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Compositio Mathematica, tome 76, n° 1-2 (1990), p. 197-201.
<http://www.numdam.org/item?id=CM_1990__76_1-2_197_0>

Compositio Mathematica **76**: 197–201, 1990.

Analytic curves in power series rings

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Received 3 December 1988; accepted 20 July 1989

Let us state a standard result on algebraic group actions:

PROPOSITION. *For an analytic map germ* $\gamma\colon S \to (V, v)$, *S a reduced analytic space germ and* (V, v) *the germ in v of a finite dimensional complex vector space V, together with an algebraic subgroup G of* $\mathrm{GL}(V)$ *the following holds:*

(i) *The germ T of points t in S for which* $\gamma(t)$ *lies in the orbit* $G \cdot v$ *of G through v is analytic.*

(ii) *There is an analytic map germ* $\phi\colon T \to (G, 1)$ *such that* $\gamma(t) = \phi(t) \cdot v$ *for all t in T.*

Indeed, the orbit $G \cdot v$ is a locally closed submanifold of V, isomorphic to the homogeneous manifold G/G_v via the orbit map $G/G_v \to G \cdot v$, and the natural map $G \to G/G_v$ admits local sections.

The second part of the Proposition asserts that every analytic curve in a G-orbit is locally induced from an analytic curve in G. The object of the present article is to establish this statement in the case where $V = \mathcal{O}_n^p$ is a finite free module over the $\mathbb{C}$-algebra $\mathcal{O}_n$ of convergent power series in n variables and $G = \mathcal{K} = \mathrm{GL}_p(\mathcal{O}_n) \rtimes \mathrm{Aut}\, \mathcal{O}_n$ is the contact group acting naturally on $\mathcal{O}_n^p$. Recall that the orbits of $\mathcal{K}$ through f in $\mathcal{O}_n^p$ just correspond to the isomorphism classes of the analytic space germs in $(\mathbb{C}^n, 0)$ defined by f. Thus we shall obtain analytic trivializations of local families of space germs whose members lie in the same isomorphism class.

DEFINITION. Let S always denote a *reduced* analytic space germ. A map germ $\gamma\colon S \to E$ with values in some subset E of $\mathcal{O}_n^p$ is called analytic if there is an analytic map germ $G\colon (\mathbb{C}^n, 0) \times S \to \mathbb{C}^p$ such that $\gamma(s)(x) = G(x, s)$.

Choosing coordinates on $(\mathbb{C}^n, 0)$, the group $\operatorname{Aut}\mathcal{O}_n$ can be considered as a subset of $\mathcal{O}_n^n$. The analyticity of a map germ with values in $\operatorname{Aut}\mathcal{O}_n$ does not depend on this choice (cf. [M, sec. 6]) and thus analytic map germs with values in $\mathcal{K}$ are defined. We then have:

THEOREM 1. *Let $\gamma: S \to \mathcal{O}_n^p$ be an analytic map germ, S reduced.*

(i) *The germ T of points t in S for which $\gamma(t)$ lies in the orbit $\mathcal{K} \cdot \gamma(0)$ is analytic.*

(ii) *There is an analytic map germ $\phi: T \to \mathcal{K}$ with $\phi(0) = 1$ such that $\gamma(t) = \phi(t) \cdot \gamma(0)$ for all t in T.*

Proof. For $m_n \subset \mathcal{O}_n$ the maximal ideal and $k \in \mathbb{N}$ consider $A_k = \mathcal{O}_n / m_n^{k+1}$ and the algebraic group $\mathcal{K}_k = \mathrm{GL}_p(A_k) \rtimes \operatorname{Aut} A_k$ acting rationally on the finite dimensional vector space $V_k = A_k^p$. The composition $\gamma_k: S \to (V_k, \gamma_k(0))$ of γ with the natural map $\mathcal{O}_n^p \to V_k$ is analytic. By the Proposition the germ T_k of points t in S with $\gamma_k(t) \in \mathcal{K}_k \cdot \gamma_k(0)$ is analytic. Clearly $T_{k+1} \subset T_k$. As $\mathcal{O}_S$ is Noetherian the sequence becomes stationary, say $T_k = T^*$ for $k \gg 0$. The Proposition gives analytic $\phi_k: T^* \to (\mathcal{K}_k, 1)$ such that $\gamma_k(t) = \phi_k(t) \cdot \gamma_k(0)$ for $t \in T^*$. By Theorem 2 below there is an analytic $\phi: T^* \to \mathcal{K}$ with $\phi(0) = 1$ such that $\gamma(t) = \phi(t) \cdot \gamma(0)$ for all $t \in T^*$. This implies $T^* \subset T$. Obviously $T \subset T^*$ and Theorem 1 is proved.

THEOREM 2. *For two analytic map germs $\gamma, \eta: S \to \mathcal{O}_n^p$, S reduced, the following conditions are equivalent:*

(i) *There exists an analytic $\phi: S \to \mathcal{K}$ with $\phi(0) = 1$ such that $\gamma(s) = \phi(s) \cdot \eta(s)$ for all $s \in S$.*

(ii) *For any $k \in \mathbb{N}$ there exist analytic $\phi_k: S \to \mathcal{K}_k$ with $\phi_k(0) = 1$ such that $\gamma_k(s) = \phi_k(s) \cdot \eta_k(s)$ for all $s \in S$.*

Proof. Embed S in $(\mathbb{C}^m, 0)$ and choose $G, H: (\mathbb{C}^{n+m}, 0) \to \mathbb{C}^p$ such that one has $\gamma(s)(x) = G(x, s)$ and $\eta(s)(x) = H(x, s)$. We have to find $u(x, s) \in \mathrm{GL}_p(\mathcal{O}_{n+m})$ and $y(x, s) \in \mathcal{O}_{n+m}^n$ such that:

$$u(x,0) = 1, \qquad y(x,0) = x, \qquad y(0,s) = 0,$$

and

$$H(y(x,s),s) \equiv u(x,s) \cdot G(x,s) \mod I(S)$$

where $I(S)$ is the ideal of $\mathcal{O}_m$ defining S in $(\mathbb{C}^m, 0)$. By condition (ii) this system of equations can be solved up to order k. A generalization of Artin's Approximation Theorem by Pfister and Popescu [P-P, Thm. 2.5] and Wavrik [W, Thm. 1] yields the solutions $u(x, s)$ and $y(x, s)$.

Let us now indicate some applications of Theorem 1. We first determine the tangent spaces to the orbits of the contact group $\mathcal{K}$ in $\mathcal{O}_n^p$:

DEFINITION. (a) The tangent vector in $\gamma(0)$ of an analytic curve $\gamma: (\mathbb{C}, 0) \to \mathcal{O}_n^p$

given by $G: (\mathbb{C}^n \times \mathbb{C}, 0) \to \mathbb{C}^p$ is defined as:

$$\frac{d\gamma}{ds}(0) = \frac{\partial G}{\partial s}\Big|_{s=0} \in \mathcal{O}_n^p.$$

(b) The tangent map at 0 of an analytic map germ $\gamma: (\mathbb{C}^m, 0) \to \mathcal{O}_n^p$ given by $G: (\mathbb{C}^n \times \mathbb{C}^m, 0) \to \mathbb{C}^p$ is defined as

$$T_0\gamma: \mathbb{C}^m \to \mathcal{O}_n^p, \quad v \to \sum_{i=1}^m v_i \cdot \frac{\partial \gamma}{\partial s_i}(0)$$

where

$$\frac{\partial \gamma}{\partial s_i}(0) = \frac{\partial G}{\partial s_i}\Big|_{s=0} \in \mathcal{O}_n^p.$$

(c) For a subset E of $\mathcal{O}_n^p$ and $g \in E$ let

$$T_g E = \left\{ \frac{d\gamma}{ds}(0) \in \mathcal{O}_n^p, \quad \gamma: (\mathbb{C}, 0) \to E \text{ analytic with } \gamma(0) = g \right\}.$$

COROLLARY. *Let $g \in \mathcal{O}_n^p$ with $\mathscr{K}$-orbit $\mathscr{K} \cdot g$.*

(i) $T_g(\mathscr{K} \cdot g) = I(g) \cdot \mathcal{O}_n^p + \mathfrak{m}_n \cdot J(g)$
where $I(g)$ is the ideal of $\mathcal{O}_n$ generated by the components of g and $J(g)$ is the $\mathcal{O}_n$-submodule of $\mathcal{O}_n^p$ generated by the partial derivatives of g.

(ii) *Let $\gamma: (\mathbb{C}^m, 0) \to \mathcal{O}_n^p$ be analytic, $\gamma(0) = g$, restricting to $\gamma_{|S}: S \to \mathscr{K} \cdot g$ for some reduced $S \subset (\mathbb{C}^m, 0)$. Then $T_0\gamma(T_0S) \subset T_g(\mathscr{K} \cdot g)$.*

Proof. (i) If $G(x, s) = u(x, s) \cdot g(y(x, s))$ with $u(x, 0) = 1, y(x, 0) = x$, then

$$\frac{\partial G}{\partial s}\Big|_{s=0} = \frac{\partial u}{\partial s}\Big|_{s=0} \cdot g + \frac{\partial g}{\partial x} \cdot \frac{\partial y}{\partial s}\Big|_{s=0}.$$

Theorem 1 therefore implies "$\subset$". The other inclusion is obvious.

(ii) Since any analytic $S \to \mathscr{K}$ can be extended to an analytic $(\mathbb{C}^m, 0) \to \mathscr{K}$, Theorem 1 gives an analytic map germ $\delta: (\mathbb{C}^m, 0) \to \mathcal{O}_n^p$, $\delta(0) = g$, such that $\delta(s) \in \mathscr{K} \cdot g$ for $s \in (\mathbb{C}^m, 0)$ and $\delta(s) = \gamma(s)$ for $s \in S$. Clearly $T_0\delta(\mathbb{C}^m) \subset T_g(\mathscr{K} \cdot g)$ and $T_0\delta(v) = T_0\gamma(v)$ for $v \in T_0S$.

Next, let us interpret Theorem 1 geometrically.

For an analytic $\gamma: S \to \mathcal{O}_n^p$ given by $G: (\mathbb{C}^n, 0) \times S \to \mathbb{C}^p$ consider the space germ X defined in $(\mathbb{C}^n, 0) \times S$ by G. For fixed $s \in S$ the vector $\gamma(s) \in \mathcal{O}_n^p$ defines the germ in $\sigma(s) = (0, s)$ of the fiber of $\pi = pr_{|X}: X \to S$ over s. Conversely, a morphism $\pi: X \to S$ of space germs with section $\sigma: S \to X$ has an embedding $X \subset (\mathbb{C}^n, 0) \times S$ over S with $\sigma(S) = 0 \times S$, see [F, 0.35]. Moreover an analytic $\Phi: S \to \operatorname{Aut} \mathcal{O}_n$ given by $\Phi(s)(x) = y(x, s)$ induces an automorphism ϕ of $(\mathbb{C}^n, 0) \times S$ over S mapping $0 \times S$ onto itself: $\phi(x, s) = (y(x, s), s)$.

Combining these remarks we get:

THEOREM 1′. *For a morphism of analytic space germs $\pi: X \to S$, S reduced, with section $\sigma: S \to X$ denote by X_t, $t \in S$, the germ in $\sigma(t)$ of the fiber of π over t.*

(i) *The germ T of points t in S with $X_t \simeq X_0$ is analytic.*

(ii) *For any base change $\alpha: S' \to S$ with S' reduced the induced morphism $\pi': X' = X \times_S S' \to S'$ is trivial along the induced section $\sigma': S' \to X'$ if and only if α maps into T.* (*We say that π' is trivial along σ' if there is an isomorphism $X' \simeq X_0 \times S'$ over S' mapping $\sigma'(S')$ onto $0 \times S'$.*)

The universal property of (ii) applies in particular to the base change $T \subset S$ and then reads as follows: A local analytic family of analytic space germs with isomorphic members is trivial. This is a local analogon of a result of Fischer and Grauert [F-G] and Schuster [Sch, Satz 4.9]: A flat analytic family of compact analytic spaces with isomorphic members is locally trivial.

Theorem 1′ can be extended to the case where π does not come with a section σ:

THEOREM 3. *For a morphism of analytic space germs $\pi: X \to S$ with S reduced denote by $X(a)$, $a \in X$, the germ in a of the fiber of π through a.*

(i) *The germ Y of points a in X with $X(a) \simeq X(0)$ is analytic.*

(ii) *The restriction $\pi_Y: Y \to S$ is a mersion (i.e., has smooth special fiber $Y(0)$ and there is a germ $T \subset S$ such that π_Y maps into T and $Y \simeq Y(0) \times T$ over T).*

(iii) *For any base change $\alpha: S' \to S$ with S' reduced the induced morphism $\pi': X' = X \times_S S' \to S'$ is trivial if and only if α maps into T.*

(iv) *There is a germ Z with $X(0) \simeq Y(0) \times Z$.*

Proof. Choose embeddings $X \subset (\mathbb{C}^n, 0) \times S$ over S and $S \subset (\mathbb{C}^m, 0)$ and let $F: (\mathbb{C}^n \times \mathbb{C}^m, 0) \to \mathbb{C}^p$ define X. Let $\gamma: (\mathbb{C}^n \times \mathbb{C}^m, 0) \to \mathcal{O}_n^p$ be given by $\gamma(a)(x) = F(x + a_1, a_2)$. For fixed $a \in X$ the germ $\gamma(a) \in \mathcal{O}_n^p$ defines $X(a)$. Hence Theorem 1 yields the analyticity of Y.

Let $Y(a)$ be the germ in a of the fiber of π_Y through a. For $a \in Y$ fixed its reduction red $Y(a)$ is the germ of those points $b \in X(a)$ with $X(b) \simeq X(a)$. As $X(a)$ and $X(0)$ are isomorphic, red $Y(a)$ and red $Y(0)$ are isomorphic. In particular, $\dim Y(a) = \dim Y(0)$ for all $a \in Y$.

By the Corollary we have for $g = \gamma(0) \in \mathcal{O}_n^p$ and $v \in T_0 Y(0) = T_0 Y \cap (\mathbb{C}^n \times 0)$:

$$\sum_{i=1}^{n} v_i \cdot \frac{\partial g}{\partial x_i} \in I(g) \cdot \mathcal{O}_n^p + \mathfrak{m}_n \cdot J(g).$$

As g defines $X(0)$ in $(\mathbb{C}^n, 0)$ this signifies that there are $d = \dim T_0 Y(0)$ vectorfields $\xi_1, \ldots, \xi_d$ on $X(0)$ with $\xi_1(0), \ldots, \xi_d(0)$ linearly independent. A Theorem of Rossi [F, 2.12] implies $X(0) \simeq (\mathbb{C}^d, 0) \times Z$ for some Z. Hence, by definition, $Y(0)$ must have dimension at least d and therefore $Y(0) \simeq (\mathbb{C}^d, 0)$. This gives (iv). Moreover π_Y is a mersion by [F, 2.19, Cor. 2]. The universal property of (iii) is then a consequence of Theorem 1′.

We conclude by some remarks: In the absolute case $S = 0$ we have recovered a result of Ephraim [E, Thm. 0.2]. Another Corollary is Teissier's economy of the semi-universal deformation: In the semi-universal deformation of an isolated singularity $X(0)$ there are no fibers isomorphic to $X(0)$, [T, Thm. 4.8.4].

Finally, it is possible to provide the germs Y and T of Theorem 3 with canonical non-reduced analytic structures. The universal property then holds for arbitrary base changes. A detailed exposition of this non-reduced case is given in [H-M]. We also refer to results of Flenner and Kosarew [F-K] and Greuel and Karras [G-K]. Using deformation theory and Banach-analytic methods they treat the case of flat morphisms.

Acknowledgements

We would like to thank G.-M. Greuel and H. Flenner for stimulating discussions and suggestions.

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