complexity of a singularity. We therefore ask: Which number tells us the "distance" of a singular point from being a non-singular point? And: Does this number drop under blowup?

There are many proposals for such a measure, and each fullfils the expectation of certain objectives. In our context, we shall confine in a first instance to measures which, at least, do not deteriorate under blowup. This is not yet sufficient for the induction argument. Nevertheless, it is an important step towards it, because in most cases the measure will drop, and the remaining cases are so special that they can be treated by another type of induction.

A suitable numeric measure will be called a resolution invariant. By this we mean that the measure belongs to a well ordered set $\Gamma$, typically $\mathbb{N}^n$, that it is attached in an upper-semicontinuous way to each point $a$ of a variety $X$, and that it does not depend on any choices: It is invariant under local isomorphisms of $X$ at $a$. Said differently, it depends only on the local ring $\mathcal{O}_{X,a}$ or its completion. This is sometimes expressed by saying that the invariant is intrinsic. Morever, if the center of blowup is chosen suitably (this will have to be specified), it decreases with respect to the ordering on $\Gamma$.

We have already seen in the exercises that regular varieties remain regular under blowup if the center itself is regular and transversal to or included in the variety. Which number expresses regularity? Of course, this is a local notion. By the implicit function theorem, a point $a$ on a hypersurface is regular, if and only if the order of vanishing of the defining polynomial $f$ at $a$ is 1. This order is just the order of the Taylor expansion of $f$ at $a$, which we will denote by $\mathrm{ord}_a f$.

For non-hypersurfaces the situation is trickier: Measuring regularity involves the dimension of the vartiety as well as the jacobian matrix of the defining equations. Hironaka used his vector $\nu$ mentioned in the last chapter. Also the local multiplicity or the Hilbert-Samuel function can be considered as invariants. For the sake of simplicity, we shall restrict in the sequel to hypersurfaces. Non-hypersurfaces are technically a little bit more complicated, but all the interesting phenomena already appear for hypersurfaces. This justifies our restriction.

*Observation*: A regular hypersurface remains regular under blowup if we consider its strict transform. This does not tell us how the transform meets the exceptional divisor, in particular, if the intersection is transversal. We could instead consider the total transform of the hypersurface, because it contains also this piece of information. But: The total transform will never be a regular variety, even if the strict transform is regular. The best we can hope for is to achieve a normal crossings singularity.

Now, it's not difficult to prove that if we blow up a regular center which is (sufficiently) transversal to the variety, a normal crossings singularity has total transform which has again only normal crossings singularities (see exercise 17 in chapter IV). This suggests to measure at a singular point instead of the distance to regularity the distance to having a normal crossings point. Algebraically, this is the distance of an ideal of polynomials from being a monomial ideal, possibly in other (formal) coordinates.

*Quandary*: A significant and genuine invariant indicating the distance of a singularity from having normal crossings at a given point has not been discovered yet.

So there is homework to do. Be aware that it may not be easy: The invariant should be intrinsic, upper-semicontinuous, and it should not increase under blowup (if the center is chosen with care). A first candidate goes as follows: Factor from your polynomial the monomial of maximal degree, over all choices of coordinates. Then take the order of the remaining factor at the point in question.

This is what (almost) everybody does, though it is rather clumsy. Essentially, it boils down to considering the order of the strict transform of the hypersurface. Main drawback: It does not work well in positive characteristic, cf. [Ha 1, Ha 4]. *End of observation.*

By lack of any better, we shall stick to the order of a polynomial and of its strict transform as our main (though very rough) measure of singularity. And the question is: Does it never increase under blowup? Some examples will lift the fog.

*Example* 1: The plane curve $X : x^m - y^k = 0$ with $m < k$ of order $m$ at 0. We blow up the origin. The tangent cone of $X$ (given by the homogeneous form of lowest degree of the defining polynomial) at 0 is the $y$-axis $x = 0$ (would not hold for $m = k$). We are therefore led to consider in the exceptional divisor $E$ only the $y$-chart, and there the point $x = 0$, i.e., the origin (prove this with care, cf. exercise 4 of chapter I). The strict transform $X'$ has equation $x^m - y^{k-m} = 0$ there. Ah! If $m < 2k$, the order has dropped to $k - m < m$, and otherwise it has remained constant equal to $m$. Easy.

*Example* 2: The surface $X : x^m - y^k z^\ell = 0$ in $\mathbb{A}^3$ with $m \leq k$ and $\ell$ of order $m$ at 0. Which blowup is suited? With center the origin? Or a line? Let us do both.

First the origin. The tangent cone of $X$ at 0 is the $yz$-plane $x = 0$ (because of $m < k+\ell$), it hence suffices to consider the $y$- and the $z$-chart of the blowup and there only the points of $E$ with $x$-coordinate equal to 0. In the $x$-chart, $X'$ does not meet the exceptional divisor. And outside of $E$, the order must have remained the same since the blowup is an isomorphism there. By symmetry, we only consider the $y$-chart with strict transform $X'$ given by $x^m - y^{k+\ell-m} z^\ell = 0$. As $k + 2\ell > 2m$, the order has remained constant equal to $m$ at the origin of $E$. Observe that $X'$ intersects $E$ in the $y$-chart along the $z$-axis and has order $m$ at each of its points. This shows that at no point of $E$ the order has increased. Fine!

So let's blow up the line $x = y = 0$, i.e., the $z$-axis. In the $y$-chart, the equation of $X'$ is $x^m - y^{k-m} z^\ell = 0$ since $m \leq k$. The order has remained constant at the origin and did not increase at the other points (check this carefully at all points of the chart and determine the respective orders). Fine again! What about the $x$-chart? It suffices to look there at the origin, the other points being covered already by the $y$-chart. The equation of $X'$ is $1 - y^{k-m} z^\ell = 0$ and $X'$ does not pass through the origin of this chart (the order is 0 there). Done!

*Example* 3: As the exercise-experienced reader already suspected, these examples are misleading. Take the preceding surface $X : x^m - y^k z^\ell = 0$ but with exponents $k < m \leq \ell$. We assume that $m \leq k + \ell$ so that $X$ has order $m$ at 0. We blow up the $z$-axis $x = y = 0$. In the $x$-chart we get $X'$ defined by $x^{m-k} - y^k z^\ell = 0$ of order $\leq m$. In the $y$-chart the equation of $X'$ is $x^m y^{m-k} - z^\ell = 0$. The order at the origin equals the minimum of $2m - k$ and $\ell$. As both $2m - k$ and $\ell$ are larger than $m$, the order has increased. Bad news!

But why did this happen? There is no apparent reason for the increase. But our algebraic manipulations tend to hide the internal structure of the singularities.

Zariski was one of the first – if not the first – to notice and understand this phenomenon. Actually, it is not very complicated: In examples 2 and 3, compare the behaviour of $X$ along the points of the $z$-axis $Z$, which was our chosen center. The relative size of the exponents $k$, $\ell$ and $m$ comes into play.

If $m \leq k, \ell$, the order of $X$ along $Z$ is constant equal to $m$, because $f = x^m - y^k z^\ell$ belongs to the $m$-th power of the ideal $(x, y)$ defining $Z$ (it suffices to have $k \geq m$). If $k < m \leq \ell$, this is not the case. Indeed, the order at 0 is $m$ but at points of $Z$ outside the origin it is $k < m$. We have a drop of the order when leaving the origin.

We look closer at the total transform $X^*$ of $X$ at the origin of the $y$-chart. In both cases it is given by $f^* = x^m y^m - y^k z^\ell = 0$. The maximal power of the exceptional variable $y$ which we can factor from $f^*$ depends on the relative size of $k$ and $m$. If $k < m$ we can factor only a power $y^k$ whose exponent $k$ is *smaller* than the order $m$ of $f$ at 0. Therefore, the strict transform $f'$ has order which is *larger* than this order. If $k \geq m$, we can factor the whole power $y^m$.

You may notice that we saw this type of question already in exercise 1 of chapter III.

*Exercise* 1: Do example 3 with center the origin.

**Fact**: *The order of a variety does not increase at any point of its strict transform if we blow up a regular center along which the variety has constant order. In particular, regular points remain regular.*

We shall establish the fact for hypersurfaces only. Let $Z \subset X$ be the center, $X$ defined in $W = \mathbb{A}^n$ by $f = 0$, and $\mathrm{ord}_a f = \mathrm{const}$ for $a \in Z$. Let $\pi : W' \to W$ be the induced blowup, and $X'$ the strict transform of $X$ in $W'$. Choose a point $a \in Z$ and a point $a'$ above $a$, i.e., so that $\pi(a') = a$. Set $o = \mathrm{ord}_a f$ and $o' = \mathrm{ord}_{a'} f'$. We wish to prove that $o' \leq o$.

This is done by specifying very nice coordinates at $a$ and $a'$. With these, the proof becomes almost automatic, so that the details can be grouped in a series of exercises. We may work in the completion of the local rings of $W$ and $W'$ at $a$ and $a'$, because completion does not alter the order. Local shall always mean formally local, i.e., working in $K[[x_1, \ldots, x_n]]$.

*Exercise* 2: Choose local coordinates $x_1, \ldots, x_n$ at $a$ so that $a = 0$ and $Z$ is defined by $x_1 = \ldots = x_k = 0$, where $k = \mathrm{codim}_W Z$.

*Exercise* 3: Permute the first $k$ coordinates so that $a'$ belongs to the $x_k$-chart of the blowup $W'$.

*Exercise* 4: Apply a lower triangular linear coordinate change in $W$ (thus preserving the ideal defining $Z$ in $W$) so that $a'$ becomes the origin of the $x_k$-chart.

*Exercise* 5: Show that in these coordinates, the local chart expression of $\pi$ at $a'$ and $a$ is given by a monomial substitution of the variables (the usual one, see chapter III).

*Exercise* 6: Show that in the coordinates chosen at $a$, the expansion of $f$ has order $\geq o$ with respect to the variables $x_1, \ldots, x_k$ (i.e., belongs to the ideal $(x_1, \ldots, x_k)^o$).

*Exercise* 7: Use this to show that the exceptional monomial $x_k^o$ can be factored from the total transform $f^*$ of $f$.

*Exercise* 8: Now use the fact that $(x_1, \ldots, x_k)^o$ is the maximal power to which $f$ belongs to show that $f'$ has order at most $o$ with respect to all variables, i.e., $f'$ does not belong to $(x_1, \ldots, x_n)^{o+1}$.

*Exercise* 9: Conclude that $o' \leq o$, say $\mathrm{ord}_{a'} f' \leq \mathrm{ord}_a f$.

*Question*: This proof is very explicit, though not really conceptual. Where did you use that the order of $f$ is constant along $Z$?

*Exercise* 10: For those familiar with the Rees algebra of blowups, find a less computational and more direct proof of the fact. Does it work also for singular centers?

*Exercise* 11: We have already seen that the total transform of a normal crossings singularity has again normal crossings, provided that the center is sufficiently transversal to the variety. Prove this again by exhibiting an invariant which measures the distance from being normal crossings and which does not increase under blowup.

*Exercise* 12: You may choose at your taste 27 different lines and 15 different planes in $\mathbb{A}^3$, all passing through the origin, no line contained in a plane. Then modify their union by blowups and taking total transforms until you get a variety with normal crossings. Intuitively, it is clear that this should work in finitely many steps. So write a computer program for it. What is the minimal number of blowups (with regular centers) needed?

## 6. The induction argument of resolution

Starting with this chapter, pace and level will go up. But before doing so, one and a half examples.

*Example* 1: Once again our lemon-tree $x^2 - y^3 z^3 = 0$. The order at 0 is 2, and this is also the case at the points of the $y$- and $z$-axis. The locus of maximal order is therefore the cross formed by these two axes. Hesitating which axis would be better as our center of blowup we take, to be on the sure side, their intersection the origin 0. The point blowup gives in the $y$-chart the transform $x^2 - y^4 z^3$ (the $z$-chart is symmetric, and the $x$-chart is irrelevant). The order has remained constant equal to 2 at the origin of the $y$-chart, not too bad, but the polynomial itself has become worse as the degree of the monomial in $y$ and $z$ has increased. How to build up an induction on this? Unclear.

*Example* $1\frac{1}{2}$: Hardly enchanted by the starter, we opt for an uncommon way to pursue: We complicate the example, taking instead the hypersurface $X$ in $\mathbb{A}^6$ defined by $x^3 - y^\alpha z^\beta u^\gamma v^\delta w^\varepsilon = 0$. The exponents $\alpha$, $\beta$, $\gamma$, $\delta$ and $\varepsilon$ may be quite arbitrary and are only subject to $\alpha+\beta+\gamma+\delta+\varepsilon \geq 3$ so as to have a singularity of order 3 at 0 (the actual value 3 does not matter here). Let us not care about symmetry when choosing the center. We just take any regular subvariety of the locus $L$ of order 3 of $f$. But we require that it has maximal possible dimension among all regular subvarieties of $L$. By the shape of $L$ in the present situation, $Z$ will be a coordinate subspace. To fix ideas, assume that we choose for $Z$ the $vw$-plane of equations $x = y = z = u = 0$.

There are four charts. The $x$-chart can be discarded by what we have already learned, and up to a permutation we may restrict our attention to the $y$-chart. There, the total transform $X^*$ is defined by $x^3y^3 - y^{\alpha+\beta+\gamma} z^\beta u^\gamma v^\delta w^\varepsilon = 0$. The strict transform is given by factoring the maximal power of $y$, which is the minimum of 3 and $\alpha+\beta+\gamma$. As $X$ has, by the very choice of the center, order 3 along $Z$, we have $\alpha+\beta+\gamma \geq 3$ and the minimum is 3. We get $X'$ of equation $x^3 - y^{\alpha+\beta+\gamma-3} z^\beta u^\gamma v^\delta w^\varepsilon = 0$, which we write as

$$x^3 - y^{\alpha-(3-\beta-\gamma)} z^\beta u^\gamma v^\delta w^\varepsilon = 0.$$

Well, where is the point in all this? It is as follows, and this will be decisive: Up to now, we have not used the hypothesis that $Z$ has *maximal possible dimension* in $L$. This assumption implies that the three-dimensional coordinate-subspace $S$ defined by $x = y = z = 0$ is not contained in $L$. Hence $f$ does not have order 3 along $S$, which, in turn, signifies that $\beta+\gamma < 3$. Therefore, in the equation of $X'$, the degree of the monomial $y^{\alpha-(3-\beta-\gamma)} z^\beta u^\gamma v^\delta w^\varepsilon$ has *decreased*. Of course, this is a general phenomenon, which we now state explicitly.

*Exercise* 1: Let $f = x^m - y_1^{\alpha_1} \cdots y_n^{\alpha_n} = x^m - y^\alpha$ be a binomial with $|\alpha| = \alpha_1 + \ldots + \alpha_n \geq m$. Then $\mathrm{ord}_0 f = m$. Let $L$ be the locus of points $a$ where $\mathrm{ord}_a f = m$, and let $Z$ be a coordinate subspace of $L$ of maximal possible dimension. Blow up $Z$ in $W = \mathbb{A}^{1+n}$ with strict transform $f'$. Then at all points $a'$ of the exceptional divisor $E$ either the order of $f'$ has dropped, or the affine chart expression of $f'$ is again a binomial $f = x^m - y_1^{\alpha_1'} \cdots y_n^{\alpha_n'} = x^m - y^{\alpha'}$ with a monomial $y^{\alpha'}$ of total degree strictly smaller than the degree of $y^\alpha$, say $|\alpha'| < |\alpha|$.

*Exercise* 2: Conclude from this that finitely many blowups in maximal dimensional centers succeed in decreasing the order of binomials $f$ as above.

Neglecting in this argument symmetry considerations is harmless as long as we work locally at the point in question (though the equivariance of the resolution with respect to group actions, say symmetries, will be lost). Globally, the center $Z$ you may have chosen may return after some promenade inside $X$ to the respective point from a different direction and thus form a normal crossings center there (recall that the center must be closed, cf. figure 11). Therefore we would have a singular center. This problem can be circumvented by applying first auxiliary blowups which help to separate the local

components of possible (big) centers $Z$. For the moment, we do not discuss this topic any further.

*Exercise* 3: Construct a surface $X$ in $\mathbb{A}^3$ whose locus $L$ of order 2 is the node $Y$ : $x^2 - y^2 - y^3 = z = 0$. Show that blowing up $\mathbb{A}^3$ in $Z = Y$ produces a singular variety. Next, blow up the origin $Z = 0$ in $\mathbb{A}^3$, and show that the locus $L'$ of the strict transform $X'$ has become either regular or a union of regular components intersecting transversally (produce examples of $X$ for both occurences). In the second case, show that there exists a resolution of $X$ (or, equivalently, of $X'$) which preserves all local and global symmetries.

*Exercise* 4: Is it possible to equip $Y$ with a non-reduced structure $Y'$ so that the blowup of $\mathbb{A}^3$ with center $Z = Y'$ is regular?

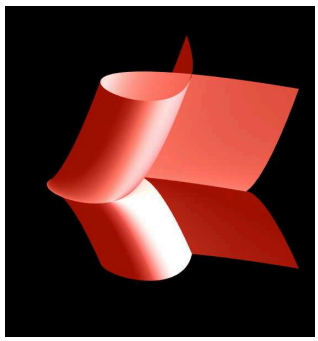

*Figure* 11: Quaste, cartesian product of cusp with node.

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Let us look more closely at the order function: Given a variety $X$ in a regular ambient space $W$, we stratify $X$ according to the local order $o = \mathrm{ord}_a I$ of the defining ideal $I = I_W(X)$ of $X$ in $W$. Similarly as for hypersurfaces, $\mathrm{ord}_a I$ denotes the maximal power of the maximal ideal of $\mathcal{O}_{W,a}$ at $a$ which contains $I$. More generally, for $S$ closed, we define $\mathrm{ord}_S I$ as the maximal power of the ideal $I_W(S)$ of $\mathcal{O}_W$ defining $S$ in $W$ which contains $I$. It can be shown that $\mathrm{ord}_S I = \min_{a \in S} \mathrm{ord}_a I$.

Note that, locally at a point $a$, the variety may be singular though the order at $a$ is 1. This happens when $X$ is not minimally embedded locally at $a$. In particular, the order depends on the embedding of $X$ in the ambient space $W$. (We are somewhat imprecise of what we understand by locally at a point: To avoid ambiguities and delicacies, you may always take the completed local rings of the variety and its ambient space. This is ok because the order is invariant under completion.)

As the order is an upper-semicontinuous function (you may convince yourself by some examples or try to prove it rigorously), the stratum $\mathrm{top}(X)$ of points where the order attains its maximum is closed. We call it the *top locus* of $X$. Recall here that a function on a topological space with values in a totally ordered set is upper-semicontinuous if the subsets of points where the function takes values $\geq$ a given value are closed.

The top locus can be singular, so we cannot take it directly as center. Instead, we may choose any regular closed subvariety $Z$ of $\mathrm{top}(X)$, preferably of maximal possible dimension. We have seen before that this choice may be non-canonical, for instance in the case where $X$ has a symmetry so that $\mathrm{top}(X)$ has two regular components intersecting transversally. Then it will be up to us to choose one of these components. This problem of choice may appear locally as well as globally.

In the published proofs of resolution in characteristic 0, this ambiguity is resolved by prescribing ab initio a uniquely determined center, similarly as in exercise 3 from above.

So let the center $Z$ inside $\mathrm{top}(X)$ be chosen, take the induced blowup $\pi : W' \to W$ of the ambient space and consider the total transform $X^*$ of $X$ in $W'$. Let $a$ be a point of $Z$, and let $a'$ be any point of $E = \pi^{-1}(Z)$ above $a$. For reasons to be explained later, we set $o = \mathrm{ord}_a X = \mathrm{ord}_Z X$ and define the *weak transform* $X^\curlyvee$ of $X$ as the ideal $I^\curlyvee = I_E^{-o} \cdot I^*$. Here, $I^*$ is the total transform (= pullback) of $I$ in $W'$ and $I_E$ is the principal ideal defining $E$ in $W'$. If you wonder whether the weak transform is well-defined (i.e., that the ideal $I_E^o$ can indeed be factored from $I^*$), you missed to do one of the earlier exercises.

*Exercise* 5: Determine the geometric difference between weak and strict transform. Exhibit this difference in three significant examples. After that you should have recognized that working algebraically with the weak transform is more pleasant than with the strict transform.

**Fact**: *The order of the weak transform does not increase under blowup,*

$$\mathrm{ord}_{a'} I^\curlyvee \leq \mathrm{ord}_a I$$

*for* $a \in Z \subset \mathrm{top}(I)$ *and* $a' \in E$.

The proof is similar to the hypersurface case.

*Observation* 1: At points $a'$ where $\mathrm{ord}_{a'} I^\curlyvee < \mathrm{ord}_a I$ induction applies to deduce that after finitely many steps the order of the weak transform has dropped to 0.

*Exercise* 6: Why 0 is better than 1?

Having order 0 implies that the total transform of $X$ has become a normal crossings variety supported by the exceptional components, provided the centers have always been choosen transversal to the exceptional locus (so that the next exceptional locus is indeed a normal crossings divisor).

*Observation* 2: At points $a'$ where $\mathrm{ord}_{a'} I^\curlyvee = \mathrm{ord}_a I$ induction does not apply. But: These *equiconstant* points are rare. In fact, they always lie inside a regular hypersurface $F = H' \cap E$ of $E$, where $H'$ is a regular hypersurface in $W'$ transversal to $E$ (cf. exercise 4 of chapter I).

This can be seen by relating the equiconstant points to the tangent cone of $X$ along $Z$. Actually, if $\tau$ is the minimal number of variables necessary to define the tangent cone of

$X$ at $a$, the equiconstant points lie inside a regular subvariety of codimension $\tau$ in the fiber $E_a \subseteq E$ of $\pi$ over $a$.

*Exercise* 7: Prove this.

What does this help, if there do exist points in $X'$ where the order did not decrease? How to apply any other type of induction?

To attack and overcome this intricacy, we consider two examples, a trivial and an almost trivial one.

*Example* 2: Let $X$ be a fat point in $\mathbb{A}^1$, say the origin, defined by $x^m = 0$ (from now on, we will allow also non-reduced varieties). You would think there is not so much to do with this fat point geometrically, but algebraically it is of some interest. Let's blow up the (reduced) origin in $\mathbb{A}^1$. This is an isomorphism $\pi : \mathbb{A}^1 \to \mathbb{A}^1$, with $E = Z = 0$. Nevertheless, the weak transform $X^\curlyvee$ of $X$ differs from $X$, because it is defined by the ideal $I_E^{-m} \cdot I^* = (x^{-m}x^m) = (1)$. So $X^\curlyvee$ is empty and the order has dropped to zero.

*Conclusion*: In ambient dimension 1, there arise no equiconstant points.

*Example* 3: What about dimension 2, say plane curves? We take again $x^m - y^k = 0$ with $k \geq m$, of order $m$ at 0. We blow up the origin. If $k < 2m$, the order drops at all points of $E \cong \mathbb{P}^1$. If $k \geq 2m$ there is precisely one point in $E$ where the order remains constant, namely the origin of the $y$-chart. There, the equation of the weak transform is $x^m - y^{k-m} = 0$. The exponent of the second monomial has dropped. How to profit of this drop?

The answer is to descend to the one-dimensional case from before. Intersect $X$ in $\mathbb{A}^2$ and $X^\curlyvee$ in the $y$-chart of $W'$ with the hyperplane $x = 0$. You get the ideals $I_1 = (y^k)$ respectively $I_1' = (y^{k-m})$ in one-dimensional affine space $\mathbb{A}^1$. There, the order has dropped. Nevertheless, this could just be by chance.

Now comes the clue: Possibly, $I_1'$ is the transform of $I_1$ under the blowup of 0 in $\mathbb{A}^1$ as in example 1. Then the drop would have been forced, and hopefully lend itself to a similar argument in higher dimensions.

Obviously, $I_1'$ is not the strict transform of $I_1$ (which is the ideal (1)) nor the total transform. But $I_1'$ can be obtained from the total transform $I_1^*$ of $I_1$ by factoring a *prescribed* power of the exceptional component. More precisely, $I_1' = I_E^{-m} \cdot I_1^*$ with $m = \mathrm{ord}_Z I$ and $I_E$ the principal ideal of the exceptional divisor $E$. Observe that $m \leq \mathrm{ord}_0 I_1 = k$. This leads to defining the *controlled transform* $I^!$ of an ideal $I$ with respect to $c \in \mathbb{N}$ under the blowup with center $Z$ more generally as $I^! = I_E^{-c} \cdot I^*$ for any *control* $c \leq \mathrm{ord}_Z I$.

Let us conceptualize this very rough proposal in a diagram for ideals $I$ in a regular ambient space $W$. Let $W' \to W$ be the blowup with center $Z$ contained in $\mathrm{top}(I)$. Let $H$ be a hypersurface of $W$ containing $Z$, with induced blowup $H' \to H$.

*Exercise* 8: Show that $H'$ is the strict transform of $H$ under the blowup $W' \to W$. Moreover, it coincides with the weak transform of $H$.

Assume now that we dispose of a sophisticated construction which associates to $I$ an ideal $I_-$ in $H$, i.e., in one variable less. We will work here always locally at a given point $a \in Z \subset H$. For plane curves this was just the passage from $I$ to $I_1$. The commutative diagram

$$\begin{array}{ccc} W' & \supset & H' \\ \downarrow & & \downarrow \\ W & \supset & H \end{array}$$

gives rise to an incomplete diagram

$$\begin{array}{ccc} I^{\curlyvee} & \rightsquigarrow & I^{?}_{?} \\ \downarrow & & \downarrow \\ I & \rightsquigarrow & I_- \end{array}$$

Here, the lower curly arrow $\rightsquigarrow$ denotes the descent in dimension, associating to $I$ the ideal $I_-$. In the upper horizontal row, we would like to replace $I^{?}_{?}$ in the upper right corner by the ideal $(I^{\curlyvee})_-$ associated to $I^{\curlyvee}$. Similarly, in the right vertical column, we would like to replace $I^{?}_{?}$ in the upper right corner by the controlled transform $(I_-)^!$ of $I_-$. So we have two choices for $I^{?}_{?}$ to complete the diagram, and we don't know which one is better. Best would be if we had equality $(I^{\curlyvee})_- = (I_-)^!$, because then the diagram would *commute*.

Unfortunately, equality does not hold. In general, the two ideals have nothing to do with each other, $(I^{\curlyvee})_- \neq (I_-)^!$.

*Exercise* 10: Verify this by a concrete example.

Yet another clue: Recall that in $E$, we are only interested in equiconstant points $a'$, say $\mathrm{ord}_{a'} I^{\curlyvee} = \mathrm{ord}_a I$. Locally at such points, there exists a hypersurface $H'$ in $W'$ which contains all equiconstant points. As $H'$ is transversal to $E$, its image $H$ in $W$ under $\pi$ is again regular, locally at $a$.

**Fact**: *Choosing such $H$ and $H'$, then, for any equiconstant point $a'$ above $a$, we have the desired equality $(I^{\curlyvee})_- = (I_-)^!$.*

Hence, at equiconstant points, we may write $I'$ for $I^{\curlyvee}$ and $I'_-$ for $(I^{\curlyvee})_- = (I_-)^!$ and get the commutative diagram

$$\begin{array}{ccc} I' & \rightsquigarrow & I'_- \\ \downarrow & & \downarrow \\ I & \rightsquigarrow & I_- \end{array}$$

This is really very pleasant. Therefore we repeat it: If the order of the weak transform did not decrease, we may descent in dimension if we allow there the controlled transform instead of the weak transform. In $H$ and $H'$, we may apply induction on the dimension on

any assertion we would like to prove – provided it holds in dimension 1 and is compatible with descent.

This principle is called – for obvious reasons – *cartesian induction.*

The descent in dimension can be performed in various ways and has been a common procedure in resolution of singularities since the work of Jung. Geometrically, one often takes a generic projection to a hypersurface and there the discriminant, see the survey of J. Lipman [Lp 3] for more details. Algebraically, Hironaka introduced the coefficient ideal as an ideal in one variable less. Although it is less suggestive than the discriminant it is easier to work with. All known proofs in characteristic 0 and for arbitrary dimension use one or the other form of coefficient ideals, cf. [EH] or [EV].

Encouraged by the perspective of nice commutative diagrams we apply induction on the dimension to show that we can resolve $I_-$ inside $H$ by a sequence of blowups with centers in $H$. If we are lucky, the centers are also contained in $\mathrm{top}(I)$ so as not to increase the order of $I$. Stop! The ideal $I_-$ does not pass in this process to its strict or weak transform, but to its controlled transform, and, as we have seen many times, its order may *increase.*

This is not a big deal, in view of what we learned in the 1.5 introductory examples of this chapter. There, the ideal $I_-$ corresponded to the monomial $y^\alpha$, and we saw that once $I_-$ is a monomial (if defined correctly), the drop of the order of $I$ can be forced by choosing centers of maximal dimension. So, in the general situation, we will try to transform $I_-$ into a monomial – instead of making the associated variety regular. With this new objective in mind, taking the controlled transform of $I_-$ makes no harm: we just factor after each blowup the maximal exceptional power $I^r_{(E\cap H')}$ from $I^!_-$, so that the remaining factor is the weak transform of $I_-$. Now, induction applies: Its order will drop eventually to 0. Therefore, vice versa, the final controlled transform of $I_-$ will have become a monomial.

In the course of this argument it is essential that the centers are chosen transversally to the already existing exceptional locus, otherwise the normal crossings property of the exceptional divisor is not ensured.

*Pause*: Before proceeding, please recapitulate what was said, read the last paragraphs again in order to digest well the various aspects of the reasoning.

*Exercise* 11: Check whether the reader followed the preceding advice.

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Why do we need a break here? Simply, because there is another obstruction to come, the most serious one. If this (last) one would not occur, we would have outlined a proof of resolution of singularities in arbitrary characteristic, despite the fact that characteristic $p > 0$ is still wide open.

So where is the trap we have overseen? It lies in the choice of the local hypersurface $H$. We have found $H$ by projecting the hypersurface $H'$ of $W'$ (which is not unique) down

to $W$, getting a regular hypersurface $H$ there. Then we take $I_-$ in $H$ and lift it to $H'$, getting $I_-^!$.

Now, most probably, the order of $I$ will not drop also in the subsequent blowup, say $W'' \to W'$. At least, this cannot be excluded and actually does happen. So the argument and the construction of the commutative diagram with the descent in dimension have to be repeated. For $a''$ an equiconstant point for $I'$ in $W''$, choose $H''$ in $W''$ at $a''$ and project it to a regular hypersurface in $W'$ containing the center $Z'$ chosen there. This hypersurface of $W'$ will in general be different from $H'$, so that the right vertical columns of our diagrams do not match. In $W'$ we will have to choose a *new* hypersurface $\widetilde{H}'$, and consequently we lose any control on $I'_-$, which may switch to anything under the change. That this really happens show examples in positive characteristic.

*Lucky stroke*: It is not completely clear who was the first to observe that in *characteristic zero* this change of hypersurfaces is superfluous: There exists a hypersurface $H$ in $W$ at $a$ whose successive transforms $H'$, $H''$, ... under blowup in centers $Z \subset H$, $Z' \subset H'$, ... contain *all* equiconstant points $a'$, $a''$, ... above $a$. Wonderful!

Such hypersurfaces are called *hypersurfaces of maximal contact* by Hironaka, and *Tschirnhaus transformations* by Abhyankar. There is an explicit construction of them.

Abhyankar says that the concept appears first in a paper of his with Zariski, and that it has been discussed, investigated and exploited during a four day visit of Hironaka and Nagata to his house at the end of the fifties. Quote: "Hironaka did not stop asking questions, and if I possibly understood better the situation at the beginning, I am not sure if it was not him who did at the end."

That's it! The way was open to the proof of resolution of singularities in any dimension over fields of characteristic zero. The rest are "technicalities".

In the next and thus last lecture we will give the precise statement for the resolution of singularities in characteristic zero. The proof we shall present is based on the concept of *mobiles* and *setups* as developed by Encinas and the author in [EH]. These two data allow to define the resolution invariant elegantly and are then used to establish the required induction.

## 7. The resolution theorem and its proof

Let $X$ be a reduced singular scheme. A *strong resolution of* $X$ is, for every closed embedding of $X$ into a regular ambient scheme $W$, a proper birational morphism $\varepsilon\delta\varepsilon\lambda\beta\varepsilon\iota\varsigma$ from a regular scheme $W'$ onto $W$ subject to the following conditions.

*Explicitness.* $\mathcal{E}\delta\varepsilon\lambda\beta\varepsilon\iota\varsigma$ is a composition of blowups of $W$ in regular closed centers $Z$ transversal to the exceptional loci.

*Embeddedness.* The strict transform $X'$ of $X$ is regular and has normal crossings with the exceptional locus in $W'$.

*Excision.* The morphism $X' \to X$ does not depend on the embedding of $X$ in $W$.

*Equivariance.* $\mathcal{E}\delta\varepsilon\lambda\beta\varepsilon\iota\varsigma$ commutes with smooth morphisms $W^- \to W$, embeddings $W \to W^+$, and separable field extensions.

Equivariance implies that $\mathcal{E}\delta\varepsilon\lambda\beta\varepsilon\iota\varsigma$ is an isomorphism outside the singular locus of $X$ and commutes with group actions. The resolution commutes with open immersions, local and global isomorphisms and taking cartesian products with regular schemes. The smooth morphisms of equivariance need not be defined over the ground field. Passage to the completions implies resolution of formal schemes. One may add *effectiveness*: The centers are equal to the top locus of an upper-semicontinuous invariant $i_a(X)$ on $W$ given by the local rings of $X$.

**Theorem.** (Hironaka 1964) *Reduced schemes of finite type over a field of characteristic zero admit a strong resolution.*

The existence of resolutions satisfying explicit- and embeddedness was established by Hironaka. His aproach was in part motivated by earlier work of Zariski and Abhyankar. Villamayor in 1989 and 1991 and Bierstone-Milman in 1997 described constructive resolution algorithms satisfying in addition excision, equivariance and effectiveness. The resolution process of Villamayor was implemented in Maple and Singular by Bodnár and Schicho in 2000. Włodarzcyk simplified in 2005 the descent in dimension by taking a different notion of coefficient ideal. Weak resolution theorems in characteristic zero have been established with different methods by Abramovich-de Jong, Abramovich-Wang and Bogomolov-Pantev.

In the following, we shall present an outline of the proof for the existence of strong resolution in characteristic 0 based on the concept of *mobiles*. This notion was introduced by Santiago Encinas and the author in [EH]. Mobiles allow to clarify substantially the structure of the induction argument of the various resolution proofs. Essentially all nowadays existing proofs (depending on the counting this is a theme with up to eleven variations) use explicit or implicitly mobiles – the only difference being that the transformation rules of the mobile under blowup may vary.

We wish to emphasize that all constructions and arguments below are very much inspired by the earlier proofs of Villamyor and Bierstone-Milman, which in turn, rely on Hironaka's work and ideas.

Throughout, we fix a regular ambient scheme $W$, always assumed to be of finite type over a field. By a *divisor* in $W$ we shall mean an effective Weil divisor $D$. A closed subscheme $D$ of $W$ has *normal crossings* if it can be defined locally by a monomial ideal. The subscheme $V$ meets $D$ *transversally* if the product of the defining ideals of $V$ and $D$ defines a normal crossings scheme.

A *stratified ideal* in $W$ is a collection of coherent ideal sheaves each of them defined on a stratum of a stratification of $W$ by locally closed subschemes. A *stratified divisor* is

defined by a stratified principal ideal. All ideals and divisors will be stratified without notice, except if said to be coherent.

A *shortcut* of a normal crossings divisor $M$ in $W$ is a divisor $N$ obtained from $M$ by deleting on each stratum of the underlying stratification of $M$ some components of $M$. The divisor $M$ is *labelled* if each shortcut $N$ comes with a different non-negative integer $\operatorname{lab} N$, its *label*. The empty shortcut has label 0.

A *handicap* on $W$ is a sequence $D = (D_n, \ldots, D_1)$ of stratified normal crossings divisors $D_i$ of $W$.

A *singular mobile* in $W$ is a quadruple $\mathcal{M} = (\mathcal{J}, c, D, E)$ with $\mathcal{J}$ a coherent nowhere zero ideal sheaf on $W$, $c$ a non-negative constant associated to $W$ and $D$ and $E$ handicaps in $W$ with $D$ labelled and $E$ reduced. We call $c$ the *control* of $\mathcal{J}$, and $D$ and $E$ the *combinatorial* and *transversal handicap* of $\mathcal{M}$.

A *strong resolution of a mobile* $\mathcal{M} = (\mathcal{J}, c, D, E)$ in $W$ with $\mathcal{J}$ a nowhere zero ideal in $W$ is a sequence of blowups of $W$ in regular closed and equivariant centers $Z$ transversal to the exceptional loci such that the ideal $\mathcal{J}'$ of the final transform $\mathcal{M}' = (\mathcal{J}', c', D', E')$ of $\mathcal{M}$ in $W'$ (as defined later) has become the locally principal monomial ideal defining $D'_n$ in $W'$.

**Theorem.** (Encinas-Hauser 2002) *Singular mobiles defined over a field of characteristic zero admit a strong resolution.*

This result implies the result of Hironaka by taking for $\mathcal{J}$ the ideal defining $X$ in $W$, as control $c$ the supremum of the orders $\operatorname{ord}_a \mathcal{J}$ of $\mathcal{J}$ on $W$ and empty handicaps $D$ and $E$. Varying suitably the definition of the transform of a mobile, different resolution algorithms can be obtained.

The resolution of mobiles is found by associating to the mobile $\mathcal{M}$ in $W$ and to each point $a \in W$ a local upper-semicontinuous invariant $i_a(\mathcal{M})$ in a well ordered set $\Gamma$. Its top locus $\operatorname{top}(\mathcal{M})$ is shown to be closed, regular and transversal to the divisors of $E$. Taking it as the center of blowup in $W$, one obtains the transform $\mathcal{M}'$ of $\mathcal{M}$ in $W'$. It is shown that for $a \in Z$ and $\mathcal{M}$ unresolved at $a$ the invariant drops at any point $a' \in W'$ above $a$,

$$i_{a'}(\mathcal{M}') < i_a(\mathcal{M}).$$

It reaches therefore in finitely many steps its minimum. This, in turn, implies that the final transform of the mobile is resolved in the sense defined above.

The invariant $i_a(\mathcal{M})$ will be a vector of non-negative numbers defined through a local analysis of the mobile. We shall therefore work from now on with the stalks of the ideals at a given point $a \in W$, denoted by roman characters. For purposes of the induction we add subscripts indicating the dimension of the ambient space in which the objects are defined (with the exception of handicaps where the subscripts indicate the dimension where the divisor is used).

A *local flag* at $a$ is a decreasing sequence $W_n \supset \ldots \supset W_1$ of closed $i$-dimensional regular subschemes $W_i$ of a neighborhood $U = W_n$ of $a$ in $W$. We shall define, for each mobile $\mathcal{M} = (\mathcal{J}, c, D, E)$ in $W$ and each chosen local flag at $a$, several ideals $J_i$, $I_i$, $P_i$, $Q_i$, and $K_i$, related to each other by various constructions. All together will form a *setup* of $\mathcal{M}$ at $a$.

Before giving the construction, we motivate the procedure by recalling the objectives of the resolution invariant $i_a(\mathcal{M})$. Its role is two-fold: First to define the center of blowup as the locus where it attains its maximum, second to decrease under blowup. Actually, these two tasks could also be fulfilled by two different invariants. Here, we will accomplish them simultaneously.

In the sequel, all constructions will be local in a sufficiently small neighborhood of a point $a$, respectively a point $a'$ above $a$.

We start with $J_n$ the stalk of $\mathcal{J}$ at $a$. Several blowups have already occurred, so $J_n$ will factor into a product $J_n = M_n \cdot I_n$ with $M_n$ a prinicipal monomial ideal supported by the exceptional locus (which we denote by $F$) and an ideal $I_n$. We use here that $F$ is a normal crossings divisor because we always chose the centers regular and transversal to the existing exceptional locus.

Our purpose is to simplify $I_n$ by further blowups (until $I_n = 1$ and $J_n = M_n$ is a monomial ideal). Therefore our center $Z$ should lie inside the top locus $\mathrm{top}(I)$ of $I_n$ consisting of those points where $I_n$ has maximal order. It is hence natural to define the first component of $i_a(\mathcal{M})$ as the order $o_n$ of $I_n$ at $a$ and to consider the vector $i_a(\mathcal{M})$ with respect to the lexicographic order. In this way the top locus of $i_a(\mathcal{M})$ will be contained in $\mathrm{top}(I_n)$ locally at $a$.

Next, we wish to ensure that the center is transversal to the already existing exceptional locus $F$. If we neglected this requirement, we could complete by descending induction on the dimension the construction of the further components of the invariant (see below) and would get for the center its top locus $\tilde{Z}$ which, then, might not be transversal to $F$. Inspection shows that the locus $\tilde{Z}$ is, however, always transversal to some of the components of $F$, namely those which appeared in the last few blowups (which can be determined explicitly). The intersection with the other components of $F$ is unknown and could be non-transversal. The first divisor $E_n$ of the exceptional handicap $E$ of $\mathcal{M}$ collects precisely these suspicious components.

As our final choice of the center $Z$ has to be transversal to all exceptional components, the simplest though brute way is to require $Z$ to lie locally at each point $a$ inside the intersection of the components of $E_n$ going through $a$. If we introduce the transversality ideal $Q_n$ of $\mathcal{M}$ as the ideal defining $E_n$ in $W_n$, its local top locus at a point is the intersection of the components of $E_n$ there. Take then the order $q_n$ of $Q_n$ as the second component of the invariant

$$i_a(\mathcal{M}) = (o_n, q_n, \ldots).$$

With this agreement, the top locus of $i_a(\mathcal{M})$ is contained in $\mathrm{top}(I_n)$ and in all components of $E_n$ it meets, locally at $a$.

It turns out that $I_n$ passes to its weak transform under blowup, so its order does not increase. If its order remains constant, the determination of the new suspicious exceptional components $E'_n$ shows that also $Q_n$ passes to its weak transform under blowup. So its orders does not increase in this case. Put together, the first two components of $i_a(\mathcal{M})$ do not increase lexicographically,

$$(o'_n, q'_n) \leq_{lex} (o_n, q_n).$$

Up to now, neither the center is specified nor the defined portion of the invariant must drop under blowup. The next step will be to descend in dimension. Consider the product $K_n = I_n \cdot Q_n$ at $a$ and its coefficient ideal $J_{n-1}$ in any local hypersurface $W_{n-1}$ of $W_n$ through $a$. Later on, $I_n$ will have to be replaced in the product by a slightly modified ideal $P_n$. We do not bother here how the coefficient ideal is constructed in reality, we shall only use its properties. For those who wish to see the details we refer to [Ha 3].

We have seen in the last chapter that the hypersurface $W_{n-1}$ can be chosen so that its strict transform $W'_{n-1}$ contains all equiconstant points of $K_n$, i.e., the points $a'$ above $a$ where the order of $K'_n$ has remained constant. Moreover, we can achieve that it maximizes the order of $J_{n-1}$ over all choices of hypersurfaces. And, finally, it is shown by a separate argument that then $\mathrm{top}(J_{n-1})$ is contained in $\mathrm{top}(K_n) = \mathrm{top}(I_n) \cap \mathrm{top}(Q_n)$.

Therefore, even though $J_{n-1}$ depends on the choice of $W_{n-1}$, its order does not. As we have mentioned in the last chapter, coefficient ideals commute with blowup at equiconstant points, provided that we take their controlled transform. In the present case, the control for $J_{n-1}$ will be $c_n = \mathrm{ord}_a K_n = o_n + q_n$.

As $J_{n-1}$ passes under blowup to its controlled transform (at equiconstant points of $K_n$, the other points need not be considered since there $(o_n, q_n)$ has already dropped), we will have again a factorization $J_{n-1} = M_{n-1} \cdot I_{n-1}$ (in $W_{n-1}$ locally at $a$) with a monomial exceptional factor $M_{n-1}$ and an ideal $I_{n-1}$. It is the second entry $D_{n-1}$ of the combinatorial handicap $D$ of $\mathcal{M}$ which prescribes the monomial factor, i.e., $M_{n-1}$ is the ideal $I_{W_{n-1}}(D_{n-1} \cap W_{n-1})$ defining $D_{n-1}$ in $W_{n-1}$. Of course, this is only a monomial ideal if $D_{n-1}$ is transversal to $W_{n-1}$. The transversality will be ensured by the transformation rules for $D_{n-1}$ and $W_{n-1}$. They are chosen so that the second factor $I_{n-1}$ of $J_{n-1}$ passes under blowup to its weak transform $I^{\Upsilon}_{n-1}$.

The story now repeats: The order of $I_{n-1}$ will not increase at the points where $(o_n, q_n)$ has remained constant, so $o_{n-1} = \mathrm{ord}_a I_{n-1}$ is the correct candidate for the third component of the invariant

$$i_a(\mathcal{M}) = (o_n, q_n, o_{n-1}, \ldots).$$

Taking this portion of the invariant, there will be again suspicious exceptional components to which the foreseen center of blowup $\tilde{Z}$ could be non-transversal. They are collected in the second entry $E_{n-1}$ of the transversal handicap, yielding an ideal $Q_{n-1}$ in $W_{n-1}$. Note

here that $E_{n-1}$ is defined globally in $W = W_n$ (though it is a stratified divisor), whereas $Q_{n-1}$ is defined only in the local hypersurface $W_{n-1}$ at $a$. Setting $q_{n-1} = \mathrm{ord}_a Q_{n-1}$ we get

$$i_a(\mathcal{M}) = (o_n, q_n, o_{n-1}, q_{n-1}, \ldots),$$

form the product $K_{n-1} = I_{n-1} \cdot Q_{n-1}$ and take the coefficient ideal $J_{n-2}$ of $K_{n-1}$ in a local hypersurface $W_{n-2}$ of $W_{n-1}$ at $a$. This yields by induction on the dimension the complete invariant

$$i_a(\mathcal{M}) = (o_n, q_n, o_{n-1}, q_{n-1}, \ldots, o_1, q_1) \in \mathbb{N}^{2n}.$$

As all ideals $I_i$ and $Q_i$ pass under blowup to their weak transforms at points where the prior portion $(o_n, q_n, o_{n-1}, q_{n-1}, \ldots, o_{i+1}, q_{i+1})$ has not dropped, we can conclude that

$$i_{a'}(\mathcal{M}') \leq_{lex} i_a(\mathcal{M}).$$

So the invariant never increases lexicographically. Assume it remained constant. Then it remained constant also in dimension 1, $(o_1', q_1') = (o_1, q_1)$. But we saw in chapter V that the order of the weak transform of an ideal in one variable always drops to 0. So it could only remain constant if it was already equal to 0, say $I_1 = 1$ ($Q_1$ can be discarded because it equals always 1 – in one variable there is no transversality problem). Now, $I_1 = 1$ signifies that $J_1 = M_1$ is a monomial ideal.

Two cases: If $M_1 = 1$, then $J_1 = 1$ and $K_2$ is generated by a power of a variable, by the very definition of coefficient ideals. In this case, either $K_2 = 1$, $o_2 = q_2 = 0$ and we go one dimension higher as before, or not both $o_2$ and $q_2$ are simultaneously zero, with a forced drop of the order by the form of $K_2$ (to prove this could be an exercise if we wanted).

To conceptualize, let $d$ be the maximal index so that $o_d = \ldots = o_1 = 0$. It can be shown that the transversality ideals $Q_d, \ldots, Q_1$ are all trivial and can therefore be discarded. As $o_d = 0$, the coefficient ideal $J_d = M_d$ of $K_{d+1}$ is a principal monomial ideal. In this case we saw at the beginning of the last chapter (example $1\frac{1}{2}$) that if we choose the center of maximal possible dimension, we can make the *degree* of this monomial drop until the order of $K_{d+1}$ has dropped. In this case, also our invariant will have dropped, so that induction applies.

In example $1\frac{1}{2}$ there occurred this ambiguity about symmetry. We now show how it is taken care of (again, not in the most economic way). Recall that $J_d = M_d$ is defined in $W_d$ by restricting the combinatorial handicap $D_d$ to $W_d$. As $D_d$ is a collection of exceptional components, and as it is labelled, each of its shortcuts $N_d$ comes with a different label (usually, the labels chosen for single components of $D$ are just their birth date in the resolution process). Once the labels are chosen (and there is a prescibed rule how to do it when passing from $D$ to $D'$), they are intrinsic information and respect any symmetry of the original mobile we wanted to resolve. Therefore, selecting the shortcut of $D_d$ with maximal label among those whose order is $\geq$ the order of $K_d$ (again, locally at $a$) and which do not have any proper shortcut of order $\mathrm{ord}_a K_d$ is a well defined process which is

compatible with the requirements of *equivariance*. This *maximal tight* shortcut $N_d$ of $D_d$ has regular top locus $\mathrm{top}(N_d)$ which will form our desired center $Z$.

By the choice of the shortcut, $Z$ has maximal dimension as a regular subscheme of $\mathrm{top}(K_{d+1})$, so the degree of $M_d$ will drop after blowup, with an eventual drop of the order of $K_{d+1}$ (if not one of the earlier components of $i_a(\mathcal{M})$ has already dropped). The center is transversal to the existing exceptional locus $F$, because it lies in $\mathrm{top}(Q_n) \cap \ldots \mathrm{top}(Q_{d+1})$ so that it is contained in all suspicious exceptional components. It lies in $\mathrm{top}(I_n) \cap \ldots \cap \mathrm{top}(I_{d+1})$ by the presence of the orders $o_n, \ldots, o_{d+1}$ in the invariant.

In all this we use the inclusions

$$\mathrm{top}(I_i) \supset \mathrm{top}(P_i),\ \mathrm{top}(E_i) = \mathrm{top}(Q_i) \supset \mathrm{top}(K_i) \qquad \text{and}$$
$$\ldots \supset \mathrm{top}(K_{i+1}) \supset \mathrm{top}(J_i, c_{i+1}) \supset \mathrm{top}(P_i) \supset \mathrm{top}(K_i) \supset \ldots \supset Z,$$

where $c_{i+1} = \mathrm{ord}_a K_i$ is the control for $J_i$ and $\mathrm{top}(J_i, c_{i+1})$ denotes the locus of points where $J_i$ has order at least $c_{i+1}$. The ideal $P_i$ is the *companion ideal* of $I_i$. It is almost always equal to $I_i$, except in case when the order of $I_i$ has become too small. Then it is introduced as a technical device to ensure the inclusion $\mathrm{top}(P_i) \subset \mathrm{top}(J_i, c_{i+1})$, and $K_i$ is defined as $P_i \cdot Q_i$ instead of $K_i = I_i \cdot Q_i$. Its precise definition is

$$P_i = I_i + M^{\frac{o_i}{c_{i+1} - o_i}} \qquad \text{if } \ 0 < o_i = \mathrm{ord}_a I_i < c_{i+1},$$
$$P_i = I_i \qquad \text{otherwise}$$

(don't worry about the rational exponent). The product $K_i$ is called the *composition ideal*.

The invariant is intrinsic (i.e., does not depend on any choices), mainly, because the local flag $W_n \supset \ldots \supset W_1$ is chosen so as to maximize the orders of the induced coefficient ideals. We say that $W_i$ has *weak maximal contact* with $K_{i+1}$.

Adding the pair $m_d$ consisting of the order and the label of the maximal tight shortcut $N_d$ of $M_d$ to our invariant $i_a(\mathcal{M})$, we conclude that its top locus $Z = \mathrm{top}(i_a(\mathcal{M}))$ has all the required properties to be chosen as center. It is independent of any choices, regular, transversal to the exceptional locus, and satisfies the requirements of equivariants. So blow up $Z$ in $W$, say $\pi : W' \to W$.

Taking the transformed mobile $\mathcal{M}' = (\mathcal{J}', c', D', E')$ of $\mathcal{M} = (\mathcal{J}, c, D, E)$ in $W'$ (where the ideal $\mathcal{J}$ passes to the controlled transform $\mathcal{J}^!$ of $\mathcal{J}$ with respect to $c$, $c$ remains constant and $D$ and $E$ transform so as to capture the exceptional monomial factors of $J_i$ respectively the suspicious exceptional components in each dimension) it follows from the various aspects described above that the invariant drops,

$$i_{a'}(\mathcal{M}') <_{lex} i_a(\mathcal{M}),$$

until it equals its minimal value $(0, \ldots, 0)$. In this case the mobile has reached its final parking position and the seat-belt signs will be switched off, $J_n = M_n = I_W(D_n)$.

All this was for motivations. It remains to define the objects systematically. For further details, see the paper [EH] of Encinas and the author, or [Ha 3].

A *punctual setup* of $\mathcal{M}$ at $a$ is a sequence $(J_n, \ldots, J_1)$ of stalks of ideals $J_i$ in a local flag $(W_n, \ldots, W_1)$ of $W$ at $a$ satisfying for all $i \leq n$

(1) $J_i = M_i \cdot I_i$ with $M_i = I_{W_i}(D_i \cap W_i)$ and $I_i$ an ideal in $W_i$ at $a$.

(2) $M_i$ defines a normal crossings divisor in $W_i$ at $a$.

(3) $W_{i-1}$ has weak maximal contact at $a$ with the composition ideal $K_i = P_i \cdot Q_i$ in $W_i$ of $(J_i, c_{i+1}, D_i, E_i)$. Here, $c_{i+1}$ is the control of $J_i$ on $W_i$, given for $i < n$ as the order of $K_{i+1}$ in $W_{i+1}$ at $a$, and setting $c_{n+1} = c$.

(4) $J_{i-1}$ is the coefficient ideal of $K_i$ in $W_{i-1}$.

Setups depend on and are determined by the choice of the local flag subject to the above conditions. They commute with the operations described in *equivariance*.

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Let $\mathcal{M} = (\mathcal{J}, c, D, E)$ be a mobile in $W$. Assume that $\mathcal{M}$ admits locally on $W$ punctual setups $(J_n, \ldots, J_1)$. Set

$$i_a(\mathcal{M}) = (t_n, \ldots, t_1) \in \mathbb{N}^{4n}$$

with $t_i = (o_i, k_i, m_i)$ the *tag* of $(J_i, c_{i+1}, D_i, E_i)$ at $a$, $o_i = \mathrm{ord}_a I_i$, $k_i = \mathrm{ord}_a K_i = \mathrm{ord}_a I_i + \mathrm{ord}_a Q_i$ and $m_i$ the order and label of the maximal tight shortcut $N_i$ of $M_i$ of order $\geq c_{i+1}$ ($m_i$ is set equal to $(0,0)$ if $i \neq d$ for $d$ maximal with $o_d = 0$). Equipping $\mathbb{N}^{4n}$ with the lexicographic order this vector satisfies the following properties.

$i_a(\mathcal{M})$ does not depend on the chosen setup of $\mathcal{M}$ at $a$ and commutes with the operations described in *equivariance*.

The map $a \to i_a(\mathcal{M})$ is upper-semicontinuous on $V$. The induced stratification of $V$ refines the stratification underlying $D$ and $E$.

The top locus $Z$ of $i_a(\mathcal{M})$ is regular. Locally, $Z$ lies in the top loci of all $I_i$, $P_i$, $Q_i$ and $K_i$. It only depends on the restriction of $i_a(\mathcal{M})$ to the support of $\mathcal{J}$.

$Z$ is transversal to all $D_i$ and $E_i$.

We have seen above that then, for the transform $\mathcal{M}'$ of $\mathcal{M}$ under blowup, our resolution invariant drops

$$i_{a'}(\mathcal{M}) < i_a(\mathcal{M})$$

as long as $\mathcal{M}$ is not resolved. Done!

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## BIBLIOGRAPHY

[Ab 1] Abhyankar, S.: Local uniformization of algebraic surfaces over ground fields of characteristic $p \neq 0$. Ann. Math. **63** (1956), 491-526.

[Ab 2] Abhyankar, S.: Resolution of singularities of arithmetical surfaces. In: Arithmetical Algebraic Geometry, Harper and Row 1965, 111-152.

[Ab 3] Abhyankar, S.: Resolution of singularities of embedded algebraic surfaces. Acad. Press 1966. 2nd edition, Springer 1998.

[Ab 4] Abhyankar, S.: An algorithm on polynomials in one determinate with coefficients in a two dimensional regular local domain. Annali Mat. Pura Appl., Serie 4, **71** (1966), 25-60.

[Ab 5] Abhyankar, S.: Three dimensional embedded uniformization in characteristic $p$. Lectures at Purdue Univeristy, Notes by M.F. Huang 1968.

[Ab 6] Abhyankar, S.: Good points of a hypersurface. Adv. Math. **68** (1988), 87-256.

[Ab 7] Abhyankar, S.: Current status of the resolution problem. Summer Institute on Algebraic Geometry 1964. Proc. Amer. Math. Soc.

[Ab 8] Abhyankar, S.: Desingularization of plane curves. In: Summer Institute on Algebraic Geometry. Arcata 1981, Proc. Symp. Pure Appl. Math. **40** Amer. Math. Soc.

[Ab 9] Abhyankar, S.: Analytic desingularization in characteristic zero. Appendix to the 2nd edition of [Ab 3].

[Ab 10] Abhyankar, S.: Algebraic Geometry for Scientists and Engineers. Math. Surveys and Monographs **35**. Amer. Math. Soc. 1990.

[Ab 11] Abhyankar, S.: Ramification theoretic methods in algebraic geometry. Princeton Univ. Press 1959.

[Ab 12] Abhyankar, S.: On the valuations centered in a local domain. Amer. J. Math. **78** (1956), 321-348.

[Ab 13] Abhyankar, S.: Resolution of Singularities of Algebraic Surfaces. In: Proceedings of the 1968 Bombay International Colloquium, Oxford University Press 1969, 1-11.

[AHV1] Aroca, J.-M., Hironaka, H., Vicente, J.-L.: The theory of the maximal contact. Memorias Mat. Inst. Jorge Juan Madrid 29 (1975).

[AHV2] Aroca, J.-M., Hironaka, H., Vicente, J.-L.: Desingularization theorems. Memorias Mat. Inst. Jorge Juan Madrid 29 (1975).

[AJ] Abramovich, D., de Jong, J.: Smoothness, semi-stability and toroidal geometry. J. Alg. Geometry **6** (1997), 789-801.

[AM] Abhyankar, S., Moh, T.-T.: Newton-Puiseux expansion and generalized Tschirnhausen transformation I, II. J. Reine Angew. Math. **260** (1973), 47-83, **261** (1973), 29-54.

[At] Artin, M.: Lipman's proof of resolution of singularities. In: Arithmetic Geometry (eds. G. Cornell, J. H. Silverman), Springer 1986.

[Bb] Brandenberg, M.: Aufblasungen affiner Varietäten. Thesis Zürich 1992.

[Be] Berthelot, P.: Altérations de variétés algébriques. Séminaire Bourbaki 1995/96. Astérisque **241** (1997), 273-311.

[BE] Ban, C., MacEwan, L.: Canonical resolution of a quasi-ordinary surface singularity. Preprint Ohio State University 1998.

[BEV] Bravo, A., Encinas, S., Villamayor, O.: A simplified proof of desingularization and applications. Preprint.

[BK] Brieskorn, E., Knörrer, H.: Ebene algebraische Kurven. Birkhäuser 1981. English translation: Plane algebraic curves. Birkhäuser 1986.

[BM 1] Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. 128 (1997), 207-302.

[BM 2] Bierstone, E., Milman, P.: Uniformization of analytic spaces. J. Amer. Math. Soc. 2 (1989), 801-836.

[BM 3] Bierstone, E., Milman, P.: A simple constructive proof of canonical resolution of singularities. In: Effective Methods in Algebraic Geometry (eds. T. Mora, C. Traverso). Progress in Math. 94, Birkhäuser 1991, 11-30.

[BM 4] Bierstone, E., Milman, P.: Semianalytic and subanalytic sets. Publ. Math. IHES 67 (1988), 5-42.

[BM 5] Bierstone, E., Milman, P.: Resolution of singularities. In: Several complex variables (Berkeley, 1995-1996), 43-78. Math. Sci. Res. Inst. Publ. 37, Cambridge Univ. Press, Cambridge, 1999.

[BM 6] Bierstone, E., Milman, P.: Desingularization algorithms. I. Role of exceptional divisors. Mosc. Math. J. **3** (2003), 751-805.

[Bn] Bennett, B.-M.: On the characteristic function of a local ring. Ann. Math. **91** (1970), 25-87.

[Bo] Bodnár, G.: Computation of blowing up centers. J. Pure Appl. Algebra **179** (2003), 221-233.

[BP] Bogomolov, F., Pantev, T.: Weak Hironaka Theorem. Math. Res. Letters **3** (1996), 299-307.

[Br 1] Brieskorn, E.: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. Math. Ann. **166** (1966), 76-102.

[Br 2] Brieskorn, E.: Die Auflösung der rationalen Singularitäten holomorpher Abbildungen. Math. Ann. **178** (1968), 255 - 270.

[BS 1] Bodnár, G., Schicho, J.: Automated Resolution of Singularities for Hypersurfaces. J. Symb. Comp. 30 (2000), 401-428.

[BS 2] Bodnár, G., Schicho, J.: A computer program for the resolution of singularities. In: Resolution of singularities (ed. H. Hauser at al.), 231-238, Progr. in Math. 181, Birkhäuser, 2000.

[BS 3] Bodnár, G., Schicho, J.: Two computational techniques for singularity resolution. Computer algebra and mechanized reasoning (St. Andrews, 2000). J. Symbolic Comput. **32** (2001), 39-54.

[BEV] Bravo A., Encinas S., Villamayor O.: A simplifed proof of desingularization and applications. Rev. Mat. Iberoamericana **21** (2005), 349-458.

[BV] Bravo, A., Villamayor, O.: A strengthening of resolution of singularities in characteristic zero. Proc. London Math. Soc. **86** (2003), 327-357.

[CGO] Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Lecture Notes in Math. vol. 1101, Springer 1984.

[Co 1] Cossart, V.: Desingularization of embedded excellent surfaces. Tôhoku Math. J. **33** (1981), 25-33.

[Co 2] Cossart, V.: Polyèdre caractéristique d'une singularité. Thèse d'Etat, Orsay 1987.

[Co 3] Cossart, V.: Uniformisation et désingularisation des surfaces d'après Zariski. In: Resolution of Singularities (ed.: H. Hauser et al.), Progress in Math. 181, Birkhäuser 2000.

[Co 4] Cossart, V.: Contact maximal en caractéristique positive et petite multiplicité. Duke Math. J. **63** (1991), 57-64.

[Co 5] Cossart, V.: Desingularization in dimension two. In: Resolution of surface singularities. Lecture Notes in Math. vol. 1101, Springer 1984.

[CP] Cano, F., Piedra, R.: Characteristic polygon of surface singularities. In: Géométrie algébrique et applications II (eds. J.-M. Aroca, T. Sánchez-Giralda, J.-L. Vicente), Proc. of Conference on Singularities, La Rábida 1984. Hermann 1987.

[Cu 1] Cutkosky, D.: When are the blow ups of two ideals the same scheme? Comm. Algebra **19** (1991), 1119-1124.

[Cu 2] Cutkosky, D.: Local factorization of birational maps. Adv. Math. **132** (1997), 167-315.

[Cu 3] Cutkosky, D.: Local factorization and monomialization of morphisms. Astérisque 260 (1999).

[Cu 4] Cutkosky, D.: Monomialization of morphisms from 3-folds to surfaces. Lecture Notes in Mathematics, 1786. Springer 2002.

[Cu 5] Cutkosky, D.: Resolution of morphisms. Valuation theory and its applications, Vol. I (Saskatoon 1999), 117-140, Fields Inst. Commun., 32, Amer. Math. Soc. 2002.

[Cu6] Cutkosky, D.: Generically finite morphisms and simultaneous resolution of singularities. In: Commutative algebra (Grenoble/Lyon, 2001), 75-99, Contemp. Math. 331, Amer. Math. Soc. 2003.

[Cu 7] Cutkosky, D.: Resolution of singularities. Graduate Studies in Mathematics, 63. Amer. Math. Soc. 2004.

[DCP] De Concini, C., Procesi, C.: Wonderful models of subspace arrangements. Selecta Math. **1** (1995), no. 3, 459-494.

[dJ] de Jong, A.J.: Smoothness, semi-stability and alterations. Publ. Math. IHES **83** (1996), 51-93.

[EiH] Eisenbud, D., Harris, J.: The geometry of schemes. Graduate Texts in Mathematics, 197. Springer 2000.

[EH] Encinas, S., Hauser, H.: Strong resolution of singularities in characteristic zero. Comment. Math. Helv. **77** (2002), 821-845.

[En] Encinas, S.: Constructive resolution of singularities of families of schemes. Thesis, Valladolid 1996.

[ENV] Encinas, S., Nobile, A., Villamayor, O.: On algorithmic equi-resolution and stratification of Hilbert schemes. Proc. London Math. Soc. **86** (2003), 607-648.

[EV 1] Encinas, S., Villamayor, O.: Good points and constructive resolution of singularities. Acta Math. **181** (1998), 109-158.

[EV 2] Encinas, S., Villamayor, O.: A course on constructive desingularization and equivariance. In: Resolution of Singularities (ed.: H. Hauser et al.), Progress in Math. 181, Birkhäuser 2000.

[EV 3] Encinas, S., Villamayor, O.: A new proof of desingularization over fields of characteristic zero. Rev. Mat. Iberoamericana **19** (2003), no. 2, 339-353.

[FK] Feichtner, E., Koslov, D.: Incidence combinatorics of resolutions. Selecta Math. **10** (2004), 37-60.

[Gi 1] Giraud, J.: Etude locale des singularités. Cours de $3^e$ Cycle, Orsay 1971/72.

[Gi 2] Giraud, J.: Sur la théorie du contact maximal. Math. Z. **137** (1974), 285-310.

[Gi 3] Giraud, J.: Contact maximal en caractéristique positive. Ann. Scient. Ec. Norm. Sup. Paris **8** (1975), 201-234.

[Gi 4] Giraud, J.: Desingularization in low dimension. In: Resolution of surface singularities. Lecture Notes in Math. vol. 1101, Springer 1984, 51-78.

[Gr] Grothendieck, A.: Traveaux de Heisouke Hironaka sur la résolution des singularités. Actes Congrès International Math., Nice 1970.

[GT] Goldin, R., Teissier, B.: Resolving singularities of plane analytic branches with one toric morphism. In: Resolution of Singularities, Progress in Math. 181, Birkhäuser 2000.

[Ha 1] Hauser, H.: Seventeen obstacles for resolution of singularities. In: The Brieskorn Anniversary volume 1996. Progress in Math. 162, Birkhäuser 1997.

[Ha 2] Hauser, H.: Excellent surfaces and their taut resolution. In: Resolution of Singularities, Progress in Math. 181, Birkhäuser 2000.

[Ha 3] Hauser, H.: The Hironaka Theorem on resolution of singularities. Bull. Amer. Math. Soc. **40** (2003), 323-403.

[Ha 4] Hauser, H.: Why Hironaka's proof of resolution of singularities fails in positive characteristic. Preprint University of Innsbruck, 51 pp.

[Ha 5] Hauser, H.: Three power series techniques. Proc. London Math. Soc. **88** (2004), 1-24.

[Ha 6] Hauser, H., Regensburger, G.: Explizite Auflösung von ebenen Kurvensingularitäten in beliebiger Charakteristik. L'Ens. Mathém. **50** (2004), 305-353.

[Ha 7] Desingularization of ideals and varieties. Lecture notes of a course at CIRM, Luminy 2005, 18 pp.

[Hi 1] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79 (1964), 109-326.

[Hi 2] Hironaka, H.: Characteristic polyhedra of singularities. J. Math. Kyoto Univ. **7** (1967), 251-293.

[Hi 3] Hironaka, H.: Certain numerical characters of singularities. J. Math. Kyoto Univ. **10** (1970), 151-187.

[Hi 4] Hironaka, H.: Desingularization of excellent surfaces. Notes by B. Bennett at the Conference on Algebraic Geometry, Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Lecture Notes in Math. 1101, Springer 1984.

[Hi 5] Hironaka, H.: Additive groups associated with points of a projective space. Ann. Math. **92** (1970), 327-334.

[Hi 6] Hironaka, H.: Idealistic exponents of singularity. In: Algebraic Geometry, the Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977.

[Hi 7] Hironaka, H.: Schemes etc. In: 5th Nordic Summer School in Mathematics (ed. F. Oort), Oslo 1970, 291-313.

[Hi 8] Hironaka, H.: Desingularization of complex analytic varieties. Actes Congrès International Math., Nice 1970, 627-631.

[Hi 9] Hironaka, H.: On the presentations of resolution data (Notes by T.T. Moh). In: Algebraic Analysis, Geometry and Number Theory 1988 (ed. J.I. Igusa), The Johns Hopkins University Press 1989.

[Hi 10] Hironaka, H.: Bimeromorphic smoothing of a complex analytic space. Mimeographed notes. University of Warwick 1971.

[Hi 11] Hironaka, H.: Stratification and flatness. In: Nordic Summer School, Oslo 1976 (ed. P. Holm), Sijthoff and Noordhoff 1977.

[Hi 12] Hironaka, H.: Preface to Zariski's collected papers I, 307-312. MIT Press 1979.

[HIO] Herrmann, M., Ikeda, S., Orbanz, U.: Equimultiplicity and blowing up. Springer 1988.

[HR] Hauser, H., Regensburger, G.: Explizite Auflösung von ebenen Kurvensingularitäten. L'Enseign. Math. (2004)

[Ju] Jung, H.: Darstellung der Funktionen eines algebraischen Körpers zweier unabhängiger Veränderlicher $x$, $y$ in der Umgebung einer Stelle $x = a$, $y = b$. J. Reine Angew. Math. **133** (1908), 289-314.

[Ko] Kollár, J.: Lectures on resolution of singularities. Preprint 2005. ArXiv: math.AG/0508332.

[KV] Kiyek, K.-H., Vicente, J. L.: Resolution of curve and surface singularities in characteristic zero. Algebras and Applications, Kluwer 2004.

[Kz] Kunz, E.: Algebraische Geometrie IV. Vorlesung Regensburg.

[LO] Lê, D.T., Oka, M.: On resolution complexity of plane curves. Kodai Math. J. **18** (1995), 1-36.

[Lp 1] Lipman, J.: Desingularization of 2-dimensional schemes. Ann. Math. **107** (1978), 151-207.

[Lp 2] Lipman, J.: Rational singularities, with applications to algebraic surfaces and unique factorization. Publ. Math. IHES **36** (1969), 195-279.

[Lp 3] Lipman, J.: Introduction to resolution of singularities. Proceedings Symp. Pure Appl. Math. **29** Amer. Math. Soc. 1975, 187-230.

[LT] Lejeune, M., Teissier, B.: Contribution à l'étude des singularités du point de vue du polygone de Newton. Thèse d'Etat, Paris 1973.

[Ma] Matsuki, K.: Notes on the inductive algorithm of resolution of singularities by S. Encinas and O. Villamayor. ArXiv: math.AG/0103120

[Mo 1] Moh, T.-T.: On a stability theorem for local uniformization in characteristic $p$. Publ. RIMS Kyoto Univ. **23** (1987), 965-973.

[Mo 2] Moh, T.-T.: Canonical uniformization of hypersurface singularities of characteristic zero. Comm. Alg. **20** (1992), 3207-3249.

[Mo 3] Moh, T.-T.: On a Newton polygon approach to the uniformization of singularities of characteristic $p$. In: Algebraic Geometry and Singularities (eds. A. Campillo, L. Narváez). Proc. Conf. on Singularities La Rábida. Birkhäuser 1996.

[Mu] Mulay, S.B.: Equimultiplicity and hyperplanarity. Proc. Amer. Math. Soc. **87** (1983), 407-413.

[MZ] Muhly, H.T., Zariski, O.: The resolution of singularities of an algebraic curve. Amer. J. Math. **61** (1939), 107-114.

[Na 1] Narasimhan, R.: Monomial equimultiple curves in positve characteristic. Proc. Amer. Math. Soc. **89** (1983), 402-413.

[Na 2] Narasimhan, R.: Hyperplanarity of the equimultiple locus. Proc. Amer. Math. Soc. **87** (1983), 403-406.

[No] Noether, M. Über die algebraischen Functionen einer und zweier Variabeln. Gött. Nachr. (1871), 267-278.

[Od 1] Oda, T.: Infinitely very near singular points. Adv. Studies Pure Math. **8** (1986), 363-404.

[Od 2] Oda, T.: Hironaka's additive group scheme. In: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 181-219.

[Od 3] Oda, T.: Hironaka group schemes and resolution of singularities. In: Proc. Conf. on Algebraic Geometry, Tokyo and Kyoto 1982. Lecture Notes in Math. 1016, Springer 1983, 295-312.

[Ok] Oka, M.: Geometry of plane curves via toroidal resolution. In: Algebraic Geometry and Singularities (eds. A. Campillo, L. Narváez). Proc. Conf. on Singularities La Rábida. Birkhäuser 1996.

[Or] Orbanz, U.: Embedded resolution of algebraic surfaces after Abhyankar (characteristic 0). In: Resolution of surface singularities. Lecture Notes in Math. vol. 1101, Springer 1984.

[Pf] Pfeifle, J.: Aufblasung monomialer Ideale. Diplomarbeit Innsbruck 1996.

[Ro] Rosenberg, J.: Blowing up nonreduced toric subschemes of $A^n$. Preprint 1998.

[RS] Rodríguez Sánchez, C.: Good points and local resolution of threefold singularities. Thesis, Univ. Léon 1998.

[Si 1] Singh, B.: Effect of permissible blowing up on the local Hilbert function. Invent. Math. **26** (1974), 201-212.

[Si 2] Singh, B.: Formal invariance of local characteristic functions. In: Seminar D. Eisenbud, B. Singh, W. Vogel, vol. 1, 44-59. Teubner-Texte zur Math. 29, 1980.

[Sp 1] Spivakovsky, M.: A solution to Hironaka's Polyhedra Game. In: Arithmetic and Geometry. Papers dedicated to I.R. Shafarevich on the occasion of his sixtieth birthday, vol II (eds.: M. Artin, J. Tate). Birkhäuser 1983, 419-432.

[Sp 2] Spivakovsky, M.: A counterexample to Hironaka's 'hard' polyhedra game. Publ. RIMS, Kyoto University **18** (1983), 1009-1012.

[Sp 3] Spivakovsky, M.: A counterexample to the theorem of Beppo Levi in three dimensions. Invent. Math. **96** (1989), 181-183.

[T] Teissier, B.: Valuations, deformations, and toric geometry. In: Valuation theory and its applications, Vol. II (Saskatoon 1999), 361-459, Fields Inst. Commun. 33, Amer. Math. Soc. 2003.

[Ur] Urabe, T.: Resolution of singularities of germs in characteristic positive associated with valuation rings of iterated divisor type. Preprint 1999.

[Va] Vasconcelos, W.: Arithmetic of blowup algebras. London Mathematical Society Lecture Note Series, 195. Cambridge University Press 1994.

[Vi 1] Villamayor, O.: Constructiveness of Hironaka's resolution. Ann. Scient. Ec. Norm. Sup. Paris **22** (1989), 1-32.

[Vi 2] Villamayor, O.: Patching local uniformizations. Ann. Scient. Ec. Norm. Sup. Paris **25** (1992), 629-677.

[Vi 3] Villamayor, O.: An introduction to the algorithm of resolution. In: Algebraic Geometry and Singularities (eds. A. Campillo, L. Narváez). Proc. Conf. on Singularities La Rábida. Birkhäuser 1996.

[Vi 4] Villamayor, O.: On equiresolution and a question of Zariski. Acta Math. **185** (2000), 123-159.

[Vi 5] Villamayor, O.: An introduction to constructive desingularization. ArXiv: math.AG/0507537.

[Wa 1] Walker, R.J.: Reduction of singularities of an algebraic surface. Ann. Math. **36** (1935), 336-365.

[Wa 2] Walker, R.J.: Algebraic Curves. Princeton Univ. Press 1950. Reprinted: Dover 1962.

[W] Włodarzcyk, H.: Simple Hironaka resolution in characteristic zero. J. Amer. Math. Soc. **18** (2005), 779-822.

[Yo] Youssin, B.: Newton polyhedra without coordinates. Memoirs Amer. Math. Soc. **433** (1990), 1-74, 75-99.

[Za 1] Zariski, O.: The reduction of singularities of an algebraic surface. Ann. Math. **40** (1939), 639-689.

[Za 2] Zariski, O.: Local uniformization on algebraic varieties. Ann. Math. **41** (1940), 852-896.

[Za 3] Zariski, O.: A simplified proof for resolution of singularities of an algebraic surface. Ann. Math. **43** (1942), 583-593.

[Za 4] Zariski, O.: Reduction of singularities of algebraic three dimensional varieties. Ann. Math. **45** (1944), 472-542.

[Za 5] Zariski, O.: The compactness of the Riemann manifold of an abstract field of algebraic functions. Bull. Amer. Math. Soc. **45** (1945), 683-691.

[Za 6] Zariski, O.: Exceptional singularities of an algebraic surface and their reduction. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., serie VIII, **43** (1967), 135-196.

[Za 7] Zariski, O.: A new proof of the total embedded resolution theorem for algebraic surfaces. Amer. J. Math. **100** (1978), 411-442.

[Za 8] Zariski, O.: Polynomial ideals defined by infinitely near base points. Amer. J. Math. **60** (1938), 151-204.

[Za 9] Zariski, O.: Algebraic Surfaces. Ergebnisse der Mathematik, vol 61. 2nd edition, Springer 1971.

[Za 10] Zariski, O.: The concept of a simple point of an abstract algebraic variety. Trans. Amer. Math. Soc. **62** (1947), 1-52.

[Za 11] Zariski, O.: Normal varieties and birational correspondences. Bull. Amer. Math. Soc. **48** (1942), 402-413.

[Za 12] Zariski, O.: Collected papers, vol. I. Foundations of algebraic geometry and resolution of singularities. MIT Press 1979.

[Za 13] Zariski, O.: Collected papers, vol. IV. Equisingularity. MIT Press 1979.

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