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Platonic Stars

ALEXANDRA FRITZ AND HERWIG HAUSER

But of beauty, I repeat again that we saw her there shining in company with the celestial forms; and coming to earth we find her here too, shining in clearness through the clearest aperture of sense. For sight is the most piercing of our bodily senses; though not by that is wisdom seen; her loveliness would have been transporting if there had been a visible image of her, and the other ideas, if they had visible counterparts, would be equally lovely. But this is the privilege of beauty, that being the loveliest she is also the most palpable to sight

Plato, *Phaedrus*

EXAMPLE 1 The picture above shows the zero set of the following equation,

$$f(u,v) = (1-u)^3 - \frac{5}{27}cu^3 + cv, \quad \text{with } c \neq 0, \qquad (1)$$

where

$$\begin{aligned} u(x,y,z) &= x^2 + y^2 + z^2, \\ v(x,y,z) &= -z(2x+z)(x^4 - x^2z^2 + z^4 + 2(x^3z - xz^3) \\ &\quad + 5(y^4 - y^2z^2) + 10(xy^2z - x^2y^2)). \qquad (2) \end{aligned}$$

For any value $c > 0$, the zero set of this polynomial, such as the one displayed in figure 1, is an example of a surface that we want to call a "Platonic star". This particular example we call a "dodecahedral star" because it has its cusps at the vertices of a regular dodecahedron and has the same symmetries. We refer to the familiar Platonic solid with 12 regular pentagons as faces, 30 edges, and 20 vertices. See figure 2e.

The following article deals with the construction of surfaces such as the one above. We will always use polynomials such as u and v from above. Their role will become clear when we introduce some invariant theory.

The general task is to construct an algebraic surface, that is, the zero set $X = V(f)$ of a polynomial $f \in \mathbb{R}[x,y,z]$, with prescribed symmetries and singularities.[1] By "prescribed symmetries" we mean that we insist the surface should be invariant under the action of some finite subgroup of the real orthogonal group $O_3(\mathbb{R})$. Most of the time we will consider the symmetry group of some Platonic solid $S \subset \mathbb{R}^3$.

The *symmetry group* of a set $A \subset \mathbb{R}^3$ is the subgroup of the orthogonal group $O_3(\mathbb{R})$, formed by all matrices that transport the set into itself, that is, $\mathrm{Sym}(A) = \{M \in O_3(\mathbb{R}),\ M(a) \in A \text{ for all } a \in A\} \subseteq O_3(\mathbb{R})$. (Often the symmetry group is defined as a subgroup of SO_3 instead of O_3. The subgroup of O_3 we consider here is referred to as the *full symmetry group*.)

A *Platonic solid* is a convex polyhedron whose faces are identical regular polygons. At each vertex of a Platonic solid the same number of faces meet. There are exactly five Platonic solids, the *tetrahedron, octahedron, hexahedron* (or *cube*), *icosahedron*, and *dodecahedron*. See figure 2. Two Platonic solids are *dual* to each other if one is the convex hull of the centers of the faces of the other. The octahedron and the cube are dual to each other, as are the icosahedron and the dodecahedron. The tetrahedron is dual to itself. Dual Platonic solids have the same symmetry group. For a more rigorous and more general definition of duality of convex polytopes see [7, p. 77].

The Platonic solids are *vertex-transitive polyhedra*: their symmetry group acts transitively on the set of vertices. This means that for each pair of vertices there exists an element of the symmetry group that transports the first vertex to the second. One says that all vertices belong to one *orbit* of the action of the symmetry group.

Citation of *Phaedrus* from [8].

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Figures 12 and 13 are generated with Wolfram Mathematica 6 for Students. All the other figures are produced with the free ray-tracing software Povray, http://www.povray.org.

[1] Of course a lot of people have been working on construction of surfaces with many singularities, also via symmetries. We want to mention, for example, Oliver Labs and Gert-Martin Greuel.

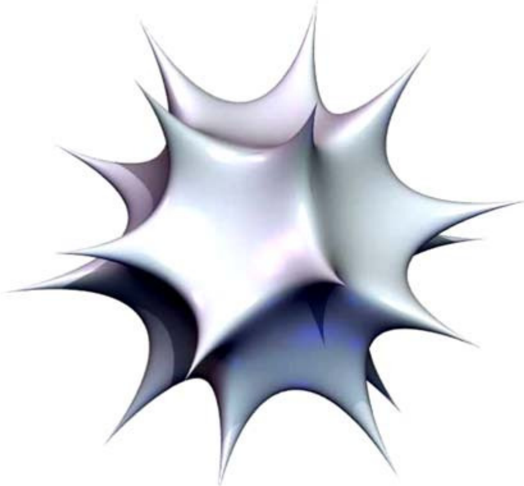

Figure 1. Dodecahedral star with parameter value $c = 81$.

A convex polyhedron that has regular polygons as faces and that is vertex-transitive is either a Platonic solid, a prism, an antiprism, or one of 13 solids called *Archimedean solids*.[2]

One can extend the notion of duality as we defined it to Archimedean solids. Their duals are not Archimedean any longer; they are called *Catalan solids*[3] or just *Archimedean duals*. Each Archimedean solid has the same symmetries as one of the Platonic solids, but with this proviso: in two cases we do not get the full symmetry group but just the rotational symmetries.

Here we will deal with just three groups: the symmetry group of the tetrahedron T_d, that of the octahedron and cube O_h, and that of the icosahedron and dodecahedron I_h. The Catalan solids are not vertex-transitive but are obviously face-transitive.

By "prescribing singularities" of a surface we mean that we insist the zero set should have a certain number of isolated singularities of fixed type, at a priori chosen locations. A *singular point*, or *singularity*, of an algebraic surface is a point where the surface is locally not a manifold. This signifies that the first partial derivatives of the defining polynomial disappear at the point. *Isolated* means that in a neighborhood of the singularity there are no other singular points.

An isolated surface singularity is said to be of type A_2 if it has (up to local analytic coordinate transformations) the equation $x^3 + y^2 + z^2 = 0$. The corresponding zero set is a two-dimensional cusp Y as displayed in figure 3a. Note that the cusp, in these coordinates, is a surface of rotation. Its axis of rotation is the x-axis. We call that axis the *tangent-line* of the cusp Y at the origin. (Clearly it is not the tangent-line in the usual, differential-geometric sense. The origin is a singularity of the cusp, that is, the surface is not a manifold there, so that differential-geometric methods fail there.) One can also view this "tangent-line" as the limit of secants of Y joining one point of intersection at the singular point 0 to another point of intersection moving toward 0. Now if X is any variety with a singularity of type A_2 at a point p, then we define the tangent-line at this point analogously. Note that we are no longer dealing with a surface of rotation.

We will choose the location of the singular points so that they all form one orbit of the action of the selected group. If we use the symmetry group of a Platonic solid, we can choose, for example, the vertices of the corresponding Platonic or Archimedean solid.

Now we are ready to define our "object of desire", the "Platonic star". We want to emphasize that the following is not a rigorous mathematical definition.

Let S be a Platonic (Archimedean) solid and m the number of its vertices. Denote its symmetry group in $O_3(\mathbb{R})$ by G. An algebraic surface X that is invariant under the action of G and has exactly m isolated singularities of type A_2 at the vertices of the solid, is called a *Platonic (Archimedean) star*. We require that the cusps point outward, otherwise we speak of an *anti-star*. In both cases for all singular points p the tangent-line of X at p should be the line joining the origin to p.

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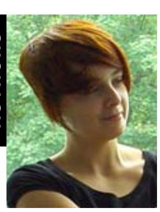

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[2] Often the Archimedean solids are defined as polyhedra that have more than one type of regular polygons as faces but do have identical vertices in the sense that the polygons are situated around each vertex in the same way. This definition admits (besides the Platonic solids, prisms, and antiprisms) an additional 14th polyhedron called the pseudo-rhombicuboctahedron. This is a fact that has often been overlooked. The sources we use, namely [3, p. 47–59] and [4, p. 156 and p. 367], are not very clear about it. See [4].

[3] Named after Eugène Charles Catalan, who characterized certain semi-regular polyhedra.

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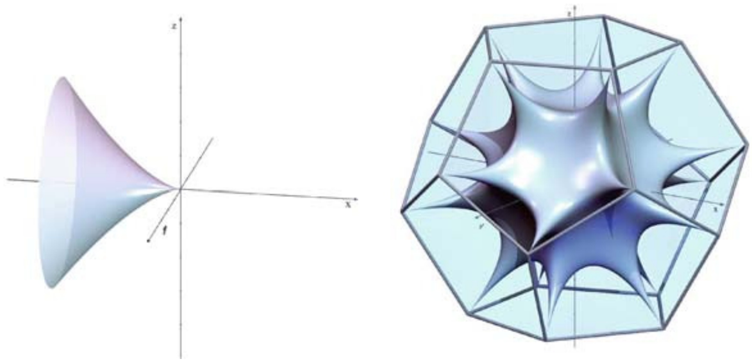
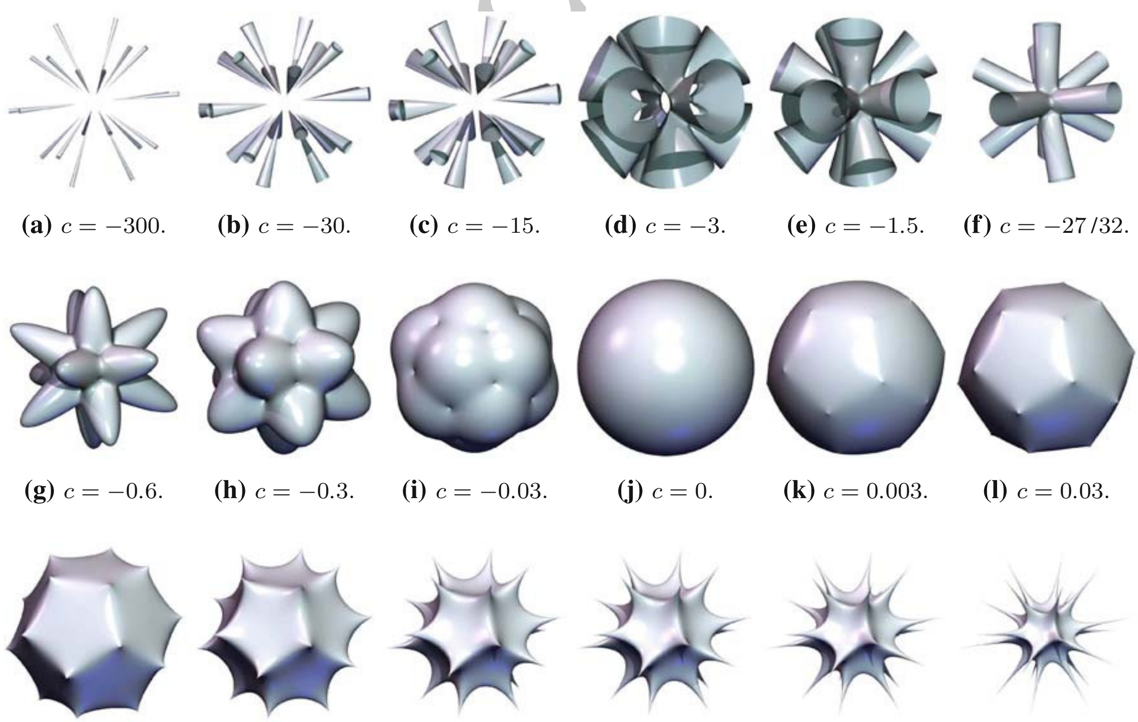
(a) Tetrahedron. **(b)** Octahedron. **(c)** Hexahedron. **(d)** Icosahedron. **(e)** Dodecahedron.

Figure 2. The five Platonic solids.



(a) The A_2-singularity, $x^3 + y^2 + z^2 = 0$.
(b) Dodecahedral star and dodecahedron.

Figure 3. The two-dimensional cusp and the dodecahedral star.



(a) $c = -300$. **(b)** $c = -30$. **(c)** $c = -15$. **(d)** $c = -3$. **(e)** $c = -1.5$. **(f)** $c = -27/32$.

(g) $c = -0.6$. **(h)** $c = -0.3$. **(i)** $c = -0.03$. **(j)** $c = 0$. **(k)** $c = 0.003$. **(l)** $c = 0.03$.

(m) $c = 0.3$. **(n)** $c = 3$. **(o)** $c = 30$. **(p)** $c = 81$. **(q)** $c = 300$. **(r)** $c = 3000$.

Figure 4. Dodecahedral star with varying parameter value c, for $c \leq -27/32$ the surfaces are clipped by a sphere of radius 4.5.

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Later we will see that the algebraic surfaces defined by equation (1) from the introductory example satisfy by construction the conditions of the definition above. For now we ask the reader to consider the illustrations, especially figure 3b, that suggest that this claim is true. If we choose $c > 0$ we get stars, for $c < 0$ anti-stars. The choice $c = 0$ yields an ordinary sphere. See figure 4 for the effect of varying the parameter c. Note that the singularities stay fixed on a sphere of radius one for all parameter values, so for $c < 0$ we have to zoom out to be able to show the whole picture. For $c = -27/32$ the anti-star has a point at infinity in the direction of the z-axis, which is among the normals of the faces of the dodecahedron. By symmetry it will also have points at infinity in the direction of the normals of the remaining faces. The pictures suggest that for $c > -27/32$ the dodecahedral anti-stars and stars are bounded while they remain unbounded for $c < -27/32$. It might be interesting to refine the definition of stars and anti-stars by demanding that the surfaces be bounded. In this article we shall not consider this question.

Some Basics from Invariant Theory

In order to explain our construction of the equations for the stars we need a few results from invariant theory. Those who are familiar with the topic can proceed to the next section; those who want to know more details than we give can refer to [10].

For ease of exposition, we work over the complex numbers $\mathbb{C}$. Let there be given a finite subgroup G of $GL_n(\mathbb{C})$. Typically, this will be the symmetry group of a Platonic solid, allowing also reflections.

The group G acts naturally on $\mathbb{C}^n$ by left-multiplication. This induces an action of G on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, via $\pi \cdot f(x) = f(\pi \cdot x)$. A polynomial f is called *invariant* with respect to G if $\pi \cdot f = f$ for all $\pi \in G$. For instance, if G is the permutation group S_n on n elements, the invariant polynomials are just the symmetric ones.

The collection of all invariant polynomials is clearly closed under addition and multiplication, and thus forms the *invariant ring*

$$\mathbb{C}[\boldsymbol{x}]^G := \{f \in \mathbb{C}[\boldsymbol{x}],\ f = \pi \cdot f, \quad \text{for all } \pi \in G\}.$$

In the nineteenth century it was a primary goal of invariant theory to understand the structure of these rings. Hilbert's Finiteness Theorem asserts that for *finite* groups, $\mathbb{C}[\boldsymbol{x}]^G$ is a finitely generated $\mathbb{C}$-algebra: There exist invariant polynomials $g_1(\boldsymbol{x}), \ldots, g_k(\boldsymbol{x})$ such that any other invariant polynomial h is a polynomial in $g_1, \ldots, g_k$, say $h(\boldsymbol{x}) = P(g_1(\boldsymbol{x}), \ldots, g_k(\boldsymbol{x}))$. Said differently,

$$\mathbb{C}[\boldsymbol{x}]^G = \mathbb{C}[g_1, \ldots, g_k].$$

In general, the generators may be algebraically dependent, that is, may satisfy an algebraic relation $R(g_1, \ldots, g_k) = 0$ for some polynomial $R(y_1, \ldots, y_k) \not\equiv 0$. It is important to understand these relations. As a first result, it can be shown that $\mathbb{C}[\boldsymbol{x}]^G$ always contains some n algebraically independent elements, say $u_1, \ldots, u_n$. These need not generate the whole ring. But it turns out that $u_1, \ldots, u_n$ can be chosen so that $\mathbb{C}[\boldsymbol{x}]^G$ is an integral ring extension of its subring $\mathbb{C}[u_1, \ldots, u_n]$. This is Noether's Normalization Lemma.

In particular, $\mathbb{C}[\boldsymbol{x}]^G$ will be a finite $\mathbb{C}[u_1, \ldots, u_n]$ -module. A theorem that probably first appeared in an article by Hochster and Eagon [5] asserts that for finite groups G, the invariant ring is even a *free* $\mathbb{C}[u_1, \ldots, u_n]$ -module (one says that $\mathbb{C}[\boldsymbol{x}]^G$ is a *Cohen-Macaulay* module). That is to say, there exist elements $s_1, \ldots, s_l \in \mathbb{C}[\boldsymbol{x}]^G$ such that $\mathbb{C}[\boldsymbol{x}]^G = \bigoplus_{j=1}^{l} s_j \cdot \mathbb{C}[u_1, \ldots, u_n]$. This decomposition is called the *Hironaka decomposition*; the u_i are called *primary invariants*[4] and the s_j *secondary invariants*.[5] Therefore each invariant polynomial f has a unique decomposition

$$f = \sum_{j=1}^{l} s_j P_j(u_1, \ldots, u_n),$$

for some polynomials $P_j \in \mathbb{C}[x_1, \ldots, x_n]$.

Things are even better if G is a reflection group. An element $M \in GL(\mathbb{C}^n)$ is called a *reflection* if it has exactly one eigenvalue not equal to one. A finite subgroup of $GL(\mathbb{C}^n)$ is called a *reflection group* if it is generated by reflections. In a reflection group, $\mathbb{C}[\boldsymbol{x}]^G$ is even generated by n algebraically independent polynomials $u_1, \ldots, u_n$ and vice versa (Theorem of *Sheppard-Todd-Chevalley*) – so that the decomposition reduces to

$$f = P(u_1, \ldots, u_n)$$

for a *uniquely* determined polynomial P.

Here is how we shall go about constructing the equations for our Platonic stars: Find a polynomial in the invariant generators such that f has the required geometric properties. (Remember that when we speak of symmetry groups we do not restrict to proper rotations. The symmetry groups of the Platonic solids as we defined them are reflection groups. By the Sheppard-Todd-Chevalley Theorem, this can be checked by calculating the primary and secondary invariants.) Even though, for each f, the polynomial P is unique, there could be several f sharing the properties. This phenomenon will actually occur; it is realized by a certain flexibility in choosing the parameters of our equations. The families of stars which are thus obtained make certain parameter values look more natural than others. This is the case for the plane symmetric star with four vertices, where only one choice of parameters yields a hypocycloid, the famous Astroid (see example 9). For surfaces, the appropriate choice of parameters is still an open problem. This raises also the question of whether (in analogy to the rolling small circle inside a larger one for the Astroid) there is a recipe for contructing the Platonic stars with distinguished parameter values. We don't know the answer.

[4]In the following chapter on the construction and in the examples, we write u, v, w instead of u_1, u_2, u_3. Note that sometimes we do not need all three of them, as in the introductory example of the dodecahedron; but a general invariant polynomial may depend on all three.

[5]There exist algorithms to calculate these invariants. One is implemented in the free Computer Algebra System SINGULAR. See http://www.singular.uni-kl.de/index.html for information about SINGULAR and http://www.singular.uni-kl.de/Manual/latest/sing_1189.htm#SEC1266 for instruction.

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Construction of Stars

In this section the group $G \subset O_3(\mathbb{R})$ we consider once again one of the three real symmetry groups of the Platonic solids. If the scalars of the input of the algorithms for the calculation of primary and secondary invariants are contained in some subfield of $\mathbb{C}$, then the scalars of the output are also contained in this subfield, see [10, p.1]. In our examples the inputs are real matrices (the generators of G) and the outputs are the primary and secondary invariants that generate the invariant ring as a subring of $\mathbb{C}[x_1, \ldots, x_n]$. They even generate the real invariant ring, $\mathbb{R}[x_1, \ldots, x_n]^G$. See the last section "Technical Details" for a proof.

The symmetry groups of the Platonic solids are reflection groups. This implies that we have primary invariants $\{u, v, w\} \subset \mathbb{R}[x, y, z]$ such that $\mathbb{R}[x, y, z]^G = \mathbb{R}[u, v, w]$. In the following we always assume that we have already constructed a set of homogeneous primary invariants $\{u, v, w\} \subset \mathbb{R}[x, y, z]$.

Our aim is to construct a polynomial f in the invariant ring of G with prescribed singularities. By the results from the previous section we may write the polynomial uniquely in the form

$$f(u, v, w) = \sum_{id_1 + jd_2 + kd_3 \le d} a_{ijk} u^i v^j w^k, \tag{3}$$

where $d_1 = \deg(u), d_2 = \deg(v), d_3 = \deg(w)$, and $a_{ijk} \in \mathbb{R}$. Such a polynomial has the desired symmetries, so we may move on and prescribe the singularities. They should lie at the vertices of a Platonic or an Archimedean solid. Let S be a fixed Platonic (or Archimedean) solid. In the introduction we mentioned that these solids are vertex-transitive. This implies that the algebraic surface corresponding to the polynomial (3), which is an element of the invariant ring of the symmetry group of S, has to have the same local geometry at each vertex of S. Therefore it is sufficient to choose one vertex and impose conditions on $f(u, v, w)$ guaranteeing an A_2-singularity there.

We can always suppose that S has one vertex at $p := (1, 0, 0)$, otherwise we perform a coordinate change to make this true. Having a singularity is a local property of the surface, so we have to look closer at f at the point p. We do that by considering the Taylor expansion at p, that is, substitute $x + 1$ for x in $f(u(x, y, z), v(x, y, z), w(x, y, z))$. We have the following necessary condition for a singularity of type A_2, with c_1 and c_2 being real constants not equal to zero, see [1, p.209].

$$\begin{aligned} &F(x, y, z) \\ &\quad := f(u_1(x+1, y, z), u_2(x+1, y, z), u_3(x+1, y, z)) \\ &\quad = c_1(y^2 + z^2) + c_2 x^3 + \text{ higher order terms.} \end{aligned} \tag{4}$$

"Higher order terms" here refers to all terms that have weighted order, with weights (1/3, 1/2, 1/2), greater than 1—that is, all monomials $x^i\, y^j\, z^k$ with $i/3 + j/2 + k/2 > 1$.

If c_1 and c_2 have the same sign, the cusps will "point outward", that is, we obtain a star. If they have different signs the cusps will "point inward".

Now expanding $F(x, y, z)$ and comparing the coefficients of x, y, and z with the right-hand side of equation (4), we obtain a system of linear equations in the unknown coefficients of f from (3), that is, in our notation the parameters a_{ijk}. Additionally we obtain inequalities that give us information about whether we will obtain a star or an anti-star. In general this system of equations will be underdetermined. We will be left with free parameters, as we already saw in the introductory example of the dodecahedral star.

Evidently, in this construction we have to choose the degree d of the indetermined polynomial f. If we choose it too small, the system of equation may not have a solution; but we want d to be as small as possible subject to this. The degree d has to be greater or equal to three, clearly. It depends on the degrees of the primary invariants u_i, as we will see in the examples.

The same construction should work for any dimension n. The case of plane curves, $n = 2$, is easier to handle. Even there the results are quite nice, as we will see in the section on "plane dihedral stars". An interesting generalization for $n = 4$ would be to calculate "Schläfli stars", corresponding to the six convex regular polytopes in four dimensions, which were classified by Ludwig Schläfli, [2, p. 142].

We now conclude this section by demonstrating the above procedure in detail in the example of the octahedron and the cube. More examples will follow in the next section, namely, the remaining Platonic stars and some Archimedean stars. We will also present some selected surfaces with dihedral symmetries in real three-space.

Example 2 (Octahedral and Hexahedral Star). The octahedron (the Platonic solid with 6 vertices, 12 edges, and 8 faces) and its dual the cube (or hexahedron - with 8 vertices, 12 edges, and 6 faces) have the same symmetry group O_h, of order 48. We choose coordinates x, y, and z of $\mathbb{R}^3$ such that in these coordinates the vertices of the octahedron are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. Then O_h is generated by two rotations σ_1, σ_2 around the x- and the y-axes by $\pi/2$, together with the reflection τ in the xy-plane:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices are the input for the algorithm implemented in `SINGULAR` that computes the primary and secondary invariants. In this example the primary invariants that generate the invariant ring are the following (although it is easy to see that these three polynomials are invariant, it is not evident that they are *primary* invariants, that is, generate the invariant ring as an algebra):

$$\begin{aligned} u(x, y, z) &= x^2 + y^2 + z^2, \\ v(x, y, z) &= x^2y^2 + y^2z^2 + x^2z^2, \\ w(x, y, z) &= x^2y^2z^2. \end{aligned} \tag{5}$$

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Now how low can the degree be of our indeterminate polynomial? (Here and in the rest of this article degree means the usual total degree in x, y, z.) Clearly it must be even. A degree four polynomial yields no solvable system of equations. Let us try a polynomial of degree six,

$$f(u,v,w) = 1 + a_1u + a_2u^2 + a_3u^3 + a_4uv + a_5v + a_6w.$$

We substitute $x+1$ for x and expand the resulting polynomial $F(x,y,z) = f(u(x+1,y,z), v(x+1,y,z), w(x+1,y,z))$. Next we collect the constant terms and the linear, quadratic, and cubic terms, and compare them with the right-hand side of (4). This yields the following system of linear equations:

$$\begin{aligned}
&\text{Constant term of } F: && 1 + a_1 + a_2 + a_3 = 0,\\
&\text{Coefficient of } x: && 2a_1 + 4a_2 + 6a_3 = 0,\\
&\text{Coefficient of } x^2: && a_1 + 6a_2 + 15a_3 = 0,\\
&\text{Coefficient of } y^2 \text{and } z^2: && a_5 + a_1 + a_4 + 2a_2 + 3a_3 = c_1,\\
&\text{Coefficient of } x^3: && 4a_2 + 20a_3 = c_2.
\end{aligned} \tag{6}$$

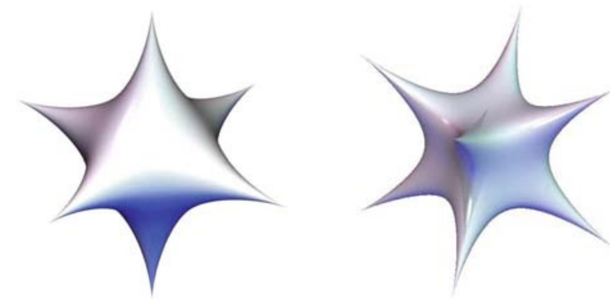
Since the monomials y, z, xy, xz, yz do not appear, we do not obtain further equations from them.

Solving the first three equations from the above system yields the polynomial (7) with three free parameters. In addition we get an inequality from the condition that the coefficient of x^3 must have the same sign as the coefficient of y^2 and z^2 if we want to obtain a star. Substituting the solution of the first three equations yields $c_1 = a_4 + a_5$ and $c_2 = -8$. We impose $a_4 + a_5 \neq 0$ to obtain a star or an anti-star:

$$f(u,v,w) = (1-u)^3 + a_4uv + a_5v + a_6w, \quad \text{with } a_4 + a_5 \neq 0. \tag{7}$$

If we allowed all three parameters to be zero we would obtain the sphere of radius one. We have already made clear that for $a_4 + a_5 = 0$ the zero set of (7) cannot have singularities of type A_2, so it must either be smooth or have singularities of a different type. If we choose $a_4 = c$, $a_5 = 0$ and $a_6 = -9c$, $c \neq 0$, the zero set is again not an octahedral star, for it has too many singularities; we will describe this phenomenon in more detail after the example of the hexahedral star. For the other choices of parameters the corresponding zero sets are octahedral stars for $a_4 + a_5 < 0$, or anti-stars for $a_4 + a_5 > 0$. See figure 5a. Sometimes additional components appear and the stars or anti-stars become unbounded. In all examples presented in this article, especially when there is more than one free parameter, special behaviors (such as additional components, unboundedness, or maybe more singularities than expected) may occur for special choices of the free parameters. Most pictures presented are merely based on (good) choices of parameters. As we already mentioned, it would be interesting to find conditions that prevent this behavior so that we could prescribe boundedness as well as irreducibility in the definition of a star.

Now we turn to the Platonic solid dual to the octahedron, namely the cube. If we use the same coordinates as before, it has vertices at $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. But as we already mentioned, we prefer to have a vertex at (1, 0, 0),



(a) Octahedral star, $a_4 = -100$, $a_5 = 0$, $a_6 = 0$. **(b)** Hexahedral star, $a_1 = -100$, $a_2 = 0$, $a_3 = 0$.

Figure 5. Octahedral and hexahedral star.

so we perform a rotation to achieve this, and write the invariants in the new coordinates. With these invariants we can proceed as in the example of the octahedron. Again we get no solution with degree four and must use a polynomial of degree six. After solving the system of equations we perform the inverse coordinate change and obtain the following polynomials (8) as candidates for hexahedral stars or anti-stars:

$$\begin{aligned} f(u,v,w) = {} & 1 - 3u + a_1u^2 + a_2u^3 + a_3uv + (9 - 3a_1)v \\ & -9(3 + a_3 + 3a_2)w, \end{aligned} \tag{8}$$

with $3a_1 + 9a_2 + 2a_3 \neq 0$. For $a_1 = 3$, $a_2 = -1$, and $a_3 = 0$ we obtain the sphere. Other choices such that $3a_1 + 9a_2 + 2a_3 = 0$ may give singularities but cannot give A_2-singularities. Again there is a choice of parameters, namely $a_1 = 3$, $a_2 = -1$ and $a_3 = c \neq 0$, for which the surface has too many singularities. We obtain the same object as in the example of the octahedral star, with equation (9) below. In the other cases we obtain a hexahedral star for $3a_1 + 9a_2 + 2a_3 < 0$ (figure 5b), or anti-star for $3a_1 + 9a_2 + 2a_3 > 0$, even though, as in the example of the octahedral stars, additional components may appear.//

Before proceeding, let us say more about the surface (9) that emerged as a special case both of the octahedral and the hexahedral stars. It has 14 singularities, exactly at the vertices of the octahedron and the cube, see figure 6,

$$f(u,v,w) = (1-u)^3 + cuv - 9cw, \quad \text{with } c \neq 0. \tag{9}$$

We will call this object a 14-*star* or 14-*anti-star* for $c < 0$ or $c > 0$, respectively. The parameter value $c = 0$ yields obviously a sphere. See figure 7 for an illustration of the dependence on the parameter.

This star does not correspond to a Platonic or Archimedean solid, but to the polyhedron S that is the convex hull of the vertices of a hexahedron and an octahedron that have all the same Euclidean diameter. This polyhedron has 14 vertices, 36 edges, and 24 faces, which are isosceles triangles. See figure 6b. It is remarkable that it appears here, for the symmetry group O_h does not act transitively on its vertices! The vertices of the hexahedron form one orbit, the vertices of the octahedron another. If we followed the program of this paper and sought such a star, we would need to fix two points, one in each orbit, and

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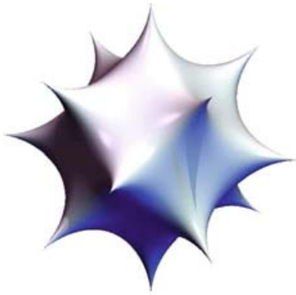
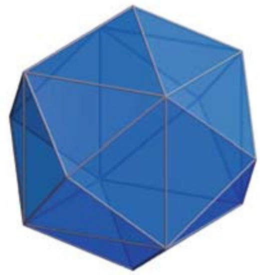


(a) 14-star, $c = -50$.



(b) Polyhedron S corresponding to the 14-star.

Figure 6. 14-star and the corresponding convex polyhedron.

prescribe singularities at both. This would lead to a larger system of linear equations.

Note, by the way, that if the vertices of the octahedron and the cube have different Euclidean norms of a certain ratio, namely $2/\sqrt{3}$, the convex hull is a Catalan solid, called the *rhombic dodecahedron* (14 vertices, only 12 faces because the triangles collapse in pairs into rhombi, and 24 edges). This is the dual of the Archimedean solid called the cuboctahedron that will be discussed later.

Further Platonic and Archimedean Stars

Example 3 (Tetrahedral star). The tetrahedron is the Platonic solid with 4 vertices, 6 edges, and 4 faces. Its symmetry group T_d has 24 elements. If we choose coordinates x, y, z such that one vertex is (1, 1, 1), the invariant ring is generated by the primary invariants displayed in (10). One could also choose (1, 0, 0) as a vertex to avoid a coordinate change, but then the invariants would be more complicated. Note how different the primary invariants are from those of the octahedron and the hexahedron (5).

$$\begin{aligned} u(x,y,z) &= x^2 + y^2 + z^2, \\ v(x,y,z) &= xyz, \\ w(x,y,z) &= x^2y^2 + y^2z^2 + z^2x^2. \end{aligned} \qquad (10)$$

Here a degree three polynomial yields no solution but degree four already suffices:

$$f(u,v,w) = 1 - 2u + cu^2 + 8v - (3c+1)w, \quad \text{with } c \neq 1. \qquad (11)$$

For $c < 1$ we obtain a star, for $c > 1$ an anti-star. Its singular points (for $c \neq 1$) are (1, 1, 1), (−1, −1, 1), (1, −1, −1), and (−1, 1, −1). If we choose $c = 1$ in (11) the polynomial f has four linear factors, see figure 8j:

$$f = (x - 1 + z - y)(x - 1 - z + y)(x + 1 - z - y) \cdot (x + 1 + z + y).$$

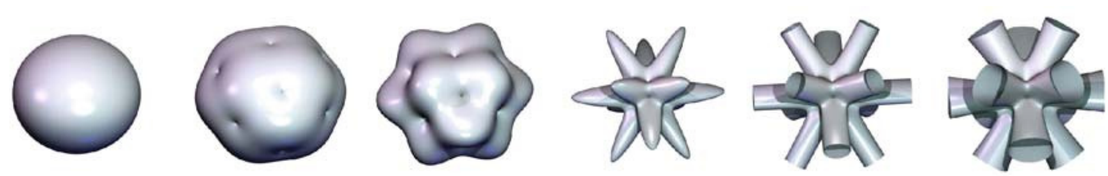
For very small c-values there seem to appear four additional cusps at the vertices of a tetrahedron dual to the first one; but these points stay smooth for all $c \in \mathbb{R}$. For $0 < c < 1$ the zero set of our polynomial has additional components besides the desired "star shape". For $c > 1$ we get anti-stars, see figure 8. Note that for $c > 0$ the surfaces are unbounded. So unlike the previous examples, there are no bounded anti-stars.

Example 4 (Icosahedral star). The icosahedron is the Platonic solid with 12 vertices, 30 edges, and 20 faces. The symmetry group I_h of the icosahedron and its dual, the dodecahedron, has 120 elements. Its invariant ring is generated by the polynomials (2) from the first example, together with a third one (12),

$$\begin{aligned} w(x,y,z) = {} & (4x^2 + z^2 - 6xz) \\ & \cdot (z^4 - 2z^3x - x^2z^2 + 2zx^3 + x^4 - 25y^2z^2 \\ & \quad - 30xy^2z - 10x^2y^2 + 5y^4) \\ & \cdot (z^4 + 8z^3x + 14x^2z^2 - 8zx^3 + x^4 - 10y^2z^2 \\ & \quad - 10x^2y^2 + 5y^4). \end{aligned} \qquad (12)$$

We point out that both invariants v and w factorize (over $\mathbb{R}$) into six, respectively ten, linear polynomials. The zero sets of these linear polynomials are related to the geometry. To explain this, we introduce a new terminology: Given a Platonic solid P, we call a plane through the origin a *centerplane*

(a) $c = -10000$. **(b)** $c = -1000$. **(c)** $c = -100$. **(d)** $c = -10$. **(e)** $c = -1$. **(f)** $c = -0.1$.



(g) $c = 0$. **(h)** $c = 0.1$. **(i)** $c = 1$. **(j)** $c = 3.5$. **(k)** $c = 4$. **(l)** $c = 5$.

Figure 7. 14-star and anti-star, for $c \geq 4$ the surfaces are clipped by a sphere with radius 5.

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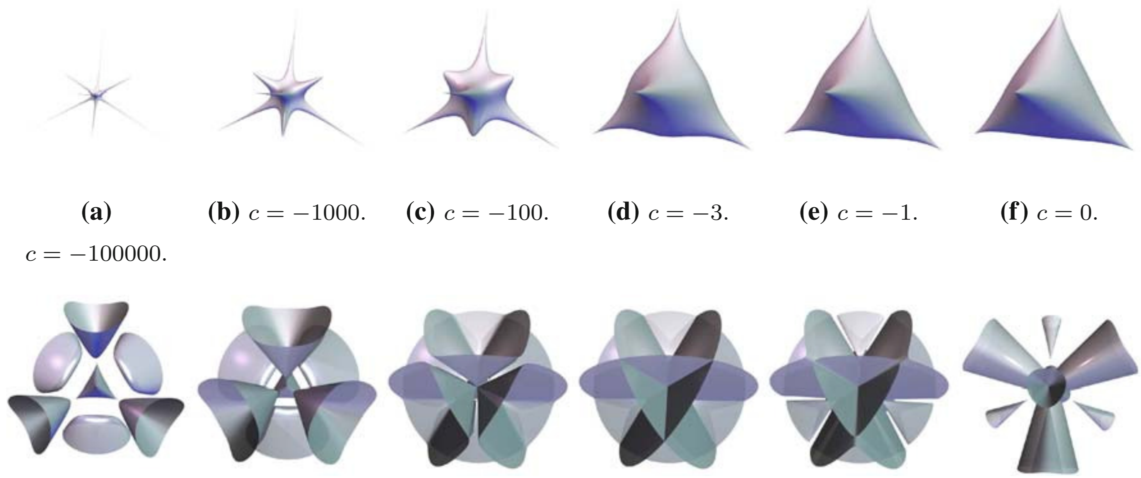


(a) $c = -100000$. (b) $c = -1000$. (c) $c = -100$. (d) $c = -3$. (e) $c = -1$. (f) $c = 0$.

(g) $c = 0.3$. (h) $c = 0.6$. (i) $c = 0.98$. (j) $c = 1$. (k) $c = 1.02$. (l) $c = 3$.

Figure 8. Tetrahedral star (and anti-star) with varying parameter value c; for $c > 0$ the images are clipped by a sphere with radius 5.

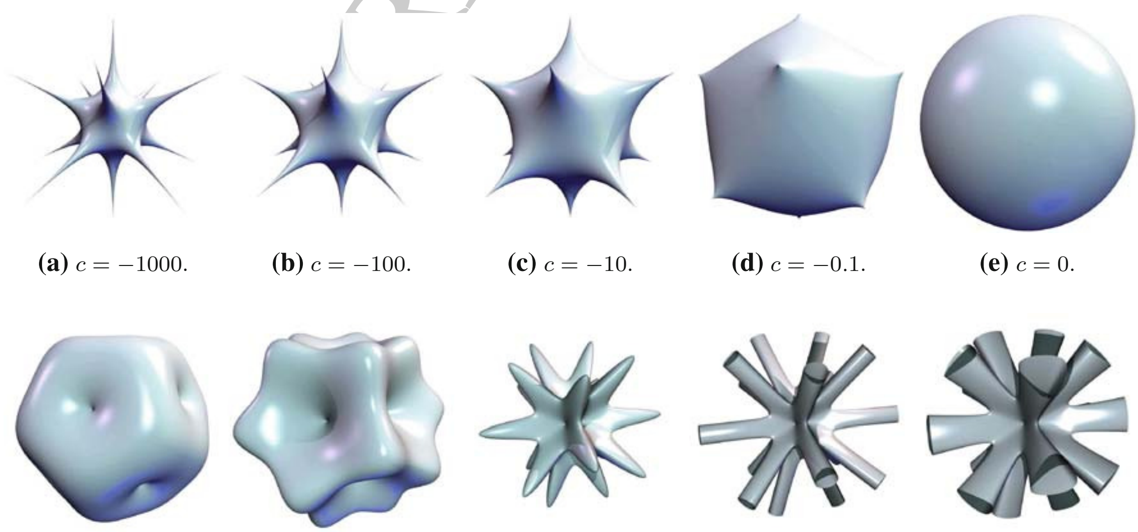
of P if it is parallel to a face of the solid. The dodecahedron has twelve faces and six pairs of parallel faces, so it has six centerplanes. They correspond to the six linear factors of the second invariant v. Analogously the icosahedron has ten centerplanes, which give the linear factors of w. We have written out the factorization in the last section, see (38).

For the dodecahedral and the icosahedral star the "smallest possible degree" is six. The third invariant has degree ten, so we do not use it in either case. An equation for the icosahedral star is the following:

$$f(u,v,w) = (1-u)^3 + cu^3 + cv, \quad \text{with } c \neq 0. \tag{13}$$

Figure (9) shows icosahedral stars ($c < 0$) and anti-stars ($c > 0$) for various c-values. For $c = 0$ we get a sphere of radius one. For all $c \neq 0$ the 12 singularities lie on this sphere. For $c = 27/32$ the surface has points at infinity in the direction of normals to the faces of the corresponding icosahedron. Note that this is just the negative value of c for which the dodecahedral stars are unbounded. The illustrations suggest that for $c > 27/32$ the surfaces become unbounded while they are bounded for $c < 27/32$.

EXAMPLE 5 (Cuboctahedral star). The cuboctahedron is the Archimedean solid with 14 faces (6 squares and 8 equilateral triangles), 24 edges, and 12 vertices. See figure 11b. Its symmetry group is that of the octahedron and cube. We use the invariants (5). Our construction yields a polynomial of degree six, with three free parameters:



(a) $c = -1000$. (b) $c = -100$. (c) $c = -10$. (d) $c = -0.1$. (e) $c = 0$.

(f) $c = 0.1$. (g) $c = 0.5$. (h) $c = 0.8$. (i) $c = 27/32$. (j) $c = 0.9$.

Figure 9. Icosahedral star and anti-star, with varying parameter c; for $c > 27/32$ the surfaces are clipped by a sphere with radius 11.

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Figure 10. The zero set of $x^3 + y^2 - z^2 = 0$.

$$f(u,v,w) = 1 - 3u + au^2 + (12 - 4a)v + bu^3 - (4 + 4b)uv + cw, \tag{14}$$

with $a + b \neq 2$ and $8(a + b) - c \neq 16$. For $a = 3$, $b = -1$, and $c = 0$ we obtain a sphere. In this example we have a new kind of behavior. We always got inequalities from the condition that the coefficients of x^3 and $y^2 + z^2$ in the Taylor expansion of f in (1, 0, 0) should have the same sign. In this case the coefficient of x^3 is -8, but y^2 and z^2 have different coefficients, namely $a + b - 2$ and $16 - 8(a + b) + c$, respectively. So if both are negative we obtain stars, see figure 11a; if both are positive, anti-stars; but if they have different signs, we will have a "new" object, whose singularities look, up to local analytic coordinate transformations[6], such as $x^3 + y^2 - z^2 = 0$, see figure 10. The singularities always lie on a sphere of radius one.

Example 6 (Soccer star). The truncated icosahedron is the Archimedean solid which is obtained by "cutting off the vertices" of an icosahedron. It is known as the pattern of a soccer ball. It has 32 faces (12 regular pentagons and 20 regular hexagons), 60 vertices, and 90 edges. See figure 11d. Its symmetry group is the icosahedral group I_h. For this example we do need the third invariant, because the first polynomial that yields a solvable system of equations is of degree ten. We obtain the following equation with four free parameters, in the invariants (2) and (12):

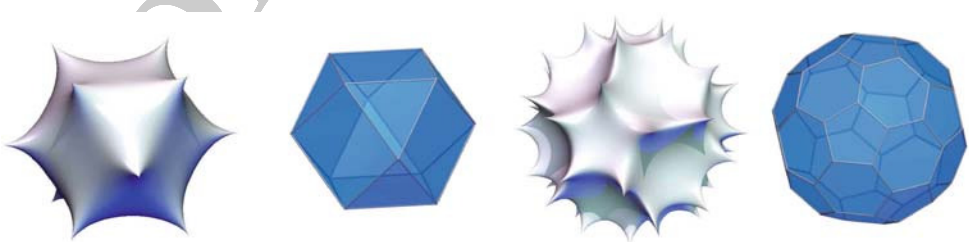
$$\begin{aligned}
&f(u,v,w)\\
&= 1 + \Bigg(\frac{128565 + 115200\sqrt{5}}{1295029}c_3 + \frac{49231296000\sqrt{5} - 93078919125}{15386239549}\\
&\qquad c_4 - c_1 - 3c_2 - 3\Bigg)u+\\
&+ \Bigg(\frac{-230400\sqrt{5} - 257130}{1295029}c_3 + \frac{238926989250 - 126373248000\sqrt{5}}{15386239549}\\
&\qquad c_4 + 3c_1 + 8c_2 + 3\Bigg)u^2+\\
&+ \Bigg(\frac{115200\sqrt{5} + 128565}{1295029}c_3 + \frac{91097280000\sqrt{5} - 172232645625}{15386239549}\\
&\qquad c_4 - 3c_1 - 6c_2 - 1\Bigg)u^3+\\
&+ \left(c_3 + \frac{121075 - 51200\sqrt{5}}{11881}c_4\right)v + \left(\frac{102400\sqrt{5} - 242150}{11881} - 2c_3\right)\\
&\cdot uv + c_1u^4 + c_2u^5 + c_3u^2v + c_4w,\\
&\quad \text{with } c_4 \neq 0 \text{ and } b(c_1, c_2, c_3, c_4)\\
&\quad := (991604250 - 419328000\sqrt{5})c_4 + 20316510c_3+\\
&\qquad + (135776068 - 121661440\sqrt{5})c_2\\
&\qquad + (33944017 - 30415360\sqrt{5})c_1 + 30415360\sqrt{5}\\
&\qquad - 33944017 \neq 0.
\end{aligned} \tag{15}$$

We obtain stars if we choose c_1, c_2, c_3, and c_4 such that c_4 and $b(c_1, c_2, c_3, c_4)$ have the same sign. Otherwise we obtain anti-stars. See figure 11c.

Plane Dihedral Stars

Analogous to the Platonic and Archimedean stars in three dimensions, we will define plane stars. Let P be a regular polygon with m vertices. Its symmetry group in $O_2(\mathbb{R})$ is the *dihedral group* denoted by D_m. It is of order $2m$. A *plane m-star* is a plane algebraic curve that is invariant under the action of the dihedral group D_m and has exactly m singularities of type A_2 (that is, with equation $x^3 + y^2 = 0$) at the vertices of P, "pointing away form the origin"; otherwise, that is, if the cusps "point towards the origin", we speak of an *m-anti-star*. In the examples presented here the singularities will be at the mth roots of unity.

If we consider the dihedral groups as subgroups of $O_2(\mathbb{R})$, they are reflection groups. This is not true if we view them as subgroups of $O_3(\mathbb{R})$, as we do in the next batch of examples.



(a) Cuboctahedral star, $a = 0$, $b = 0$, $c = -100$. **(b)** Cuboctahedron. **(c)** Soccer star, $c_1 = -100$, $c_2 = -100$, $c_3 = -100$, $c_4 = -100$. **(d)** Truncated icosahedron.

Figure 11. Two Archimedean solids and stars.

6FL01 [6]But if we allow *complex* local analytic coordinate transformations, the singularities are still of type A_2.

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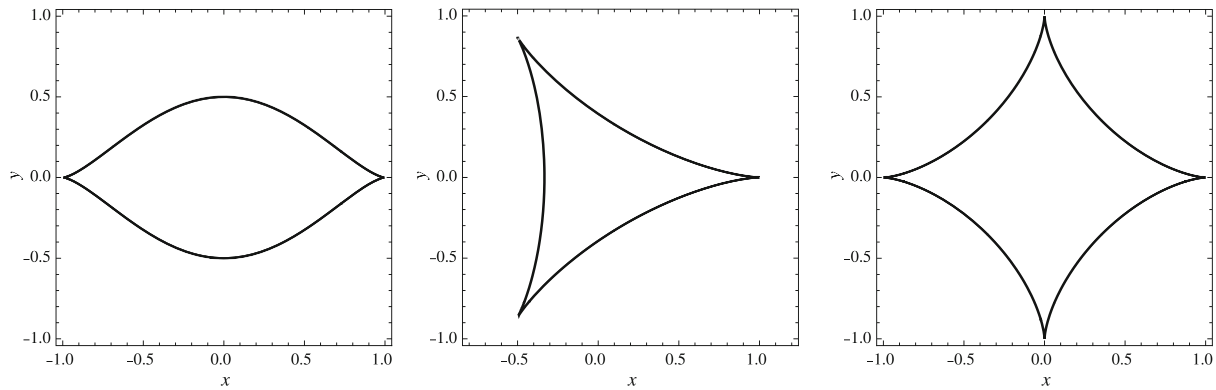

(a) A 2-star (19), $c_1 = -4$. **(b)** The Deltoid. **(c)** The Astroid.

Figure 12. Some plane dihedral stars.

There is another way to construct plane stars, namely as hypocycloids. A hypocycloid is the trace of a point on a circle of radius r that is rolling within a bigger circle of radius R. If the ratio of the radii is an integer, $R : r = k$, then the curve is closed and has exactly k cusps but no self-intersections. Hypocycloids have a quite simple trigonometric parametrization (16):

$$\varphi \mapsto \big((k-1)r\cos\varphi + r\cos(k-1)\varphi,\\ (k-1)r\sin\varphi - r\sin(k-1)\varphi\big), \quad \varphi \in [0, 2\pi]. \tag{16}$$

There are algorithms for the implicitization of trigonometric parametrization, see [6]. It turns out that hypocycloids are stars in our sense: they have the correct symmetries and singularities of type A_2. In the construction of stars via primary invariants we always try to find a polynomial of minimal degree that satisfies these properties. We will see that sometimes the hypocycloids coincide with the stars we obtain that way. In one of the examples presented here, namely the 5-star, the degree of the implicitization of the hypocycloid is higher than the degree of the polynomial our construction yields. This suggests that we define a "star" as the zero set of the polynomial of minimal degree satisfying all other conditions.

Example 7 (2-star). The group D_2 has primary invariants

$$\begin{aligned} u(x,y) &= x^2,\\ v(x,y) &= y^2. \end{aligned} \tag{17}$$

Our constructions yields the degree six polynomial with six free parameters:

$$f(u,v) = (1-u)^3 + c_1v + c_2uv + c_3v^2 + c_4uv^2 + c_5u^2v \\ + c_6v^3, \quad \text{with } c_1 + c_2 + c_5 \neq 0. \tag{18}$$

The choice $c_1 \neq 0$ and the remaining parameters equal to zero yield the simple equation

$$f(u,v) = (1-u)^3 + c_1v, \quad \text{with } c_1 \neq 0. \tag{19}$$

For $c_1 < 0$ we obtain a 2-star. The corresponding curve runs through the points $(0, \pm\frac{1}{\sqrt{-c_1}})$ and is bounded. See figure 12a. For $c_1 > 0$ it is an unbounded anti-star. In both cases it has two singularities at $(\pm 1, 0)$.

The hypocycloid for $k = 2$ is parametrized by $(2r \cos\varphi, 0)$ where φ is in $[0, 2\pi]$. So it is not an algebraic curve but an interval on the x-axis.

Example 8 (3-star). The primary invariants of D_3 are

$$\begin{aligned} u(x,y) &= x^2 + y^2,\\ v(x,y) &= x^3 - 3xy^2. \end{aligned} \tag{20}$$

In this case a degree four polynomial suffices to generate a star, see figure 12b. The polynomial (21) is completely determined, we have no free parameters. It coincides with the hypocycloid for $k = 3$, which is also called Deltoid:

$$f(u,v) = 1 - 6u - 3u^2 + 8v. \tag{21}$$

Example 9 (4-star). The dihedral group of order eight, D_4, has primary invariants

$$\begin{aligned} u(x,y) &= x^2 + y^2,\\ v(x,y) &= x^2y^2. \end{aligned} \tag{22}$$

Our construction yields the following polynomial of degree six with two free parameters:

$$f(u,v) = (1-u)^3 + c_1v + c_2uv, \quad \text{with } c_1 + c_2 \neq 0, \tag{23}$$

For $c_1 + c_2 < 0$ we obtain stars, for $c_1 + c_2 > 0$ anti-stars. In both cases additional components might appear. The curves become unbounded for $c_2 > 4$.

The hypocycloid with four cusps is also called Astroid. Its implicit equation is $(1-u)^3 - 27v = 0$. So if we choose $c_1 = -27$ and $c_2 = 0$ in (23) we obtain the same curve. See figure 12c.

Example 10 (5-star). The primary invariants of D_5 are

$$\begin{aligned} u(x,y) &= x^2 + y^2,\\ v(x,y) &= x^5 - 10x^3y^2 + 5xy^4. \end{aligned} \tag{24}$$

If we try a degree five polynomial, we obtain (25) with no free parameters. It only permits anti-stars.

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$$f(u,v) = 1 - \frac{10}{3}u + 5u^2 - \frac{8}{3}v. \quad (25)$$

So let us use degree six. This yields the following polynomial for plane 5-stars or anti-stars:

$$f(u,v) = 1 - \frac{c+10}{3}u + (2c+5)u^2 - \frac{8}{3}(1+c)v + cu^3, \quad \text{with } c \neq -1,\ 5. \quad (26)$$

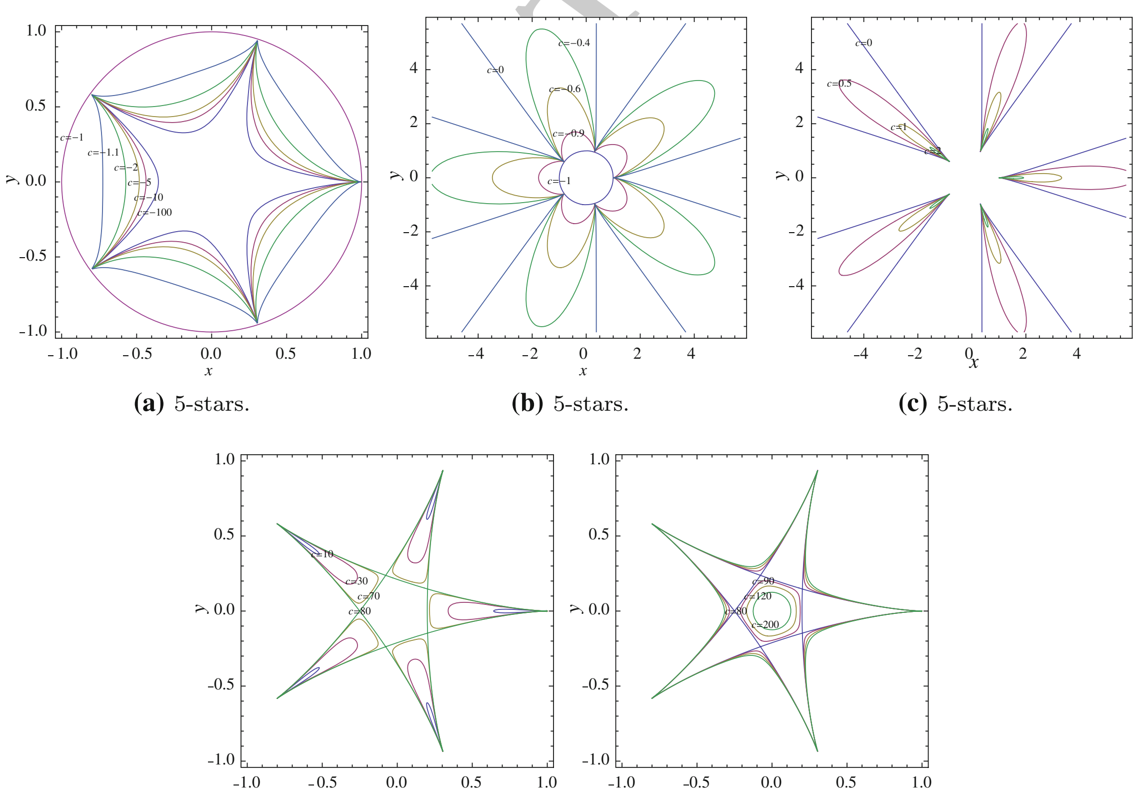
Here, as the parameter value c varies we observe a quite curious behavior. For $c < -1$ one obtains a star, the smaller c gets, the smaller is its "inner radius", see figure 13a. The choice $c = -1$ yields a circle with radius one—the circle containing the five singularities of the (26) for other c. For $-1 < c < 5$ the cusps of (26) point inward, that is, we have anti-stars. For $-1 < c < 0$ the curve has one bounded component; for $c = 0$, it is unbounded with five components, figure 13b; for $0 < c < 5$ the curve is again bounded, but has five components, like drops falling away from the center, figure 13c. For $c = 5$ only finitely many points satisfy the equation, the five points that are singular in the other cases. If we choose $c > 5$ we obtain stars again, that is, the cusps point outward, even though for $5 < c < 80$ the curve also has five components, like drops falling towards the origin, figure 13d. The curve we obtain for $c = 80$ is special in that it has self-intersections, that is, five additional singularities. They lie on a circle with radius one quarter, on a regular pentagon. These "extra singularities" are of type A_1, that is, they have, up to analytic coordinate transformations, equation $x^2 + y^2 = 0$. One could call this curve an *algebraic pentagram*. For $c > 80$ the curve has two components, see figure 13e.

The implicit equation of the hypocycloid with five cusps is already of degree eight, while the polynomial we found with our construction has degree six. The two equations cannot coincide for any choice of the free parameter c.

Dihedral "Pillow Stars" in $\mathbb{R}^3$

If we consider the dihedral groups D_m as subgroups of $O_3(\mathbb{R})$, they cease to be reflection groups, so we have to consider the secondary invariants as well. The number of secondary invariants depends on the order of the group and the degrees of the primary invariants, see [10, p.41]. In the examples we give here there are always two secondary invariants. The first one, s_1, is always 1, so we do not mention it every time but just give the second one, s_2.

In this section our aim is to construct surfaces that are invariant under the action of D_m with singularities at the mth roots of unity in the xy-plane, and which in addition pass through the points $(0, 0, \pm c)$ with $0 \neq c \in \mathbb{R}$. Intuitively the resulting surface should look like a pillow. To lead to such a shape, more conditions, such as boundedness and connectedness, would have to be imposed. We do not have a systematic theory, but our experimental results seem promising. In these examples we have a large



(a) 5-stars. **(b)** 5-stars. **(c)** 5-stars.

(d) 5-anti-stars. **(e)** 5-anti-stars.

Figure 13. 5-stars and anti-stars.

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(a) D_3-star. **(b)** D_4-star. **(c)** D_5-star.

Figure 14. Pillow stars.

Figure 15. Zitrus for $c = -4$.

number of free parameters, unfortunately. We have tried to choose values giving attractive results.

Example 11 (D_3). The primary invariants of $D_3 \subset O_3(\mathbb{R})$ are

$$\begin{aligned} u(x,y,z) &= z^2, \\ v(x,y,z) &= x^2 + y^2, \\ w(x,y,z) &= x^3 - 3xy^2; \end{aligned} \tag{27}$$

its secondary invariant is

$$s_2(x,y,z) = 3x^2yz - y^3z. \tag{28}$$

A polynomial of degree three yields no solution. The general equation of a degree four polynomial in the invariant ring of D_3 is $f_1(u, v, w) + b\, s_2$, where $f_1(u, v, w)$ is an indeterminate polynomial of degree four in $\mathbb{R}[u, v, w]$ as in the previous examples, and b is a constant. A degree four polynomial suffices to obtain a solvable system of equations. It yields the following polynomial with three free parameters:

$$f(u,v,w) = 1 - \frac{1 + c_1c^4}{c^2}u + c_1u^2 + c_2uv - 6v - 3v^2 + 8w, \tag{29}$$

with $-(1 + c_1c^4) + c^2c_2 < 0$. Note that the secondary invariant s_2 does not appear in the above polynomial, its coefficient b is zero. We obtain a nice result for $c_1 = c_2 = 0$, $c = 1/3$, see figure 14a.

Example 12 (D_4). The group D_4 has the following primary and secondary invariants:

$$\begin{aligned} u(x,y,z) &= z^2, \\ v(x,y,z) &= x^2 + y^2, \\ w(x,y,z) &= x^2y^2, \end{aligned} \tag{30}$$

$$s_2(x,y,z) = x^3yz - xy^3z. \tag{31}$$

Our construction yields a degree six polynomial; as in the previous example, the secondary invariant s_2 happens to drop out:

$$\begin{aligned} f(u,v) = 1 &- \frac{1 + c_1c^4 + c_4c^6}{c^2}u - 3v + c_1u^2 + c_2uv + 3v^2 \\ &+ c_3w + c_4u^3 - v^3 + c_5uw + +c_6vw + c_7uv^2 \\ &+ c_8u^2v, \end{aligned} \tag{32}$$

with $c_3 + c_6 < 0$ and $-(1 + c_1c^4 + c_4c^6) + c^2(c_2 + c_7) < 0$. See figure 14b for the resulting surface, where we chose $c = 1/3$, $c_3 = -27$ and set all the other parameters to zero.

Example 13 (D_5). The primary invariants of D_5 are

$$\begin{aligned} u(x,y,z) &= 0z^2, \\ v(x,y,z) &= x^2 + y^2, \\ w(x,y,z) &= x^5 - 10x^3y^2 + 5xy^4. \end{aligned} \tag{33}$$

Its secondary invariant is

$$s_2(x,y,z) = 5x^4yz - 10x^2y^3z + y^5z. \tag{34}$$

A degree five polynomial already produces a solvable system of equations, but the resulting polynomial with three free parameters only permits anti-stars. So we choose a polynomial of degree six; again s_2 does not appear:

$$\begin{aligned} f(u,v,w) = 1 &- \frac{1 + c_1c^4 + c_3c^6}{c^2}u - \frac{10 + c_4}{3}v + c_1u^2 \\ &+ c_2uv + (5 + 2c_4)v^2 - \frac{8}{3}(1 + c_4)w + \\ &+ c_3u^3 + c_4v^3 + c_5uv^2 + c_6u^2v. \end{aligned} \tag{35}$$

The zero sets of these polynomials are stars for

$$\begin{aligned} &c_4 + 1 < 0 \text{ and } -(1 + c_1c^4 + c_3c^6) + c^2(c_2 + c_5) < 0, \quad \text{or} \\ &c_4 - 5 > 0 \text{ and } -(1 + c_1c^4 + c_3c^6) + c^2(c_2 + c_5) > 0. \end{aligned} \tag{36}$$

A nice choice for the free parameters is $c = 1/3$, $c_4 = -3$, setting all other parameters equal to zero. See figure 14c.

Example 14 (Zitrus). The last example we want to present is the surface Zitrus. It is the plane 2-star rotated around the x-axis (figure 15). Its equation is

$$f(x,y,z) = (1 - (x^2 + y^2 + z^2))^3 + c(y^2 + z^2), \quad \text{with } c < 0. \tag{37}$$

Outlook

In all the examples presented above we observed "unwanted" behavior for special choices of the free parameters: the surfaces became unbounded at some point, or additional components appeared. Sometimes we even had more singularities, or singularities of a different type than we expected. Further investigations would be necessary to find conditions preventing such behavior. After this is done, one could refine the definition of "stars" and "anti-stars" by demanding that the surfaces be bounded and irreducible.

Dual (Platonic) solids have the same symmetry group, hence the same primary invariants were used to construct the corresponding stars. But there seems to be no obvious duality between the stars such as occurs for dual (Platonic) solids.

Technical Details

Factorization of the primary invariants of I_h

In example (4) of the icosahedral stars, we claimed that two of the primary invariants of I_h factor into linear polynomials corresponding to the centerplanes of the dodecahedron and icosahedron, respectively, and we promised to give the factorizations explicitly. Here they are:

$$
\begin{aligned}
&v(x,y,z)\\
&= -\frac{1}{16}z(2x+z)\left((\sqrt{5}+1)x-\sqrt{10-2\sqrt{5}}y-2z\right)\\
&\quad\cdot\left((\sqrt{5}+1)x+\sqrt{10-2\sqrt{5}}y-2z\right)\\
&\quad\cdot\left((\sqrt{5}-1)x-\sqrt{10+2\sqrt{5}}y+2z\right)\\
&\quad\cdot\left((\sqrt{5}-1)x+\sqrt{10+2\sqrt{5}}y+2z\right),\\
&w(x,y,z) = -\frac{1}{20250000}\left(-3x+x\sqrt{5}+z\right)\left(3x+x\sqrt{5}-z\right)\\
&\quad\cdot\left(-2x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}+5\sqrt{3}y\right.\\
&\qquad\left.-\sqrt{75+30\sqrt{5}z}\right)\\
&\quad\cdot\left(-2x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\cdot\sqrt{5}\right.\\
&\qquad\left.-5\sqrt{3}y-\sqrt{75+30\sqrt{5}z}\right)\\
&\quad\cdot\left(2x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}\right.\\
&\qquad\left.-5\sqrt{3}y+\sqrt{75-30\sqrt{5}z}\right)\\
&\quad\cdot\left(2x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}\right.\\
&\qquad\left.+5\sqrt{3}y+\sqrt{75-30\sqrt{5}z}\right)\\
&\quad\cdot\left(-x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}\right.\\
&\qquad\left.-5y\sqrt{5}\sqrt{3}+5\sqrt{3}y+2\sqrt{75+30\sqrt{5}z}\right)\\
&\quad\cdot\left(-x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}\right.\\
&\qquad\left.+5y\sqrt{5}\sqrt{3}-5\sqrt{3}y+2\sqrt{75+30\sqrt{5}z}\right)\\
&\quad\cdot\left(x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}+5y\sqrt{5}\sqrt{3}\right.\\
&\qquad\left.+5\sqrt{3}y-2\sqrt{75-30\sqrt{5}z}\right)\\
&\quad\cdot\left(x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}-5y\sqrt{5}\sqrt{3}\right.\\
&\qquad\left.-5\sqrt{3}y-2\sqrt{75-30\sqrt{5}z}\right).
\end{aligned}
\tag{38}
$$

The invariant ring $\mathbb{R}[\mathbf{x_1},\ldots,\mathbf{x_n}]^G$

Let $G \subset GL(\mathbb{R}^n)$ be a finite subgroup. Then there exist n homogeneous, algebraically independent polynomials $u_1,\ldots,u_n \in \mathbb{C}[x_1,\ldots,x_n]$ (called the primary invariants of G) and l (depending on the cardinality of G and the degrees of the u_i) polynomials $s_1,\ldots,s_l \in \mathbb{C}[x_1,\ldots,x_n]$ (the secondary invariants of G) such that the invariant ring decomposes as $\mathbb{C}[x_1,\ldots,x_n]^G = \bigoplus_{j=1}^{l} s_j\mathbb{C}[u_1,\ldots,u_n]$. There are algorithms to calculate these primary and secondary invariants, see [10, p.25]. Also in [10, p.1] it is claimed that if the scalars of the input for these algorithms are contained in a subfield K of C, then all the scalars in the output will also be contained in K. So in our case with $G \subset GL(\mathbb{R}^n)$, the primary and secondary invariants will be real polynomials: $u_1,\ldots,u_n,s_1,\ldots,s_l \in \mathbb{R}[x_1,\ldots,x_n]$.

Now the claim is, in the notation above: $\mathbb{R}[x_1,\ldots,x_n]^G = \bigoplus_{j=1}^{l} s_j\mathbb{R}[u_1,\ldots,u_n]$.

The first inclusion $\mathbb{R}[x_1,\ldots,x_n]^G \supset \bigoplus_{j=1}^{l} s_j\mathbb{R}[u_1,\ldots,u_n]$ is trivial. We prove the opposite inclusion: Let $f \in \mathbb{R}[x_1,\ldots,x_n]^G \subset \mathbb{C}[x_1,\ldots,x_n]^G$ be an invariant polynomial. As $\mathbb{C}[x_1,\ldots,x_n]^G$ equals $\bigoplus_{j=1}^{l} s_j\mathbb{C}[u_1,\ldots,u_n]$, we can write f uniquely in the following form:

$$f(x_1,\ldots,x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha\in A} c_{j\alpha}u^\alpha,$$

where $c_{j\alpha} = d_{j\alpha} + ie_{j\alpha}$ are complex constants, and A is some finite subset of $\mathbb{N}^n$. Then

$$
\begin{aligned}
f(x_1,\ldots,x_n) &= \sum_{j=1}^{l} s_j\left(\sum_{\alpha\in A} d_{j\alpha}u^\alpha + i\sum_{\alpha\in A} e_{j\alpha}u^\alpha\right)\\
&= \sum_{j=1}^{l} s_j\sum_{\alpha\in A} d_{j\alpha}u^\alpha + i\sum_{j=1}^{l} s_j\sum_{\alpha\in A} e_{j\alpha}u^\alpha\\
&= f_1(x_1,\ldots,x_n) + if_2(x_1,\ldots,x_n).
\end{aligned}
\tag{39}
$$

Here f_1 and f_2 are real polynomials. Since f is also contained in the real polynomial ring, f_2 must be equal to zero. But from $f_2(x_1,\ldots,x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha\in A} e_{j\alpha}u^\alpha = \sum_{\alpha\in A}\left(\sum_{j=1}^{l} s_j e_{j\alpha}\right) u^\alpha = 0$ it would follow that for all $\alpha \in A$ the sum $\sum_{j=1}^{l} s_j e_{j\alpha}$ must be equal to zero, because the u_i are algebraically independent. Hence $f = f_1(x_1, \ldots, x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha \in A} d_{j\alpha} u^{\alpha} \in \bigoplus_{j=1}^{l} s_j \mathbb{R}[u_1, \ldots, u_n]$.

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