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A game for the resolution of singularities

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Abstract

We propose a combinatorial game on finite graphs, called *Stratify*, that is played by two protagonists, $\mathcal{A}$ and $\mathcal{B}$. The game captures the logical structure of a proof of the resolution of singularities. In each round, the graph of the game is modified by the moves of the players. When it assumes a final configuration, $\mathcal{A}$ has won. Otherwise, the game goes on forever, and nobody wins. In particular, $\mathcal{B}$ cannot win the game, but can only prevent $\mathcal{A}$ from winning.

We show that $\mathcal{A}$ always possesses a winning strategy, regardless of the initial shape of the graph and of the moves of $\mathcal{B}$. This implies, translating the game back to algebraic geometry, that there is a choice of centres for the blowup of singular varieties in characteristic zero which eventually leads to their resolution. The algebra needed for this implication is elementary. The transcription from varieties to graphs and from blowups to modifications of the graph thus axiomatizes the proof of the resolution of singularities. In principle, the same logic could also work in positive characteristic, once an appropriate descent in dimension is settled.

1. *Introduction*

The existence of resolutions of singularities in characteristic zero has been proved by Hironaka in his landmark paper [19]. Since then, several authors have given variations and simplifications of Hironaka's proof. We mention Refs. [8, 30, 31], which described algorithms that are natural in the sense that they lead to a resolution that is invariant under any group of automorphisms: Ref. [14], which conceptualized the proof by introducing mobiles; Ref. [33], which simplified the proof by introducing homogenized ideals and Refs. [11, 17], which describe implementations of the resolution algorithm.

All these proofs have to deal with the complication that arises from the fact that the construction of coefficient ideals or hypersurfaces of maximal contact is only locally possible. This then requires to prove that, despite the local choices, the final resolution is independent of these choices.

In the present paper, we combine many of the ideas of earlier papers in order to introduce a new type of resolution datum that seems to be well suited for the inductive definition of the resolution invariant. At any step of the algorithm, our resolution problems are globally defined; each construction is deterministic and does not depend on any local choices. Consequently, the complication of proving that all centres of blowup are globally defined does not arise.

Instead of defining an invariant, we interpret resolution as a game between two players. The first attempts to improve the singularities. The second is some malevolent adversary who tries to keep the singularities alive as long as possible. The first player chooses the centres of the blowups, the second provides new order functions after each blowup. The order function defines a stratification of the ambient space. The stratification can be described as a labelled graph. Choosing the centre corresponds to choosing a node of this graph. Each blowup operation

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produces a new labelled graph satisfying various restrictions. In this way, we may define the moves of the game as modifications of the labelled graph according to those restrictions.

After the rules of the game are fixed, the proof of Hironaka's result can be divided into two parts which are logically independent from each other. First, one has to prove that the game is general enough to model resolution problems. This part requires algebra, but the proof is not very complex. Second, one has to prove that there exists a winning strategy. This part requires to solve a complicated combinatorial problem, but it does not need any algebra; in principle, it is an argument about labelled graphs.

Apart from the logical advantage, this approach makes it also easier to think of other resolution strategies. Hopefully, the algebraic properties of ideals and blowups which make Hironaka's proof work become more lucid, especially for non-experts. In addition, the logical argument could in principle also work in the still unsolved case of positive characteristic.

As a matter of historical correctness, it should be mentioned that the idea of formulating games related to the resolution problem is due to Hironaka himself [**20**]: he introduced two such games, the simple and the hard polyhedral game. The first models resolution for hypersurfaces with generic coefficients, the second local uniformization of arbitrary hypersurfaces. Spivakovsky gave a winning strategy for the first game [**29**] and a counter-example for the second, indicating a game that cannot be won [**28**]. This has been extended by Bloch [**10**] and Levine [**26**] in order to bring morphisms of schemes of finite type in good position. A recent account on games and resolution has been given in [**34**].

The paper consists of six parts: The second section compares the various existing proofs of resolution. The third section introduces the game *Stratify*. Its main ingredient are *Scenarios*, which are certain labelled graphs. No algebraic structures appear. The fourth section introduces the concept of *singularity datum*, which is our version of a resolution datum. This part uses only basic algebra. The fifth section establishes the transcription between singularity data and Scenarios: it is shown how to view the resolution process of a singularity datum inside the game as an evolution of a Scenario, and, conversely, how any winning strategy for the game implies the existence of a resolution algorithm for singular varieties in characteristic zero. The last section finally shows that the game has a winning strategy, thus completing the proof of resolution.

2. *Comparison of existing proofs*

In the literature, there appear at least nine proofs for the resolution of singularities of varieties of arbitrary dimension defined over fields of characteristic zero. In this section, we shall sketch some of the differences between these proofs.

The original proof of Hironaka [**19**] is some 200 pp. long. It introduces the key ideas and techniques for all subsequent proofs. As such it has always worked as the principal source of inspiration for mathematicians working in the field. The article is the first paper treating systematically varieties of arbitrary codimension (not just hypersurfaces); it establishes the principle of descent in dimension via local hypersurfaces of maximal contact and coefficient ideals, considers embedded resolution (that is, aims at normal crossings for the total transform), and develops a multiple, interwoven induction argument between various resolution statements [**19**]. Note that Hironaka's proof is existential.

In a collaboration with Aroca and Vicente, Hironaka then adapted the arguments to the resolution of complex analytic varieties, introducing on the way further techniques [**2**, **3**]. The use of the Hilbert–Samuel function as an invariant was made available by Bennett [**5**] and then used via normal flatness in several later papers of Hironaka and other authors. In 1989 and 1992, Villamayor [**30**, **31**] published two papers which provided a constructive proof of resolution (that is, indicating the centres of blowups) and added equivariance as a natural further requirement. These two papers were still very complicated and are hard to read, but the main technical advances already appear there (see [14, Appendix] for precise references). About the same time, Bierstone and Milman started to present their approach to resolution. Whereas the first papers [6, 7] developed the general ideas and concepts, the article [8] offers a complete and a thorough presentation with all technical details. It is still complicated to read, in part due to the consideration of certain equivalence relations for ideals in order to construct global objects from local data. For an extensive comparison of the papers of Villamayor and Bierstone–Milman, see their featured Mathematical Reviews by Lipman and Hauser, respectively.

The Working Week on Resolution of Singularities in Obergurgl 1997 enforced the renewed interest and activity in the field. Soon after, Bodnár and Schicho [11] came up with an implementation of Villamayor's algorithm in Maple, Encinas and Villamayor [15] succeeded in clarifying further the algorithm and Encinas and Hauser [14] gave a very succint proof (just 20 pp.) relying on the language of mobiles. There, for the first time, a global resolution datum was constructed without using equivalence relations. This was appropriate to define in an intrinsic way the local resolution invariant for the induction. Even though the definition of a mobile and of the invariant is somewhat involved, the advantage is convincing since the resulting proofs become very short and almost automatic. Moreover, the paper clearly distinguishes the places where the characteristic zero assumption enters the scene.

Next, Cutkosky [13] published a book on resolution of singularities, taking up Villamayor's approach. At about the same time, Włodarczyk [33] proposed a variation of the descent in dimension, using homogenized coefficient ideals. Up to analytic isomorphism, they are independent of the local choice of a hypersurface of maximal contact. This allowed to show by different methods from those in [8, 14, 31] that the order of the coefficient ideal was well defined. Kollár [25] used this construction to eliminate the use of a resolution invariant from the proof he presented in his book (nevertheless, it is used implicitly). In 2003, Frühbis-Krüger and Pfister [17] published a refined version of the implementation of Bodnár and Schicho in Singular.

Recently, Villamayor and his collaborators Bravo, Benito and Encinas have developed a new descent in dimension, following Jung's method [4, 12, 16]. They replace restrictions by projections. The use of elimination algebras and differential operators provides a particularly elegant argument in characteristic zero, and opens some options for positive characteristic, as is shown in the resulting proof of resolution for surfaces [4]. On the other side, Bierstone, Milman, and Temkin [9] extended resolution to a quite general setting with a strong focus on functorial properties.

All this activity has been complemented in the last years by various proposals and attacks in characteristic $p > 0$, which will not be commented upon here. We refer the interested reader to [18].

Let us briefly describe how the present paper embeds into this landscape: From the reasoning in [14], it became clear that the logical part of the argument lives somewhat separated from the algebraic part. Said differently, the actual algebraic construction of the descent in dimension (via hypersurfaces of maximal contact and coefficient ideals) as well as the construction of the transversality ideal (to ensure that the chosen centre is always transversal to the exceptional divisor) and the companion ideal (to ensure that the singular locus of the coefficient ideal is contained in the singular locus of the original ideal) did not matter so much as long as these objects satisfied some specific relations between them. It was then a natural step to isolate these properties and to formulate our game in a purely combinatorial manner (even though working out the technical details is somewhat intricate). The game shows perfectly the logical structure of Hironaka's proof, which is, at least to us, of dazzling beauty.

Of course, the game becomes only valid if it can be shown that the actual resolution process for a singularity is mimicked by a winning strategy for it. This goes in two directions: First, one has to translate the algebraic situation to the context of the game, and second, the winning strategy of the game has to be translated back to prove that the resolution process terminates. This is done in the section *Transcription*. In principle, also other algebraic formulations of resolution problems may fit into the game, even in positive characteristic.

Let us make clear that we do not overestimate the impact or importance of the present paper. After all, it is just another reading of the existing proofs for resolution (of which we have taken up freely many ideas and concepts). But as the combinatorial and algebraic part can be accessed easily even by non-experts, the paper may help to understand better the existing proofs of resolution of singularities.

3. *Stratify*

In this section, we will introduce a combinatorial game, called *Stratify*. It exhibits the axiomatic and logical structure of the existing proofs for the resolution of singularities of algebraic varieties in characteristic zero. The resolution is typically built on a sequence of blowups in smooth centres which are chosen as the smallest stratum of a suitable stratification of the variety. The choice of the stratification and the proof of termination of the resolution procedure are both established by induction on the ambient dimension. The main focus lies here on the scrutiny of the ideal defining the variety in the ambient space, together with its transforms under blowup.

In the terminology of the game, there will be no references to algebraic concepts such as ideals, varieties or blowup maps. Only labelled graphs, which evolve along with the game, appear. We will indicate in paralipomena the respective algebraic analogues of the various tokens of the game.

In the section *Transcription* we show how to pass from the algebraic setting of a resolution problem for singular varieties, encoded in a resolution datum, to the game Stratify and how, going back, a winning strategy for the game ensures the termination of the resolution algorithm of the variety.

Let us first sketch the overall idea of the game. The precise description will start with the subsection *Boards*.[†]

Our game carries on a collection of *scenarios* C (finite, directed graphs whose nodes come with certain labels) which evolve with the moves of the two players, $\mathcal{A}$ and $\mathcal{B}$. A round of the game consists in a move of $\mathcal{A}$, followed by a move of $\mathcal{B}$. The combination of both moves modifies the actual scenarios, introduces new or deletes existing ones. The rules of the game and the moves of the players in one round determine the collection of scenarios for the next round.

The game starts with a single scenario, the *initial* scenario C. It is provided by the Umpire. After inspection of the scenario, $\mathcal{A}$ announces her move, to which $\mathcal{B}$ responds with his move. This constitutes the first round. There are two types of moves for $\mathcal{A}$, *blowups* and *quests*.

3.1. *Blowups*

A blowup move of $\mathcal{A}$ is given by the selection of a certain node of the graph, the *centre*. The *response move* of $\mathcal{B}$ consists in modifying all existing scenarios. This response depends on the chosen centre, but allows $\mathcal{B}$ some flexibility on how to change the scenarios. Each round with blowup moves produces a modification C' of all scenarios C, called the *blowup transform* of C. These transforms are then the *actual* scenarios of the game in round 2. If the next move of $\mathcal{A}$ is again a blowup, then they will be modified once more. The evolution of a scenario C

[†]As a general guideline, we try to keep the exposition slim so as to transmit the essential flavour of the various constructions instead of hiding them behind tedious technicalities. It is therefore preferable to accept a rough understanding on a first reading.

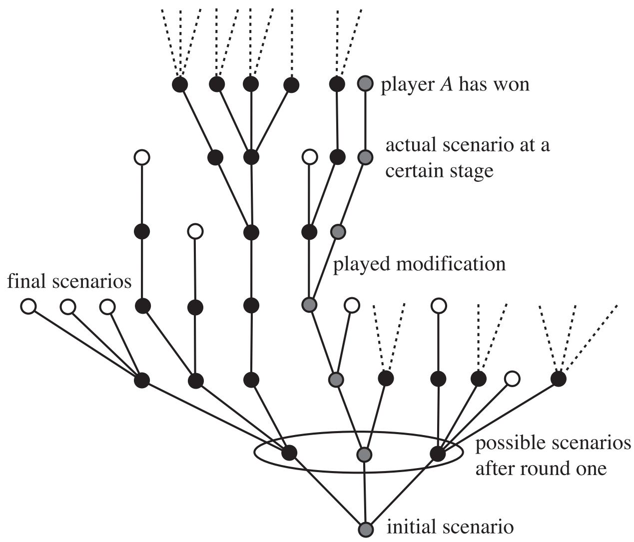

FIGURE 1. *A quest is a tree of scenarios related by blowup moves.*

continues like this either forever or until it reaches a *final* configuration, which is characterized by certain properties. In this case, $\mathcal{A}$ has won. If the game does not come to an end, neither player has won (so that $\mathcal{B}$ can never win, he can only prevent $\mathcal{A}$ from winning).

Final scenarios can be characterized by the absence of singularities. In order to survive, $\mathcal{B}$ needs to claim that there are still singularities, no matter what the other player is doing.

3.2. *Quests*

The second type of player $\mathcal{A}$'s moves (*calling* or *opening a quest*) results in the creation of new scenarios. A call can only be placed by $\mathcal{A}$, and player $\mathcal{B}$'s response to it is a scenario of a new, subordinate *quest*. A quest can be intuitively thought of as a subgame of the main game.† In our context, a quest is abstractly defined as the, possibly infinite, tree of all scenarios that can be obtained from one given scenario by blowup moves of the players.‡ Two nodes of the tree are connected by a directed edge if the scenario corresponding to the second node is an allowed blowup transform of the scenario of the first (see Figure 1).

Player $\mathcal{B}$'s move, which is considered as a response to player $\mathcal{A}$'s call, consists in providing a new scenario for the quest just opened by $\mathcal{A}$, as well as to all quests opened at earlier stages of the game. The response scenarios must satisfy certain rules according to the type of the call. In addition, they are subject to certain commutativity rules with respect to the scenarios that existed before the last blowup move of the foregoing rounds.

The responses of $\mathcal{B}$ are in general not unique, with the exception of a few auxiliary quests, called *one-way quests*, for which $\mathcal{B}$ has no choice and for which the response scenario is already completely determined by the type of the call and the scenario of the superordinate quest.§

If $\mathcal{A}$ chooses the first type of move, *blowups*, the actual scenarios (each one being the previous response of $\mathcal{B}$ to a quest) are split into two sets: the first set consists of the scenarios for which

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†For instance, taking a pawn may be considered as a subgame in chess, similarly as winning a set in lawn tennis.

‡See the respective subsection *Transforms* below.

§The responses to one-way quests can equally be provided by the Umpire.

the blowup is *admissible* in a specified sense. They undergo transformations chosen by $\mathcal{B}$ within certain rules. The remaining scenarios do not transform; their quests are deleted from the game.

It may happen that a subordinate quest reaches a final configuration earlier than a superordinate one, in which case the resolved quest is discarded from the game. We say that $\mathcal{A}$ has won the subgame. The rules of the game ensure that the superordinate scenario has then come closer (in a precise way) to its final configuration, and so winning a subgame helps $\mathcal{A}$ to win the overall game.

The collection of all quests that are at stake before/after a move is called the set of open quests. Player $\mathcal{B}$ has to provide responses, say scenarios, for all open quests. These scenarios are related to each other by precise rules. They share a common underlying structure, the *board*, which is a labelled graph. In contrast to other board games, the board in the resolution game will change in each move of $\mathcal{B}$.

From a different perspective, opening a quest can also be seen as an operational instruction: it is a move of $\mathcal{A}$ that forces $\mathcal{B}$ to respond by scenarios of a given type as long as the quest is open. These responses may provide useful information for $\mathcal{A}$ on how to select her next move.

At each stage of the game, the players have some flexibility of how to choose their moves (there are different rules for each of them). The combination of player $\mathcal{A}$'s and player $\mathcal{B}$'s moves defines in a unique way the modification of the actual scenarios, the creation of the new scenarios and the deletion of certain scenarios (the game is deterministic, chance is excluded).

Note the difference between the overall game (that is, the prescription of the initial scenario and the collection of rules for transformations and calls) and the actually played game (that is, the sequence of moves applied to the initial scenario). We start with a couple of basic ingredients of the game.

3.3. *Boards*

Player $\mathcal{B}$'s move in each round consists in responding to all open quests by scenarios. These responses share the same underlying structure, the *board*. This is a directed annotated graph Γ, which is the Hasse diagram of a finite partially ordered set.[†] We say that a node s is *below* a node t if $s \leqslant t$ with respect to the order relation on Γ; and that s and t are *remote* from each other if there is no common node u below both of them.[‡] The annotation consists of a non-negative integer $\dim(s)$, the *dimension*, for each node $s \in \Gamma$. It is supposed to define a strictly monotonously increasing function on Γ. The nodes of Γ are supposed to have a largest element; its dimension is denoted by $n = \dim(\Gamma)$.[§]

3.4. *Scenarios*

A *scenario* C on a board Γ of dimension n consists of

(1) two integers $0 \leqslant d \leqslant n$ and $0 < b$, the *dimension* and the *bound*;
(2) subsets $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{T}$ of the set of nodes of Γ, called the *handicap*, the *singular set* and the *transversal set*;
(3) a function $\mathrm{ord} : \mathcal{S} \to (1/b)\mathbb{Z} \cup \{\infty\}$, the *order function*;

[†] The Hasse diagram of a partially ordered set is the graph whose nodes are the elements of the set, and where two nodes s and t are connected by a directed edge if $s \leqslant t$ and if there is no node r in between them, say $s \leqslant r \leqslant t$ with $r \neq s, t$.

[‡] Seen geometrically, the nodes s of Γ correspond to the strata of a stratification of a manifold W by locally closed subsets. The order relation $s \leqslant t$ between nodes corresponds to the adjacency of strata, defined by s being in the closure of t. Two nodes have a common node below them if and only if the closures of the respective strata intersect.

[§] The number n corresponds to the dimension of the smooth ambient variety of the singularity to be resolved.

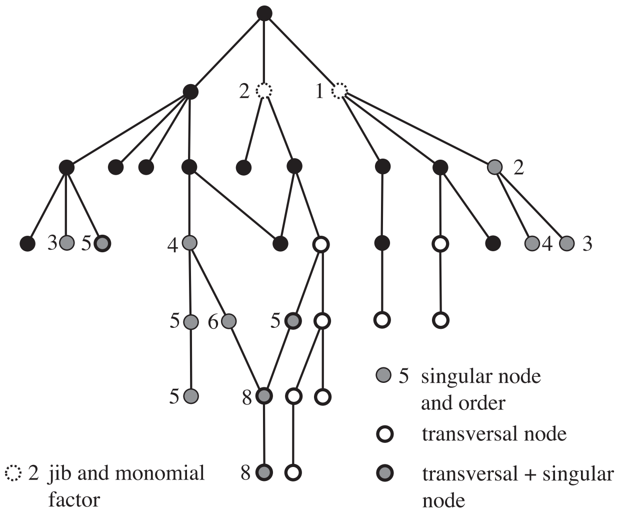

FIGURE 2. *Scenario of dimension* 6 *with* $(d, b) = (6, 1)$, *order function on* $\mathcal{S}$ *and single monomial factor* m. *The dimensions* $\dim(s)$ *of the nodes are indicated by height.*

(4) a non-empty set $\mathcal{M}$ of functions $m : \mathcal{H} \to \mathbb{Q}_{\geqslant 0}$, the *monomial factors*. They are extended to Γ by $m(s) = \sum_{h \geqslant s} m(h)$.

The elements in $\mathcal{H}$ are called *jibs*, those in $\mathcal{S}$ and $\mathcal{T}$ *singular*, respectively, *transversal nodes*.[†] See Figure 2 for an example of a scenario with just one monomial factor.

RULE 1. A scenario is subject to the following rules:

(1) every jib $h \in \mathcal{H}$ has dimension $n - 1$;
(2) the set $\mathcal{S}$ is downward closed;
(3) the maximal nodes among those with order ∞ have dimension d. If $\mathcal{H}$ is empty, then all these maximal nodes with order ∞ are in $\mathcal{T}$;
(4) for all $s \in \mathcal{S}$, we have $\dim(s) \leqslant d$. If $\dim(s) = d$, then $\operatorname{ord}(s) = \infty$;
(5) if $K \subset \mathcal{H}$, $s \in \mathcal{S}$ and $s \leqslant h$ for all $h \in K$, then $\dim(s) \leqslant d - \operatorname{card}(K)$. If $\dim(s) = d - \operatorname{card}(K)$, then $s \in \mathcal{T}$;
(6) $\operatorname{ord}(s) \geqslant m(s)$ for all $s \in \mathcal{S}$ and $m \in \mathcal{M}$;
(7) $\mathcal{M}$ is downward closed with respect to $\leqslant$;[‡]
(8) $\operatorname{ord}(s) - m(s)$ is weakly monotonously decreasing on $\mathcal{S}$ for all $m \in \mathcal{M}$;[§]
(9) if $\mathcal{K}$ is a set of jibs such that $\sum_{h \in \mathcal{K}} m(h) \geqslant 1$ for some $m \in \mathcal{M}$, then, for any node s in $\mathcal{S}$ that lies below all nodes in $\mathcal{K}$, there exists exactly one node $t \geqslant s$ in $\mathcal{S}$ which lies below all nodes in $\mathcal{K}$ of dimension $d - \operatorname{card}(\mathcal{K})$.[¶]

[†]In the transcription of algebraic resolution problems to the game, the singular set and the order function will come from a singularity datum. The jibs correspond to the exceptional hypersurfaces, and $\mathcal{T}$ describes choices of centres that are transversal to these hypersurfaces. The monomial factors stem from exceptional factors that were created by earlier blowups.

[‡]If $m_1(h) \leqslant m(h)$ for all $h \in \mathcal{H}$ and some $m \in \mathcal{M}$, then $m_1 \in \mathcal{M}$.

[§]Note that if $\operatorname{ord}(s) - m(s)$ is weakly monotonously decreasing for some $m \in \mathcal{M}$, then $\operatorname{ord}(s) - m_1(s)$ is automatically weakly monotonously decreasing for any $m_1 \leqslant m$; indeed, $\operatorname{ord}(s) - m_1(s) = \operatorname{ord}(s) - m(s) + (m(s) - m_1(s))$ and $m(s) - m_1(s)$ is given by its values on $h \in \mathcal{H}$.

[¶]Note that any node in this set has dimension at most $d - \operatorname{card}(\mathcal{K})$ by Rule 1, Issue (5).

A scenario is *tight* if its order function is constant equal to 1 on whole $\mathcal{S}$. It is *resolved* if $\mathcal{S}$ is empty. A monomial factor m is called *complete* if $\mathrm{ord}(s) = m(s)$ for all $s \in \mathcal{S}$. We then also say that C is a *monomial* scenario.

3.5. *Transform of boards*

Player $\mathcal{B}$ may change the board by two types of modifications, called *refinement* and *blowup*. In both cases, he has to give a new board Γ' with new dimension labels. On the level of nodes, $\mathcal{B}$ provides an embedding $i : \Gamma \to \Gamma'$, and a retract $u : \Gamma' \to \Gamma$ such that $u \circ i = \mathrm{id}_\Gamma$.[†] The following rule must be fulfilled.

RULE 2. The joint rules for both operations (refinement and blowup) are:

(1) $i(s)$ is the unique maximal element of $u^{-1}(s)$, for any $s \in \Gamma$;
(2) $i(s) < i(t)$ if and only if $s < t$, for any $s, t \in \Gamma$;
(3) u is weakly monotonously increasing.

Here is the additional rule for refinements:

(4) $\dim(i(s)) = \dim(s)$ for $s \in \Gamma$.

Here are the additional rules for blowups:

(5) every blowup is determined by a node $z \in \Gamma$, its centre, selected by $\mathcal{A}$;
(6) if $s \in \Gamma$ does not lie below z, then $\dim(i(s)) = \dim(s)$;
(7) if $s \leqslant z$, then $\dim(i(s)) = \dim(s) + \dim(\Gamma) - 1 - \dim(z)$.

3.6. *Transform of scenarios*

Scenarios evolve under the moves of the players. If some quest remains open after a round of the game, then a scenario C and its successor C', chosen by $\mathcal{B}$ on a board Γ, respectively, its refinement or blowup board Γ', are not independent, but have to fulfil the following rules.[‡]

RULE 3. Let the board Γ' be obtained from the board Γ by a refinement or a blowup, with an embedding $i : \Gamma \to \Gamma'$ and a retract $u : \Gamma' \to \Gamma$. The rules for both, refinements and blowups of scenarios, are as follows:

(1) the dimension d and the bound b remain unchanged;
(2) for any $s \in \Gamma$, we have $i(s) \in \mathcal{T}'$ if and only if $s \in \mathcal{T}$.[§]

Here are the additional rules for refinements:

(3) $\mathcal{S}' = u^{-1}(\mathcal{S})$;
(4) $\mathrm{ord}(s') = \mathrm{ord}(u(s'))$ for any $s' \in \mathcal{S}'$;

[†] The map i sends a stratum to a dense open subset of this stratum, the inverse image $u^{-1}(s)$ consists of strata whose union is s. A refinement corresponds to a refinement of a stratification in the classical sense (strata are replaced by unions of strata), and a blowup corresponds to a stratification such that all inverse images of strata are unions of strata.

[‡] We may think of two consecutive scenarios of a blowup move as being vertically related. There is also a horizontal relationship between scenarios on the same board; it is evoked by $\mathcal{A}$ when she issues a call and thus creates a subordinate quest to which $\mathcal{B}$ responds by a scenario. The horizontal relation lasts until one of the two quests becomes invalid (either because $\mathcal{B}$ loses it or because $\mathcal{A}$ gives it up). Deliberate divorce between horizontally related quests is not allowed in the game.

[§] There may be new transversal nodes in $\mathcal{T}'$ which are not in the image of i.

(5) $\mathcal{H}' = i(\mathcal{H})$;
(6) $m' \in \mathcal{M}$ if $m'(h') = m(u(h'))$ for some $m \in \mathcal{M}$ and all $h' \in \mathcal{H}'$.

Here are the additional rules for blowups. Let $z \in \Gamma$ be the centre and $e := i(z)$.[†]

(7) $z \in \mathcal{T}$, and either $z \in \mathcal{S}$ or z is remote from $\mathcal{S}$. Any node that satisfies these two conditions is called *admissible* for the scenario;
(8) $\mathcal{S}' \subseteq u^{-1}(\mathcal{S})$;
(9) if $\text{ord}(z) < 2$ or $z \notin \mathcal{S}$, then $e \notin \mathcal{S}'$. Otherwise, $\text{ord}(e) = \text{ord}(z) - 1$;
(10) for any $s' \in \mathcal{S}'$ which is not less than or equal to e, we have $\text{ord}(s') = \text{ord}(u(s'))$;
(11) if C is tight, then so is C';
(12) $\mathcal{H}' = i(\mathcal{H}) \cup \{e\}$;
(13) if $K \subseteq \mathcal{H}$ with $z \leqslant h$ for all $h \in K$ and $\dim(z) = d - \text{card}(K)$, then there exists no node s' in $\mathcal{S}'$ such that $s' \leqslant i(h)$ for all $h \in K$;
(14) $m' \in \mathcal{M}'$ if $m'(e) \leqslant \text{ord}(z) - 1$ and $z \in \mathcal{S}$; if $z \notin \mathcal{S}$, then $m'(e) = 0$ and there exists an $m \in \mathcal{M}$ so that $m'(i(h)) = m(h)$ for all $h \in \mathcal{H}$;[‡]
(15) if m is a complete monomial factor of C, then m' defined by $m'(e) = \text{ord}(z) - 1$ and $m'(i(h)) = m(h)$ for $h \in \mathcal{H}$ is a complete monomial factor of C'.[§]

We call C' a *transform* of C under a refinement, respectively, blowup move. It is not unique, so that $\mathcal{B}$ has some freedom of how to choose its items.[¶]

3.7. *Quests*

A *quest* is the collection of all scenarios that can be obtained from an initial scenario by blowup moves. It is thus a tree, where the directed edges connect scenarios that are related by a blowup. According to player $\mathcal{A}$'s flexibility in choosing the centre of the blowup, and player $\mathcal{B}$'s flexibility in providing transformed scenarios, the tree may ramify considerably. However, playing a quest (that is, applying concrete blowup moves) yields a sequence of transforms of the initial scenario which corresponds to a specific path in this tree. If a transform of a scenario reaches a final shape, then the quest is won and discarded from the game (provided that it is not the main quest).

The overall game is constituted by several and interrelated quests, created at different moments (together with their initial scenario) and with possibly different life times. We say that a quest is *open* as long as it forms part of the game. When it is *closed*, it will be discarded from the game. The game starts with a single quest, the main quest and an initial scenario thereof, which is provided by the Umpire. In the course of the game, other quests may open and close. The game ends when the main quest is won.

3.8. *Responses*

Whenever a quest is open, $\mathcal{B}$ has to respond to it by a scenario when it is his turn to move. The scenario has to fulfil certain properties according to the type of the quest and the stage of the game. These are specified in the description below of the various quests and in the commutativity rules for blowup moves.

[†]The node e has dimension $n - 1$, by Rule 2, Issue (7).

[‡]The transformation rule for m corresponds to the way the exceptional factor of an ideal transforms under blowup; cf. the combinatorial handicap in [**14**].

[§]So $\mathcal{B}$ has in this case no choice of how to choose C'.

[¶]But as C' has to be again a scenario, $\mathcal{B}$ has to choose for a blowup transform of a scenario the order function so that Rule 1, Issues (6) and (8), is satisfied. Observe that $\mathcal{B}$ has no choice of how to choose $\mathcal{H}'$ and $\mathcal{M}'$.

3.9. *Calls*

In addition to playing a blowup, $\mathcal{A}$ may also open at any time a new quest; we also say: she *places a call*. This move introduces a new quest, considered as being *subordinate* to a quest specified by $\mathcal{A}$; it will be part of the game until it is closed or deleted. Its scenarios are related to the scenarios of the superordinate quest by certain rules depending on the type of the call.

The calls produce two types of quests, with different objectives: the first type are the relaxation and descent quest (for which $\mathcal{B}$ has some freedom of how to respond to them by scenarios), the second type are *one-way* quests (for which the response scenario is uniquely determined by the superordinate quest). The latter calls are thus just commands without choice; they are needed for $\mathcal{A}$ in order to be able to add and factorize scenarios. It is irrelevant whether $\mathcal{B}$ or the Umpire provides the corresponding response. The one-way quests are the transversality and the monomial factorization quest.

Under a blowup move, a quest and its subordinate quest, created by a call prior to the blowup, preserve their relation defined by the type of the call (that is, the respective scenarios have to obey the corresponding rules); see the subsection *Commutativity Relations*.†

3.10. *Relaxation quest*

Let $\mathfrak{Q}$ be a quest, and assume that $\mathcal{J} \subseteq \mathcal{H}$ is a chosen set of jibs of a scenario C of $\mathfrak{Q}$ with board Γ. Then $\mathcal{A}$ may issue the call: 'Release $\mathcal{J}$!' This creates a *relaxation quest*.

RULE 4. A scenario C_1 on Γ is a response scenario for the relaxation quest that releases $\mathcal{J}$ if the following rules hold:

(1) the dimension d and the bound b are the same for C and C_1;
(2) $\mathcal{S}_1 = \mathcal{S}$ and $\mathrm{ord}_1 = \mathrm{ord}$;
(3) $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{J}$;
(4) $\mathcal{T} \subseteq \mathcal{T}_1$. If $z \leqslant h$ for all $h \in \mathcal{J}$, and $z \in \mathcal{T}_1$, then $z \in \mathcal{T}$. If z is remote from $\mathcal{J}$, and $z \in \mathcal{T}_1$, then $z \in \mathcal{T}$;
(5) $\mathcal{M}_1$ consists of the restrictions to $\mathcal{H}_1$ of the monomial factors $m \in \mathcal{M}$.

3.11. *Descent quest*

Let $\mathfrak{Q}$ be a quest, and assume that C is a tight scenario of $\mathfrak{Q}$. By Rule 3, Issues (4) and (11), the future responses to $\mathfrak{Q}$ will again be tight, so we may call the quest itself tight. The tightness property of a quest may be acquired during the game by some blowup, but once a quest is tight, it will stay so as long as it is open.

Assume, additionally, that the handicap $\mathcal{H}$ is empty. Then $\mathcal{A}$ may issue the call: 'Step down!' This creates a *descent quest*.‡ The response scenarios are subject to the following rules.

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†But it may happen that the subordinate quest closes under the blowup move and is thus discarded from the game.

‡In the geometric situation of varieties, a descent in dimension to a hypersurface of maximal contact may create a transversality problem with the exceptional divisor. If the hypersurface is not transversal, a subordinate resolution problem in smaller ambient dimension is formulated to separate the hypersurface from the exceptional components; see the notion of transversality ideal in [**14**]. Here, we handle this difficulty by only allowing a descent call for scenarios with empty handicap. After blowup, this assumption is no longer required, as the new jibs of the transformed scenario will automatically be transversal, by Rule 3, Issues (2) and (12).

RULE 5. Let C be a tight scenario for $\mathfrak{Q}$ on a board Γ. A scenario C_1 on a refinement Γ' of Γ is a response scenario for the descent quest if the following rules hold:

(1) $d_1 = d - 1$;
(2) $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{T}_1 = \mathcal{T}$.[†]

3.12. *One-way quests*

The next two quests are one-way quests, the response scenario provided by $\mathcal{B}$, or the Umpire, is always uniquely determined. It only depends on the superordinate scenario and the type of the quest.

3.13. *Transversality quest*

Let $\mathfrak{Q}$ be a quest, with scenario C on a board Γ, and let $\mathcal{K} \subseteq \mathcal{H}$ be a set of jibs. Then $\mathcal{A}$ may construct the *transversality quest* $\mathfrak{Q}_1$ with respect to $\mathcal{K}$.[‡]

RULE 6. The response scenario of a transversality quest is given by the following rule:

(1) $d_1 = d$, $b_1 = b$;
(2) $\mathcal{H}_1 = \mathcal{H}$;
(3) $\mathcal{S}_1 = \bigcap_{h \in \mathcal{K}} \{s \in \mathcal{S} \mid s \leqslant h\}$;[§]
(4) $\operatorname{ord}_1(s) = 1$ for all $s \in \mathcal{S}_1$, except for $\mathcal{K} = \emptyset$, where $\operatorname{ord}_1 = \operatorname{ord}$;
(5) $\mathcal{T}_1 = \mathcal{T}$;
(6) $\mathcal{M}_1 = \{0\}$, except for $\mathcal{K} = \emptyset$, where $\mathcal{M}_1 = \mathcal{M}$, and for $\mathcal{K} = \{h\}$, where $\mathcal{M}_1$ contains the function mapping h to 1 provided that it belongs to $\mathcal{M}$.

3.14. *Quotient quest*

Let $\mathfrak{Q}$ be a quest, with scenario C on a board Γ. Let m be a monomial factor of C and let $q > 0$ be a positive rational number, the *scale*. Both are chosen by $\mathcal{A}$. The response scenario C_1 to the *quotient quest* is called the *q-quotient* of C with respect to m.[¶]

RULE 7. The response scenario to a quotient quest is given by the following rule:

(1) $d_1 = d$, $b_1 \cdot \mathbb{Z} = \mathbb{Z} \cap b/q \cdot \mathbb{Z}$;
(2) $\mathcal{H}_1 = \mathcal{H}$;
(3) $\mathcal{S}_1 = \{s \in \mathcal{S} \mid \operatorname{ord}(s) - m(s) \geqslant q\}$;
(4) $\operatorname{ord}_1(s) = \min\{\operatorname{ord}(s), 1/q \cdot (\operatorname{ord}(s) - m(s))\}$;
(5) $\mathcal{T}_1 = \mathcal{T}$;
(6) $\mathcal{M}_1 = \{1/q \cdot (f(s) - m(s)) \mid f \in \mathcal{M}\}$.

[†] Recall that only at the moment of the call, the actual scenario C of $\mathfrak{Q}$ is required to have empty handicap.

[‡] In contrast to [**14**] where products of ideals are taken to deal with the transversality problem, we axiomatize here a transversality ideal that is given by the restriction of the singular ideal to the intersection of certain exceptional hypersurfaces. This corresponds to the sum of ideals, respectively, singularity data.

[§] In the case where $\mathcal{K} = \emptyset$, this signifies that $\mathcal{S}_1 = \mathcal{S}$.

[¶] Monomial factors and quotients axiomatize the combinatorial handicap of a mobile from [**14**]; they enable $\mathcal{A}$ to split off from an ideal the exceptional monomial factor. The q-quotient corresponds to the remaining non-monomial factor, and the scale q adjusts its control.

If $q > \mathrm{ord}(s) - m(s)$ for all $s \in \mathcal{S}$, then the singular set $\mathcal{S}_1$ is empty, the quotient quest resolved and discarded. So $\mathcal{A}$ will preferably choose smaller values of q. If q equals the maximal value of $\mathrm{ord}(s) - m(s)$ on S, the response scenario is tight.

3.15. *Commutativity relations*

New quests are created by calls. This establishes an asymmetric relation between the superordinate and the subordinate quest. The relation is specified by the type of the call. Along a sequence of blowups, the relation has to be maintained by the players. We specify this persistence by listing for each call the required properties between scenario and subordinate scenario.

We consider blowups in centres z contained in the singular and transversal sets of a scenario C of a quest $\mathfrak{Q}$, and of a scenario C_1 of some subordinate quest $\mathfrak{Q}_1$. Both scenarios are defined on the same board Γ. Their blowup transforms will be chosen by $\mathcal{B}$. Let $e = i(z)$ be the exceptional node of the blowup, and let $i : \Gamma \to \Gamma'$ and $u : \Gamma' \to \Gamma$ be the associated inclusion and retract, respectively; see Rules 2 and 3.

We denote by C' a blowup transform of C, according to Rule 3. Let $(C')_1$ be player $\mathcal{B}$'s response scenario to C' within the quest $\mathfrak{Q}_1$. Then, in order to be an allowed response of $\mathcal{B}$, $(C')_1$ has to satisfy the following properties.

Rule 8. (1) If C_1 is a relaxation scenario of C with respect to $\mathcal{J} \subseteq \mathcal{H}$, then $(C')_1$ is a relaxation scenario of C' with respect to $\mathcal{J}' = i(\mathcal{J})$ *and* a transform $(C_1)'$ of C_1.

(2) If C_1 is a descent scenario of C, then $(C')_1$ is a descent scenario of C' *and* a transform $(C_1)'$ of C_1.

(3) If C_1 is the transversality scenario of C with respect to $\mathcal{K} \subseteq \mathcal{H}$, then $(C')_1$ is the transversality scenario of C' with respect to $\mathcal{K}' = i(\mathcal{K})$ *and* a transform $(C_1)'$ of C_1.

(4) If C_1 is the q-quotient of C with respect to a monomial factor m, then $(C')_1$ is the q-quotient of C' with respect to m' defined by $m'(e) = m(z) + q - 1$ and $m'(i(h)) = m(h)$ for $h \in \mathcal{H}$, *and* $(C')_1$ is a transform $(C_1)'$ of C_1.[†]

These relations are schematized by the following diagram:[‡]

$$\begin{array}{rcccl} & & \text{call} & & \\ & C' & \rightsquigarrow & (C')_1 = (C_1)' & \\ \text{blowup} & \downarrow & & \downarrow & \text{blowup} \\ & C & \rightsquigarrow & C_1 & \\ & & \text{call} & & \end{array}$$

Observe here that the transforms of a scenario are not uniquely prescribed; but player $\mathcal{B}$'s responses have to be chosen so that commutativity holds.

[†]Observe that $m(z) + q \leqslant \mathrm{ord}(z)$ if z belongs to the singular set $\mathcal{S}_1$ of C_1, by Rule 7, Issue (4). If the centre is not admissible for C_1, say, if z is not in $\mathcal{T}$ or z is not in $\mathcal{S}_1$ and not remote from $\mathcal{S}_1$, the quotient quest is discarded after the blowup.

[‡]The response scenarios of the call after a blowup have to be transforms of the response scenarios of the call before the blowup.

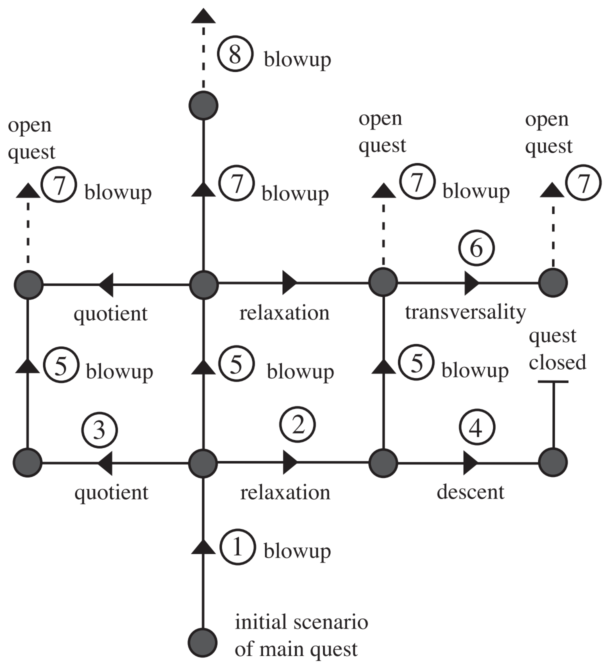

FIGURE 3. *Playing the game* Stratify.

3.16. *Playing the game*

The game starts with a board Γ carrying a single scenario C, the *initial* scenario. It is provided by the Umpire and represents the first scenario of the main quest. In the course of the game, this scenario will transform by blowups in a way that is governed by the moves of the players. The game is finished, or the main quest is *won*, when the singular set of the transform of the initial scenario has become empty.

The two protagonists $\mathcal{A}$ and $\mathcal{B}$ play their moves alternately. The moves affect and modify the actual scenarios of all open quests.

During the game, other quests may be opened by $\mathcal{A}$. Each of them remains valid until it is won or given up. Player $\mathcal{A}$ gives up a quest by choosing a centre that is not admissible for the scenario of the quest. In this case, there is no response scenario for the quest after the blowup, according to Rule 3, Issue (7), and the quest is discarded from the game.

A possible evolution of the game is depicted in Figure 3. The dots represent the various scenarios provided by $\mathcal{B}$, blowups are drawn vertically, calls horizontally. The numbering indicates the sequence of rounds formed by player $\mathcal{A}$'s move (blowup or call) and player $\mathcal{B}$'s response (scenarios).[†]

In each round of the game, all scenarios of open quests are built on the same board Γ. When a new quest is opened by $\mathcal{A}$, $\mathcal{B}$ may refine in his response the underlying board. Similarly, under a blowup move of $\mathcal{A}$, the board Γ transforms into a board Γ'. In both cases, the new board will be the common underlying graph for the scenarios of player $\mathcal{B}$'s responses.

[†]The arrows labelled with 5 correspond all to a round given by a blowup move: it applies to all quests for which the centre is admissible, and $\mathcal{B}$'s responses have to provide scenarios for all these quests. Observe that the transversality quest labelled with 6 already belongs to the next round.

If the singular set of a scenario C is empty, then the respective quest is closed, and $\mathcal{B}$ has lost this quest. So assume that some quests are still open, in particular, the main quest. It is player $\mathcal{A}$'s turn to move. She has three choices.

(1) She plays a blowup by nominating a node z in the actual board Γ as its centre. If z is not admissible for the actual scenario of a quest, then the quest is closed and will be discarded; the other quests remain open.
(2) She selects one of the actual scenarios and calls a one-way quest for it. The call will be a transversality or a quotient quest. In case of a transversality quest, a set of jibs has to be specified by $\mathcal{A}$. In case of a quotient quest, the monomial factor and the scale have to be specified by her.
(3) She selects one of the actual scenarios and calls a relaxation or descent quest for it. This opens a new quest, which will be subordinate to the quest of the chosen scenario. The new quest is then played simultaneously with the other, already open quests.

Now it is player $\mathcal{B}$'s turn. First, assume that $\mathcal{A}$ has played a blowup with node $z \in \Gamma$. In this case, $\mathcal{B}$ provides a blowup transform Γ' of the actual board Γ, together with an embedding $i : \Gamma \to \Gamma'$ and a retract $u : \Gamma' \to \Gamma$. Then, he responds to all open quests by specifying appropriate scenarios on Γ'. For each such quest, the response scenario C' is related to the previous scenario C of the quest by the transformation laws of Rule 3. In addition, the responses of $\mathcal{B}$ need to respect the *Commutativity Relations* from Rule 8.

Second, assume that $\mathcal{A}$ has placed a one-way call, say a transversality or a monomial factorization quest. In this case, $\mathcal{B}$ has no choice and he or the Umpire provide the uniquely determined scenario.[†]

Finally, assume that $\mathcal{A}$ has placed a relaxation or descent call, thus creating a new quest to which $\mathcal{B}$ has to respond. He provides first a refinement Γ' of the actual board Γ, together with an embedding $i : \Gamma \to \Gamma'$ and a retract $u : \Gamma' \to \Gamma$. Then he responds to all open quests by specifying the respective scenarios on Γ'. In his choice of scenarios, he has to respect three rules:

(1) The scenarios of all open quests except of the new one must be refinements of the previous scenarios (that is, those provided by $\mathcal{B}$ before the call) in the sense of Rule 2.[‡]
(2) The scenarios of the quests that were already open before the last blowup move of $\mathcal{A}$ must be a transform of the respective scenario of the quest before the blowup as indicated in Rule 3.[§]
(3) The responses of $\mathcal{B}$ need to respect the *Commutativity Relations* from Rule 8.

The next round proceeds as before with player $\mathcal{A}$'s move followed by player $\mathcal{B}$'s response. The game continues like this until $\mathcal{B}$ loses or forever. $\mathcal{A}$ cannot lose, because even if there is no admissible centre of blowup to be chosen, there are infinitely many ways to open new quests. But $\mathcal{A}$ wants more: she wants to win. Her goal is to choose her moves so that, regardless of the responses of $\mathcal{B}$, the resulting path of scenarios in the tree of the main quest leads from the initial scenario to a final scenario. Along the way, she may (and will) win subordinate quests which she has opened by her calls.

4. *Singularity data*

In his proof for the resolution of singularities over fields of characteristic zero, Hironaka used a multiple induction between various resolution statements carrying on different types

[†]Recall here that the factor of an initial scenario is set equal to 0.

[‡]The initial scenario of the new quest has no predecessor.

[§]The initial scenario of the new quest cannot be a transform. Its factor is 0.

of resolution data [19, Chapter I, Section 2]. Similarly, Abhyankar [1] introduced several types of resolution data for proving resolution in small dimensions and positive characteristic; see also the last section of Lipman's survey article [27]. Later on, many variants of resolution data have been proposed and used in the literature [8, 14, 23, 24, 30, 31, 33]. They all share the necessity of book-keeping the configuration of the exceptional divisors as well as certain numerical data like orders and Hilbert–Samuel functions of ideals.†

In this section, we introduce yet another type of resolution datum. Its structure allows an induction in dimension that does not depend on any local choice of hypersurfaces. The global definition then permits a significant simplification of the induction argument.‡

DEFINITION 1. A *habitat* is a pair $W = (W, E)$ consisting of a connected algebraic manifold W over a field of characteristic zero (that is, a non-singular algebraic variety), together with a finite set $E = \{E_1, \ldots, E_r\}$ of non-singular hypersurfaces forming a normal crossings divisor. A habitat restricts to open subsets U of W by taking $\{E_1 \cap U, \ldots, E_r \cap U\}$.

A non-empty closed subvariety $Z \subset W$ is called *transversal* to (W, E) if, for every point $p \in Z$, there is a system of regular local parameters such that Z is defined by a subset of these parameters and such that every hypersurface E_i containing p is defined by one of these parameters. Equivalently, for every point p, there exists a set of regular functions whose differentials freely generate the module of Kähler differentials, such that Z and the E_i are defined by them. The set E may be empty. In this case, a subvariety is transversal if and only if it is non-singular.§

PROPOSITION 1. *The blowup of a habitat along a transversal subvariety, with the proper transforms of hypersurfaces in E supplemented by the exceptional divisor is again a habitat.*

Proof. Let $f : W' \to W$ be the blowup. Let $p' \in W'$ and let $p = f(p')$. Let $n := \dim(W)$ and $m := \dim(Z)$, where Z is the blowup centre. Assume that $u_1, \ldots, u_n$ is a set of local parameters at p such that Z is locally defined by $u_1, \ldots, u_m$, and each of the hypersurfaces in E is locally defined by u_i for some i, $1 \leqslant i \leqslant n$. Then any point in $f^{-1}(p)$ is contained in some affine open chart of W' with regular functions $u_j, u_1/u_j, \ldots, u_m/u_j, u_{m+1}, \ldots, u_m$ for some j, $1 \leqslant j \leqslant m$. The differentials of these functions freely generate the module of Kähler differentials and shows that W' is a habitat. □

DEFINITION 2. Let (W, E) be a habitat. We say that an ideal sheaf $I \subset \mathcal{O}_W$ of the structure sheaf $\mathcal{O}_W$ of regular functions on W is *principal monomial* if I is a tensor product of the invertible ideal sheaves of E_i for $E_i \in E$. A resolution of an ideal sheaf I is a finite sequence of

†Most of the notions in this section have been proposed, often in a slightly different manner, by Hironaka and/or Villamayor. Our terminology does not intend to overrule their notions. But due to the various differences, a systematic new terminology was preferable.

‡Originally, local data had to be glued together via complicated equivalence relations in order to perform the descent in dimension [8, 21, 30, 31]. In [14], a global resolution datum, called *mobile*, was defined without any gluing. The local resolution invariant could then be defined directly by a local surgery, and Hironaka's trick showed its independence from any choices. In the present paper, the construction of mobiles is refined even further, combining it with ideas from [25, 33] and, most essentially, from [22, 32].

§Geometry: One should think of the set E as the collection of the exceptional divisors created by the blowups so far; they will be used later on to identify and separate the combinatorial portion of a resolution problem.

blowups of the habitat along transversal centres $W_r \to \cdots \to W_1 \to W$ such that the pullback of I in $\mathcal{O}_{W_r}$ is principal monomial.[†]

4.1. *Ideals and algebras*

In the next few pages, we define the relevant algebraic objects of our study. As the role of the set of hypersurfaces E is only to keep track of the transversality of the possible centres of blowup with the components of the exceptional divisor, we will not mention E until explicitly needed.

DEFINITION 3. An *ideal with control* (I, c) on a manifold W is an ideal $I \subset \mathcal{O}_W$ together with an integer $c > 0$, the *control*.[‡] The *singular locus* of (I, c) is the set of all points $p \in W$ such that the order of I at p is at least c. The *sum* of two ideals with control (I_1, c_1) and (I_2, c_2) on the same manifold is defined as the ideal $I_1^{c_2} + I_2^{c_1}$ with control $c_1 c_2$. We say that (I_1, c_1) and (I_2, c_2) are *equivalent* if there exists a positive integer k such that $I_1^{kc_2} = I_2^{kc_1}$.

DEFINITION 4. Let $Z \subset W$ be a transversal subvariety contained in the singular locus of (I, c), and let $f : W' \to W$ be the blowup of W along Z. Any element in the pullback $f^*(I)$ is divisible by the cth power of a local generator of the exceptional divisor. Therefore, $f^*(I)$ can be written as a product $M^c \cdot I'$, where M is the ideal of the exceptional divisor $D = f^{-1}(Z)$ and I' is an ideal sheaf on W'. The pair $(I', c') = (I', c)$ is called the *transform* of (I, c) under f or the *controlled transform* of I.

A *resolution* of an ideal I with control c is a finite sequence of blowups with transversal centres contained in the singular locus such that the last singular locus is empty.[§]

REMARK 1. The resolution of the singularities of a given variety is a consequence of the resolution of ideals with control. The argument is as follows. If the variety is a hypersurface, then one simply resolves the ideal with control $(I, 2)$, where I is the ideal of the hypersurface. In general, one can resolve the ideal with control $(I, 1)$ and take only the part of the resolution of this ideal where the proper transform of the variety is not blown up. In the next step, when this variety is blown up, the proper transform must be a non-singular subvariety, which means by this stage we have achieved resolution of singularities of the given variety; cf. [**14**].

PROPOSITION 2. *Equivalent ideals with control have the same singular locus, equivalent transforms and the same resolutions.*

PROPOSITION 3. *The sum operation is commutative and associative. The singular locus is the intersection of the singular loci of the summands. If a transversal subvariety is contained in the singular locus of the sum, then the transform of the sum is the sum of the transforms of the summands.*

[†] If I is the ideal sheaf of a hypersurface X, then a resolution of I is a resolution of X, in the sense that the inverse image of X is the union of non-singular hypersurfaces forming a normal crossings divisor.

[‡] We follow here and in the sequel Hironaka and his notion of *idealistic exponent* [**21**].

[§] The existence of resolutions of ideals can easily be reduced to the existence of resolutions of ideals with control: one just needs to resolve the ideal with control equal to 1. Then the last transform is $(\mathcal{O}_{W_r}, 1)$, and since the pullback of I can always be written as a product of its controlled transform and a monomial ideal, the last pullback is a monomial.

Definition 5. A *Rees algebra* A on a manifold W is a coherent sheaf of locally finitely generated subalgebras of $\mathcal{O}_W[T]$, where T is a variable. We will write $A = \bigoplus A_i$ for the decomposition into homogeneous components. The A_i are ideals in $\mathcal{O}_W$, and will be considered with control i. The *singular locus* of a Rees algebra is the set of all points where A_i has order at least i, for each $i \geqslant 0$.

If Z is a transversal subvariety of (W, E) inside the singular locus of A, then the *transform* of A under the blowup of W along Z is the algebra generated by all controlled transforms of A_i for all $i > 0$.

A *resolution* of a Rees algebra is a finite composition of blowups along transversal centres inside the singular locus such that the last singular locus is empty.

Let A and B be two Rees algebras. We write $A \subseteq B$ if and only if $A_i \subseteq B_i$ for each $i > 0$; we say that A and B are *equivalent* if and only if there exists a positive integer k such that $A_{ki} = B_{ki}$ for all $i > 0$; the *sum* of A and B is the smallest Rees algebra containing both.

Proposition 4. *Equivalent Rees algebras have the same singular locus, equivalent transforms under blowup and the same resolutions.*

Definition 6. If K is a finite set of positive integers and B_k is an $\mathcal{O}_W$-ideal for all $k \in K$, we define the Rees algebra $A = \bigoplus_{i=0}^{\infty} A_i$ *generated* by B_k, $k \in K$, by taking for A_i the ideal generated by all products $B_{k_1} B_{k_2} \ldots B_{k_m}$ with $k_1 + \cdots + k_m = i$. Thus, A is the smallest Rees algebra containing B_k in degree k for all $k \in K$. If K is a finite set such that $A = \bigoplus_{i=0}^{\infty} A_i$ is the Rees algebra generated by A_k, $k \in K$, then we say that K is a *set of generating degrees*.[†]

Both Rees algebras and ideals with control are algebraic realizations of 'resolution problems'. It is possible to go back and forth between them.

Definition 7. Let A be a Rees algebra and let I be an ideal with control c. We say that A and (I, c) are *associated* if and only if there exists a positive integer k such that $A_{kci} = I^{ki}$ for all $i > 0$.

Proposition 5. *Every Rees algebra is associated to an ideal with control, which is unique up to equivalence. Conversely, every ideal with control is associated to a Rees algebra, which is unique up to equivalence.*

Associated ideals with control and Rees algebras have the same singular locus, and associated transforms under blowup. The sums of associated pairs of ideals with control and Rees algebras are again associated.

Proof. The Rees algebra associated to the ideal I with control c is the algebra generated by I in degree c. Conversely, let A be a Rees algebra with generating set K, and let n be an integer that is divisible by all integers in K. Then the ideal A_n with control n is associated to A. □

[†]Any Rees algebra A has a set of generating degrees: Take a finite cover of W by affine open subsets such that A is finitely generated on each of these. If N is the maximal degree of all these local generators, then $\{1, \ldots, N\}$ is a set of generating degrees for A.

Definition 8. Denote by Δ the operator that takes an ideal I in $\mathcal{O}_W$ and produces the ideal $\Delta(I)$ generated by I and all first-order partial derivatives of sections of I. For any Rees algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with a set of generating degrees K, the algebra $\tilde{A}$ generated by $\Delta^j(A_k)$ in degree $k-j$ for $k \in K$ and $j < k$ is called the *differential closure* of A.[†]

A *Villamayor algebra* is a differentially closed Rees algebra $A = \tilde{A}$. Equivalently, it suffices to require that $\Delta(A_{i+1}) \subseteq A_i$ for all $i \geqslant 0$.[‡] A set K such that $\Delta^j(A_k)$ generates A in degree $k-j$ for $k \in K$ and $j < k$ is called a set of generating degrees of A as a Villamayor algebra.

Definition 9. Let A be a Rees algebra. Then the *interior* A° of A is defined as the Villamayor algebra generated by all Villamayor algebras B contained in A.

Remark 2. The interior algebra can be constructed by induction on the degree i. We set $A_0^\circ = A_0 = \mathcal{O}_W$ and $A_1^\circ = A_1$. For $i > 1$, the sections of A_i° are the sections f of A_i that satisfy $\Delta(f) \subseteq A_{i-1}^\circ$. For obtaining a set of generating degrees of A°, it suffices to do the construction up to the largest degree of a set of generating degrees of A.

An essential step of Hironaka's resolution proof is induction on the dimension. We will prepare such an induction by defining restrictions of Rees algebras to smooth, locally closed subvarieties. Even though the restrictions may be local in nature, the induction argument will remain global: restriction plays only an auxiliary role.

Definition 10. Let (W, E) be a habitat and let V be a transversal subvariety of W. The algebra $A(V)$ is generated in degree 1 by the ideal defining V in W. If A is a Villamayor algebra on W such that $A(V) \subseteq A$, then the restriction $A|_V$ is defined as the Villamayor algebra on V with the ith component the image of A_i in the quotient ring $\mathcal{O}_V$ of $\mathcal{O}_W$.

Definition 11. A *singularity datum of dimension* d in a habitat (W, E) is a pair $G = (A, d)$, where A is a Villamayor algebra on W and d is an integer $0 \leqslant d \leqslant \dim(W)$, such that every point p in the singular locus of A has a neighbourhood U and a transversal subvariety $V \subseteq U$ of dimension d not contained in any hypersurface of E so that $A(V) \subseteq A$ holds in U. Such a local transversal subvariety of W is called a *zoom* for (A, d) at p.[§]

The *singular locus*, the *transform* under blowup and the *resolution* of a singularity datum (A, d) are defined as the respective items of A.

4.2. *Transforms*

We next describe the behaviour of the various algebraic items under blowup. We start with Rees algebras.

[†]The definition does not depend on the choice of the set K of generating degrees.

[‡]The differential closure is the smallest Villamayor algebra containing A. Note that if A is a Villamayor algebra, then we have $A_{i+1} \subseteq A_i$ for each $i > 0$. This is a consequence of $I \subseteq \Delta(I)$.

[§]As a first example of a singularity datum, take any Villamayor algebra A and set $d = \dim(W)$. Then W itself is a zoom for any point $p \in W$. More generally, zooms mimic the notion of hypersurfaces of maximal contact, and the restriction of singularity data to zooms captures the passage from ideals to coefficient ideals.

Lemma 6. *The singular locus of $A(V)$ is equal to V. If we blow up a proper subvariety $Z \subset V$, the transform of $A(V)$ is equal to $A(V')$, where V' is the strict transform of V. If we blow up $Z = V$, then the transform is the trivial algebra generated by $\mathcal{O}_W$ in degree* 1.

Proof. The proof is clear. □

Lemma 7. *Let A contain $A(V)$. The singular loci of A and $A|_V$ coincide. If $Z \subset V$ is a transversal subvariety contained in the singular locus, then the transform of the restriction $A|_V$ under the blowup of V along Z is equal to the restriction of the transform of A under the blowup of W along Z to the strict transform V' of V.*

Proof. We can choose regular local parameters such that V is given by a subset of these, and identify A with the result of the first description of the extension of $A|_V$. The computation is straightforward. □

Lemma 8 (Giraud). *Let I be an ideal with control c on W and let Z be a transversal subvariety of (W, E) contained in the singular locus of (I, c). Let I' be the controlled transform of (I, c) under the blowup of W along Z. Then the controlled transform of the ideal $\Delta(I)$ with control $c - 1$ is contained in $\Delta(I')$.*

Proof. Let $f : W' \to W$ be the blowup. Let $p' \in W'$ be a point on the exceptional divisor D, and set $p = f(p')$. Let $x \in \mathcal{O}_{W',p'}$ be a local equation of D. Let $\eta \in \mathrm{Der}(\mathcal{O}_{W,p})$ be a derivation. Then $f^*(\eta)$ has at most a simple pole along D, so $\eta' := xf^*(\eta)$ is a derivation in $\mathrm{Der}(\mathcal{O}_{W',p'})$. It is tangential to D. If we replace x by a suitable analytic generator of D, then we may assume $\eta'(x) = 0$.

The controlled transform of $(\Delta(I), c - 1)$ is generated by elements of the form $x^{1-c} \cdot f^*(\eta(a))$, with $a \in I$. We compute

$$\frac{f^*(\eta(a))}{x^{c-1}} = \frac{f^*(\eta)(f^*(a))}{x^{c-1}} = \frac{\eta'(f^*(a))}{x^c} = \eta'\left(\frac{f^*(a)}{x^c}\right),$$

hence $f^*(\eta(a)) \in \Delta(I')$. Since the elements of the form $\eta(a)$ generate I at p, it follows that $f^*(\Delta(I))\hat{\mathcal{O}}_{W',p'} \subseteq \Delta(I')\hat{\mathcal{O}}_{W',p'}$. Completion is a faithfully exact functor, hence the statement is also true for the ideals in the local rings. □

Lemma 9 (Villamayor). *Let A be a Rees algebra and let B be its differential closure. Then the singular loci of A and B are equal. If Z is a transversal subvariety contained in the singular set, and A' and B' are the transforms of A and B, then A' and B' have the same differential closure.*

Proof. The singular locus is equal to the intersection of all zero sets of all $(i - 1)$st derivatives of A_i for $i > 0$. But these are all in B_1, hence the singular locus of B is contained in the singular locus of A. The other direction is obvious, because A is a subset of B.

Let $f : W' \to W$ be the blowup along Z. The differential closure of A' is clearly contained in the differential closure of B'. For the converse, it suffices to show that B' is contained in the differential closure of A'. Let us denote the differential closure of A' by C'.

We prove by induction over j that, for all $i > 0$, the ideal of the controlled transform of $\Delta^j(A_{i+j})$ with control i is contained in C'_i. Since B_i is generated by the ideals $\Delta^j(A_{i+j})$, $j \geqslant 0$, it will follow that $B' \subseteq C'$.

The case $j = 0$ is obvious. Assume $j > 0$. Let $i > 0$, and set $I = \Delta^{j-1}(A_{i+j})$ with control $i+1$. By induction hypothesis, the transform I' of I with control $i+1$ is contained in C'_{i+1}. By Lemma 8, the controlled transform of $\Delta(I)$ with control i is contained in $\Delta(C'_{i+1})$. And $\Delta(C'_{i+1}) \subseteq C'_i$, because C' is differentially closed. □

PROPOSITION 10. *If (A, d) is a singularity datum, Z is a transversal subvariety of (W, E) in the singular locus of (A, d) and $\tilde{A}' = \widetilde{(A')}$ is the differential closure of the transform A' of A under the blowup of W along Z, then $(\tilde{A}', d)$ is again a singularity datum.*

Proof. Let $f : W' \to W$ be the blowup. Let $p' \in W'$. Then it is clear that the strict transform of any zoom at $f(p')$ is a zoom at p'. □

4.3. *Restriction*

An important property of Villamayor algebras is their 'stability' under restriction. This is made precise in the following statement.

THEOREM 11. *Let (W, E) be a habitat and let $V \subset W$ be a transversal subvariety. Then the restriction operator from Villamayor algebras on W containing $A(V)$ to Villamayor algebras on V is bijective.*

The inverse operator, called extension, can be constructed in two ways. For the first construction, we assume that we have a left inverse $\beta : W \to V$ to the inclusion map $i : V \to W$. Then the extension of A from V to W is equal to $\beta^*(A) + A(V)$.

The second construction is to take the interior $i_*(A)^\circ$ of $i_*(A)$.

Proof. The statement is local, hence it suffices to show it on the stalks at some point $p \in V$. Even more, we may pass to the completion, because completion of local rings is faithfully exact and therefore preserves equality of ideals. We will show that both constructions above are inverse to the restriction. This also shows that the second construction is an inverse in the Zariski-local case, when there is not necessarily a left inverse β.

Set $B = \beta^*(A) + A(V)$ and $C = i_*(A)^\circ$. We can choose a system of regular local parameters such that V is the zero set of a subset of these parameters, and the images of β^* are constant on this subset of parameters. Then B is a Villamayor algebra, and $B|_V = A$. Hence, the first construction is a right inverse operator for restriction.

Since B is a Villamayor algebra contained in $i_*(A)$, we also have $B \subseteq C$. Let T be a Villamayor algebra contained in $i_*(A)$. We prove that $T_r \subseteq B_r$ for all $r \geqslant 0$, by induction on r. For $r = 0$, the statement is trivially true. Let $r > 0$. Let a be an element of T_r. Then $a - \beta^*(i_*(a))$ is a sum of elements in $I(V)^j \Delta^j(T_r)$ for $j = 1, \ldots, r$ and $I(V)$ the ideal defining V in W, by Taylor expansion in the variables vanishing along V. This sum is in B_r because B is a Villamayor algebra and by induction hypothesis. But $\beta^*(i_*(a))$ is also in B_r, hence $a \in B_r$. It follows that $T \subseteq B$ and consequently $B = C$.

Now let T be a Villamayor algebra on W such that $T|_V = A$. Then $T \subseteq i_*(A)$, and it follows that $T \subseteq C$. We prove that $\beta^*(A_r) \subseteq T_r$ for all $r \geqslant 0$, by induction on r. For $r = 0$, the statement is trivially true. Let $r > 0$. Let b be an element of A_r. Because $T|_V = A$, there is an element $a \in T_r$ such that $i_*(a) = b$. By Taylor expansion again, $a - \beta^*(b)$ is a sum of elements in $I(V)^j \Delta^j(T_r)$ for $j = 1, \ldots, r$. This sum is in T_r, and therefore $\beta^*(b) \in T_r$. It follows that $T = B = C$. This shows that the second construction is a left inverse operator for restriction. □

The theorem shows that the Villamayor algebra on W does not carry more information than its restriction to a subvariety. A consequence is that different choices of subvarieties lead to isomorphic restrictions. The following theorem is inspired by Włodarczyk [33], where a similar statement is shown for hypersurfaces.

THEOREM 12. *Let W be a manifold. Let A be a Villamayor algebra on W. Let $V_j \subseteq W$, $j = 1, 2$, be two submanifolds of the same dimension. Assume that there exist left inverses $\beta_j : W \to V_j$ for the inclusion maps $i_j : V_j \to W$, together with isomorphisms $\phi : V_1 \to V_2$ and $\psi : W \to W$ such that $i_2 \circ \phi = \psi \circ i_1$ and $\phi \circ \beta_1 = \beta_2 \circ \psi = \beta_2$. Then $\phi^*(A|_{V_2}) = A|_{V_1}$ and $\psi^*(A) = A$.*

Proof. We first observe that $\psi^*(A)$ is the algebra generated by $\psi^*(\beta_2^*(A|_{V_2}))$ and $I(V_2)$ in degree 1. But $\psi^* \circ \beta_2^* = \beta_2^*$, hence this is the algebra generated by $\beta_2^*(A|_{V_2})$ and $I(V_2)$ in degree 1, and this is exactly A. Then it follows also that $\phi^*(A|_{V_2}) = \phi^*(i_{2*}(A)) = i_{1*}(\psi^*(A)) = i_{1*}(A) = A|_{V_1}$. □

REMARK 3. The maps β_j, ϕ and ψ in Theorem 12 need not exist in general, even in the Zariski-local case. But they always exist in the étale topology, that is, we have such left inverses and isomorphisms for the spectra of complete local rings. Therefore, we will use the result only for proving that certain algebras are equal after completion, but never for their construction.

4.4. *Orders*

The stability of Villamayor algebras under restriction allows to define order functions by their restriction to zooms.

DEFINITION 12. Let I be an ideal in W with control c and stalk I_p at $p \in W$, let A be a Rees algebra in W and let (A, d) be a singularity datum in (W, E). Define *order functions* with values in $\mathbb{Q} \cup \{+\infty\}$ by

$$\begin{aligned} \operatorname{ord}_{I,c}(p) &:= \frac{\operatorname{ord}_p(I_p)}{c}, \\ \operatorname{ord}_A(p) &:= \min_{i>0} \operatorname{ord}_{A_i,i}(p), \\ \operatorname{ord}_{A,d}(p) &:= \operatorname{ord}_B(p), \end{aligned}$$

where ord_p denotes the order of ideals in the local ring $\mathcal{O}_{W,p}$, and where B is the restriction of A to a zoom at p. The first two orders are defined on W, the third on the singular locus of (A, d).

REMARK 4. The next proposition guarantees that the definitions make sense. The order of an ideal is ∞ at p if and only if the local ideal at p is zero. The order function of a singularity datum is ∞ at p if and only if the algebra A is locally equal to $A(V)$ for one or equivalently for any zoom V at p.

PROPOSITION 13. *Equivalent, respectively, associated ideals with control and Rees algebras have the same order function.*

The minimum in the definition of the order of a Rees algebra is attained for a degree i in some set of generating degrees.

If B is the differential closure of A, then $\mathrm{ord}_A(p) = \mathrm{ord}_B(p)$ for all p in the common singular locus of A and B.

If V_1 and V_2 are two zooms, and $B_1 := A|_{V_1}$ and $B_2 := A|_{V_2}$ are the two restrictions, then $\mathrm{ord}_{B_1}(p) = \mathrm{ord}_{B_2}(p)$ for all p in the singular locus of (A, d).

The order function of a singularity datum has values at least 1. The denominator of any such value belongs to a set of generating degrees.

Proof. The first two statements are straightforward. The third statement follows from the fact that the set of all points p with order at least some rational number q is equal to the intersection of the zero sets of the ideals $\Delta^j(A_i)$ with $qi > j$. For the order function of B, we have to take all ideals of the form $\Delta^j(B_i)$ with $qi > j$, or, equivalently, all ideals of the form $\Delta^{j+k}(A_{i+k})$ with $i > qj$ and $k \geqslant 0$, because B_i is the sum of all $\Delta^k(A_{i+k})$, $k > 0$. But if $q \geqslant 1$, then the inequality $qi > j$ implies $q(i + k) > (j + k)$, hence the two intersections are equal.

The fourth statement is an immediate consequence of Theorem 12.

The last statement is a consequence of the second and third statement. □

REMARK 5. By Lemma 9, any set of generating degrees for A as a Villamayor algebra is also a set of generating degrees for A' as a Villamayor algebra (but in general not as a Rees algebra). As a consequence, the maximum of the order function cannot drop infinitely often.

DEFINITION 13. A Rees algebra or a singularity datum is called *tight* if its order function is at most 1; in the singularity datum case, this implies that it is equal to 1.

DEFINITION 14. If (A, d) is a singularity datum in a habitat (W, E) so that $(A, d - 1)$ is again a singularity datum in (W, E), then $(A, d - 1)$ is called the *descent* of (A, d).

The next lemma is an immediate consequence of Proposition 10.

LEMMA 14. *If $(A, d - 1)$ is the descent of (A, d), and $\tilde{A}'$ is the differential closure of the transform A' of A under a blowup of W along a transversal centre in the singular locus, then $(\tilde{A}', d - 1)$ is the descent of $(\tilde{A}', d)$.*

LEMMA 15. *If a singularity datum has a descent, then it is tight. Conversely, if E is empty, then every tight singularity datum with $d > 0$ has a descent. The transform of a tight singularity datum under blowup is tight.*

Proof. Assume that (A, d) has a descent. Let p be a point in the singular locus. Then there exists a zoom V_1 of dimension $d - 1$ at p. It is easy to construct a zoom V of dimension d: just leave away one of the defining equations in a system of regular parameters that defines V_1 and such that every hypersurface E_i in E containing p is defined by one of the parameters; such a system exists by transversality of V_1. Let $a \in \mathcal{O}_{W,p}$ be the parameter which has been left out to define V. Then a considered as an element in $\mathcal{O}_{V,p}$ is an element of order 1 in the restriction $A|_V$. Hence, $\mathrm{ord}_{A,d}(p) = 1$ and it follows that (A, d) is tight.

Now, assume that E is empty, and that (A, d) is tight with $d \geqslant 1$. Let p be a point in the singular locus. Let V be a zoom at p, closed in some open neighbourhood U of p. Since $A|_V$ has order 1 at p (note that this can only happen for $d \geqslant 1$), there exists an element $a \in A_1$ such that a restricted to V has order 1. The zero set Y of a in V is then non-singular at p. After shrinking U we may assume that $Y \cap U$ is non-singular, and therefore transversal, since E is empty. Obviously, $A(Y)$ is contained in A restricted to U, hence Y is a zoom at p of dimension $d-1$. It follows that $(A, d-1)$ is a descent.

In order to prove the last statement, assume again that (A, d) is tight. This condition is not related to E, so we may assume that E is empty. Therefore, (A, d) has a descent $(A, d-1)$. If $\tilde{A}'$ is the differential closure of the transform A' of A, then $(\tilde{A}', d-1)$ is the descent of $(\tilde{A}', d)$, by Lemma 14. But then $(\tilde{A}', d)$ is tight, as we have shown above. □

4.5. *Monomial factors*

Descent only works for tight singularity data. It is possible to produce tight singularity data from non-tight ones. The easiest way to see this is in terms of ideals with control: when the control c is replaced by the maximum order of the ideal, then the ideal with the new control is tight. Changing the control also changes the transform of the ideal under blowup: the new ideal differs from the old one by a monomial factor supported on the exceptional divisor. This leads us to the problem of defining monomial factors for singularity data.

Definition 15. Let (W, E) be a habitat. A *monomial* on (W, E) is a formal sum S of the hypersurfaces in E with non-negative rational coefficients. A monomial induces a *monomial function* $s : W \to \mathbb{Q}$ taking p to the sum of the coefficients of the hypersurfaces through p.

Let (I, c) be an ideal with control with order function $\mathrm{ord} : W \to \mathbb{Q}$, and let S be a monomial with monomial function $s : W \to \mathbb{Q}$. We say that S is a *monomial factor* of (I, c) if and only if $\mathrm{ord}(p) \geqslant s(p)$ for all $p \in W$. In the same way, we define monomial factors of Rees algebras.

Lemma 16. *Let (I, c) be an ideal with control. Let S be a monomial factor of (I, c). Then there exists an ideal with control (J, b), an integer $k > 0$, and a monomial ideal N, such that the order function of N is $k \cdot s$ and $I^{bk} = N^{bc} J^{ck}$. The ideal (J, b) is unique up to equivalence. Its order function is $\mathrm{ord}_{I,c} - s$.*

Definition 16. The ideal (J, b) is called the *quotient* of (I, c) by S. The quotient of a Rees algebra by a monomial factor is defined by passing to the associated ideal with control, taking the quotient and returning to the associated Rees algebra.†

Proof. Let k be a common denominator of all coefficients appearing in S. Let N be the monomial ideal defined by the integer-valued monomial kS. For $E_i \in E$, let r/k be the coefficient of E_i in S. Then $r/k \leqslant \mathrm{ord}_p(I_p)/c$ for all $p \in E_i$. It follows that $\mathrm{ord}_p(I_p^k) \geqslant rc$ for all $p \in E_i$. Then the ideal $I({E_i}^{rc})$ of ${E_i}^{rc}$ is a monomial factor of every section in I^k. The product of all these monomial factors is N^c, and by dividing them out we get J. Finally, we set $b = ck$; then the desired equations for ideals and order functions are fulfilled for (J, b). □

Lemma 17. *Let (I, c) be an ideal with control. Let S be a monomial factor of (I, c) and let (J, b) be the quotient. Let Z be a transversal subvariety contained in the singular locus of both (I, c) and (J, b). Let I' and J' be the controlled transforms on W' under the blowup of*

†It seems mandatory here to pass from algebras to associated ideals in order to define monomial factors.

W along Z. Then there is a monomial factor S' of (I', c) such that (J', b) is equivalent to the quotient.

Similarly, the quotient of a Rees algebra A by some monomial factor transforms under blowup to a quotient of the transform of A.[†]

Proof. By pulling back the two sides of the equation $I^{bk} = N^{bc}J^{ck}$, we obtain $(I')^{bk}M^{cbk} = f^*(N)^{bc}(J')^{ck}M^{bck}$, where M is the ideal sheaf of the exceptional divisor D. Since M is invertible, we may cancel it on both sides. Then the monomial of $f^*(N)$ divided by k is the desired monomial factor of (I', c). □

DEFINITION 17. Let (A, d) be a singularity datum. Let S be a monomial. We say that S is a monomial factor of (A, d) if and only if for any p in the singular locus, there exists a zoom V such that S considered as a monomial in V is a monomial factor of $A|_V$.

REMARK 6. If V_1 and V_2 are two zooms at p, then there exist analytic left inverses $\beta_j : W \to V_j$ for the inclusion maps $i_j : V_j \to W$, and analytic isomorphisms $\phi : V_1 \to V_2$ and $\psi : W \to W$ such that $i_2 = \psi \circ i_1$ and $\phi \circ \beta_1 = \beta_2 \circ \psi = \beta_2$ and such that ψ fixes the hypersurfaces in E, as a consequence of transversality. By Theorem 12, it follows that the property for some monomial being a monomial factor of a singularity datum does not depend on the choice of the zoom.

REMARK 7. It is not possible to recognize monomial factors of a singularity datum purely by looking at the order function. The reason is that the domain of the order function is only the singular locus (which may be empty), and this is too small for this purpose.

DEFINITION 18. A monomial factor S of (A, d) with monomial function s is called *exhaustive* if and only if $s(p) = \mathrm{ord}(p)$ for all p in the singular locus.[‡]

LEMMA 18. *Let S be an exhaustive monomial factor of the singularity datum (A, d), with monomial function s and monomial ideal (N, k) (that is, k is a common denominator of all values and N is the monomial ideal with order function ks). Let (A', d) be the transformed singularity datum under some blowup along a transversal centre inside the singular locus. Let S' be (the ideal of) the transform of (N, d). Then S' is an exhaustive monomial factor of (A', d).*

Proof. Let p be a singular point, and let V be a zoom at p. Let (I, c) be the ideal associated to $A|_V$ for some zoom V. Then (I, c) is equivalent to $(N|_V, k)$ locally at p. It follows that the transforms (I', c) and $(N'|_{V'})$ are also equivalent (where V' is the strict transform of V). Therefore, the order functions of (I', c) and of (N', k) are equal. □

An analogue of Lemma 16 for singularity data would certainly be useful for dividing out monomial factors. Extra care is necessary to make the construction independent of the choice of the zoom.

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†Compare this with the transformation formula under blowup for the combinatorial handicap of a mobile in [**14**].

‡This notion parallels the concept of complete monomial factors from Section 3.

THEOREM 19. *Let $M = (W, E)$ be a habitat. Let (A, d) be a singularity datum on M. Let S be a monomial factor of (A, d) with monomial function s. Let $q > 0$ be a positive number. Then there exists a singularity datum (B, d) such that*

$$\mathrm{Sing}(B, d) = \{p \in \mathrm{Sing}(A, d) \mid \mathrm{ord}_{A,d}(p) - s(p) \geqslant q\}$$

and

$$\mathrm{ord}_{B,d}(p) = \min\left(\mathrm{ord}_{A,d}(p), \frac{\mathrm{ord}_{A,d}(p) - s(p)}{q}\right)$$

for all p in $\mathrm{Sing}(B, d)$.

We call (B, d) the *quotient* of (A, d) by S scaled by q.

Proof. We construct the sheaf B locally, so we assume that we have a zoom V. Then S (considered as a monomial in V) is a monomial factor of $A|_V$. Let (I, c) be the ideal associated to $A|_V$. By Lemma 16, there is a quotient ideal (J, c). Let C be the differential closure of the Rees algebra associated to (J^n, cm), where $m/n = q$. Then we define B as the extension of $A|_V + C$. The claimed equalities of the singular loci and order functions are then fulfilled.

We need to show that the result does not depend on the choice of the zoom V (otherwise we do not get a sheaf). Let V_1 and V_2 be two zooms at p, where p is a point in the singular locus. We denote the two results of the above construction by B_1 and B_2. Let us assume that there exist analytic left inverses $\beta_1 : W \to V_1$ and $\beta_2 : W \to V_2$ for the inclusion maps $i_1 : V_1 \to W$ and $i_2 : V_2 \to W$, as well as isomorphisms $\phi : V_1 \to V_2$ and $\psi : W \to W$ such that $i_2 = \psi \circ i_1$ and $\phi \circ \beta_1 = \beta_2 \circ \psi = \beta_2$ and ψ maps any divisor in E into itself. By Theorem 12, the isomorphism ϕ^* takes $A|_{V_2}$ to $A|_{V_1}$. Let $C_i = B_i|_{V_i}$ for $i = 1, 2$. Then C_i may be obtained as above (passing to ideals with control, taking quotients changing the control, passing back to algebras, adding $A|V_i$), and it follows that ϕ^* takes C_2 to C_1.

By the first construction of the extension operator in Theorem 11, $B_1 = \beta_1^*(C_1) + A(V_1)$ and $B_2 = \beta_2^*(C_2) + A(V_2)$. By $A|_{V_1} \subseteq C_1$ and monotonicity of extension, we have $B_1 \supseteq A \supseteq A(V_2)$ and similarly $B_2 \supseteq A \supseteq A(V_1)$. For the other summands, we have

$$\beta_2^*(C_2) = \beta_1^* \circ \phi^*(C_2) = \beta_1^*(C_1),$$

and it follows that $B_1 = B_2$.

In general, there do not always exist left inverses β_1 and β_2 and isomorphisms ϕ and ψ as above, not even in the local rings. But they do exist after completion. As completion is faithfully flat, we get $B_1 = B_2$ also in this case. □

LEMMA 20. *Let (A, d) be a singularity datum. Let S be a monomial factor of (A, d). Let $q > 0$ be a rational number. Let (B, d) be the quotient of (A, d) by S scaled by q. Let Z be a transversal subvariety contained in the singular locus of B, and let (A', d) and (B', d) be the transformed singularity data. Then there is a monomial factor S' of (A', d) such that (B', d) is the quotient of (A', d) by S' scaled by q.*

Proof. This is a straightforward consequence of Lemmas 7 and 17. □

4.6. *Axioms*

We conclude this section by resuming the main properties of singularity data and operations on them. This review may also be seen as an 'axiomatic characterization' of singularity data. It might well be that one can replace the algebraic realization of singularity data presented here by a different one, but as long as the axiomatic characterization is fulfilled, the construction can be used for translating it into the combinatorial game *Stratify*.

THEOREM 21. *Any singularity datum of dimension d on a habitat (W, E) defines a closed subset, the singular locus and an upper semicontinuous order function from the singular locus to $\mathbb{Q} \cup \{\infty\}$. Its values are at least 1.*

If Z is a transversal subvariety inside the singular locus, then the transform of the singularity datum under the blowup of W along Z exists on the transformed habitat. The transform of a tight singularity datum is tight.

If the singularity datum is tight and E is empty, it admits a descent. The descent singularity datum has the same singular locus, but possibly a different order function; its dimension is $d-1$. The transform under blowup does not change the dimension d. The transform of the descent is the descent of the transform.

The subset of the singular locus with order ∞ is non-singular of pure dimension d. If a d-dimensional component is transversal to E, then the transform under the blowup along this component has no singular points in the exceptional divisor.

The existence of a monomial factor S of a singularity datum implies that the singular locus contains the set of all points where the monomial function is at least 1, and that the order function is greater than or equal to the order function. If equality holds everywhere, that is, if the monomial factor is exhaustive, then the transform of the monomial factor is an exhaustive monomial factor of the transform of the singularity datum. The trivial monomial 1 is always a monomial factor.

If a singularity datum G with order function ord_G has a monomial factor S, and $q > 0$ is a rational number, then there exists the quotient singularity datum scaled by q. Its singular locus is the set of all points p in the singular locus of G such that $\mathrm{ord}_G(p) - s(p) \geqslant q$, where s is the monomial function defined by S. Its order function is the minimum of $(\mathrm{ord}_G - s)/q$ and ord_G. The transform of the quotient is a quotient of the transform by some monomial, scaled by q.

Proof. Almost everything has been already proved, and the remaining assertions are easy consequences: the singular locus of the singularity datum (A, d) is the singular locus of A as defined in Definition 5. It is closed because it is the zero set of A_1 (see the proof of Lemma 9). The order function is defined in Definition 12 and Proposition 13. The existence of the transform is Proposition 10. The stability of tightness under blowup is in Lemma 15. The existence of the descent in case E is empty is also stated in Lemma 15. The equality of the singular loci is obvious, because the singular locus depends only on A. Commutativity of descent and transform is stated in Lemma 14. The structure of the set of points with order ∞ follows from the remark before Proposition 13. The non-existence of singular points in the exceptional divisor under blowup of a component follows from the same remark together with the remark preceding Theorem 11. The implications of the existence of a monomial factor on the singular locus, on the order function and the fact that 1 is always a monomial factor are easy consequences of the definition. Stability of exhaustive monomial factors is stated in Lemma 18. The existence and properties of the quotient is Theorem 19, and commutativity of quotient and transform is Lemma 20. □

5. *Transcription between game and resolution*

We show in this section how the resolution problem represented by a singularity datum can be transcribed into a scenario of our resolution game Stratify so that the transforms of the singularity datum under blowups, descents and quotients correspond to the respective transforms of the scenario. This permits us to transfer the search for a resolution algorithm for a given singularity datum to the search of a winning strategy for $\mathcal{A}$ for the associated scenario. By applying the reverse transcription from scenarios to singularity data, $\mathcal{A}$ resolves unconsciously a singularity datum, despite the fact that she has never heard of algebraic varieties, ideal sheaves and blowups.

The transcription has to be done in both directions. Player $\mathcal{A}$'s moves in the game will be reread geometrically: In particular, the choice of the nodes which represent the centres of her blowup move will be interpreted as the choice of a transversal subvariety of the habitat of a singularity datum, that is, the centre of the blowup of the habitat, which is contained in the singular locus and induces the transform of the singularity datum.

If $\mathcal{A}$ opens a quest, then $\mathcal{B}$ has to respond with a scenario. In this, he just has to obey the rules of the game. But for the transcription between the resolution of singularity data and the winning strategy for the game, player $\mathcal{B}$'s answer will be given as the scenario defined by the respective modification of the singularity datum which is specified by the quest (for example, for a descent quest, it will be the scenario corresponding to the descent singularity datum). It then has to be checked that this answer also obeys the rules of the game. In this sense, whenever one wishes to prove resolution via the transcription to the combinatorial game, player $\mathcal{B}$'s moves are dictated by the geometric reality. The moves and the rules of the game Stratify constitute an essentially minimal catalogue of axioms satisfied by all presently known resolution algorithms.

As resolved scenarios correspond to conquered quests, it follows that a winning strategy for the game induces a resolution strategy for singularity data.

A *stratification* of a topological space X is a finite partition of X into locally closed, irreducible subsets, called *strata*, such that the closure of any stratum is a union of strata. A stratum A is *adjacent* to a stratum B if it is contained in the closure of B. A stratification T' is a refinement of T if every stratum of T is a union of strata of T'.

For any finite collection of closed sets, there is coarsest stratification such that any closed set in the collection is a union of strata. Its strata are the irreducible components of all intersections of the closed sets in the collection and their complements. We call it the stratification induced by the collection of closed sets. Similarly, any upper semicontinuous function from X to a finite, partially ordered set induces a stratification.

DEFINITION 19. Let (W, E) be a habitat, and let (A, d) be a singularity datum on W. Let T be a stratification of W such that any divisor in E is a union of strata and such that the order function of (A, d) is constant in each stratum. Then the *induced scenario* $C = C(T, A, d)$ is defined as follows:

(1) the nodes of the underlying graph Γ are the strata of T, and the directed edges between nodes are given by the adjacency relation;
(2) the dimension of a node is the dimension of the stratum;
(3) the singular set is the set of strata with order at least 1;
(4) the order of a node is the order of the stratum;
(5) the jibs are the dense strata of the divisors E_i in E;
(6) the transversal nodes are the strata whose closure is transversal to E;
(7) the bound b is a least common multiple for the degrees of elements in a finite generating set of the Villamayor algebra A;
(8) the dimension is d;
(9) the set $\mathcal{M}$ is the set of all monomial factors of A.

LEMMA 22. *Let Z be a subvariety of W which is transversal to E and contained in the singular locus of (A, d). Assume that Z is a union of strata in T. Let (A, d) be the transformed* *singularity datum under the blowup $f : W' \to W$ of W along Z, and let T' be a stratification of W' at least as fine as the one induced by the set of exceptional divisors, the singular locus, the order function and the preimages of strata in T. Then the scenario $C(T', A', d)$ is a transform of $C(T, A, d)$ under the blowup with centre the node z corresponding to the dense stratum of Z.*

Proof. We have to check that Rules 2 and 3 are fulfilled. Most issues are straightforward, so we treat here only the more interesting ones. Rule 2, Issue (7) says that $\dim(f^{-1}(Y)) = \dim(Y) + \mathrm{codim}(Z) - 1$, if Y is a locally closed subvariety of the centre Z. This is a well-known property of blowups along non-singular subvarieties. Rule 3, Issue (2) says that the strict transform of a transversal subvariety or the total transform of a subvariety contained in the centre is transversal to the exceptional divisor. This is a well-known property of blowups along centres transversal to an already existing exceptional divisor. Item (11) is an immediate consequence of Theorem 21, stating that the transform of a tight singularity datum is tight. □

LEMMA 23. *Let C be the scenario induced by a singularity datum (A, d) on a habitat (W, E). Let C' be the scenario induced by the singularity datum (A, d) considered on the habitat (W, E'), where $E' = E \setminus \{H\}$ for some $H \in E$. Then C' is a relaxation scenario for C, releasing the node corresponding to H.*

Proof. By comparing with Rule 4, it is clear that it suffices to check the following property: if Z is transversal to $E \setminus \{H\}$, and Z is either contained in H or disjoint to it, then Z is transversal to E. But this is obvious. □

LEMMA 24. *Let C be the scenario induced by a tight singularity datum (A, d) on a habitat (W, E). Let C' be the scenario induced by the descent singularity datum $(A, d - 1)$. Let T' be refinement of the stratification T of C such that the order of $(A, d - 1)$ is constant along each stratum. Then $C(T', A, d - 1)$ is a descent scenario for $C(T, A, d)$.*

Proof. Obvious. □

LEMMA 25. *Let (A, d) be a singularity datum on a habitat (W, E), and let T be a stratification of W such that all hypersurfaces of E and the singular locus are unions of strata, and such that the order function is constant in each stratum. Let S be a monomial factor of (A, d). Let $q \in \mathbb{Q}_{\geqslant 0}$, and let (A', d) be the quotient singularity datum. Then $C(T, A', d)$ is a q-quotient scenario for $C(T, A, d)$), and S is the divided monomial factor.*

Proof. This is a direct consequence of Theorem 21. □

LEMMA 26. *Let (A, d) be a singularity datum on a habitat (W, E), and let T be a stratification of W such that all hypersurfaces of E and the singular locus are unions of strata, and such that the order function is constant in each stratum. Let E_+ be a subset of E. Let A' be the sum of A and the Villamayor algebra generated in degree 1 by the sum of the ideals in E_+. If $C(T, A, d)$ is a response for some quest, then $C(T, A', d)$ is the response of its transversality quest with respect to the set of jibs corresponding to E_+.*

Proof. This is an easy consequence of definition of transversality quests and the fact that the order function of the Villamayor algebra generated in degree 1 by the sum of the ideals in E_+ has value 1 in the intersection of these hypersurfaces. $\square$

THEOREM 27. *If there is a winning strategy for $\mathcal{A}$ in the game Stratify, singularity data admit a resolution.*

Proof. Let (A, d) be a singularity datum on an habitat (W, E). Let G be the set containing as a single element this singularity datum. Throughout, it will be a set of singularity data on the same manifold W (the set E may vary). Let T be a stratification of W induced by the divisors in E, by the singular loci of the singularity data in B and by the order functions of the singularity data in B. Let $C = C(T, A, d)$ be the induced scenario, and let Γ be the underlying board. This is the initial scenario that the Umpire provides for the game. By assumption, $\mathcal{A}$ has a winning strategy, and she uses it to win this game.

In her move, $\mathcal{A}$ may either specify a transversal and singular node z for a blowup move, apply a one-way quest or place a call for a quest. In the case of blowup, the closure of the stratum corresponding to z is taken as the centre Z of a blowup $W' \to W$ of W. Let G' be the set of all transformed singularity data (A', d). Let T' be the stratification induced by the divisors in E', the singular loci, the order functions and the preimages of strata in T. Then, for any singularity datum (A', d) in G', let $C' = C(A', d, T')$ be the induced scenario. Then C' is a transform of C by Lemma 22, and player $\mathcal{B}$'s move is given by the blowup transforms of the board and the scenarios C'.

If the move of $\mathcal{A}$ was a relaxing call releasing the set $\mathcal{J}$ of jibs, the respective hypersurfaces H are removed from E providing a new habitat (W', E') and a new singularity datum (A', d), which is just (A, d) considered on this new habitat. It is added to G. The stratification T is also compatible with (A', d), and by Lemma 23, the scenario $C(T, A', d)$ is a relaxation response for $C(T, A, d)$. Player $\mathcal{B}$'s move consists in giving the trivial refinement, the unaltered scenarios for all quests in G, the scenario C' for the relaxation quest and the scenarios C for the main quest and C' for the relaxation quest.

If the move was a call for a descent quest, then (A, d) must have been tight and $(A, d-1)$ is a descent singularity datum. It is added to G. Let T' be the refinement induced by the new order function. By Lemma 24, $C(T', A, d-1)$ is a descent scenario for C. Player $\mathcal{B}$'s move consists in refining the board and the scenarios for the old quests and providing the response $C(T', A, d-1)$ for the descent quest.

If the move was the construction of a transversality quest of some set $\mathcal{K}$ of jibs, then let A' be the sum of A and the Villamayor algebra generated in degree 1 by the sum of the ideals of the hypersurfaces corresponding to the jibs in $\mathcal{K}$. By Lemma 26, $C(T, A', d)$ is a response scenario.

If the move was a monomial factorization call with divided monomial factor m, then m is also a monomial factor of the singularity datum (A, d). Let (A', d) be the quotient singularity datum, appropriately scaled. By Lemma 25, $C(T, A', d)$ is a scaled quotient scenario.

Now it is again player $\mathcal{A}$'s turn. Since $\mathcal{A}$ has a winning strategy, she wins the game after a finite number of steps. By then, the singular locus of the transforms of (A, d) must have become empty, and the singularity datum is resolved. $\square$

6. *A winning strategy*

In this section, we prove that $\mathcal{A}$ has a winning strategy for the resolution game Stratify defined in Section 3. Again, there are no references to algebraic concepts such as ideals, varieties etc.

We say that a quest is *strictly* won if the blowup centres chosen by $\mathcal{A}$ always lie in the singular set $\mathcal{S}$.[†] Our goal is to show that any quest admits a strict winning strategy. Along the way, it will be necessary to introduce and play various auxiliary quests, whose winning strategies require centres which may lie outside the singular set of some of the other auxiliary quests. It will, however, be ensured that the centres always lie in the singular set of the main quest.

LEMMA 28 (Hironaka). *There is a strict winning strategy for monomial scenarios.*[‡]

Proof. Let $\mathfrak{Q}$ be a quest with a complete monomial factor $m : \mathcal{H} \to \mathbb{Q}$. Let C be a scenario for $\mathfrak{Q}$ on some board Γ, and assume that $\mathfrak{Q}$ is not won yet (that is, the singular set $\mathcal{S}$ of C is not empty). We define a critical set for C as a subset $\mathcal{K}$ of $\mathcal{H}$ such that there is a singular node of C smaller than or equal to every node in $\mathcal{K}$.[§] For monomial scenarios as here (that is, those with a complete monomial factor m), this is the case if and only if $\sum_{h\in\mathcal{K}} m(h) \geqslant 1$. Observe that as long as $\mathcal{S}$ is non-empty there exist critical sets. Let $N = N_{\mathcal{K}}$ be the set of singular nodes of C that are maximal among the singular nodes lying below all $h \in \mathcal{K}$. By Rule 1, Issue (9), the nodes in N have dimension $d - \mathrm{card}(\mathcal{K})$, and any two nodes in N have no node in N below both of them. By Rule 3, Issue (5), it follows that the nodes in N are transversal, that is, belong to $\mathcal{T}$. By Rule 3, Issue (13), the iterated blowup transform of C with centres all nodes in N (in any order; the individual blowups do not effect each other because there is no common node below two nodes in N) achieves that the transforms of the jibs in $\mathcal{K}$ do not have a common node below them.

An *elementary step* for a chosen minimal critical set $\mathcal{K}$ consists in blowing up successively all nodes in $N = N_{\mathcal{K}}$ (this, of course, represents a sequence of moves of $\mathcal{A}$, with respective responses by $\mathcal{B}$). We claim that quests with a complete monomial factor can be won by a concatenation of suitable elementary steps.

Define the *multiplicity* of a critical set $\mathcal{K}$ as $\sum_{h\in\mathcal{K}} m(h)$. As long as $\mathfrak{Q}$ is not yet resolved, $\mathcal{A}$ can and will choose a minimal critical set $\mathcal{K}$ for the actual scenario C of $\mathfrak{Q}$. She then applies the corresponding elementary step to C. Let us assume that the elementary step consists only of one blowup; the general case is similar. Let C' be the blowup transform of C. Then any critical set of C' is either the set of transforms of the jibs of some critical set $\mathcal{K}_1$ of C which is not a superset of $\mathcal{K}$, we call this an 'old critical set' or the set of transforms of the jibs of some critical set $\mathcal{K}_2$ of C which is a superset of $\mathcal{K}$, where one of the jibs in $\mathcal{K}$ is replaced by the exceptional jib $e = i(z)$, we call this a 'new critical set'. The multiplicity of an old critical set of C' is equal to the multiplicity of the preceding critical set of C; the multiplicity of a new critical set of C' is smaller than the multiplicity of the preceding critical set of C, because

$$m(e) = \sum_{h'\in\mathcal{K}} m(h') - 1 = m(h) + \sum_{h'\in\mathcal{K}\setminus\{h\}} m(h') - 1 < m(h),$$

where the last inequality is a consequence of the minimality of $\mathcal{K}$. If we identify the old critical sets of C' with the corresponding critical sets on C, then we can say that an elementary step replaces some critical sets by new critical sets, each of them of smaller multiplicity than the

[†] So being remote from $\mathcal{S}$ is excluded. To have the centres inside $\mathcal{S}$ is required for the induction argument to work, see the proof of Lemma 30.

[‡] The lemma asserts that a quest can strictly be won if its initial scenario is monomial, that is, has a complete monomial factor. This result corresponds to the so-called resolution of varieties in the monomial case.

[§] Singular nodes are expected candidates for centres of blowups. $\mathcal{A}$ has to ensure by auxiliary blowups that the centre of preference becomes transversal to all hypersurfaces of $\mathcal{H}$ it meets, that is, those from nodes in $\mathcal{K}$. Only after this preparation it can be chosen as a centre. The first step in the resolution is therefore to separate the stratum $\mathcal{A}$ would like to choose as centre from the hypersurfaces from $\mathcal{K}$. In the geometric situation, this is done by means of the transversality ideal [**14**, **15**].

one which was replaced. It follows that in any sequence of elementary steps all critical sets disappear after finitely many iterations. This implies that the singular set $\mathcal{S}$ of C has become empty, that is, that C is resolved and $\mathfrak{Q}$ is won. □

LEMMA 29 (Relaxation). *Let $n \geqslant 0$ be an integer. If there is a strict winning strategy for tight quests of dimension n, then there is a strict winning strategy for all quests of dimension n.*

Proof. Let $\mathfrak{Q}$ be a quest of dimension n. In order to win $\mathfrak{Q}$, $\mathcal{A}$ keeps track of a monomial factor $m : \mathcal{H} \to \mathbb{Q}$ of the scenarios of $\mathfrak{Q}$, for which the bound b of C is a common denominator of all values of m. Initially, m may be chosen as zero; this is always a monomial factor. Assume that we are at a certain stage of the game, with actual scenario C and monomial factor m of $\mathfrak{Q}$.

We distinguish two cases: First, m is a complete monomial factor. Then $\mathfrak{Q}$ can be won by Lemma 28.

Second, m is not a complete monomial factor. Then $\operatorname{ord}(s) > m(s)$ for at least one singular node s of C. Let $q > 0$ be the maximal value of $\operatorname{ord}(s) - m(s)$ for $s \in \mathcal{S}$. We will show that $\mathcal{A}$ has a strategy to make this maximum drop.

If $q = \infty$, then the maximal singular nodes for which $\operatorname{ord}(s) = \infty$ have dimension d and are transversal, by Rule 1, Issues (3) and (5). By Rule 3, Issue (13), the blowup of one of these nodes reduces by 1 the number of d-dimensional nodes with order ∞. By a finite number of steps, $\mathcal{A}$ reaches a scenario C of $\mathfrak{Q}$ where q is finite. Along the way, she lifts the monomial factor m according to Rule 8, Issue (4).

Player $\mathcal{A}$ calls now a monomial factorization quest creating thus the q-quotient $\mathfrak{Q}_1$ of $\mathfrak{Q}$ with respect to the monomial factor m. Let C_1 be the obtained scenario. It is tight, and its singular set consists of all nodes s with $\operatorname{ord}(s) - m(s) = q$. After any blowup along a centre which is in the singular set $\mathcal{S}_1$ of $\mathfrak{Q}_1$, the transform $(C_1)'$ of C_1 is tight, and it is also a q-quotient scenario for C', the transform of C, by some monomial factor m'; see Rule 8, Issue (4). Note that the bound $b' = b$ is again a common denominator for the values of m'. It follows that $\operatorname{ord}(s') - m'(s') \leqslant q$ for all s' in the singular set of C', and equality holds for the singular nodes of $(C_1)'$.

By assumption, $\mathfrak{Q}_1$ can be won by a strict winning strategy, so $\mathcal{A}$ uses it to win $\mathfrak{Q}_1$. After these blowups the singular set of the scenario of $\mathfrak{Q}_1$ has become empty. Denote by C again the scenario of $\mathfrak{Q}$ obtained by the blowups, and by m the lifted monomial factor of C as prescribed by Rule 8, Issue (4).†

As the scenario of $\mathfrak{Q}_1$ is empty, Rule 7, Issue (4), implies that $\operatorname{ord}(s) < m(s) + q$ for all singular nodes s of C. Hence, the maximal value of $\operatorname{ord}(s) - m(s)$ has dropped below q. As it is a multiple of $1/b$, it may only drop a finite number of times; eventually it reaches zero, which means that m has become a complete monomial factor of the scenario C of $\mathfrak{Q}$. Then $\mathfrak{Q}$ can be won by Lemma 28. □

REMARK 8. If $n = 0$, then there are no tight scenarios of dimension n, by Rule 1, Issue (4). The proof of Lemma 29 can easily be adapted to show that there is always a strict winning strategy for quests of dimension 0.

LEMMA 30 (Descent). *Let $n > 0$. If there is a strict winning strategy for quests of dimension $n - 1$, then there is a strict winning strategy for tight quests of dimension n.*

† Observe that C is not unique but given by player $\mathcal{B}$'s responses to player $\mathcal{A}$'s blowup moves.

Proof. Let $\mathfrak{Q}$ be a tight quest of dimension n, with scenario C. Assume that C has singular nodes, that is, that the quest is not yet won. Recall that a critical set for C is a subset $\mathcal{K}$ of $\mathcal{H}$ for which there is a singular node of C below all nodes in $\mathcal{K}$. Let now G be the collection of all critical sets of C, and let $\mathcal{L} = \mathcal{H}$ be the set of all jibs of C.[†] For any set $\mathcal{K} \in G$, we define successively three quests: the transversality quest $\mathfrak{P}_{\mathcal{K}}$ of $\mathfrak{Q}$, the relaxation quest $\mathfrak{R}_{\mathcal{K}}$ of $\mathfrak{P}_{\mathcal{K}}$ obtained by releasing $\mathcal{L}$ from $\mathfrak{P}_{\mathcal{K}}$ and the descent quest $\mathfrak{Q}_{\mathcal{K}}$ of $\mathfrak{R}_{\mathcal{K}}$.

Note that, by assumption, there exists a winning strategy for $\mathfrak{Q}_{\emptyset}$. But it would be too hasty to try to win $\mathfrak{Q}_{\emptyset}$ now, because many nodes that are admissible for $\mathfrak{Q}_{\emptyset}$ and might be needed as centres to win it may not be admissible for $\mathfrak{Q}$, so that the winning strategy for $\mathfrak{Q}_{\emptyset}$ would lead to the loss of $\mathfrak{Q}$.

In view of this, we follow a procedure which strictly wins the quests $\mathfrak{Q}_{\mathcal{K}}$ for $\mathcal{K} \in G$ one by one, starting with a maximal $\mathcal{K}$. In the course of this procedure, it may be necessary to blow up nodes that are not singular but only admissible for the involved quests. In other words, it is not possible to achieve simultaneous strict winnings for all quests.[‡]

As transversality and descent quests do not affect the transversal nodes, $\mathfrak{Q}$ and $\mathfrak{P}_{\mathcal{K}}$ have the same transversal set $\mathcal{T}$, as well as $\mathfrak{R}_{\mathcal{K}}$ and $\mathfrak{Q}_{\mathcal{K}}$, whose transversal set will be denoted by $\mathcal{T}_{\mathcal{K}}$. Similarly, $\mathfrak{P}_{\mathcal{K}}$, $\mathfrak{R}_{\mathcal{K}}$ and $\mathfrak{Q}_{\mathcal{K}}$ all have the same singular set $\mathcal{S}_{\mathcal{K}}$.

Assume that $\mathcal{K}$ is maximal and let $z \in \mathcal{T}_{\mathcal{K}} \cap \mathcal{S}_{\mathcal{K}}$. We claim that z is an admissible centre for $\mathfrak{Q}$ and for the quests $\mathfrak{P}_{\mathcal{K}'}$, $\mathfrak{R}_{\mathcal{K}'}$ and $\mathfrak{Q}_{\mathcal{K}'}$, for all $\mathcal{K}' \in G$.

Since $z \in \mathcal{S}_{\mathcal{K}}$, it follows that $z \leqslant h$ for all $h \in \mathcal{K}$. By maximality of $\mathcal{K}$, z is remote from $\mathcal{L} \setminus \mathcal{K}$. Hence $z \in \mathcal{T}$, by Rule 4, Issue (4). From $\mathcal{S}_{\mathcal{K}} \subseteq \mathcal{S}$, it follows that z is admissible for $\mathfrak{Q}$.

Let now $\mathcal{K}' \in G$ be some other critical set. Then $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{K}'}$, hence $z \in \mathcal{T}_{\mathcal{K}'}$. We distinguish two cases. If $\mathcal{K}' \subseteq \mathcal{K}$, then $\mathcal{S}_{\mathcal{K}} \subseteq \mathcal{S}_{\mathcal{K}'}$, hence $z \in \mathcal{S}_{\mathcal{K}'}$ and z is admissible for $\mathfrak{P}_{\mathcal{K}'}$, $\mathfrak{R}_{\mathcal{K}'}$ and $\mathfrak{Q}_{\mathcal{K}'}$. Otherwise, choose some $h' \in \mathcal{K}' \setminus \mathcal{K}$. By maximality of $\mathcal{K}$, the node z is remote from h', and therefore it is also remote from $\mathcal{S}_{\mathcal{K}}$. Again, it follows that z is admissible for $\mathfrak{P}_{\mathcal{K}'}$, $\mathfrak{R}_{\mathcal{K}'}$ and $\mathfrak{Q}_{\mathcal{K}'}$.

By assumption, there is a strict winning strategy for $\mathfrak{Q}_{\mathcal{K}}$. Player $\mathcal{A}$ applies it to strictly win $\mathfrak{Q}_{\mathcal{K}}$. By the above observations (still assuming that $\mathcal{K}$ is maximal), the chosen centres are also admissible for the quests $\mathfrak{Q}$, $\mathfrak{P}_{\mathcal{K}'}$, $\mathfrak{R}_{\mathcal{K}'}$ and $\mathfrak{Q}_{\mathcal{K}'}$ for all $\mathcal{K}' \in G$. Therefore, they remain open while winning $\mathfrak{Q}_{\mathcal{K}}$. And when $\mathfrak{Q}_{\mathcal{K}}$ is won, there will be no node below all jibs in $\mathcal{K}$ which is singular for $\mathfrak{Q}$. At this point, we remove $\mathcal{K}$ from the collection G of critical sets, take another maximal critical set in G in place of $\mathcal{K}$ and repeat. The reasoning that none of the remaining quests becomes invalid still applies.

The last maximal set which $\mathcal{A}$ removes from G using the above strategy will be the emptyset $\emptyset$. By then, there is no singular node at all, which means that $\mathfrak{Q}$ is resolved. □

THEOREM 31. *There is a strict winning strategy for Stratify.*

Proof. This is now an obvious consequence of the two lemmas and the remark above. □

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[†] We need to introduce a new letter here because $\mathcal{H}$ will change under blowup, whereas $\mathcal{L}$ will not change.

[‡] Compare this argument with the treatment of the transversality problem in [**14**] where the transversality ideal is defined as a product of ideals. Here, however, we take a construction which mimics the sum of ideals.

through a careful reading of an earlier draft, several ambiguities which by now should be eliminated.

References

1. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Academic Press, 1966, 2nd edn (Springer, Berlin, 1998).
2. J. M. Aroca, H. Hironaka and J. L. Vicente, *The theory of the maximal contact*, Memorias de Matemática del Instituto 'Jorge Juan' de Matemáticas, Consejo Superior de Investigaciones Científicas, Madrid, No. 29, 1975.
3. J. M. Aroca, H. Hironaka and J. L. Vicente, *Desingularization theorems*, Memorias de Matemática del Instituto 'Jorge Juan' de Matemáticas, Consejo Superior de Investigaciones Científicas, Madrid, No. 30, 1977.
4. A. Benito and O. Villamayor, 'Singularities in positive characteristic: elimination and monoidal transformations', Preprint, 2008, arXiv:math/0811.4148.
5. B. Bennett, 'On the characteristic function of a local ring', *Ann. of Math.* 91 (1970) 25–87.
6. E. Bierstone and P. Milman, 'Local resolution of singularities', *Real analytic and algebraic geometry (Trento, 1988)*, Lecture Notes in Mathematics 1420 (Springer, Berlin, 1990) 42–64.
7. E. Bierstone and P. Milman, 'A simple constructive proof of canonical resolution of singularities', *Effective methods in algebraic geometry* (eds T. Mora and C. Traverso; Birkhäuser, Basel, 1991) 11–30.
8. E. Bierstone and P. Milman, 'Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant', *Invent. Math.* 128 (1997) 207–302.
9. E. Bierstone, P. Milman and M. Temkin, 'Q-universal desingularization', Preprint, 2009, arXiv:0905.3580.
10. S. Bloch, 'The moving lemma for higher Chow groups', *J. Algebraic Geom.* 3 (1994) 537–568.
11. G. Bodnár and J. Schicho, 'Automated resolution of singularities for hypersurfaces', *J. Symbolic Comput.* 30 (2000) 401–428.
12. A. Bravo and O. Villamayor, 'Singularities in positive characteristic, stratification and simplification of the singular locus', *Adv. Math.* 224 (2010) 1349–1418.
13. D. Cutkosky, *Resolution of singularities*, Graduate Studies in Mathematics 63 (American Mathematical Society, Providence, RI, 2004).
14. S. Encinas and H. Hauser, 'Strong resolution of singularities in characteristic zero', *Comment. Math. Helv.* 77 (2002) 421–445.
15. S. Encinas and O. Villamayor, 'A course on constructive desingularization and equivariance', *Resolution of singularities (Obergurgl, 1997)* (eds H. Hauser et al; Birkhäuser, Basel, 2000) 147–227.
16. S. Encinas and O. Villamayor, 'Rees algebras and resolution of singularities', *Proc. XVI-Coloquio Latinoamericano de Álgebra*, 2006, Revista Matemática Iberoamericana.
17. A. Frühbis-Krüger and G. Pfister, 'Computational aspects of singularities', *Singularities in geometry and topology* (World Scientific Publishing, Hackensack, 2007) 253–327.
18. H. Hauser, 'On the problem of resolution of singularities in positive characteristic (Or: a proof we are still waiting for)', *Bull. Amer. Math. Soc.* 47 (2010) 1–30.
19. H. Hironaka, 'Resolution of singularities of an algebraic variety over a field of characteristic 0', *Ann. of Math.* 79 (1964) 109–326.
20. H. Hironaka, 'Characteristic polyhedra of singularities', *J. Math. Kyoto Univ.* 7 (1967) 251–293.
21. H. Hironaka, *Idealistic exponents of singularity*, The Johns Hopkins Centennial Lectures (Johns Hopkins University Press, Baltimore, 1977).
22. H. Hironaka, 'Theory of infinitely near singular points', *J. Korean Math. Soc.* 40 (2003) 901–920.
23. H. Kawanoue, 'Toward resolution of singularities over a field of positive characteristic. Part I', *Publ. Res. Inst. Math. Sci.* 43 (2007) 819–909.
24. H. Kawanoue and K. Matsuki, 'Toward resolution of singularities over a field of positive characteristic (The Idealistic Filtration Program). Part II. Basic invariants associated to the idealistic filtration and their properties', *Publ. Res. Inst. Math. Sci.* 46 (2010) 819–909.
25. J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies 166 (Princeton University Press, Princeton, NJ, 2007).
26. M. Levine, 'Blowing up monomial ideals', *J. Pure Appl. Algebra* 160 (2001) 67–103.
27. J. Lipman, 'Introduction to resolution of singularities', *Proc. Sympos. Pure Appl. Math. Amer. Math. Soc.* 29 (1975) 187–230.
28. M. Spivakovsky, 'A counterexample to Hironaka's "hard" polyhedral game', *Publ. Res. Inst. Math. Sci.* 18 (1982) 1009–1012.
29. M. Spivakovsky, 'A solution to Hironaka's polyhedral game', *Arithmetic and geometry* (eds M. Artin and J. Tate; Birkhäuser, Basel, 1983) 419–432.
30. O. Villamayor, 'Constructiveness of Hironaka's resolution', *Ann. Sci. École Norm. Sup.* (4) 22 (1989) 1–32.
31. O. Villamayor, 'Patching local uniformizations', *Ann. Sci. École Norm. Sup.* 25 (1992) 629–677.
32. O. Villamayor, 'Hypersurface singularities in positive characteristic', *Adv. Math.* 213 (2007) 687–733.

33. J. Włodarczyk, 'Simple Hironaka resolution in characteristic zero', *J. Amer. Math. Soc.* 18 (2005) 779–822.
34. D. Zeillinger, 'Polyhedral games and resolution of singularities', PhD Thesis, Univ. Innsbruck, 2006.

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