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AFFINE VARIETIES AND LIE ALGEBRAS OF VECTOR FIELDS

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In this article, we associate to affine algebraic or local analytic varieties their tangent algebra. This is the Lie algebra of all vector fields on the ambient space which are tangent to the variety. Properties of the relation between varieties and tangent algebras are studied. Being the tangent algebra of some variety is shown to be equivalent to a purely Lie algebra theoretic property of subalgebras of the Lie algebra of all vector fields on the ambient space. This allows to prove that the isomorphism type of the variety is determined by its tangent algebra.

INTRODUCTION

Algebraic geometry relies on associating to an affine variety its ideal of vanishing functions. Hilbert's Nullstellensatz asserts that this defines a one to one correspondence between affine varieties and radical ideals.

We propose to associate to an affine algebraic or local analytic variety a different algebraic object, its tangent algebra. This is the Lie algebra formed by all vector fields defined on the affine ambient space and tangent to the variety.

In the first part of this paper we explore the correspondence between varieties and their tangent algebra. Various fundamental properties are described. This leads to a characterization of tangent algebras as subalgebras of the Lie algebra of all vector fields in purely Lie algebra theoretic terms. Subalgebras satisfying this Lie algebra theoretic property will be called geometric. Being geometric can be viewed as an integrability condition on Lie algebras of vector fields: For every geometric subalgebra there is a unique variety, called the integral variety, with tangent algebra equal to the given algebra.

Tangent algebras and integral varieties thus define a one to one correspondence between subvarieties of affine space and geometric subalgebras of the Lie algebra of all vector fields, the Gröbner correspondence. To our knowledge, Gröbner initiated the study of tangent algebras of varieties considering vector fields tangent to projective varieties [10].

Disposing of the correspondence, one is led to express geometric properties of varieties through Lie algebra properties of the tangent algebra. In this way we shall characterize smooth and irreducible varieties or varieties with isolated singularities in terms of their tangent algebra. Also it is possible to describe the singular locus and the irreducible components of the variety.

All these results hold true for affine algebraic varieties considering polynomial vector fields as well as for germs of analytic subvarieties of complex space and analytic vector fields. The proofs differ only slightly.

In the second part the Gröbner correspondence will be applied to prove that two varieties embedded in the same affine space are isomorphic if and only if their tangent algebras are isomorphic (in the local analytic case, the tangent algebra is viewed as a topological Lie algebra, and the ambient space has dimension at least three.) This type of result has first been established in 1954 by Shanks and Pursell [23] for compact differentiable manifolds and later on in various other situations by numerous authors, e.g. [8, 16, 19, 25, 26, 30]. Our result was known to hold true in the following special cases: Omori [19] treated the case of weighted homogeneous singularities. A result of Skryabin [26] combined with Proposition 5.3 of part I shows that cartesian products of analytic germs $X = X' \times (\mathbb{C}, 0)$ are determined by the abstract tangent algebra without topology. Similar results but only valid for weighted homogeneous isolated complete intersection singularities can be found in articles of Martin and Siebert [17, 24, 25].

In the subsequent paper [15] we shall prove that isolated hypersurface singularities are determined by the Lie algebra of derivations of their local ring, i.e. vector fields *on* the variety. In the algebraic case, this has been proven for normal varieties by Siebert [25].

In forthcoming work we shall use our main result to prove that analytic hypersurface singularities are determined by their infinite dimensional Lie group of embedded automorphisms.

The work on the present paper was motivated by our reading of Omori's article [19] which served us as a valuable source of inspiration. Our results were announced in [14]. The first author thanks the Universidad Complutense de Madrid and the Universidad de Valladolid, the second author the Universität Innsbruck for their hospitality during part of the work on this paper.

PART I: THE GRÖBNER CORRESPONDENCE

1. RESULTS

Let $\mathbf{A}^n$ denote the n-dimensional affine space over an algebraically closed field K of characteristic 0 and $(\mathbb{C}^n, 0)$ the germ at 0 of complex affine space. By $\mathbb{D}$ we shall either denote the Lie algebra of polynomial vector fields on $\mathbf{A}^n$ or of germs of analytic vector fields on $(\mathbb{C}^n, 0)$. We identify $\mathbb{D}$ with the Lie algebra of derivations of the polynomial ring $K[x]$ or of the convergent power series ring $\mathcal{O}_n$ respectively.

By variety we shall always understand an algebraic subvariety of $\mathbf{A}^n$ or the germ of an analytic subvariety of $(\mathbb{C}^n, 0)$, both reduced but possibly reducible. Given a variety X, let I_X denote its ideal of vanishing functions. The *tangent algebra* $\mathbb{D}_X$ of X of vector fields on the ambient space tangent to the variety equals

$$\mathbb{D}_X = \{D \in \mathbb{D},\ D(I_X) \subset I_X\}.$$

Let A be a subalgebra of $\mathbb{D}$. The *integral variety* X_A of A is defined as the smallest subvariety of the ambient space such that all vector fields vanishing on X_A belong to A. We shall prove:

Narkissos' Theorem. *Tangent algebras are geometric in* $\mathbb{D}$.

Echo's Theorem. *Every geometric subalgebra of* $\mathbb{D}$ *is the tangent algebra of a variety. This variety is unique and given as the integral variety of the algebra.*

Being geometric is a property of subalgebras of a Lie algebra which is defined in purely Lie algebra theoretic terms. It is as follows: A subalgebra A of a Lie algebra B will be called *balanced in* B if A contains no non-zero ideal of B but an element $a \neq 0$ such that

$$[a, B] \subset A \quad \text{and} \quad [[a, B], B] \subset A.$$

It is called *geometric in* B if every chain

$$A \subset A_k \subset \ldots \subset A_0 = B$$

with $A_i \subset A_{i-1}$ maximal balanced can be completed to a chain

$$A = A_m \subset \ldots \subset A_k \subset \ldots \subset A_0 = B$$

with all inclusions maximal balanced. For a tangent algebra $A = \mathbb{D}_X$ such chains can be constructed using the irreducible components of X and its singular locus Sing X, see part (5) of the Theorem below.

The relation between geometric properties of a variety and algebraic properties of its tangent algebra can be seen from:

Dictionary Theorem. *Let X be a non-empty proper subvariety of the affine ambient space.*

1. *X is smooth and irreducible if and only if its tangent algebra $\mathbb{D}_X$ is maximal geometric (= maximal balanced) in $\mathbb{D}$.*

2. *X is irreducible if and only if $\mathbb{D}_X$ is maximal geometric in $\mathbb{D}_{\mathrm{Sing}\ X}$.*

3. *Assume X irreducible. Then X is singular in 0 and smooth else if and only if $\mathbb{D}_X$ is maximal geometric in $\mathbb{D}_0$, the Lie algebra of vector fields vanishing at 0.*

4. *Assume X irreducible. Then Sing X equals the integral variety of the unique geometric subalgebra A of $\mathbb{D}$ containing $\mathbb{D}_X$ for which $\mathbb{D}_X \subset A$ is maximal geometric.*

5. *The number of irreducible components of X equals the number of geometric subalgebras $A \subset \mathbb{D}$ containing $\mathbb{D}_X$ as a maximal geometric subalgebra. Any such algebra is the tangent algebra of the union of the singular locus of one component of X with the union of the remaining components.*

Example. Let $X \subset \mathbf{A}^2$ be the union of the y-axis Y and the cusp $Z = \{(x,y),\ x^3 = y^2\}$. Then $\mathbb{D}_X = \mathbb{D}_Y \cap \mathbb{D}_Z$, where $\mathbb{D}_Y$ is generated as $K[x,y]$-module by $x\partial_x$ and ∂_y, and $\mathbb{D}_Z$ is generated by the Euler vector field $2x\partial_x + 3y\partial_y$ and the Hamiltonian $2y\partial_x + 3x^2\partial_y$. There are two chains of maximal balanced subalgebras between $\mathbb{D}_X$ and $\mathbb{D}$, namely

$$\mathbb{D}_X \subset \mathbb{D}_Y \subset \mathbb{D} \quad \text{and} \quad \mathbb{D}_X \subset \mathbb{D}_Z \subset \mathbb{D}_0 \subset \mathbb{D}.$$

Nota. With very few exceptions all results of part I hold true in the algebraic as well as in the local analytic context. The statements are identical or differ slightly. To straighten the exposition we shall only treat in this part the local analytic case. The corresponding assertions and proofs for the algebraic case can be mostly obtained by obvious minor changes. If not so, explicit mention is made.

2. THE TANGENT ALGEBRA OF AN ANALYTIC GERM

From now on variety shall always mean germ of an analytic subvariety of $(\mathbb{C}^n, 0)$. In this section we describe tangent algebras $\mathbb{D}_X$ geometrically (Proposition 2.1), express them in terms of the tangent algebras of the irreducible components of X (Seidenberg's Theorem), reconstruct X from $\mathbb{D}_X$ through Fitting ideals (Proposition 2.2) and show how to compute $\mathbb{D}_X$ from X in special cases (Aleksandrov's and Kersken's Theorem).

Let $D : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be an analytic vector field on $(\mathbb{C}^n, 0)$, naturally identified with a $\mathbb{C}$-linear derivation $D : \mathcal{O}_n \to \mathcal{O}_n$ of the $\mathbb{C}$-algebra $\mathcal{O}_n$. We shall write

$$D = (a_1, ..., a_n) = \sum a_i \partial_{x_i},$$
$$Dg = \sum a_i \partial_{x_i} g \quad \text{and} \quad D(p) = (a_1(p), ..., a_n(p)) \in \mathbb{C}^n$$

for coordinates $x = (x_1, ..., x_n)$ on $(\mathbb{C}^n, 0)$, functions $g \in \mathcal{O}_n$ and points $p \in (\mathbb{C}^n, 0)$. We denote by $\mathbb{D} = \mathbb{D}_n = \operatorname{Der} \mathcal{O}_n$ the space of all analytic vector fields on $(\mathbb{C}^n, 0)$. This is an $\mathcal{O}_n$-module as well as a Lie algebra (with the usual bracket $[D, E] = DE - ED$.) As an $\mathcal{O}_n$-module it is generated by the partial derivatives $\partial_{x_1}, ..., \partial_{x_n}$. There is a basic identity relating $\mathcal{O}_n$-multiplication with the bracket, namely

$$[D, gE] = Dg \cdot E + g \cdot [D, E],$$

where $D,\ E \in \mathbb{D}$ and $g \in \mathcal{O}_n$. This identity will be used at various occasions.

For an analytic germ $X \subset (\mathbb{C}^n, 0)$ denote by $\mathrm{T}_p X$ its Zariski tangent space at a point $p \in X$. Here point of a germ means point of a suitable representative of X on a small neighborhood of 0 in $\mathbb{C}^n$. We shall say that a property holds for all points in X if there is some representative of X on which the property is valid.

We collect some elementary facts about $\mathbb{D}_X$:

Proposition 2.1. (Description of $\mathbb{D}_X$) *Let $X \subset (\mathbb{C}^n, 0)$ be analytic.*
(a) *$D \in \mathbb{D}$ is tangent to X if and only if $D(p) \in \mathrm{T}_p X$ for all points p of a dense subset of X, e.g., for all smooth points p of X.*
(b) *$\mathbb{D}_X$ is a Lie submodule of $\mathbb{D}$, i.e., a Lie subalgebra as well as an $\mathcal{O}_n$-submodule. The set $I_X \cdot \mathbb{D}$ of vector fields vanishing on X is an ideal of $\mathbb{D}_X$.*

Remarks. (a) For hypersurfaces, Saito [21] interpreted the tangent algebra as the dual of the module of logarithmic differential forms having poles only on the variety. He called tangent vector fields logarithmic. See also [2, 6, 27].

(b) We stress that $\mathbb{D}_X$ is defined using the embedding of X in $(\mathbb{C}^n, 0)$. It must not be confused with the Lie algebra $\theta_X = \mathrm{Der}\ \mathcal{O}_X = \mathrm{Der}\ (\mathcal{O}_n/I_X)$ of vector fields *on* X.

(c) For germs of analytic hypersurfaces we can prove that $\mathbb{D}_X$ equals the Lie algebra of the infinite dimensional Lie group $\mathrm{Aut}_X(\mathbb{C}^n, 0)$ of analytic automorphisms of $(\mathbb{C}^n, 0)$ stabilizing X.

Example. Consider the Whitney umbrella $X \subset (\mathbb{C}^3, 0)$ defined by the polynomial equation $x^2 - y^2 z = 0$. Its tangent algebra $\mathbb{D}_X$ is generated as an $\mathcal{O}_n$-module by the Euler vector fields $2x\partial_x + y\partial_y + 2z\partial_z$ and $y\partial_y - 2z\partial_z$ and the Hamiltonians $2yz\partial_x + 2x\partial_y$, $y^2\partial_x + 2x\partial_z$, $y^2\partial_y - 2yz\partial_z$.

Proof of Proposition 2.1. Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}^m$ be a map germ with components f_j generating I_X. Thus $\mathrm{T}_p X$ can be identified with $(\mathrm{T}_p f)^{-1}(0) \subset \mathbb{C}^n$. Let $D \in \mathbb{D}$, $D = \sum a_i \partial_{x_i}$, and $p \in X$. Then $D(p) \in \mathrm{T}_p X$ if and only if

$$(\textstyle\sum a_i \partial_{x_i} f)(p) = (\mathrm{T}_p f)(a_1(p), ..., a_n(p)) = 0.$$

By continuity, this holds for all p of a dense subset of X if and only if $Df_j \in I_X$ for all j, which in turn is equivalent to $D(I_X) \subset I_X$. This is (a). The first part of (b) is immediate from the definition of $\mathbb{D}_X$. Finally for $D \in \mathbb{D}_X$, $E \in \mathbb{D}$ and $g \in I_X$:

$$[D, gE] = Dg \cdot E + g \cdot [D, E] \in I_X \cdot \mathbb{D}$$

proving that $I_X \cdot \mathbb{D}$ is an ideal.

Seidenberg's Theorem. [22, Thm. 1] (a) *Let $X \subset (\mathbb{C}^n, 0)$ be analytic. A vector field is tangent to X if and only if it is tangent to all irreducible components $X_1, ..., X_m$ of X:*

$$\mathbb{D}_X = \bigcap\ \mathbb{D}_{X_i}.$$

(b) *Let $I \subset \mathcal{O}_n$ be an arbitrary ideal with radical $\sqrt{I} = \{g \in \mathcal{O}_n,\ g^k \in I$ for some $k \in \mathbb{N}\}$. Then $D(I) \subset I$ implies $D(\sqrt{I}) \subset \sqrt{I}$.*

Remark. One might possibly want to consider vector fields tangent to non-reduced analytic space germs. Seidenberg's Theorem asserts that any vector field tangent to an analytic space germ is also tangent to its reduction. Actually, it may happen that a non-reduced germ and its reduction have the same tangent algebra. For instance, let $X \subset (\mathbb{C}^n, 0)$ be a hypersurface defined by $f \in \mathcal{O}_n$ with prime factor decomposition $f = f_1^{k_1} \cdots f_m^{k_m}$. Then the reduction of X is defined by $g = f_1 \cdots f_m$. If $Dg \in (g)$ then $Df_i \in (f_i)$ for all i by (a) above. The product rule implies $Df \in (f)$. Hence $\mathbb{D}_X = \mathbb{D}_{\mathrm{red}(X)}$. For this reason we shall only consider *reduced* analytic space germs, viz germs of analytic sets.

The following result will be of frequent use (the second assertion does not hold in the algebraic case.)

Rossi's Theorem. [20, Cor. 3.4] *Let $X \subset (\mathbb{C}^n, 0)$ be analytic and x_i coordinates on $(\mathbb{C}^n, 0)$.*
(a) *If $\partial_{x_1}, ..., \partial_{x_k} \in \mathbb{D}_X$ then $X = (\mathbb{C}^k, 0) \times (X \cap (\mathbb{C}^{n-k}, 0))$.*
(b) *If there are $D_1, ..., D_k \in \mathbb{D}_X$ such that $D_1(0), ..., D_k(0)$ are $\mathbb{C}$-linearly independent there is an analytic germ Y such that $X \cong (\mathbb{C}^k, 0) \times Y$.*
(c) *If there are $D_1, ..., D_k \in \mathbb{D}_X$ such that $D_1(0), ..., D_k(0)$ are $\mathbb{C}$-linearly independent X has dimension $\geq k$.*

Having defined tangent algebras one wants to know whether it is possible to recover the germ X from its tangent algebra $\mathbb{D}_X$? This depends on the type of information about $\mathbb{D}_X$ available. If one knows $\mathbb{D}_X$ as an $\mathcal{O}_n$-submodule of $\mathbb{D}$ the answer is yes and goes as follows: Define for arbitrary $\mathcal{O}_n$-submodules M of $\mathbb{D}$ and points $p \in (\mathbb{C}^n, 0)$ the span $M(p)$ of M in p as the $\mathbb{C}$-subspace of $\mathrm{T}_p(\mathbb{C}^n, 0) = \mathbb{C}^n$ generated by the evaluations $D_1(p), ..., D_m(p)$ of $\mathcal{O}_n$-generators $D_1, .., D_m$ of M. This definition does not depend on the choice of the D_i's if p is sufficiently close to 0. We then have:

Proposition 2.2. (Fitting ideals) *Let $X \subset (\mathbb{C}^n, 0)$ be analytic, different from $(\mathbb{C}^n, 0)$, and let $d = \dim X$.*
(a) *$I_X = \sqrt{F_{n-d-1}} = ... = \sqrt{F_0}$ where F_j is the j-th Fitting ideal of the $\mathcal{O}_n$-module $\mathbb{D}/\mathbb{D}_X$.*
(b) *$X = \{p \in (\mathbb{C}^n, 0),\ \dim \mathbb{D}_X(p) \leq \dim X\}$.*
(c) *For irreducible X one has* $\mathrm{Sing}\ X = \{p \in (\mathbb{C}^n, 0), \dim \mathbb{D}_X(p) < \dim X\}$.

Remarks. (a) It is necessary to exclude $X = (\mathbb{C}^n, 0)$ since $\emptyset$ and $(\mathbb{C}^n, 0)$ have the same tangent algebra $\mathbb{D}_\emptyset = \mathbb{D}_{(\mathbb{C}^n, 0)} = \mathbb{D}$ and $F_0 = \mathcal{O}_n$.

(b) Setting $V_j(\mathbb{D}_X) = \{p \in (\mathbb{C}^n, 0), \dim \mathbb{D}_X(p) < n - j\}$ for the zero set of F_j one has $\emptyset = V_n(\mathbb{D}_X) \subset ... \subset V_0(\mathbb{D}_X)$ and the Proposition asserts that

$$V_{n-d-1}(\mathbb{D}_X) = ... = V_0(\mathbb{D}_X) = X.$$

And
$$V_{n-d'-1}(\mathbb{D}_X) = ... = V_{n-d}(\mathbb{D}_X) = \mathrm{Sing}\ X$$

for $d' = \dim \mathrm{Sing}\ X$ and X irreducible.

(c) A description of projective algebraic varieties similar to Proposition 2.2 already appears in the work of Gröbner [10].

Proof. By definition of the j-th Fitting ideal of $\mathbb{D}/\mathbb{D}_X$ as the ideal generated by the $(n-j)$-minors of the matrix defining an $\mathcal{O}_n$-resolution

$$\mathcal{O}_n^m \to \mathcal{O}_n^n \to \mathbb{D}/\mathbb{D}_X \to 0$$

the assertions of the Proposition and of remark (b) are equivalent. Let $p \notin X$ be a point off X. Choose $f \in I_X$ with $f(p) \neq 0$. Then $f\partial_{x_1}, ..., f\partial_{x_n} \in I_X \cdot \mathbb{D} \subset \mathbb{D}_X$ are linearly independent in p. Hence $p \notin V_0(\mathbb{D}_X)$. This gives $V_0(\mathbb{D}_X) \subset X$. Conversely, if $V_{n-d-1}(\mathbb{D}_X)$ were strictly contained in X we could find points $p \in X$ arbitarily close to 0 with $p \notin V_{n-d-1}(\mathbb{D}_X)$. Hence $\dim \mathbb{D}_X(p) \geq d+1$. Part (c) of Rossi's Theorem then implies that X has dimension $\geq d+1$ in p. As p was arbitrarily close to 0, we get $\dim X \geq d+1$, contradiction. This proves (a) and (b).

For (c) we refer to Proposition 5.1 proven later and independent of Proposition 2.2. It asserts that $\mathbb{D}_X \subset \mathbb{D}_{\text{Sing } X}$. Hence we have $V_j(\mathbb{D}_{\text{Sing } X}) \subset V_j(\mathbb{D}_X)$ for all j. Part (a) above implies

$$\text{Sing } X = V_{n-d'-1}(\mathbb{D}_{\text{Sing } X}) \subset V_{n-d'-1}(\mathbb{D}_X).$$

It remains to show $V_{n-d}(\mathbb{D}_X) \subset \text{Sing } X$. But if $p \notin \text{Sing } X$ one can construct d vector fields in $\mathbb{D}_X$ which are linearly independent in p, see the Lemma below. Hence $p \notin V_{n-d}(\mathbb{D}_X)$.

Existence Lemma. (existence of vector fields) *Let $X \subset (\mathbb{C}^n, 0)$ be analytic and p a smooth point of X. Let d be the dimension of the component of X containing p. There exist d vector fields D_i tangent to X which are linearly independent in p.*

Proof. Denote $X_1, \ldots, X_r$ the components of X and assume that $p \in X_1$. Let $f_1, \ldots, f_m$ be generators of the ideal I_{X_1} of X_1 and let k be the codimension of X_1. The matrix

$$\begin{pmatrix} \partial_{x_1} f_1 & \ldots & \partial_{x_n} f_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \partial_{x_1} f_m & \ldots & \partial_{x_n} f_m \end{pmatrix}$$

has rank $\leq k$ in every point of X_1, and Sing X_1 is defined by the vanishing of its k-minors. As $p \notin \text{Sing } X$ we may assume that the k-minor

$$\begin{vmatrix} \partial_{x_1} f_1 & \ldots & \partial_{x_k} f_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \partial_{x_1} f_k & \ldots & \partial_{x_k} f_k \end{vmatrix}$$

does not vanish in p. Moreover one can choose some $g \in \mathcal{O}_n$ vanishing on $X_2, \ldots, X_r$ but not in p. Consider the d vector fields ($i = k+1, \ldots, n$)

$$D_i = g \cdot \begin{vmatrix} \partial_{x_1} & \ldots & \partial_{x_k} & \partial_{x_i} \\ \partial_{x_1} f_1 & \ldots & \partial_{x_k} f_1 & \partial_{x_i} f_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \partial_{x_1} f_k & \ldots & \partial_{x_k} f_k & \partial_{x_i} f_k \end{vmatrix}$$

given by the cofactor expansion along the first row. By the choice of g they vanish on $X_2 \cup \ldots \cup X_r$ and are hence tangent to this union. On the other hand $D_i(f_j) = 0$ trivially for $j = 1, \ldots, k$. And for $j = k+1, \ldots, m$ the functions $D_i f_j$ vanish on X_1 because the resulting $(k+1) \times (k+1)$-matrix has rank $\leq k$ on X_1. As the f_j's generate I_{X_1} we get $D_i \in \mathbb{D}_{X_1}$. Therefore $D_i \in \mathbb{D}_X$. By construction $D_{k+1}, \ldots, D_n$ are linearly independent in p.

We now ask for the converse: Is it possible to compute $\mathbb{D}_X$ from X? In general this is very hard. Also it is not clear what is meant by "computing" the infinite dimensional Lie algebra $\mathbb{D}_X$. Even how to find $\mathcal{O}_n$-module generators of $\mathbb{D}_X$ is only known in special cases. One has to determine the module of relations between f, $\partial_{x_1} f, \ldots, \partial_{x_n} f$:

Aleksandrov's and Kersken's Theorem. [1, Thm. 6.1, 29, Thm. 2.8] *Let X be a complete intersection of codimension k in $(\mathbb{C}^n, 0)$ with isolated singularity at 0 such that I_X can be generated by weighted homogeneous polynomials $f_1, \ldots, f_k$ (all w.r.t. the same weights $w_1, \ldots, w_n$.) Then $\mathbb{D}_X$ is generated as an $\mathcal{O}_n$-module by the ideal $I_X \cdot \mathbb{D}$, the Euler vector field $E = \sum w_i x_i \partial_{x_i}$ and the trivial vector fields which are zero on f_j and given by the cofactor expansion along the first row of the $(k+1)$-minors of the matrix*

$$\begin{pmatrix} \partial_{x_1} & \ldots & \partial_{x_n} \\ \partial_{x_1} f_1 & \ldots & \partial_{x_n} f_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \partial_{x_1} f_k & \ldots & \partial_{x_n} f_k \end{pmatrix}.$$

3. ANALYTIC FILES

It will be necessary to consider intersections of tangent algebras, i.e. vector fields tangent to several analytic germs. These germs may be contained in each other. Typical examples are vector fields which are tangent to a germ X *and* to its singular locus Sing X. When treating collections of germs, Seidenberg's Theorem allows to restrict attention to collections of irreducible germs. Since these will appear frequently a proper terminology is convenient:

A *file of analytic germs in* $(\mathbb{C}^n, 0)$ is a finite set $\mathbf{X} = \{X_1, \ldots, X_m\}$ of irreducible analytic germs in $(\mathbb{C}^n, 0)$, possibly contained in each other. We shall usually simply speak of an *analytic file.* The X_i will be called the components of $\mathbf{X}$. To any file $\mathbf{X}$ we associate its underlying germ $|\mathbf{X}| = \bigcup X_i$ which is the germ of an analytic set. The irreducible components of $|\mathbf{X}|$ are just the components X_i of $\mathbf{X}$ which are not contained in any other.

For an analytic file $\mathbf{X} = \{X_1, \ldots, X_m\}$ we define its tangent algebra $\mathbb{D}_{\mathbf{X}}$ as the Lie algebra of vector fields tangent to all X_i, i.e. $\mathbb{D}_{\mathbf{X}} = \bigcap \mathbb{D}_{X_i}$. This is again a Lie submodule of $\mathbb{D}$ and Seidenberg's Theorem yields $\mathbb{D}_{\mathbf{X}} \subset \mathbb{D}_{|\mathbf{X}|}$.

On the other hand, given an analytic germ X in $(\mathbb{C}^n, 0)$ its irreducible components form a file $\mathbf{X}$ with $|\mathbf{X}| = X$ and $\mathbb{D}_{\mathbf{X}} = \mathbb{D}_X$. We agree that the empty germ $\emptyset$ has no components and hence its file is empty.

The reader is invited to extend suitably the results of the last section to files of analytic germs.

4. INTEGRAL VARIETIES

The purpose of this section is to associate to any subalgebra A of $\mathbb{D}$ an analytic germ X_A, resp. an analytic file $\mathbf{X}_A$, such that all vector fields in A are tangent to X_A, resp. $\mathbf{X}_A$. This will be achieved by the concept of integral variety. In contrast to the Fitting ideal description of X from $\mathbb{D}_X$ of Proposition 2.2, the present construction will only rely on the assumption that A is a Lie subalgebra of $\mathbb{D}$. It does not require that A is an $\mathcal{O}_n$-submodule. However, it does involve the $\mathcal{O}_n$-multiplication in $\mathbb{D}$. Quite generally, we shall consider inclusions $A \subset \mathbb{D}'$ where A is a subalgebra of a Lie submodule $\mathbb{D}'$ of $\mathbb{D}$. In all applications $\mathbb{D}'$ will be the tangent algebra $\mathbb{D}_{\mathbf{Y}}$ of some analytic file $\mathbf{Y}$.

Let $\mathbb{D}' \subset \mathbb{D}$ be a Lie submodule. For a $\mathbb{C}$-subspace $A \subset \mathbb{D}'$ we define analogously to section 1

$$I_A = I_A(\mathbb{D}') = \{g \in \mathcal{O}_n,\ g \cdot \mathbb{D}' \subset A\}.$$

As $\mathbb{D}'$ is an $\mathcal{O}_n$-module, I_A is an ideal of $\mathcal{O}_n$, in fact the largest ideal of $\mathcal{O}_n$ such that $I_A \cdot \mathbb{D}' \subset A$. The zero set $X_A = X_A(\mathbb{D}') \subset (\mathbb{C}^n, 0)$ of I_A will be called the *integral variety of A relative to $\mathbb{D}'$*. This name is justified by:

Proposition 4.1. (integral variety) *Let $A \subset \mathbb{D}'$ be a subalgebra of a Lie submodule $\mathbb{D}' \subset \mathbb{D}$. Any vector field in A is tangent to the integral variety X_A:*

$$A \subset \mathbb{D}_{X_A}.$$

Proof. It suffices to show that any $D \in A$ satisfies $D(I_A) \subset I_A$. For, by Seidenberg's Theorem, we then know $D(\sqrt{I_A}) \subset \sqrt{I_A}$, and $\sqrt{I_A}$ defines X_A. So let $g \in I_A$. For $E \in \mathbb{D}'$ we have

$$Dg \cdot E = [D, gE] - g \cdot [D, E] \in A$$

because A and $\mathbb{D}'$ are Lie algebras. Hence $Dg \in I_A$.

Remarks. (a) In the next section it will be shown that the integral variety of a tangent algebra $\mathbb{D}_X \subset \mathbb{D}$ equals the germ X. It is the largest germ strictly contained in $(\mathbb{C}^n, 0)$ to which all vector fields in $\mathbb{D}_X$ are tangent.

(b) The integral variety can be trivial, i.e., empty or equal to the ambient space. These non-geometric cases do not occur if $A \subset \mathbb{D}'$ is a geometric subalgebra (see Proposition 6.2 for a proof.)

Proposition 4.2. (codimension) *A finite codimensional submodule A of $\mathbb{D}'$ has empty or zero-dimensional integral variety. An infinite codimensional $\mathbb{C}$-subspace A of $\mathbb{D}'$ has positive dimensional integral variety.*

Proof. If A has finite codimension in $\mathbb{D}'$, the quotient module $\mathbb{D}'/A$ has finite length and thus its annihilator I_A has Krull dimension < 1 [18, sec. 12.B]. Conversely, if I_A has Krull dimension < 1 it has finite codimension as a linear subspace. Hence $I_A \cdot \mathbb{D}'$ and A have finite codimension in $\mathbb{D}'$.

Example. Consider the analytic file $\mathbf{X}$ with components $X_1 = \{x = 0\} \subset (\mathbb{C}^2, 0)$ and $X_2 = \{0\} \subset (\mathbb{C}^2, 0)$. Then $A = \mathbb{D}_{\mathbf{X}} \subset \mathbb{D}$ is the $\mathcal{O}_2$-module generated by the vector fields $x\partial_x$, $x\partial_y$, $y\partial_y$. We have $X_A(\mathbb{D}) = \{x = 0\} = X_1$. The "lost" embedded component X_2 can be recovered by iteration: $\mathbb{D}_{X_A}$ is the $\mathcal{O}_2$-module

generated by $x\partial_x$, ∂_y. And $X_2 = \{0\} = X_A(\mathbb{D}_{X_A})$, the integral variety of A relative to $\mathbb{D}_{X_A}$. Therefore we define:

Definition. (integral file) Let $\mathbb{D}' \subset \mathbb{D}$ be a Lie submodule. For a subalgebra $A \subset \mathbb{D}'$ define inductively a sequence X_A^i of germs of analytic sets in $(\mathbb{C}^n, 0)$ by

$$X_A^1 := X_A(\mathbb{D}')$$
$$X_A^i := X_A(\mathbb{D}' \cap \mathbb{D}_{X_A^1} \cap \ldots \cap \mathbb{D}_{X_A^{i-1}}).$$

The defining ideals I_A^i of X_A^i are given by

$$I_A^i = \{g \in \mathcal{O}_n,\ g \cdot (\mathbb{D}' \cap \mathbb{D}_{X_A^1} \cap \ldots \cap \mathbb{D}_{X_A^{i-1}}) \subset A\}.$$

If confusion is likely we indicate dependence on $\mathbb{D}'$ by $X_A^i(\mathbb{D}')$. As the algebras $\mathbb{D}' \cap \mathbb{D}_{X_A^1} \cap \ldots \cap \mathbb{D}_{X_A^{i-1}}$ are decreasing the ideals I_A^i increase, the germs X_A^i decrease. Since $\mathcal{O}_n$ is Noetherian the sequence of X_A^i becomes stationary for some k:

$$X_A^k = X_A^{k+1} = \ldots$$

The analytic file $\mathbf{X}_A$ formed by all irreducible components of $X_A^1, \ldots, X_A^k$ will be called the *integral file of A relative to* $\mathbb{D}'$. Proposition 4.1 guarantees that $A \subset \mathbb{D}_{\mathbf{X}_A}$.

In the example above we have $\{X_1, X_2\} = \{X_A(\mathbb{D}), X_A(\mathbb{D}_{X_A})\} = \{X_A^1, X_A^2\}$.

5. INVARIANCE AND IRREDUNDANCE

We ask whether different germs or files can have the same tangent algebra. In general, this is only possible for files: there exist files $\mathbf{X}'$ and $\mathbf{X}$ contained in each other (i.e., any component of $\mathbf{X}'$ is one of $\mathbf{X}$) with $\mathbb{D}_{\mathbf{X}'} = \mathbb{D}_{\mathbf{X}}$. This signifies that the components of $\mathbf{X}$ not lying in $\mathbf{X}'$ have no effect on the tangent algebra. They are superfluous. We shall study these components more closely.

Definition. (invariance) Given analytic files $\mathbf{X}$ and $\mathbf{Y}$ in $(\mathbb{C}^n, 0)$ we say that $\mathbf{X}$ is $\mathbf{Y}$-*invariant* if any vector field tangent to $\mathbf{Y}$ is also tangent to $\mathbf{X}$, i.e., $\mathbb{D}_{\mathbf{Y}} \subset \mathbb{D}_{\mathbf{X}}$. Analogously, invariant germs are defined.

Of course, $\emptyset$ and $(\mathbb{C}^n, 0)$ are invariant w.r.t. any file $\mathbf{Y}$. Seidenberg's Theorem asserts that the irreducible components of a germ X are X-invariant. Here are more examples (it is not clear whether the second assertion has an analogue in the algebraic case):

Proposition 5.1. (examples of invariant germs) *Let $X \subset (\mathbb{C}^n, 0)$ be an analytic germ.*
(a) *The singular locus* Sing X *of X is invariant w.r.t. X, $\mathbb{D}_X \subset \mathbb{D}_{\mathrm{Sing}X}$. More generally:*
(b) *Analytic subsets Z of X stable under all automorphisms of $(\mathbb{C}^n, 0)$ stabilizing X are X-invariant.*

Remark. Assertion (b) cannot be reversed: it may happen that an invariant Z is not stable under all automorphisms of $(\mathbb{C}^n, 0)$ which stabilize X. For instance, if X is the union of the coordinate axes in $(\mathbb{C}^2, 0)$ then $(x, y) \to (y, x)$ stabilizes

X but permutes the components of X. But those are invariant by Seidenberg's Theorem.

Proof. (a) This is a special case of (b). Nevertheless we provide a direct proof (for a purely algebraic proof see [25, Kor. 3.16].) If $D \in \mathbb{D}_X$ is non-singular, $D(0) \neq 0$, a change of coordinates in $(\mathbb{C}^n, 0)$ allows to assume $D = \partial_{x_1}$, [5, chap. III, sec. VII, Lemma 1]. Rossi's Theorem implies

$$X = (\mathbb{C}, 0) \times X' \quad \text{with} \quad X' = X \cap (\mathbb{C}^{n-1}, 0).$$

Then Sing $X = (\mathbb{C}, 0) \times$ Sing X' and thus $D = \partial_{x_1}$ is tangent to Sing X.

If D is singular, $D(0) = 0$, this argument applies to all points $p \in$ Sing X where $D(p) \neq 0$ and gives $D(p) \in \mathrm{T}_p(\text{Sing } X)$ for all these p. For the other points $D(p) = 0$. Thus $D(p) \in \mathrm{T}_p(\text{Sing } X)$ for all $p \in$ Sing X.

(b) We assume first that X does not split off a smooth factor: $X \not\cong (\mathbb{C}, 0) \times X'$ for any germ X'. Let $D \in \mathbb{D}_X$. Rossi's Theorem gives $D(0) = 0$. In this case D can be integrated to a one parameter group ϕ_t of automorphisms of $(\mathbb{C}^n, 0)$, [28, sec. 5]. As

$$f(\phi_t(x)) = \textstyle\sum_{k=0}^{\infty} \frac{t^k}{k!}(D^k f)(x)$$

for $f \in \mathcal{O}_n$ we see that all ϕ_t stabilize X. By assumption all ϕ_t stabilize Z. From $D(p) = \partial_t \phi_t(p)_{|t=0}$ we conclude that D is tangent to Z.

Secondly, consider the case where X splits off a smooth factor. Choose $k \in \mathbb{N}$ maximal such that $X \cong (\mathbb{C}^k, 0) \times X'$ for some X'. We may assume $X = (\mathbb{C}^k, 0) \times X'$ with $X' = X \cap (\mathbb{C}^{n-k}, 0)$. Points in $(\mathbb{C}^n, 0)$ will be written (p, q) according to $(\mathbb{C}^n, 0) = (\mathbb{C}^k, 0) \times (\mathbb{C}^{n-k}, 0)$. For any analytic map germ $\psi : (\mathbb{C}^{n-k}, 0) \to (\mathbb{C}^k, 0)$ the automorphism

$$\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) : (x, y) \to (x + \psi(y), y)$$

stabilizes X. Thus if $q \neq 0$ with $(0, q) \in Z$ then $(\mathbb{C}^k, 0) \times \{q\} \subset Z$. This implies $Z = (\mathbb{C}^k, 0) \times Z'$ for some germ $Z' \subset (\mathbb{C}^{n-k}, 0)$. Clearly Z' is stabilized by all automorphisms of $(\mathbb{C}^{n-k}, 0)$ which stabilize X'. One has $\mathrm{T}_{(p,q)}X = \mathbb{C}^k \times \mathrm{T}_q X'$ for $(p, q) \in X$ and $\mathrm{T}_{(p,q)}Z = \mathbb{C}^k \times \mathrm{T}_q Z'$ for $(p, q) \in Z$. Fix $p \in (\mathbb{C}^k, 0)$ and let $D \in \mathbb{D}_X$. By restriction to $\{p\} \times (\mathbb{C}^{n-k}, 0)$ and projection to the tangent spaces $\mathrm{T}_q(\mathbb{C}^{n-k}, 0)$ one obtains a vector field on $\{p\} \times (\mathbb{C}^{n-k}, 0)$ tangent to $\{p\} \times X'$. By the case discussed before it must be tangent to $\{p\} \times Z'$. This implies that D is tangent to Z and proves the assertion.

Proposition 5.1 and Seidenberg's Theorem are complemented by:

Proposition 5.2. (description of invariant germs) *Assume that the irreducible germ $X \subset (\mathbb{C}^n, 0)$ is invariant w.r.t. some analytic file $\mathbf{Y}$ in $(\mathbb{C}^n, 0)$, $\mathbb{D}_{\mathbf{Y}} \subset \mathbb{D}_X$. Then either $X = (\mathbb{C}^n, 0)$ or X is contained in at least two components of $\mathbf{Y}$ or X is contained in the singular locus of a component of $\mathbf{Y}$ or X equals a component of $\mathbf{Y}$.*

Proof. We assume that none of the first three possibilities holds and prove that X equals a component of $\mathbf{Y}$. By Proposition 2.2. and since $X \neq (\mathbb{C}^n, 0)$ we can write $X = V_0(\mathbb{D}_X)$. The inclusion $\mathbb{D}_{\mathbf{Y}} \subset \mathbb{D}_X$ gives $V_0(\mathbb{D}_X) \subset V_0(\mathbb{D}_{\mathbf{Y}})$ and hence $X \subset V_0(\mathbb{D}_{\mathbf{Y}})$. But $V_0(\mathbb{D}_{\mathbf{Y}}) \subset |\mathbf{Y}|$ because any function f vanishing on $|\mathbf{Y}|$

induces vector fields $f\partial_{x_1}, ..., f\partial_{x_n}$ tangent to $\mathbf{Y}$ and linearly independent off the zero set of f. Therefore $X \subset |\mathbf{Y}|$. Being irreducible, X must be contained in some component of $\mathbf{Y}$, say $X \subset Y_1$. We shall show $X = Y_1$ by comparing dimensions. Set $k = \operatorname{codim} Y_1$.

We assumed that X is not contained in two components of $\mathbf{Y}$ nor in the singular locus of one. This allows to choose points p in X outside $Y_2 \cup ... \cup Y_r \cup \operatorname{Sing} Y_1$ and arbitrarily close to 0. By the Existence Lemma of section 2 (extended to the case of files) there exist $n-k$ vector fields D_i in $\mathbb{D}_{\mathbf{Y}}$ linearly independent in p. As $\mathbb{D}_{\mathbf{Y}} \subset \mathbb{D}_X$ they belong to $\mathbb{D}_X$. By Rossi's Theorem the dimension of X in p must be greater or equal $n-k$. Since p was arbitrarily close to 0 we conclude that $\dim X \geq n-k$. Together with $X \subset Y_1$ and $k = \operatorname{codim} Y_1$ this implies $X = Y_1$. The assertion is established.

Definition. (irredundance) Given analytic files $\mathbf{X}$ and $\mathbf{Y}$ in $(\mathbb{C}^n, 0)$ we say that $\mathbf{X}$ is *irredundant* w.r.t. $\mathbf{Y}$ if deleting any component from $\mathbf{X}$ alters $\mathbb{D}_{\mathbf{X},\mathbf{Y}} := \mathbb{D}_{\mathbf{X}} \cap \mathbb{D}_{\mathbf{Y}}$. Equivalently, no component Z of $\mathbf{X}$ is invariant w.r.t. $\mathbf{X}^- \cup \mathbf{Y}$ where $\mathbf{X}^-$ is the file obtained from $\mathbf{X}$ by deleting Z. For analytic germs we make the analogous definition.

If $\mathbf{X}$ is redundant w.r.t. $\mathbf{Y}$ then clearly by deleting some of the components of $\mathbf{X}$ one can obtain a file $\mathbf{X}^0$ with $\mathbb{D}_{\mathbf{X}^0,\mathbf{Y}} = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ and such that $\mathbf{X}^0$ is irredundant w.r.t. $\mathbf{Y}$. It will follow from the next proposition that $\mathbf{X}^0$ is uniquely determined by $\mathbf{X}$ and $\mathbf{Y}$, namely as the integral file $\mathbf{X}_A$ of $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ relative to $\mathbb{D}_{\mathbf{Y}}$.

Example. (1) For $\mathbf{Y} = \emptyset$ and any germ X the file $\mathbf{X} := \{(\mathbb{C}^n, 0),\ X,$ components of $X\}$ is redundant.

(2) For a germ Y the file $\mathbf{X} := \{Y, \operatorname{Sing} Y,\ \operatorname{Sing}(\operatorname{Sing}(Y))\}$ is redundant w.r.t. Y.

The next result describes the integral file of a tangent algebra:

Proposition 5.3. (integral file of tangent algebra) *Given analytic files* $\mathbf{X}$ *and* $\mathbf{Y}$ *in* $(\mathbb{C}^n, 0)$ *assume that* $\mathbf{X}$ *is irredundant w.r.t.* $\mathbf{Y}$. *Then the integral file of* $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ *relative to* $\mathbb{D}_{\mathbf{Y}}$ *equals* $\mathbf{X}$: $\mathbf{X}_A = \mathbf{X}$.

Proof. Let $X_1, ..., X_k$ be the irreducible components of the germ $|\mathbf{X}|$. We are going to show that $|\mathbf{X}|$ equals the integral *variety* of A relative to $\mathbb{D}_{\mathbf{Y}}$: $|\mathbf{X}| = X_A$. Repeating the argument with $\mathbf{Y}$ replaced by $\mathbf{Y}^+ = \mathbf{Y} \cup \{X_1, ..., X_k\}$ and $\mathbf{X}$ by $\mathbf{X}^- = \mathbf{X} \setminus \{X_1, ..., X_k\}$ will give the assertion by induction.

Observe first that any g vanishing on $|\mathbf{X}|$ satisfies $g \cdot \mathbb{D}_{\mathbf{Y}} \subset \mathbb{D}_{\mathbf{X},\mathbf{Y}} = A$. Hence $I_{|\mathbf{X}|} \subset I_A$ and $X_A \subset |\mathbf{X}|$. As $\mathbf{X}$ is irredundant w.r.t. $\mathbf{Y}$ there are vector fields $D_j \in \mathbb{D}_{\mathbf{Y}}$ with $D_j \notin \mathbb{D}_{X_j}$ for $j = 1, ..., k$. Hence there are $h_j \in I_{X_j}$ with $D_j h_j \notin I_{X_j}$. Take an arbitrary $g \in I_A$. As the vector fields gD_j belong to A by definition of I_A and since $A \subset \mathbb{D}_{X_j}$ we obtain $gD_jh_j \in I_{X_j}$. The X_j's are irreducible and $D_jh_j \notin I_{X_j}$, thus $g \in X_j$ for all j. We have shown $I_A \subset \bigcap I_{X_j} = I_{|\mathbf{X}|}$. This gives $|\mathbf{X}| \subset X_A$.

Remark. It follows from Propositions 4.1, 5.2 and 5.3 that the integral variety X_A of a tangent algebra $A = \mathbb{D}_X$ (as defined in section 1) is in fact the largest germ strictly contained in $(\mathbb{C}^n, 0)$ to which all vector fields in A are tangent.

6. THE TRANSPORTER SERIES

This section is devoted to the study of balanced subalgebras of a Lie algebra introduced in section 1. Propositions 6.1 and 6.2 below present key ingredients in the proof of Narkissos' and Echo's Theorem. They establish that tangent algebras are balanced and describe the integral variety of balanced algebras.

For any inclusion of Lie algebras $A \subset B$ we define a decreasing series $A^{[i]}$ of subalgebras of A, the *transporter series* of A relative to B, as follows:

$$A^{[1]} := \{D \in A,\ [D,B] \subset A\},$$
$$A^{[i]} := (A^{[i-1]})^{[1]} = \{D \in A^{[i-1]},\ [D,B] \subset A^{[i-1]}\},$$
$$A^{[\infty]} := \bigcap A^{[i]}.$$

In case confusion is likely we write $A^{[i]} = A_B^{[i]}$ to indicate dependence on B. The transporter series allows to characterize balanced algebras:

Lemma. (balanced algebras) *Let $A \subset B$ be Lie algebras.*
(i) *$A^{[\infty]}$ is the largest ideal of B contained in A.*
(ii) *A is balanced if and only if $A^{[\infty]} = 0$ and $A^{[2]} \neq 0$.*

Proof. (i) To see that $A^{[\infty]}$ is an ideal of B let $D \in A^{[\infty]}$ and $E \in B$. For all i we have $D \in A^{[i]}$, hence $[D,E] \in A^{[i-1]}$ by definition of $A^{[i]}$. Thus $[D,E] \in A^{[\infty]}$. Conversely let $A' \subset A$ be an ideal of B. Then $A' \subset A^{[1]}$. If $A' \subset A^{[i]}$ for some i then $[A',B] \subset A' \subset A^{[i]}$, i. e. $A' \subset A^{[i+1]}$. This proves $A' \subset A^{[\infty]}$.

(ii) This is clear from the definition since $[D,B] \subset A^{[1]}$ is equivalent to $[[D,B],B] \subset A$.

Remarks. (a) The transporter series also appears in the work of Omori [19] as well as in the theory of Guillemin and Sternberg on transitive Lie algebras [11, 12, 13].

(b) Camacho and Sad [4] prove that any analytic vector field on $\mathbb{C}^2$ has a separatrix. By this they mean a possibly singular analytic curve germ through 0 to which the given vector field is tangent in the sense of the present paper. Gómez-Mont and Luengo [7] show that such a curve generally does not exist for vector fields on $\mathbb{C}^3$. They do not disprove that a higher dimensional analytic germ, namely a hypersurface in $\mathbb{C}^3$, with this property exists. One is tempted to apply Echo's Theorem to prove the existence of "integral varieties" of appropriate dimension for *single* vector fields: It would suffice to embed the vector field in a geometric or a maximal balanced subalgebra of $\mathbb{D}$. The problem in doing this is related to the condition $A^{[2]} \neq 0$. We can show that if this inequality were automatically fulfilled for maximal Lie-submodules of $\mathbb{D}$ the existence of integral varieties for single vector fields would follow.

The condition $A^{[2]} \neq 0$ is automatic in case A is of finite codimension in the infinite dimensional Lie algebra B (since $A^{[1]}$ is the kernel of the natural map $A \to \mathrm{End}\ B/A$ given by the adjoint representation.) In our situation, however, A will be of infinite codimension in B.

Before stating the main assertions of this section we prove two auxiliary results. The first shows how to construct elements in I_A from elements in $A^{[1]}$:

Amemiya's Lemma. [3, p. 547] *Let A be a subalgebra of a Lie submodule $\mathbb{D}'$ of $\mathbb{D}$ and $A^{[1]} = A^{[1]}_{\mathbb{D}'}$. If $D \in A^{[1]}$ and $hD \in A^{[1]}$ for some $h \in \mathcal{O}_n$ then $(Dh)^2 \in I_A$.*

Proof. For arbitrary $E \in \mathbb{D}'$ we have $Dh \cdot E + h[D,E] = [D, hE] \in A$ and $-Eh \cdot D + h[D,E] = [hD, E] \in A$. Hence $Dh \cdot E + Eh \cdot D \in A$. Replacing E by $Eh \cdot D$ and $Dh \cdot E$ in turn we obtain $Dh \cdot Eh \cdot D \in A$ and $(Dh)^2 E + Dh \cdot Eh \cdot D \in A$. This yields $(Dh)^2 E \in A$ for all $E \in \mathbb{D}'$, i.e., $(Dh)^2 \in I_A$ proving the Lemma.

Amemiya's Lemma can be sharpened as follows:

Omori's Lemma. [19, Lemma 3.4] *Let $0 \neq A \subset \mathbb{D}'$ be Lie submodules of $\mathbb{D}$. Then $A^{[1]}$ is a submodule too. Moreover $A^{[1]} = 0$ if and only if $I_A = 0$.*

Proof. Consider the equation

$$[fD, E] = -Ef \cdot D + f \cdot [D, E].$$

If $f \in \mathcal{O}_n$, $D \in A^{[1]}$, $E \in \mathbb{D}'$ then $[fD, E] \in A$ and $fD \in A^{[1]}$. Next suppose $A^{[1]} = 0$. Let $f \in I_A$ and choose $D \in A$, $D \neq 0$. Then $[fD, E] \in A$ for all $E \in \mathbb{D}'$, i.e., $fD \in A^{[1]}$ and $f = 0$. Finally suppose $I_A = 0$. Let $D \in A^{[1]}$. For all $f \in \mathcal{O}_n$ we have $fD \in A^{[1]}$. Amemiya's Lemma gives $(Df)^2 \in I_A$, hence $Df = 0$. This proves $D = 0$ and the Lemma.

The next two results are mutually symmetric. The first asserts that tangent algebras are balanced algebras if the corresponding file satisfies certain conditions. The second shows that the integral files of balanced algebras satisfy these conditions.

Proposition 6.1. (tangent algebras are balanced) *Let $\mathbf{X}$ and $\mathbf{Y}$ be analytic files in $(\mathbb{C}^n, 0)$ with tangent algebra $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ considered as a subalgebra of $\mathbb{D}' = \mathbb{D}_{\mathbf{Y}}$. Then:*

(a) $A^{[2]} \neq 0$.

(b) $A^{[\infty]} = 0$ *provided* $|\mathbf{X}|$, $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$ *and* $|\mathbf{X}| \not\subset |\mathbf{Y}|$.

(c) *If* $|\mathbf{X}|$, $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$ *and* $|\mathbf{X}| \not\subset |\mathbf{Y}|$ *the tangent algebra* $\mathbb{D}_{\mathbf{X},\mathbf{Y}}$ *is a balanced subalgebra of* $\mathbb{D}_{\mathbf{Y}}$.

Proposition 6.2. (integral variety of balanced algebra) *Let $\mathbb{D}' \subset \mathbb{D}$ be a Lie submodule and $A \subset \mathbb{D}'$ a Lie subalgebra with integral variety X_A relative to $\mathbb{D}'$.*

(a) *If $A^{[2]} \neq 0$ then $X_A \neq (\mathbb{C}^n, 0)$.*

(b) *Assume $\mathbb{D}' = \mathbb{D}_{\mathbf{Y}}$ for some file $\mathbf{Y}$ in $(\mathbb{C}^n, 0)$ with $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$. If $A^{[\infty]} = 0$ then $X_A \not\subset |\mathbf{Y}|$.*

(c) *Every balanced subalgebra A of a tangent algebra $\mathbb{D}_{\mathbf{Y}}$ of some file $\mathbf{Y}$ in $(\mathbb{C}^n, 0)$ with $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$ has integral variety X_A different $(\mathbb{C}^n, 0)$ and not contained in $|\mathbf{Y}|$.*

Examples. (1) Concerning Proposition 6.1 (b): Let $Y = \{x = 0\} \subset (\mathbb{C}^2, 0)$ and $X = \{0\} \subset Y$, $A = \mathbb{D}_{X,Y} = (x\partial_x, x\partial_y, y\partial_y)$ and $\mathbb{D}' = \mathbb{D}_Y = (x\partial_x, \partial_y)$. Then $I_Y \cdot \mathbb{D} = (x\partial_x, x\partial_y)$ is an ideal of $\mathbb{D}_Y$ contained in $\mathbb{D}_{X,Y}$, hence $A^{[\infty]} \neq 0$.

(2) Concerning Proposition 6.2 (b): Let $Y = \{x^2 + y^3 = 0\} \subset (\mathbb{C}^2, 0)$ and $A = I_Y \cdot \mathbb{D} = ((x^2 + y^3)\partial_x, (x^2 + y^3)\partial_y)$ be an ideal of $\mathbb{D}_Y$. Then $A^{[\infty]} = A \neq 0$. Using that $\mathbb{D}_Y = (3x\partial_x + 2y\partial_y, 3y^2\partial_x - 2x\partial_x)$ by Aleksandrov's and Kersken's Theorem one obtains $X_A = Y$.

Proof of Proposition 6.1. (a) We may exclude the case where $(\mathbb{C}^n, 0)$ is a component of $\mathbf{X}$, i.e., assume that the underlying germ $|\mathbf{X}|$ of $\mathbf{X}$ is different from $(\mathbb{C}^n, 0)$. Choose $g \in I_{|\mathbf{X}|}$, $g \neq 0$. Then $g \cdot \mathbb{D}' \subset A$ and hence $g \in I_A$. For D, $E \in \mathbb{D}'$ we have

$$[g^2 D, E] = g^2[D, E] - 2g \cdot Eg \cdot D \in A,$$

hence $g^2 \cdot D \in A^{[1]}$ and $g^2 \in I_{A^{[1]}}$. Omori's Lemma implies $A^{[2]} = (A^{[1]})^{[1]} \neq 0$.

(b) We may assume by section 5 that $\mathbf{X}$ is irredundant w.r.t. $\mathbf{Y}$. Observe that by Proposition 5.2 the condition $|\mathbf{X}| \not\subset |\mathbf{Y}|$ persists if superfluous components of $\mathbf{X}$ are deleted.

So let us prove $A^{[\infty]} = 0$. This signifies that for every $D \in A$, $D \neq 0$ there exist vector fields $E_1, ..., E_k \in \mathbb{D}_{\mathbf{Y}}$ such that $[...[D, E_1], ..., E_k] \notin A$.

Case 1: The argument is easy if D does not vanish on $X = |\mathbf{X}|$, i.e. $D \notin I_X \cdot \mathbb{D}$. Choose $g \in \mathcal{O}_n$ with $Dg \notin I_X$. Since $X_A = X$ by Proposition 5.3 and irredundance of $\mathbf{X}$ we have $I_A \subset \sqrt{I_A} = I_X$. Therefore $Dg \notin I_A$, i.e., $Dg \cdot E \notin A$ for a suitable $E \in \mathbb{D}_{\mathbf{Y}}$. If $[D, E] \notin A$ we are done. If $[D, E] \in A$ we use $[D, gE] = Dg \cdot E + g \cdot [D, E] \notin A$ proving the assertion.

Case 2: We are left with elements $D \in I_X \cdot \mathbb{D}$. The argument is a bit more complicated. By (a) it is sufficient to find $E_1, ..., E_k \in \mathbb{D}_{\mathbf{Y}}$ such that

$$[...[D, E_1], ..., E_k] \notin I_X \cdot \mathbb{D}.$$

The case $n = 1$ being easy we may assume $n \geq 2$. It is convenient to express D in suitable coordinates on $(\mathbb{C}^n, 0)$: Choose a smooth hypersurface H in $(\mathbb{C}^n, 0)$ to which D is not tangent. Such a hypersurface exists for $n \geq 2$. Indeed, either D is not tangent to some linear hyperplane. Otherwise, a simple exercise shows that it must be of the form $D = a \cdot \sum x_i \partial_{x_i}$ with $a \in \mathcal{O}_n$. Then take $H = \{x_n + x_1^d = 0\}$ with $d \geq 2$ such that a does not vanish on H. After changing coordinates we may assume that H is linear and given by $x_n = 0$. Decompose $Dx_n = g + x_n h$ with $g \in \mathbb{C}\{x_1, ..., x_{n-1}\} = \mathcal{O}_{n-1}$ and $h \in \mathcal{O}_n$. As $Dx_n \notin (x_n)$ by the choice of H we have $g \neq 0$. Changing the first $n-1$ coordinates $x_1, ..., x_{n-1}$ linearly we may obtain $g(x_1, 0, ..., 0) \neq 0$. Thus we have found coordinates on $(\mathbb{C}^n, 0)$ for which a monomial x_1^e appears for some $e \geq 0$ in the power series expansion of Dx_n. Now use $X \not\subset |\mathbf{Y}|$ to find an $f \in \mathcal{O}_n$ vanishing on $|\mathbf{Y}|$ but not on X. We claim that the vector field $E = f\partial_{x_1} \in \mathbb{D}_{\mathbf{Y}}$ satisfies $[...[D, E], ..., E] \notin I_X \cdot \mathbb{D}$ (e-fold bracket.) Using induction and the product rule one proves for $a, b \in \mathcal{O}_n$, $F \in \mathbb{D}$ and $m \in \mathbb{N}$:

$$(aF)^m b \equiv a^m \cdot F^m b \mod (aFb, ..., a^{m-1}F^{m-1}b).$$

Applied to $a = f$, $b = Dx_n$, $F = \partial_{x_1}$ this gives for the m-fold bracket:

$$[f\partial_{x_1}, ..., [f\partial_{x_1}, D]...](x_n) \equiv f^m \partial_{x_1}^m Dx_n \mod (f\partial_{x_1} Dx_n, ..., f^{m-1}\partial_{x_1}^{m-1} Dx_n).$$

If the left hand side is not in I_X for some m we are done. If it were in I_X for all m we could prove inductively $f^m \partial_{x_1}^m Dx_n \in I_X$ for all m. But $\partial_{x_1}^e Dx_n$ with e as above is a unit in $\mathcal{O}_n$, hence $f^e \in I_X$ and $f \in I_X$, contradiction. (In the algebraic case Dx_n is a polynomial and one derives until it becomes a non-zero constant.)

(c) Follows from (a) and (b) by definition of balanced algebras.

Proof of Proposition 6.2. (a) We have to construct a non-zero element in I_A. It is not possible to apply Omori's Lemma by using the weaker assumption $A^{[1]} \neq 0$ since we do not know whether $A^{[1]}$ is a submodule. Instead we refer to Amemiya's Lemma. Take $D \in A^{[2]}$, $D \neq 0$. Then the differential operator D^2 obtained by composing D with itself is non-zero. Choose $g \in \mathcal{O}_n$ with $D^2 g \neq 0$. We have $D \in A^{[1]}$ and $Dg \cdot D = [D, gD] \in A^{[1]}$. Amemiya's Lemma gives $(D^2 g)^2 \in I_A$.

(b) We have to show $I_Y \not\subset \sqrt{I_A}$ for $Y = |\mathbf{Y}|$. Otherwise $I_Y^k \subset I_A$ for some $k \in \mathbb{N}$. For $g \in I_Y^k$ and $D,\ E \in \mathbb{D}_{\mathbf{Y}}$ one has

$$[gD, E] = -Eg \cdot D + g \cdot [D, E] \in I_Y^k \cdot \mathbb{D}_{\mathbf{Y}}.$$

Hence $I_Y^k \cdot \mathbb{D}_{\mathbf{Y}} \subset I_A \cdot \mathbb{D}_{\mathbf{Y}} \subset A$ is a non-zero ideal of $\mathbb{D}_{\mathbf{Y}}$, contradiction.

(c) Follows from (a) and (b) by definition of balanced algebras.

7. VISIBLE ALGEBRAS ARE TANGENT ALGEBRAS

In this section we formulate and prove Echo's Theorem of section 1 for analytic files. A subalgebra A of a Lie algebra B is called *visible*, if there is a chain

$$A = A_m \subset \ldots \subset A_0 = B$$

with $A_i \subset A_{i-1}$ maximal balanced for all i. Of course, geometric implies visible.

Echo's Theorem. (visible implies tangent, file version) *Let* $\mathbf{Y}$ *be an analytic file in* $(\mathbb{C}^n, 0)$ *with* $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$. *Every visible subalgebra* A *of* $\mathbb{D}_{\mathbf{Y}}$ *is a tangent algebra* $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ *where* $\mathbf{X}$ *is an analytic file in* $(\mathbb{C}^n, 0)$ *with no component of* $\mathbf{X}$ *contained in* $|\mathbf{Y}|$. *The file* $\mathbf{X}$ *is unique if assumed irredundant w.r.t.* $\mathbf{Y}$ *and then given as the integral file* $\mathbf{X}_A$ *of* A *relative to* $\mathbb{D}_{\mathbf{Y}}$.

Proof. Uniqueness of $\mathbf{X}$ in the irredundant case follows from Proposition 5.3. To prove existence we proceed by induction. By definition, A admits a chain

$$A = A_m \subset \ldots \subset A_1 \subset A_0 = \mathbb{D}_{\mathbf{Y}}$$

with $A_i \subset A_{i-1}$ maximal balanced. In particular, A_1 is maximal balanced and hence maximal visible in $\mathbb{D}_{\mathbf{Y}}$. If we show existence for A_1, say $A_1 = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ for some file $\mathbf{X}$, we may replace $\mathbf{Y}$ by $\mathbf{Y}^+ = \mathbf{X} \cup \mathbf{Y}$ and consider

$$A = A_m \subset \ldots \subset A_1 = \mathbb{D}_{\mathbf{Y}^+}.$$

The length of the chain having dropped, the induction hypothesis applies and gives the assertion. Thus we may assume from the beginning that $A \subset \mathbb{D}_{\mathbf{Y}}$ is maximal visible. In this situation, the existence of a file $\mathbf{X}$ with $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ can be described more explicitly:

Proposition 7.1. (maximal visible implies tangent) *Let* $\mathbf{Y}$ *be an analytic file in* $(\mathbb{C}^n, 0)$ *with* $|\mathbf{Y}| \neq (\mathbb{C}^n, 0)$. *Every maximal visible (= maximal geometric) subalgebra* A *of* $\mathbb{D}_{\mathbf{Y}}$ *is a tangent algebra* $A = \mathbb{D}_{X,\mathbf{Y}}$ *where* X *is an irreducible analytic germ different* $(\mathbb{C}^n, 0)$ *and not contained in* $|\mathbf{Y}|$ *but with* $\operatorname{Sing} X \subset |\mathbf{Y}|$. *This germ* X *is unique and given as the integral variety* X_A *of* A *relative to* $\mathbb{D}_{\mathbf{Y}}$.

Proof. Uniqueness follows again from Proposition 5.3 for X is irreducible, $\neq (\mathbb{C}^n, 0)$ and not contained in $|\mathbf{Y}|$, hence irredundant w.r.t. $\mathbf{Y}$. To prove existence note that by Proposition 6.2 X_A is different from $(\mathbb{C}^n, 0)$ and not contained in $|\mathbf{Y}|$. To see that $A = \mathbb{D}_{X_A,\mathbf{Y}}$ choose an irreducible component X of X_A not contained in $|\mathbf{Y}|$. Proposition 4.1 and Seidenberg's Theorem yield inclusions

$$A \subset \mathbb{D}_{X_A,\mathbf{Y}} \subset \mathbb{D}_{X,\mathbf{Y}}.$$

By choice of X Proposition 6.1 applies. It shows that $\mathbb{D}_{X,\mathbf{Y}}$ is balanced in $\mathbb{D}_{\mathbf{Y}}$. In particular $\mathbb{D}_{X,\mathbf{Y}} \neq \mathbb{D}_{\mathbf{Y}}$. Then $A \subset \mathbb{D}_{X,\mathbf{Y}} \subset \mathbb{D}_{\mathbf{Y}}$ implies $A = \mathbb{D}_{X,\mathbf{Y}}$ because A is maximal visible, hence maximal balanced in $\mathbb{D}_{\mathbf{Y}}$. As X is irreducible and strictly contained in $(\mathbb{C}^n, 0)$ Proposition 5.3 applies again and gives $X_A = X$. Hence X_A itself is irreducible and $A = \mathbb{D}_{X_A,\mathbf{Y}}$. All properties except Sing $X_A \subset |\mathbf{Y}|$ are shown. To prove this inclusion, a similar argument as for X_A works. Assume that Sing $X_A \not\subset |\mathbf{Y}|$. Choose an irreducible component Z of Sing X_A not contained in $|\mathbf{Y}|$. Proposition 5.1 and Seidenberg's Theorem yield

$$\mathbb{D}_{X_A,\mathbf{Y}} \subset \mathbb{D}_{\mathrm{Sing} X_A,\mathbf{Y}} \subset \mathbb{D}_{Z,\mathbf{Y}}.$$

By Proposition 6.1 $\mathbb{D}_{Z,\mathbf{Y}} \subset \mathbb{D}_{\mathbf{Y}}$ is balanced. Then $A = \mathbb{D}_{X_A,\mathbf{Y}} \subset \mathbb{D}_{Z,\mathbf{Y}} \subset \mathbb{D}_{\mathbf{Y}}$ implies $A = \mathbb{D}_{Z,\mathbf{Y}}$ for A is maximal balanced in $\mathbb{D}_{\mathbf{Y}}$. We get $\mathbb{D}_{X_A,\mathbf{Y}} = \mathbb{D}_{Z,\mathbf{Y}}$. Uniqueness of X_A gives $Z = X_A$ and contradiction to $Z \subset$ Sing X_A. This concludes the proof of the Proposition and of Echo's Theorem.

Remark. Echo's Theorem implies in particular that every visible subalgebra of a tangent algebra must also be an $\mathcal{O}_n$-submodule. This will be a central point in proving that up to analytic isomorphism a singularity is determined by the isomorphism class of its tangent algebra.

8. TANGENT ALGEBRAS ARE VISIBLE

In this section we formulate and prove Narkissos' Theorem of section 1 for analytic files.

Narkissos' Theorem. (tangent implies visible, file version) *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be analytic files in* $(\mathbb{C}^n, 0)$ *with no component of* $\mathbf{X}$ *contained in* $|\mathbf{Y}|$. *Then the tangent algebra* $\mathbb{D}_{\mathbf{X},\mathbf{Y}}$ *is a visible subalgebra of* $\mathbb{D}_{\mathbf{Y}}$.

Proof. We shall use double induction. The first one is on the number of components of $\mathbf{X}$. Let the components X_i of $\mathbf{X}$ be numbered such that $X_i \not\subset X_1$ for $i \geq 2$. Set $\mathbf{Y}^+ = \mathbf{Y} \cup \{X_1\}$, $\mathbf{X}^- = \mathbf{X} \setminus \{X_1\}$. Then

$$\mathbb{D}_{\mathbf{X},\mathbf{Y}} = \mathbb{D}_{\mathbf{X}^-,\mathbf{Y}^+} \subset \mathbb{D}_{\mathbf{Y}^+} = \mathbb{D}_{X_1,\mathbf{Y}} \subset \mathbb{D}_{\mathbf{Y}}.$$

Since visibility is transitive, induction reduces to the case where $\mathbf{X}$ consists of a single irreducible germ X. We have to show that $\mathbb{D}_{X,\mathbf{Y}}$ is visible in $\mathbb{D}_{\mathbf{Y}}$. Consider the iterated singular loci $\mathrm{Sing}^k X = \mathrm{Sing}(...(\mathrm{Sing}(X))...)$ of X. They form a strictly decreasing sequence of analytic subsets of X. As $X \not\subset |\mathbf{Y}|$ by assumption there is a maximal integer $k = k_X$ such that $\mathrm{Sing}^k X \not\subset |\mathbf{Y}|$. The second induction now goes simultaneously on k and on the number l of components of $\mathrm{Sing}^k X$ not contained in $|\mathbf{Y}|$. Take such a component $Z \not\subset |\mathbf{Y}|$ of $\mathrm{Sing}^k X$. Proposition 5.1 gives $\mathbb{D}_X \subset \mathbb{D}_Z$ and hence $\mathbb{D}_X = \mathbb{D}_{X,Z}$. Setting $\mathbf{Y}^+ = \mathbf{Y} \cup \{Z\}$ we get

$$\mathbb{D}_{X,\mathbf{Y}} = \mathbb{D}_{X,\mathbf{Y}^+} \subset \mathbb{D}_{\mathbf{Y}^+} \quad \text{and} \quad \mathbb{D}_{Z,\mathbf{Y}} = \mathbb{D}_{\mathbf{Y}^+} \subset \mathbb{D}_{\mathbf{Y}}.$$

In the first inclusion the number of components of $\mathrm{Sing}^k X$ not contained in $|\mathbf{Y}^+|$ is smaller than l. In case it has become zero one has $\mathrm{Sing}^k X \subset |\mathbf{Y}^+|$ and there is some $m < k$ maximal with $\mathrm{Sing}^m X \not\subset |\mathbf{Y}^+|$. Thus either l has dropped and remained positive or k has dropped. In the second inclusion the irreducible germ Z satisfies $Z \not\subset |\mathbf{Y}|$ but $\mathrm{Sing}\ Z \subset \mathrm{Sing}(\mathrm{Sing}^k(X)) \subset |\mathbf{Y}|$. This corresponds to $k_Z = 0$.

Using transitivity of visibility induction on pairs (k, l) ordered lexicographically now allows to reduce to the case $k = 0$ and $l = 1$. In this situation the visibility of $\mathbb{D}_{X,\mathbf{Y}}$ in $\mathbb{D}_{\mathbf{Y}}$ follows from:

Proposition 8.1. (tangent implies maximal visible) *Let* $\mathbf{Y}$ *be an analytic file in* $(\mathbb{C}^n, 0)$ *and* X *an irreducible germ different* $(\mathbb{C}^n, 0)$ *with* $X \not\subset |\mathbf{Y}|$ *but* $\mathrm{Sing}\ X \subset |\mathbf{Y}|$. *Then the tangent algebra* $\mathbb{D}_{X,\mathbf{Y}}$ *is a maximal visible (= maximal geometric) subalgebra of* $\mathbb{D}_{\mathbf{Y}}$.

Proof. By Proposition 6.1 $A = \mathbb{D}_{X,\mathbf{Y}}$ is balanced in $\mathbb{D}_{\mathbf{Y}}$ and we only have to test A for maximality. So let $B \subset \mathbb{D}_{\mathbf{Y}}$ be balanced with $A \subset B$. Proposition 6.2 implies that the integral variety X_B of B is different $(\mathbb{C}^n, 0)$ and not contained in $|\mathbf{Y}|$. Choose an irreducible component Z of X_B with $Z \not\subset |\mathbf{Y}|$. Proposition 4.1 and Seidenberg's Theorem yield inclusions

$$A = \mathbb{D}_{X,\mathbf{Y}} \subset B \subset \mathbb{D}_{X_B,\mathbf{Y}} \subset \mathbb{D}_{Z,\mathbf{Y}}.$$

Proposition 5.2 implies that $Z \subset \mathrm{Sing}\ X$ or $Z = X$. The first is impossible since $Z \not\subset |\mathbf{Y}|$ but $\mathrm{Sing}\ X \subset |\mathbf{Y}|$. Thus $Z = X$ and $A = \mathbb{D}_{X,\mathbf{Y}} = B$ proving maximality of A.

Proof of Dictionary Theorem. The first three assertions follow from Proposition 7.1 and 8.1 with $\mathbf{Y} = \emptyset$, $\mathbf{Y}$ the file formed by the components of $\mathrm{Sing}\ X$ and $\mathbf{Y} = \{0\}$. The last two will be proven in the next section.

9. THE GRÖBNER CORRESPONDENCE

All ingredients are collected to prove the results announced in the first section.

Narkissos' and Echo's Theorem. (germ version, relative case) *For any analytic file* $\mathbf{Y}$ *in* $(\mathbb{C}^n, 0)$ *the Gröbner correspondence relative to* $\mathbb{D}_{\mathbf{Y}}$ *is a bijection:*

$$\begin{array}{ccc} \{X \subsetneq (\mathbb{C}^n, 0) \text{ analytic germ}, X_i \not\subset |\mathbf{Y}|\} & \longleftrightarrow & \{A \subset \mathbb{D}_{\mathbf{Y}} \text{ geometric subalgebra}\} \\ X & \rightarrow & \mathbb{D}_{X,\mathbf{Y}} \\ X_A & \leftarrow & A \end{array}$$

Remark. Specialisation to the absolute case $\mathbf{Y} = \emptyset$, viz $\mathbb{D}_{\mathbf{Y}} = \mathbb{D}_{\emptyset} = \mathbb{D}$, gives Narkissos' and Echo's Theorem of section 1. In case $\mathbf{Y}$ is an irreducible analytic germ $Y \subset (\mathbb{C}^n, 0)$ one can even characterize Lie algebra theoretically the tangent algebras $\mathbb{D}_{X,Y}$ for all germs $X \subset (\mathbb{C}^n, 0)$ having possibly components contained in

Y but not in Sing Y: By (4) of the Dictionary Theorem $\mathbb{D}_{\mathrm{Sing}\ Y}$ can be characterized Lie algebra theoretically as a subalgebra of $\mathbb{D}$ in terms of $\mathbb{D}_Y$. As $\mathbb{D}_{X,\mathrm{Sing}Y}$ is a geometric subalgebra of $\mathbb{D}_{\mathrm{Sing}\ Y}$ the intersection $\mathbb{D}_{X,Y} = \mathbb{D}_{X,\mathrm{Sing}Y} \cap \mathbb{D}_Y$ gives the characterization.

Proof. (a) The correspondence is well defined: Let $A = \mathbb{D}_{X,\mathbf{Y}}$ be the tangent algebra associated to some germ X in $(\mathbb{C}^n, 0)$ and to $\mathbf{Y}$ and assume that no component of X is contained in $|\mathbf{Y}|$. By definition of geometric algebras, we have to show that A is visible in every visible subalgebra B of $\mathbb{D}_{\mathbf{Y}}$ containing A. The file version of Echo's Theorem gives $B = \mathbb{D}_{\mathbf{Z},\mathbf{Y}}$ for some analytic file $\mathbf{Z}$ in $(\mathbb{C}^n, 0)$. We may choose $\mathbf{Z}$ irredundant w.r.t. $\mathbf{Y}$. Now assume that there is a component X_1 of X contained in $|\mathbf{Y}| \cup |\mathbf{Z}|$. Then $X_1 \subset Z_1$ for some component Z_1 of $\mathbf{Z}$ and, in particular, $Z_1 \not\subset |\mathbf{Y}|$. Proposition 5.2 applied to $\mathbb{D}_{X,\mathbf{Y}} \subset \mathbb{D}_{Z_1}$ implies $X_1 = Z_1$, for there cannot be strict inclusions between the components of one germ. But then we can delete X_1 from X and $\mathbf{Z}$ and add it to $\mathbf{Y}$ without altering the inclusion $\mathbb{D}_{X,\mathbf{Y}} \subset \mathbb{D}_{\mathbf{Y},\mathbf{Z}}$. Note that this change does not effect the condition of no component of X being contained in $|\mathbf{Y}|$.

This argument can be repeated until no component of X is contained in $|\mathbf{Y}| \cup |\mathbf{Z}|$. Then the file version of Narkissos' Theorem implies that

$$A = \mathbb{D}_{X,\mathbf{Y}} = \mathbb{D}_{X,\mathbf{Y},\mathbf{Z}} \subset \mathbb{D}_{\mathbf{Y},\mathbf{Z}} = B$$

is visible. Therefore $\mathbb{D}_{X,\mathbf{Y}}$ is geometric in $\mathbb{D}_{\mathbf{Y}}$.

(b) Injectivity: Follows from Proposition 5.3. since X having no component in $|\mathbf{Y}|$ is irredundant w.r.t. $\mathbf{Y}$ by Proposition 5.2.

(c) Surjectivity: Let $A \subset \mathbb{D}_{\mathbf{Y}}$ be a geometric subalgebra. As A is then visible in $\mathbb{D}_{\mathbf{Y}}$ the file version of Echo's Theorem yields an analytic file $\mathbf{X}$ in $(\mathbb{C}^n, 0)$ with no component contained in $|\mathbf{Y}|$ and such that $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$. Of course we can choose $\mathbf{X}$ irredundant w.r.t. $\mathbf{Y}$. We have to show that $\mathbf{X}$ has no "embedded" components, i.e., that $\mathbf{X}$ equals the analytic file associated to its underlying germ $X := |\mathbf{X}|$. If this were not the case irredundance of $\mathbf{X}$ would imply that $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}}$ were strictly contained in $B = \mathbb{D}_{X,\mathbf{Y}}$. This latter is a visible subalgebra of $\mathbb{D}_{\mathbf{Y}}$ by the file version of Narkissos' Theorem. As A is geometric in $\mathbb{D}_{\mathbf{Y}}$ it is visible in B. And being different from B there exists a balanced subalgebra of B containing A. This implies that A cannot contain a non zero ideal of B. But $I_X \cdot \mathbb{D}_{X,\mathbf{Y}}$ is such an ideal of B contained in A, contradiction. Therefore $\mathbf{X}$ is the file of irreducible components of $X = |\mathbf{X}|$ and $A = \mathbb{D}_{\mathbf{X},\mathbf{Y}} = \mathbb{D}_{X,\mathbf{Y}}$. This proves the Theorem.

Proof of Dictionary Theorem. We are left to show the last two assertions. Observe that (4) is a special case of (5). Let X^- denote the union of the singular locus of one component, say X_1, with the remaining components $X_2, \ldots, X_m$. Of course, $A = \mathbb{D}_{X^-}$ is geometric in $\mathbb{D}$. The components of X^- are $X_2, \ldots, X_m$ together with those components of Sing X_1 which are not contained in $X_2 \cup \ldots \cup X_m$. By Proposition 5.1 and Seidenberg's Theorem we have $\mathbb{D}_X = \mathbb{D}_{X_1,X^-}$. Since $X_1 \not\subset X^-$ but Sing $X_1 \subset X^-$ Proposition 8.1 implies that $\mathbb{D}_X = \mathbb{D}_{X_1,X^-}$ is maximal geometric in $\mathbb{D}_{X^-}$.

Conversely, let $A \subset \mathbb{D}$ be geometric containing $\mathbb{D}_X$ as a maximal geometric subalgebra. We know by Echo's Theorem that $A = \mathbb{D}_Y$ for some proper subvariety Y of $(\mathbb{C}^n, 0)$. By deleting suitably components of X we can obtain a germ $X_1 \subset X$ with $\mathbb{D}_{X_1,Y} = \mathbb{D}_{X,Y} = \mathbb{D}_X$ and such that X_1 is irredundant w.r.t. Y. Then X_1 is the integral variety of $\mathbb{D}_X$ relative to $\mathbb{D}_Y$, see Proposition 5.3. Now Proposition 7.1 shows that X_1 is irreducible with $X_1 \not\subset Y$ but Sing $X_1 \subset Y$.

It remains to show that Y equals the germ X^- corresponding to the component X_1. Apply Proposition 5.2 to the components Y_j of Y and to the remaining components $X_2, ..., X_m$ of X. Then $\mathbb{D}_X \subset \mathbb{D}_{Y_j}$ yields: Y_j is contained in two components of X or Y_j is contained in the singular locus of a component of X or Y_j equals a component of X. And $\mathbb{D}_{X_1,Y} \subset \mathbb{D}_{X_k}$ for $k \geq 2$ yields: X_k is contained in a component of Y. Combining these informations we conclude that each X_k, $k \geq 2$, is a component of Y. Let Y_j be one of the remaining components of Y. Such a Y_j cannot be contained in any of the X_k, $k \geq 2$, and $X_1 \not\subset Y$ implies $Y_j \neq X_1$. Hence $Y_j \subset$ Sing X_1. All this together with Sing $X_1 \subset Y$ gives $Y = X_2 \cup ... \cup X_m \cup$ Sing $X_1 = X^-$. This proves the Theorem.

PART II: VARIETIES ARE DETERMINED BY THEIR TANGENT ALGEBRA

1. THE LOCAL ANALYTIC CASE

Let $X, Y \subset (\mathbb{C}^n, 0)$ be two germs which are analytically isomorphic: $X \xrightarrow{\simeq} Y$. The isomorphism can be extended to an analytic automorphism φ of $(\mathbb{C}^n, 0)$:

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & Y \\ \downarrow & & \downarrow \\ (\mathbb{C}^n, 0) & \xrightarrow{\varphi} & (\mathbb{C}^n, 0). \end{array}$$

The corresponding algebra automorphism

$$\varphi^* : \mathcal{O}_n \to \mathcal{O}_n : f \to f \circ \varphi$$

maps I_Y onto I_X. It induces a Lie algebra automorphism (interpreting vector fields as derivations of $\mathcal{O}_n$)

$$\Phi : \mathbb{D} \to \mathbb{D} : D \to \varphi^* \circ D \circ (\varphi^*)^{-1}$$

which maps $\mathbb{D}_Y$ onto $\mathbb{D}_X$ by definition of tangent algebras. We thus obtain a Lie algebra isomorphism

$$\varphi^\sharp = \Phi : \mathbb{D}_Y \xrightarrow{\simeq} \mathbb{D}_X.$$

Now consider $\mathcal{O}_n$ with the topology of coefficientwise convergence of power series. It thus becomes a topological $\mathbb{C}$-algebra and the automorphism $\varphi^* : \mathcal{O}_n \to \mathcal{O}_n$ is continuous. Identifying $\mathbb{D}$ with the free module $\mathcal{O}_n^n$ by taking the coefficients of a vector field, $\mathbb{D}$ is provided with the induced topology.

Fact 1. $\mathbb{D}$ is a topological Lie algebra. Indeed, for $D = \sum a_i \partial_{x_i}$ and $E = \sum b_i \partial_{x_i}$ in $\mathbb{D}$, their bracket $[D, E] = \sum c_i \partial_{x_i}$ is given by

$$c_i = \sum_k (a_k \partial_{x_k} b_i - b_k \partial_{x_k} a_i).$$

Taking partial derivatives is a continuous map from $\mathcal{O}_n$ to $\mathcal{O}_n$ and the assertion follows.

Fact 2. For $\varphi \in \mathrm{Aut}(\mathbb{C}^n, 0)$, the induced automorphism $\Phi = \varphi^\sharp : \mathbb{D} \to \mathbb{D}$ is continuous. Indeed, for $D = \sum a_i \partial_{x_i} = a \cdot \partial_x$ in $\mathbb{D}$, its image is given by

$$\Phi(D) = (a \circ \varphi) \cdot (\partial\varphi)^{-1} \cdot \partial_x,$$

where $\partial\varphi$ denotes the jacobian matrix of φ and ∂_x the column vector of components ∂_{x_i}. This shows that the tangent algebras of isomorphic germs are isomorphic as topological Lie algebras. We assert that the converse is true as well:

Theorem. *Let X and Y be reduced analytic germs in $(\mathbb{C}^n, 0)$ different from $\emptyset$. Assume that $n \geq 3$. Then X and Y are analytically isomorphic if and only if $\mathbb{D}_X$ and $\mathbb{D}_Y$ are isomorphic as topological Lie algebras. More precisely, for every isomorphism $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ of topological Lie algebras there is a unique automorphism φ of $(\mathbb{C}^n, 0)$ mapping X onto Y and such that $\Phi = \varphi^\sharp$.*

Comments. (a) One has to assume that X and Y are reduced as it may happen that a non-reduced germ and its reduction have equal tangent algebra, see part I, section 2.

(b) The cases $X = \emptyset$ or $Y = \emptyset$ are excluded since $\emptyset$ and $(\mathbb{C}^n, 0)$ have equal tangent algebra. One may equally allow $X,\ Y = \emptyset$ but exclude $X,\ Y = (\mathbb{C}^n, 0)$.

(c) The assumption $n \geq 3$ is used in two places of the proof (Proposition 3.1 and 4.1). We do not know whether the Theorem becomes false for $n = 2$.

(d) Denote by $\mathbb{D}_{X,0}$ the subalgebra of $\mathbb{D}_X$ of vector fields vanishing in 0, $\mathbb{D}_{X,0} = \mathbb{D}_X \cap \mathbb{D}_0$. The Theorem also holds true with $\mathbb{D}_X$ and $\mathbb{D}_Y$ replaced by $\mathbb{D}_{X,0}$ and $\mathbb{D}_{Y,0}$. However, one has to assume X and Y different from $\emptyset$ *and* $(\mathbb{C}^n, 0)$. Compare the remark at the end of section 5.

(e) The choice of topology on $\mathcal{O}_n$ and hence on $\mathbb{D}$ is rather arbitrary. For instance, the analytic topology on $\mathcal{O}_n$ defined by pseudonorms [9, Kap. I] works as well. Again $\mathbb{D}$ becomes a topological Lie algebra with continuous automorphisms $\Phi = \varphi^\sharp : \mathbb{D} \to \mathbb{D}$. The statement of the Theorem remains valid for this topology. Actually, the only properties of the topology on $\mathcal{O}_n$ which are used are that the units are dense in $\mathcal{O}_n$ and that $\mathcal{O}_n$ is Hausdorff, see the proof of Proposition 4.1. We do not know whether the Theorem still holds true if we omit completely the topology and consider only the abstract Lie algebra structure of $\mathbb{D}_X$.

(f) Observe that the Theorem contains as a particular case that any bicontinuous Lie algebra automorphism $\Phi : \mathbb{D} \to \mathbb{D}$ is induced from an analytic automorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$.

2. OUTLINE OF PROOF

For a given analytic automorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ it is easy to construct the Lie algebra automorphism $\Phi = \varphi^\sharp : \mathbb{D} \to \mathbb{D}$ by defining $\Phi(D) = \varphi^* \circ D \circ (\varphi^*)^{-1}$. The problem is to do the converse: Given $\Phi : \mathbb{D} \to \mathbb{D}$ construct a $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\Phi = \varphi^\sharp$. Even in case that one knows that Φ is induced from some φ as above, it is not clear how to recover this φ from Φ. Namely, write $\Phi(D) = \varphi^* \circ D \circ (\varphi^*)^{-1}$ as

$$\Phi(a \cdot \partial_x) = (a \circ \varphi) \cdot (\partial\varphi)^{-1} \cdot \partial_x,$$

where $D = a \cdot \partial_x$. This equation is explicit in Φ, but implicit in φ, actually a partial differential equation in φ. One gets stuck. But for $\Phi = \varphi^\sharp$ there exists another identity relating Φ and φ which makes things much easier: Take a power series $f \in \mathcal{O}_n$ and a vector field $D \in \mathbb{D}$. Then $\Phi^{-1}(D) \in \mathbb{D}$ and hence $f \cdot \Phi^{-1}(D) \in \mathbb{D}$. As $\Phi = \varphi^\sharp$ and φ^* is an algebra homomorphism we obtain

$$\Phi(f \cdot \Phi^{-1}(D)) = \varphi^* \circ (f \cdot ((\varphi^*)^{-1} \circ D \circ \varphi^*)) \circ (\varphi^*)^{-1} = \varphi^*(f) \cdot D.$$

This is now an identity implicit in Φ but quite explicit in φ. As a first thing it shows - since $\mathbb{D}$ is a free $\mathcal{O}_n$-module - that there can only be one φ inducing Φ. Moreover it indicates how one may try to construct φ for an arbitrary topological Lie algebra automorphism Φ of $\mathbb{D}$. Define a $\mathbb{C}$-linear map $\mathrm{Twist}_{\Phi,f} : \mathbb{D} \to \mathbb{D}$ by

$$\mathrm{Twist}_{\Phi,f}(D) := \Phi(f \cdot \Phi^{-1}(D)).$$

Then, for $f \in \mathcal{O}_n$, $\varphi^*(f)$ should be given as the factor by which D is multiplied when one applies $\mathrm{Twist}_{\Phi,f}$ to D. This attempt for recovering φ involves several obstacles one has to overcome. These are:

(1) Given Φ, f and D, why should $\mathrm{Twist}_{\Phi,f}(D)$ be a multiple of D?

(2) Assume it is. Why should $c(f, D)$ in $\mathrm{Twist}_{\Phi,f}(D) = c(f, D) \cdot D$ be independent of D?

(3) Assume it is. Why should the map $\varphi^* : \mathcal{O}_n \to \mathcal{O}_n$ defined by $\varphi^*(f) = c(f, D) = c(f)$ be a $\mathbb{C}$-algebra automorphism?

(4) Assume it is. Why should the argument work as well for topological Lie algebra isomorphisms $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ and then induce an isomorphism $\varphi : X \to Y$?

The affirmative answers to these four questions cover the next three sections. We start directly with $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$. Since $\mathbb{D}_Y$ is an $\mathcal{O}_n$-submodule, $f \cdot \Phi^{-1}(D)$ again belongs to $\mathbb{D}_Y$ and $\mathrm{Twist}_{\Phi,f} : \mathbb{D}_X \to \mathbb{D}_X$ is well defined. Question (1) is by far the hardest part and requires the whole characterization of tangent algebras as geometric subalgebras of $\mathbb{D}$ developped in part I. With this the argument for (1) is geometric and consists in studying the effect of Lie algebra isomorphisms on proportional vector fields (section 3). Once this is settled a comparison test allows to establish (2): One has to choose carefully different non proportional vector fields to show that $c(f, D)$ actually does not depend on D (section 4). Here the continuity assumption gets involved. The proof of (3) will be a quite forward computation. Finally, to show that indeed $\varphi(X) = Y$, the concept of integral variety is used (section 5).

3. LIE ALGEBRA ISOMORPHISMS RESPECT PROPORTIONALITY

Let $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ be a Lie algebra isomorphism, $f \in \mathcal{O}_n$ a power series and

$$\mathrm{Twist}_{\Phi,f} : \mathbb{D}_X \to \mathbb{D}_X : D \to \Phi(f \cdot \Phi^{-1}(D))$$

the associated $\mathbb{C}$-linear map. Our aim in this section is to show that $\mathrm{Twist}_{\Phi,f}(D)$ is a multiple of D. Observe that if we knew that Φ^{-1} maps the $\mathcal{O}_n$-module (D) generated by D onto a submodule of $\mathbb{D}_Y$ this were immediate. Indeed, $f \cdot \Phi^{-1}(D)$ would then also belong to this submodule and $\Phi(f \cdot \Phi^{-1}(D)) \in (D)$ would follow. But there is no a priori reason for a Lie algebra isomorphism to send modules to modules, i.e. to be "compatible" with the $\mathcal{O}_n$-module structure of $\mathbb{D}$.

Actually, we shall be able to prove this statement for certain submodules of $\mathbb{D}_X$, but the argument makes some detour before arriving there. We give a brief heuristic description valid for nonsingular vector fields D, i.e. $D(0) \neq 0$. Then $E \in (D)$, say $E = a \cdot D$ with some $a \in \mathcal{O}_n$, if and only if E is proportional to D on some neighborhood of 0 in $\mathbb{C}^n$. By this we mean that for $p \in \mathbb{C}^n$ close to 0, the evaluations $D(p)$ and $E(p)$ point in the same direction as vectors of $\mathbb{C}^n$.

Now assume that we can show that Φ maps proportional vector fields onto proportional vector fields. As $\Phi^{-1}(D)$ and $f \cdot \Phi^{-1}(D)$ are trivially proportional, it would follow that $D = \Phi(\Phi^{-1}(D))$ and $\mathrm{Twist}_{\Phi,f}(D) = \Phi(f \cdot \Phi^{-1}(D))$ are proportional. For nonsingular D we could conclude that

$$\mathrm{Twist}_{\Phi,f}(D) \in (D)$$

as required. Therefore one is led to investigate whether Lie algebra isomorphisms preserve proportionality of vector fields. The idea is that two nonsingular vector fields D, E are proportional if and only if they are tangent to the same collection of subvarieties $Z \subset (\mathbb{C}^n, 0)$:

$$\{Z \subset (\mathbb{C}^n, 0),\ D \in \mathbb{D}_Z\} = \{Z \subset (\mathbb{C}^n, 0),\ E \in \mathbb{D}_Z\}.$$

Hence it would be useful to know whether the image of a tangent algebra $\mathbb{D}_Z$ under a Lie algebra isomorphism Φ is again a tangent algebra, $\Phi(\mathbb{D}_Z) = \mathbb{D}_{Z'}$. For this purpose we use the characterization of tangent algebras proven in part I, section 9, and conclude:

Corollary. *Let $X, Y, Z \subset (\mathbb{C}^n, 0)$ be analytic germs, $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ an isomorphism of Lie algebras. If no component of Z is contained in Y there is a germ $Z' \subset (\mathbb{C}^n, 0)$ with no component contained in X such that*

$$\Phi(\mathbb{D}_{Y,Z}) = \mathbb{D}_{X,Z'}.$$

In particular, $\Phi(\mathbb{D}_{Y,Z})$ is a submodule of $\mathbb{D}_X$.

The statement of the Corollary is crucial for our argument. In order to prove that $\mathrm{Twist}_{\Phi,f}(D) \in (D)$ we shall combine this result with the following description of principal modules: Denote by $\mathbf{H}_D$ the set of all irreducible hypersurfaces $Z \subset (\mathbb{C}^n, 0)$ to which D is tangent:

$$\mathbf{H}_D = \{Z \subset (\mathbb{C}^n, 0) \text{ irreducible hypersurface with } D \in \mathbb{D}_Z\}.$$

Proposition 3.1. *Assume $n \geq 3$. Let $X \subset (\mathbb{C}^n, 0)$ be an analytic germ and $D \in I_X \cdot \mathbb{D}$ a reduced vector field, i.e. $D = g \cdot D'$ with $g \in I_X$ without multiple factors and $D' \in \mathbb{D}$ with $D'(0) \neq 0$. Assume that D' is not tangent to any component of the zero set V of g in $(\mathbb{C}^n, 0)$. Then*

$$(D) = \bigcap_{Z \in \mathbf{H}_D} \mathbb{D}_{X,Z}.$$

Note that this is false for $n = 2$. Take e.g. $D = \partial_{x_1}$. Then $Z = \{x_2 = 0\}$ is the only hypersurface to which D is tangent, $\mathbf{H}_D = \{Z\}$. But $E = x_2 \cdot \partial_{x_2}$ is tangent to Z, $E \in \mathbb{D}_Z$, whereas $E \notin (D)$.

The proof will show that it suffices to take in the intersection above for Z only the irreducible components of V and smooth hypersurfaces. Before proving Proposition 3.1 we deduce:

Corollary. *Assume $n \geq 3$. Let $X, Y \subset (\mathbb{C}^n, 0)$ be analytic germs different from $(\mathbb{C}^n, 0)$ and $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ an isomorphism of Lie algebras. Let $D \in I_X \cdot \mathbb{D}$ be as in Proposition 3.1 and $M = (D) \subset \mathbb{D}_X$ the submodule generated by D. Then $\Phi^{-1}(M)$ is a submodule of $\mathbb{D}_Y$. In particular, for all $f \in \mathcal{O}_n$ one has* $\mathrm{Twist}_{\Phi,f}(D) \in (D)$.

Proof of Corollary. If some $Z \in \mathbf{H}_D$ is contained in X, it must be a component of X by comparison of dimensions and since $X \neq (\mathbb{C}^n, 0)$. But a vector field is tangent to a germ if and only if it is tangent to all of its components. Therefore $\mathbb{D}_{X,Z} = \mathbb{D}_X$ and Z can be dropped from the intersection in Proposition 3.1. The assertion now follows from the Corollary preceding the Proposition applied to Φ^{-1}.

Proof of Proposition 3.1. We only have to show that

$$\bigcap_{Z \in \mathbf{H}_D} \mathbb{D}_Z \subset (D).$$

(a) Let $g = g_1 \cdots g_m$ be the decomposition into prime factors and $V = \bigcup V_i$ the corresponding decomposition into irreducible components. We first prove that $h \cdot D' \in \bigcap \mathbb{D}_{V_i}$ for some $h \in \mathcal{O}_n$ implies $h \in (g)$. But $h \cdot D' \in \mathbb{D}_{V_i}$ signifies that

$$h \cdot D' g_i \in (g_i).$$

As $D' g_i \notin (g_i)$ by assumption on D we obtain $h \in (g_i)$. Thus $h \in \bigcap (g_i) = (g)$, for g has no multiple factors.

(b) By (a) and since all $V_i \in \mathbf{H}_D$ we are reduced to prove that

$$\bigcap_{Z \in \mathbf{H}_D} \mathbb{D}_Z \subset (D').$$

As $D'(0) \neq 0$ a change of coordinates in $(\mathbb{C}^n, 0)$ allows to write $D' = \partial_{x_1}$ [5, chap. III, sec. VII, Lemma 1]. Take any $E \in \bigcap_{Z \in \mathbf{H}_D} \mathbb{D}_Z$ and an arbitrary irreducible hypersurface $Z' \subset (\mathbb{C}^{n-1}, 0)$. Clearly $D' = \partial_{x_1}$ is tangent to $Z = (\mathbb{C}, 0) \times Z'$, thus $D \in \mathbb{D}_Z$, $Z \in \mathbf{H}_D$ and consequently $E \in \mathbb{D}_Z$. Restricting E to $\{p\} \times (\mathbb{C}^{n-1}, 0)$ for $p \in \mathbb{C}$ close to 0 and projecting onto its tangent space one obtains a vector field E' on $(\mathbb{C}^{n-1}, 0)$ which is tangent to Z'. As this holds for all Z' the assumption $n \geq 3$

forces E' to be 0, see part I, section 6. For this argument it suffices to consider only smooth hypersurfaces. But $p \in \mathbb{C}$ was arbitrary. Hence $E \in (\partial_{x_1}) = (D')$ as required.

4. INDEPENDENCE

In this section it is shown that $\mathrm{Twist}_{\Phi,f}(D)$ is a multiple of D with factor independent of D.

Proposition 4.1. *Let $X, Y \subset (\mathbb{C}^n, 0)$ be analytic germs different from $(\mathbb{C}^n, 0)$. Assume $n \geq 3$. Let $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ be an isomorphism of topological Lie algebras. For every $f \in \mathcal{O}_n$ there is an element $c = c(f) \in \mathcal{O}_n$ such that for all $D \in I_X \cdot \mathbb{D}$:*

$$\mathrm{Twist}_{\Phi,f}(D) = c \cdot D.$$

Proof. Choose generators $g_1, \ldots, g_m$ of I_X without multiple factors and coordinates $x_1, \ldots, x_n$ such that none of the coordinate axes is contained in any of the hypersurfaces V_i defined by g_i. Rossi's Theorem shows that none of the vector fields $\partial_{x_1}, \ldots, \partial_{x_n}$ is tangent to any of the components of the V_i's. By the Corollary of Proposition 3.1 the modules M_{ij} generated by $g_i \partial_{x_j}$ are mapped by $\mathrm{Twist}_{\Phi,f}$ into themselves. Hence for every unit $u \in \mathcal{O}_n$ there are unique $c_{ij}(u) \in \mathcal{O}_n$ with

$$\mathrm{Twist}_{\Phi,f}(u g_i \partial_{x_j}) = c_{ij}(u) \cdot u \cdot g_i \cdot \partial_{x_j}.$$

We shall show that $c_{ij}(u)$ does not depend on i, j and u. So let us write $D_{ij} = g_i \partial_{x_j}$. Choose $k \neq j$ and consider

$$D = u \cdot D_{ij}, \; E = a \cdot D_{ik} \quad \text{with} \quad u \in \mathcal{O}_n^*, \; a \in \mathbb{C}^*,$$

$$D' = u \cdot \partial_{x_j}, \quad E' = a \cdot \partial_{x_k}.$$

We claim that there is an $a \in \mathbb{C}^*$ such that $D' + E'$ is not tangent to any component of V_i. Indeed, otherwise $D' + E'$ were tangent to the same component of V_i for at least two different values of a. Taking differences, E' were tangent contradicting our choice of coordinates. Thus the Corollary to Proposition 3.1 can also be applied to $D + E = g_i(D' + E')$. All together we get:

$$\mathrm{Twist}_{\Phi,f}(D) = c_{ij}(u) \cdot D, \quad \mathrm{Twist}_{\Phi,f}(E) = c_{ik}(a) \cdot E,$$

$$\mathrm{Twist}_{\Phi,f}(D + E) \in (D + E).$$

As D, E are linearly independent over $\mathcal{O}_n$ one deduces (see the Lemma after the proof) that

$$c_{ij}(u) = c_{ik}(a).$$

As $a \in \mathbb{C}^*$ and $\mathrm{Twist}_{\Phi,f}$ is $\mathbb{C}$-linear we get $c_{ik}(a) = c_{ik}(1)$. Hence $c_{ij}(u)$ is independent of j and u:

$$\mathrm{Twist}_{\Phi,f}(u \cdot D_{ij}) = c_i \cdot u \cdot D_{ij}.$$

But $\mathrm{Twist}_{\Phi,f}$ is continuous, the units are dense in $\mathcal{O}_n$ and $\mathcal{O}_n$ is Hausdorff. Therefore

$$\mathrm{Twist}_{\Phi,f}(h \cdot D_{ij}) = c_i \cdot h \cdot D_{ij}$$

for all $h \in \mathcal{O}_n$. In particular, setting $h = g_k$ for $k \neq i$ we see that c_i does not depend on i. The Proposition is proven.

Lemma. *Let $L : \mathbb{D}_X \to \mathbb{D}_X$ be $\mathbb{C}$-linear, $D, E \in \mathbb{D}_X$ linearly independent over $\mathcal{O}_n$. Assume*

$$L(D) = c_D \cdot D, \quad L(E) = c_E \cdot E, \quad L(D+E) = c_{D+E} \cdot (D+E)$$

with coefficients in $\mathcal{O}_n$. Then $c_D = c_E = c_{D+E}$.

Proof. Linearity gives $(c_D - c_{D+E}) \cdot D = (c_{D+E} - c_E) \cdot E$. Then $c_D = c_{D+E} = c_E$ as D, E are linearly independent.

5. X AND Y ARE ISOMORPHIC

The proof of the Theorem of section 1 is now completed by

Proposition 5.1. *Let X and Y be reduced analytic germs in $(\mathbb{C}^n, 0)$ different from $(\mathbb{C}^n, 0)$. Let $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ be an isomorphism of Lie algebras. Assume that for every $f \in \mathcal{O}_n$ there are elements $a_f, c_f \in \mathcal{O}_n$ such that*

$$\mathrm{Twist}_{\Phi,f}(D) = a_f \cdot D$$
$$\mathrm{Twist}_{\Phi^{-1},f}(E) = c_f \cdot E$$

for all $D \in I_X \cdot \mathbb{D}$ and all $E \in I_Y \cdot \mathbb{D}$. Then $f \to a_f$ and $f \to c_f$ define mutually inverse automorphisms of $\mathcal{O}_n$. The corresponding analytic automorphism φ of $(\mathbb{C}^n, 0)$ maps X onto Y and induces Φ as $\Phi = \varphi^\sharp$.

Proof. (a) Let $\alpha : \mathcal{O}_n \to \mathcal{O}_n$ be the map given by $\alpha(f) = a_f$. Clearly α is $\mathbb{C}$-linear and injective, since $I_X \neq 0$. It is an algebra homomorphism, since for $f, g \in \mathcal{O}_n$ and $D \in I_X \cdot \mathbb{D}$ computation gives $\alpha(f \cdot g) \cdot D = \alpha(f) \cdot \alpha(g) \cdot D$.

(b) α is bijective: Let $\gamma : \mathcal{O}_n \to \mathcal{O}_n$ be the map defined by $\gamma(f) = c_f$. For $0 \neq E \in I_Y \cdot \mathbb{D}$ and $0 \neq g \in I_X$ one has

$$0 \neq D := g \cdot \Phi(E) \in I_X \cdot \mathbb{D} \quad \text{and} \quad \Phi^{-1}(D) = \gamma(g) \cdot E \in I_Y \cdot \mathbb{D}.$$

For $f \in \mathcal{O}_n$ we get by computation

$$f \cdot D = \alpha(\gamma(f)) \cdot D$$

proving $\gamma = \alpha^{-1}$.

(c) Φ is induced from α: Choose any $D \in I_X \cdot D$, $D \neq 0$. For arbitrary $E \in \mathbb{D}_Y$, $f \in \mathcal{O}_n$ we have

$$\begin{aligned}\Phi([E, f \cdot \Phi^{-1}(D)]) &= [\Phi(E), \Phi(f \cdot \Phi^{-1}(D))] \\ &= [\Phi(E), \alpha(f) \cdot D] \\ &= \Phi(E)(\alpha(f)) \cdot D + \alpha(f) \cdot [\Phi(E), D].\end{aligned}$$

On the other hand

$$\begin{aligned}\Phi([E, f\cdot\Phi^{-1}(D)]) &= \Phi(Ef\cdot\Phi^{-1}(D) + f\cdot[E,\Phi^{-1}(D)])\\ &= \alpha(Ef)\cdot D + \Phi(f\cdot\Phi^{-1}([\Phi(E),D]))\\ &= \alpha(Ef)\cdot D + \alpha(f)\cdot[\Phi(E),D].\end{aligned}$$

For the last equality note that $[\Phi(E), D] \in I_X \cdot \mathbb{D}$ since $I_X \cdot \mathbb{D}$ is an ideal of $\mathbb{D}_X$, cf. Proposition 2.1 of part I. We conclude that $\Phi(E) \circ \alpha = \alpha \circ E$ for all $E \in \mathbb{D}_Y$ as claimed.

(d) It remains to show that $\varphi(X) = Y$. We shall use Proposition 5.3 of part I. To apply this result in our situation, let $X' = \varphi^{-1}(Y)$. Then $\mathbb{D}_{X'} = \Phi(\mathbb{D}_Y) = \mathbb{D}_X$. Setting $\mathbf{Y} = \emptyset$ in the Proposition gives $X' = X$.

Remark. Assume that X and Y are different from $\emptyset$ and $(\mathbb{C}^n, 0)$. Then Proposition 5.1 holds true with $\mathbb{D}_X$ and $\mathbb{D}_Y$ replaced by $\mathbb{D}_{X,0}$ and $\mathbb{D}_{Y,0}$. Indeed $I_X \cdot \mathbb{D}$ is an ideal of $\mathbb{D}_{X,0}$ since $X \neq \emptyset$ and the remainder of the proof remains unchanged. For (d) only note that $\varphi(X) = Y$ is easily seen if both X and $X' = \varphi^{-1}(Y)$ are different from 0, apply Proposition 5.3, part I, with $\mathbf{Y} = \{0\}$. In case $X = 0$ and $X' \neq 0$ we have $\mathbb{D}_{X',0} = \mathbb{D}_{X,0} = \mathbb{D}_{\emptyset,0}$ and the same Proposition yields $X' = \emptyset$ contradiction. The case $X = X' = 0$ is trivial.

6. THE AFFINE ALGEBRAIC CASE

Theorem. *Let X and Y be reduced non-empty algebraic subvarieties of affine space $\mathbf{A}^n$ over an algebraically closed field K of characteristic zero. For every isomorphism $\Phi : \mathbb{D}_Y \to \mathbb{D}_X$ of abstract Lie algebras there is a unique algebraic automorphism φ of $\mathbf{A}^n$ mapping X onto Y and inducing Φ.*

Proof. Again we may assume that X and Y are different from $\mathbf{A}^n$. To a point $p \in \mathbf{A}^n$ associate the Lie algebra $\mathbb{D}_{X,p}$ of vector fields tangent to X and vanishing in p. By Propositions 4.2, 7.1 and 8.1 of part I this sets up a one to one correspondence between the points outside X and the maximal geometric subalgebras of $\mathbb{D}_X$ having finite codimension. Hence for any p in the complement of X there is a unique point $q = \varphi(p)$ in the complement of Y such that $\Phi(\mathbb{D}_{Y,q}) = \mathbb{D}_{X,p}$. This defines a bijection $\varphi : \mathbf{A}^n \setminus X \to \mathbf{A}^n \setminus Y$. We claim that φ is the restriction of a biregular map $\bar{\varphi} : \mathbf{A}^n \to \mathbf{A}^n$. One first shows that φ is regular on $\mathbf{A}^n \setminus X$. Taking up an argument already employed by Shanks and Pursell [23] and Omori [19], let x_i denote the coordinate functions on $\mathbf{A}^n$ and let $D \in \mathbb{D}_X$. Set

$$D_i = \mathrm{Twist}_{\Phi,x_i}(D) = \Phi(x_i \cdot \Phi^{-1}(D)).$$

Then
$$\Phi^{-1}(D_i - \varphi_i(p)\cdot D) = (x_i - \varphi_i(p))\cdot\Phi^{-1}(D).$$

This vector field vanishes in $q = \varphi(p)$, i.e. belongs to $\mathbb{D}_{Y,q}$. Hence, if p is outside X, its image $D_i - \varphi_i(p)\cdot D$ lies in $\mathbb{D}_{X,p}$. Setting $D = \sum b_j\partial_{x_j}$, $D_i = \sum a_{ij}\partial_{x_j}$ with b_j, $a_{ij} \in K[x]$ and varying p in $\mathbf{A}^n \setminus X$ we obtain that $a_{ij} = \varphi_i \cdot b_j$ on $\mathbf{A}^n \setminus X$ for all i, j.

For any point p in the complement of X one may choose $b \in I_X$ not vanishing in p. Since $D = b\cdot\partial_{x_1}$ is contained in $\mathbb{D}_X$ this shows that φ is regular on $\mathbf{A}^n\setminus X$. If X has

codimension ≥ 2, φ extends to a regular map $\bar{\varphi} : \mathbf{A}^n \to \mathbf{A}^n$. The same argument applied to φ^{-1} proves that $\bar{\varphi}$ is biregular, i.e. an algebraic automorphism. It is then clear that $\bar{\varphi}$ maps X onto Y and induces Φ.

In general, X is the union of a variety X_1 of codimension ≥ 2 and a hypersurface $X_2 = \{h = 0\}$. Choosing $b \in I_{X_1}$ the vector fields $b \cdot (\partial_{x_j} h \cdot \partial_{x_k} - \partial_{x_k} h \cdot \partial_{x_j})$ belong to $\mathbb{D}_{X_1} \cap \mathbb{D}_{X_2} = \mathbb{D}_X$. Taking these for D one concludes that φ is the restriction of a regular map on $\mathbf{A}^n \setminus (X_1 \cup \operatorname{Sing} X_2)$. But as $X_1 \cup \operatorname{Sing} X_2$ has codimension ≥ 2 one can proceed as before.

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