Singularities, etc.

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To Heisuke Hironaka, with great respect

Monomials

(1) *Monomial algebras.* Let A be a set of vectors of non-negative integers in $\mathbb{Z}^n$, and let K be a field. For variables $s = (s_1, \ldots, s_n)$, write s^α for $\prod_i s_i^{\alpha_i}$.
(a) Give a criterion in terms of A for $K[A] = K[s^\alpha,\ \alpha \in A]$ to be a regular ring.
(b)* Let $\mathcal{A}$ denote the set of all finite and non-empty subsets $A \subset \mathbb{Z}^n$. Define a map $C : \mathcal{A} \to \mathcal{A}$ with $C_A \subset A$ such that the following holds. For any choice of map $\eta : \mathcal{A} \to \mathbb{Z}^n$ with $\eta_A \in C_A$ and any $A \in \mathcal{A}$, denoting by A' the transform

$$A' = A \cup \{\alpha - \eta_A,\ \alpha \in C_A\}.$$

of A with respect to C and η, the sequence of transforms $K[A]$, $K[A']$, $K[A'']$, ... becomes eventually regular.

(2)* *Monomial Jacobian ideals.* Characterize the hypersurface singularities whose Jacobian ideal is a monomial ideal.

(3) *Integral closure of monomial ideals.* For an ideal I of a commutative ring R, define the integral closure $\overline{I}$ of I as the elements x of R which satisfy an integrality relation

$$x^k + a_1 x^{k-1} + \cdots + a_{k-1} x + a_k = 0$$

with a_i in the i-th power I^i of I.
(a) Show that the integral closure is again an ideal.
(b) Let $R = \mathbb{C}\{x\}$ be the ring of convergent power series in n variables, and let I be generated by $g_1, \ldots, g_m$. Show that $f \in \overline{I}$ if and only if there is a constant $C > 0$ and a neighborhood U of 0 in $\mathbb{C}^n$ so that

$$|f(a)| < C \cdot \max\{|g_1(a)|, \ldots, |g_m(a)|\}$$

for all $a \in U$.
(c) Determine the integral closure of an ideal generated by monomials.

(4) *Ordering monomial ideals.* Consider $\mathbb{N}^n$ with a total order ε compatible with addition, and let $<_\varepsilon$ be the induced order on the monomials in n variables.
(a) Show that $<_\varepsilon$ induces naturally an order, also denoted by $<_\varepsilon$, on the set of monomial ideals in $K[x_1, \ldots, x_n]$.
(b) Show that this order is a well-ordering.
(c) For an ideal I of the formal power series ring $K[[x_1, \ldots, x_n]]$, let $\mathrm{in}_x(I)$ be the initial ideal of I, given as the ideal generated by all minimal monomials of the expansions of elements of I. Show that the minimum $\min(I)$ and the maximum $\max(I)$ of $\mathrm{in}_x(I)$ over all coordinates $x_1, \ldots, x_n$ exist.

Supported by project P-21461 of the Austrian Science Fund. The questions and problems collected in this article are to a large extent inspired by Hironaka's work. They often refer to results or techniques established by him. Of course, only a few aspects of his œuvre could be covered. For more background we recommend "On the Mathematical Work of Professor Heisuke Hironaka", by Lê and Teissier, Publ. RIMS 44 (2008), 165 - 177.

(d) Show that $K[[x_1,\dots,x_n]]/I$ is isomorphic as a K-vectorspace to $K[[x_1,\dots,x_n]]/\mathrm{in}_x(I)$.
(e) Let $<_\varepsilon$ be compatible with the total degree. Let $g_1,\dots,g_m$ be generators of I whose initial monomials generate $\mathrm{in}_x(I)$. Show that the strict transform I' of I under the blowup of $\mathbb{A}^n$ with center 0 at any point a' of the exceptional divisor is generated by the strict transforms of $g_1,\dots,g_m$.
(f) Show that $\min(I') \leq_\varepsilon \min(I)$.

Polyhedra

(5) *Lattice points in simplices.* For integers $m, c \geq 1$ consider the simplex $\Delta = \{\alpha \in \mathbb{N}^n,\ |\alpha| = m\}$ and the lattice $L = c \cdot \mathbb{N}^n$. Characterize the vectors $r \in \mathbb{Z}^n$ such that $|(r+\Delta) \cap L| = |(\Delta \cap L)|$.

(6) *Newton polyhedra with compact facets.* Let $P \in K[x]$ be a polynomial in n variables and N its Newton polyhedron at a given point $a \in \mathbb{A}^n$ (i.e., the positive convex hull in $\mathbb{R}^n$ of the exponents of the expansion of P at a). For a blowup of $\mathbb{A}^n$ along a coordinate subspace of codimension ≥ 2, consider the total transforms P^* of P in any of the the affine coordinate charts, together with their associated Newton polyhedra N^*, taken at the origin of the respective chart.
(a) Determine N^* in terms of N.
(b) Show that for any sequence of such blowups the transformed polyhedra have eventually no compact facet.
(c)* For a hypersurface $f = 0$ in $\mathbb{A}^n$, consider the locus of points where the Newton polyhedron of f has in all local coordinates at least one compact facet. Show that this locus is finite.

(7)* *Polyhedral game.* Find a winning strategy for Hironaka's polyhedral game without using induction on the dimension.

Transversality

(8) *Analytic irreducibility.* (a) Find an effective criterion for checking whether a polynomial in n variables over $\mathbb{C}$ is analytically irreducible at 0.
(b) Find an effective criterion for checking whether a polynomial in n variables over $\mathbb{C}$ defines locally at 0 a normal crossings divisor.
(c)* Is it true that a free divisor in $(\mathbb{C}^n, 0)$ has normal crossings if and only if the singular locus is reduced?
(d)* Which hypersurface singularities have a reduced singular locus?
(e)* Classify all (not necessarily reduced) curves which may appear as the singular locus of a surface.

(9)* *Distance to normal crossings.* Find a (significant) measure for the distance of a hypersurface singularity from having normal crossings. This measure should not increase under point blowup.

(10) *Mikado singularities.* A union of smooth varieties is called *mikado* if all intersections of its components are scheme-theoretically smooth.
(a) Construct an example of two analytically non-isomorphic mikado hypersurface singularities of the same dimension and having the same number of components.
(b) Construct a union of smooth hypersurfaces whose pairwise scheme-theoretic intersections are smooth, but which is not mikado.

(c) Find a mikado hypersurface singularity which is not analytically isomorphic to a hyperplane arrangement.
(d)* Find a characterization of mikado singularities through unions of linear spaces.

(11) *Blowup of mikado.* Show that the blowup of a mikado scheme in $\mathbb{A}^n$ along a center $Z \subset \mathbb{A}^n$ transversal to all its components need not be again mikado.

(12)* *Resolution of mikado.* Find for any mikado scheme an (explicitly given) birational proper map which transforms the scheme into normal crossings.

(13) *Non normal crossings locus.* (a) Find an example of a singularity where the algebraic and the analytic non normal crossings locus differ.
(b)* Show that any variety admits an embedded resolution with centers inside the algebraic non normal crossings locus.
(c) Show that this is not the case for the analytic non normal crossings locus.

(14)** *Casas-Alvero conjecture.* Let P be a univariate polynomial over a field of characteristic zero. Assume that each (non constant) derivative shares a divisor with P.
(a) Is P a monomial $(ax+b)^k$?
(b) What could be the respective statement for multivariate polynomials?

Singular locus

(15) *Singular singular locus.* (a) Construct a complex algebraic surface X whose reduced singular locus S equals the curve parametrized by (t^3, t^4, t^5).
(b)* Answer (a) by a surface whose generic transversal plane section along S is an ordinary cusp $x^2 = y^3$.
(c) Is it possible to realize (a) by a surface whose singular locus is reduced?

(16) *Locally symmetric singularities.* (a) Construct an algebraic surface in $\mathbb{A}^3$ whose singular locus is a node $x^2 = y^2 + y^3$ and which has at 0 a local symmetry of order 2.
(b) Can you achieve that the local multiplicity is constant equal to 2 along the singular curve?
(c) Is it possible to embed the Cartesian product of the node $x^2 = y^2 + y^3$ with the cusp $x^2 = y^3$ into $\mathbb{A}^3$?
(d)* Find a non reduced ideal structure on a given union of coordinate subspaces of $\mathbb{A}^n$ so that the blowup of $\mathbb{A}^n$ with center this ideal gives a smooth transform.

(17) *Smooth singular locus.* Characterize the hypersurface singularities $f = 0$ whose singular locus (defined by f and its derivatives) is scheme-theoretically smooth.

Resolution

(18) *Affine real resolution.* (a)* Classify the real algebraic surfaces X in $\mathbb{A}^3_{\mathbb{R}}$ which admit an affine resolution, i.e., a smooth algebraic surface X' in $\mathbb{A}^3_{\mathbb{R}}$ and a birational map $\pi : X' \to X$ which is "real proper", i.e., the induced map on subsets of $\mathbb{R}^3$ is proper in the Euclidean topology.
(b)* Then try to interpret X' as a smoothing of X.

(19) *Drop of order under blowup.* Let X be a plane curve singularity and consider a sequence of monomial point blowups (i.e., the successive centers are the origins of the affine charts of the preceding blowup). Show that the local multiplicity of X drops at least to its half if there appears in the sequence of blowups at least one chart change.

(20) *Separation of varieties.* Let X and Y be algebraic, possibly singular affine varieties in $\mathbb{A}^n_{\mathbb{C}}$. Find a sequence of blowups in smooth centers contained in $X \cap Y$ (respectively the transforms of the intersection) which separates X and Y.

(21)** *Higher Nash modifications.* Nash modifications are obtained by replacing the singular locus of a hypersurface by the projectivized first jet space. Extend this to higher jet spaces.

(22)* *Global descent for resolution.* Construct a reasonable globally defined descent in dimension for the resolution of singularities in characteristic zero.

(23)* *Failure of maximal contact.* Characterize all hypersurfaces in $\mathbb{A}^n_K$, K a field of positive characteristic, whose locus of points of maximal multiplicity has local embedding dimension n at 0.

(24)** *Quings.* (a) Let K be a field of characteristic $p > 0$. Develop a reasonable concept of local multiplicity for equivalence classes of power series in $K[[x]]/K[[x]]^p$.
(b) Resolve "plane curves" f in $K[[y,z]]/K[[y,z]]^p$.

(25)** *Monomialization of morphisms.* Show that any morphism of varieties over a field of characteristic zero can be transformed via blowups of source and target into a monomial morphism, i.e., a morphism which can be expressed in suitable local coordinates by monomials.

Algebraic series

(26) *Algebraic series.* (a) Show that any complex algebraic power series is convergent.
(b)* Show that for any algebraic power series h with $h(0) = 0$ there is a polynomial mapping $F : \mathbb{C}^{n+p} \to \mathbb{C}^p$, $F(0) = 0$, with non zero Jacobian determinant $\det(D_y F(0)) \neq 0$ (where $y = (y_1, \ldots, y_p)$ are the coordinates on $\mathbb{C}^p$) such that the first component y_1 of the implicit solution $y(x)$ of $F(x,y) = 0$ equals h.

(27) *Recursions for algebraic series.* (a) Let $S = \sum_k a_k x^k$ be a formal power series. Express the fact that S is an algebraic power series through a recursion formula for the coefficients a_k.
(b) Same problem for D-finite power series.

(28) *Artin Approximation.* (a) Let X and Y be hypersurface singularities in $(\mathbb{C}^n, 0)$. Show that if X and Y are isomorphic up to a sufficiently high power of the maximal ideal, then X and Y are analytically isomorphic.
(b)** Find a proof for the Artin Approximation Theorem which does not use induction on the dimension. Such a proof should make explicit why the counterexamples of Gabrielov (separate variables condition for solutions of polynomial equations) and Becker (nested subring condition for solutions of non-algebraic series) cannot be avoided.

(29) *Division of algebraic series.* (a) Divide xy by $(x - y^2)(y - x^2)$ as power series in $K[[x,y]]$, taking for the divisor the initial monomial xy. What do you observe?
(b)* Let $f \in K[[x_1, \ldots, x_n]]$ be an x_n-regular algebraic series (i.e., $f(0, \ldots, 0, x_n) \neq 0$). Show that the Weierstrass division of any algebraic series by f yields an algebraic quotient and remainder.

(30)** *Algebraic solutions of ODE's.* Let $a_i(x) \in \mathbb{Q}[x]$ be polynomials in one variable x, and consider the ordinary differential equation

$$D: \; a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Then D has a complete set of algebraic series solutions if and only if, for almost all primes p, the reduction D_p of D modulo p has a complete set of rational solutions in $\mathbb{F}_p(x)$.

Symmetry

(31) *Symmetries of singularities.* (a) Determine the local symmetries of the Cartesian square $C \times C$ of the cusp $C : x^2 = y^3$ at 0.
(b)* Is it possible to construct the automorphism group of a Cartesian product from the automorphisms of the factors?
(c)* Describe the symmetries of finite order of a hypersurface singularity.

(32)** *Cauchy-Riemann automorphisms.* (a) Let X be the germ of a smooth real analytic hypersurface in $(\mathbb{C}^n, 0)$. Describe the group of local biholomorphic automorphisms of $(\mathbb{C}^n, 0)$ fixing X.
(b) Determine the relation of the holomorphic automorphism group of X with its Lie algebra.
(c) Same problem as in (a), with X singular.

(33) *Platonic stars.* The Astroid $x^{2/3} + y^{2/3} = 1$ is the real plane curve (hypocycloid) parametrized by $(\cos(3t), \sin(3t))$. It equals the trajectory of a point on a small circle rolling inside a larger circle.
(a) Determine the ratio of radii and the polynomial equation of the Astroid.
(b) Find a compact real algebraic surface in $\mathbb{R}^3$ which generalizes the Astroid.
(c) Answer (b) by requiring the surface to have isolated singularities.
(d)* Find a construction recipe for this surface analogous to the rolling circles.

Miscellanea

(34) *Polynomially defined singularities.* (a) Show that any complex analytic hypersurface singularity can be defined by a polynomial in as many variables as the codimension of the singular locus indicates, with coefficients convergent power series in the remaining variables.
(b)* Is any complex analytic hypersurface singularity algebraic, i.e., analytically isomorphic to a singularity that can be defined by a polynomial?

(35)** *Plain varieties.* Call a complex algebraic variety *plain* if it is everywhere locally (in the Zariski topology) isomorphic to an open subset of affine space. Is plain equivalent to smooth and rational?

(36)** *Strong factorization.* Show that any projective birational morphism between non singular varieties is a sequence of blowups along non singular subvarieties followed by a sequence of blowdowns to non singular subvarieties.

(37)** *Normalization.* Find a refinement of the notion of integral closure of rings and normal varieties which ensures smoothness in codimension 2.

(38) *Multiplicity and localization.* Let (R, m) be a regular local ring and I a non zero ideal in R. Let $\widehat{R}$ denote the completion of R and set $\widehat{I} = I \cdot \widehat{R}$. For a prime ideal p in R, denote by S the localization of R at p, and by J the ideal generated by I in S. Define the order $\mathrm{ord}(I)$ of I as the maximal power of m containing I, and similarly for J.
(a) Show that $\mathrm{ord}(\widehat{I}) = \mathrm{ord}(I)$.
(b)* Show that $\mathrm{ord}(J) \leq \mathrm{ord}(I)$.

(39)* *Adjoints.* Let $X \subset \mathbb{P}^n$ be a projective hypersurface over a field K, with structure sheaf $\mathcal{O}_X$. Compute the subsheaf $\mathcal{A} \subset \mathcal{O}_X$ of adjoint functions, without computing a resolution of the singularities of X.

(40) *Topology of plane curve singularities.* Intersecting a complex curve singularity $(X,0) \subset (\mathbb{C}^2,0)$ with a sufficiently small ε-sphere around p, one obtains a link that classifies the topological type of the singularity. The links that can be obtained in this way are called algebraic links.
(a)** Can the link become non-algebraic when the radius of the sphere becomes larger?
(b)** Assume that $(X,0)$ intersects a sphere with radius $\varepsilon > 0$ in an algebraic link L. Can we continously deform $(X,0)$ to a curve singularity with link L, such that the induced deformation of the intersection with the sphere is an isotopy of links?

For the bus-ride. There are only finitely many points in the real plane $\mathbb{R}^2$ with integer distance to three given points not on a line.

Notes

(1) The ring $K[A]$ defines an affine toric, not necessarily normal variety. The purpose of the exercise is to establish resolution for these without passing to polyhedral cones, fans and subdivisions, as is done classically. The map C will prescribe the center, the map η chooses the chart of the blowup. The transformation rule between A and A' is a version of the Euclidean division for integer vectors.

Example: $A = \{(1,1),(1,0),(0,2)\}$ so that $K[A] = K[st, s, t^2]$ is non-regular. If $C_A = A$ and $\eta_A = (0,2)$, then $K[A'] \cong K[A]$, whereas for $C_A = \{(1,1),(1,0)\}$ and arbitrary η_A, $K[A']$ is regular.

References: Cox, D., Little, J., Schenck, H.: Toric Geometry. Forthcoming.
Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings. Lecture Notes in Math. 339, Springer 1973.

(2) By the Mather-Yau Theorem and its generalization by Gaffney-Hauser, any germ of a complex analytic hypersurface (of isolated singularity type) is determined up to analytic isomorphism by its singular locus, provided this locus is considered with the non reduced structure given by the ideal generated by the defining function and its partial derivatives. Therefore, an answer to the problem also yields a classification of hypersurface singularities for which there is purely combinatoric description. The simplest examples of singularities with monomial Jacobian ideal are Brieskorn singularities $x_1^{a_1} + \ldots + x_n^{a_n} = 0$.

References: Gaffney, T., Hauser, H.: Characterizing singularities of varieties and of mappings. Invent. Math. 79 (1985), 427 - 447 .
Hauser, H., Müller, G.: Harmonic and dissonant singularities. In: Proc. Conf. Algebraic Geometry, Berlin 1985 (ed. Kurke et al.), 123 - 134. Teubner 1986.
Mather, J., Yau, S. S.-T.: Classification of isolated hypersurface singularities by their moduli algebras. Invent. Math. 69 (1982), 243 - 251.

(3) An answer to (c) was given by Howald in terms of Newton polyhedra. An alternative, analytic proof of this characterization was found by McNeal and Zeytuncu.

References: Howald, J.: Multiplier ideals of monomial ideals. Trans. Amer. Math. Soc. 353 (2001), 2665 - 2671.

Lejeune-Jalabert, M., Teissier, B.: Clôture intégrale des idéaux et équisingularité. Ann. Fac. Sci. Toulouse Math. 17 (2008), 781 - 859. With an appendix by J.-J. Risler.
Lipman, J., Teissier, B.: Pseudo-rational singularities and a theorem of Briançon-Skoda about integral closures of ideals. Michigan Math. J. 28 (1981), 97 - 116.
McNeal, J., Zeytuncu, Y.: Multiplier ideals and integral closure of monomial ideals: an analytic approach. Preprint 2010, Schrödinger Institute Vienna.

(4) Generators of ideals as in (e), but taking instead of the initial monomials the initial forms with respect to the natural grading, were called by Hironaka standard basis (and are called nowadays Macaulay basis). Hironaka developed the extension of the Weierstrass Division Theorem to ideals as in (d) in order to construct reduced standard bases whose orders were then used to define his invariant ν^*. At about the same time, Grauert established the Division Theorem for convergent power series, using it for the construction of the semi-universal deformation of an isolated singularity. Proofs for (c) and (f) can be found in Hauser's paper. The inequality of (f) is the analog of Bennett's Theorem asserting the non-increase of the Hilbert-Samuel function under blowup with center along which normal flatness holds.

References: Bennett, B.: On the characteristic function of a local ring. Ann. Math. 91 (1970), 25 - 87.
Hauser, H.: Three power series techniques. Proc. London Math. Soc. 88 (2004), 1 - 24.
Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79 (1964), 109 - 326.
Singh, B.: Effect of permissible blowing up on the local Hilbert function. Invent. Math. 26 (1974), 201 - 212.

(5) Counting lattice points in simplices and their translates appears in the theory of wild singularities and kangaroo points. These singularities represent one of the main obstructions to the resolution of singularities in positive characteristic. See also Abhyankar's notion of good points.

References: Abhyankar, S: Good points of a hypersurface. Adv. Math. 68 (1988), 87 - 256.
Hauser, H.: Wild singularities and kangaroo points for the resolution of singularities in positive characteristic. Preprint 2009.

(6) Newton polyhedra have been studied by Hironaka (and others) in order to control the improvement of singularities under suitable blowups. A hypersurface singularity has normal crossings if and only if its Newton polyhedron is in some local coordinates an orthant. Hypersurface singularities whose Newton polyhedron has a compact facet in all local coordinate systems can be seen as far away from being normal crossings. It is not clear how to define an invariant which is able to capture this distance from normal crossings in a coordinate independent manner and which improves under blowup until all compact faces have disappeared. Considering only monomial blowups in fixed coordinates, such a measure can be defined and used to prove (b).

References: Bruschek, C., D. Wagner, D.: Some constructions in the étale topology. Preprint 2010.
Cossart, V.: Sur le polyèdre caractéristique. Thèse d'État. Univ. Paris-Sud, Orsay 1987.
Hauser, H.: Three power series techniques. Proc. London Math. Soc. 88 (2004), 1 - 24.
Hauser, H., Wagner, D.: Alternative invariants for the embedded resolution of surfaces in positive characteristic. Preprint 2009.

Hironaka, H.: Characteristic polyhedra of singularities. J. Math. Kyoto Univ. 7 (1967), 251 - 293.
Hironaka, H.: Certain numerical characters of singularities. J. Math. Kyoto Univ. 10 (1970), 151 - 187.
Youssin, B.: Newton polyhedra without coordinates. Memoirs Amer. Math. Soc. 433 (1990), 1 - 74, 75 - 99.

(7) The polyhedral game appears in various disguises within resolution of singularities, with various proposals how to win it, the first going probably back to Zariski. When Hironaka formulated his polyhedral game – with a succinct winning strategy proposed soon afterwards by Spivakovsky – it was quickly seen that this game is too coarse to imply resolution of singularities. There then appeared a more exigent version of the game, the hard polyhedral game, which turned out to be too hard – it does not admit a winning strategy. It is believed today that resolution of singularities cannot be reduced to a purely combinatoric problem.

References: Spivakovsky, M.: A solution to Hironaka's Polyhedra Game. In: Arithmetic and Geometry. Papers dedicated to I.R. Shafarevich, vol. II (eds. M. Artin, J. Tate). Birkhäuser 1983, 419 - 432.
Spivakovsky, M.: A counterexample to Hironaka's 'hard' polyhedra game. Publ. RIMS 18 (1983), 1009 - 1012.
Zeillinger, D.: A short solution to Hironaka's polyhedra game. L'Enseign. Mathém. 52 (2006), 143 - 154.

(8) Clearly, (b) is an easy consequence of (a). In (c), the singular locus is equipped with the Jacobian ideal, generated by the partial derivatives of the defining function. A divisor is free if the module of analytic vector fields on $(\mathbb{C}^n, 0)$ which are tangent to the divisor is a free module. Such divisors have been introduced and studied by Saito. Tangent vector fields are also called logarithmic, being dual to the logarithmic differential forms. They appear in Deligne's work on regular singular points of differential equations.
As for (a), one can consider the normalization of the divisor defined by the polynomial. We have irreducibility if and only if the preimage of zero under the normalization map has exactly one closed point. A more direct approach is the Artin Approximation Theorem. There exists, for any integer m and any polynomial f, a number k such that if $f \equiv \overline{g} \cdot \overline{h}$ modulo terms of degree $> k$ for some polynomials $\overline{g}$ and $\overline{h}$ then $f = g \cdot h$ for convergent series g and h coinciding up to degree m with $\overline{g}$ and $\overline{h}$. The problem is to determine the bound k explicitly in terms of m and f. This is related to the more general problem of the existence and the computability of the Artin function.
Problem (e) is related to the integral of an ideal studied by Pelikaan. This is the ideal of functions all whose derivatives belong to the given ideal, or, alternatively, the symbolic square of the ideal.

References: Artin, M.: On the solution of analytic equations. Invent. Math. 5 (1968), 277 - 291.
Deligne, P.: Equations differentielles à points singuliers réguliers. Lecture Notes in Math. 163. Springer 1970.
Hickel, M.: Un cas de majoration affine pour la fonction d'approximation d'Artin. C. R. Acad. Sci. Paris 346 (2008), 753 - 756.
Pelikaan, R.: Finite determinacy of funtions with non-isolated singularities. Proc. London Math. Soc. 57 (1988), 357 - 382.

Saito, K.: Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo 27 (1980), 265 - 291.
Spivakovsky, M.: Non-existence of the Artin function for Henselian pairs. Math. Ann. 299 (1994), 727 - 729.
Wavrik, J.: A theorem on solutions of analytic equations with applications to deformations of complex structures. Math. Ann. 216 (1975), 127 - 142.

(9) This problem is related to problem (6). Any reasonable resolution invariant should not increase under point blowup (though, in general, it will only decrease if the center of the blowup is sufficiently large). Most proposals in the literature try to capture the monomiality of a function by factoring it into a monomial part (usually given as the defining equation of the exceptional divisor produced by earlier blowups) and a singular non monomial part, whose multiplicity is then taken as the required measure. It is known by examples of Moh that in positive characteristic this invariant does not behave well under descent in dimension (i.e., when passing to coefficient ideals) and taking the transform of the coefficient ideal under blowup.
Already for plane curves the problem is interesting and consists in finding a substitute for the usual invariant given by the lexicographic pair formed by the local multiplicity and the maximal slope of the fist segment of the Newton polygon.

References: Moh, T.-T.: On a stability theorem for local uniformization in characteristic p. Publ. RIMS 2 (1987), 965 - 973.
Moh, T.-T.: On a Newton polygon approach to the uniformization of singularities of characteristic p. In: Algebraic Geometry and Singularities. Proc. Conference on Singularities, La Rábida. Birkhäuser 1996.

(10) Mikado is a natural generalization of normal crossings. It is inspired by the geometry of hyperplane arrangements. Clearly, any union of linear spaces is mikado, as well as any union of smooth plane curves meeting pairwise transversally. An answer to (a) was given by Whitney, taking unions of four lines in the plane passing through a given point, with different cross ratios.

References: De Concini, D., Procesi, C.: Wonderful models of subspace arrangements. Selecta Mathematica 3 (1995), 459 - 494.
Li, L.: Wonderful compactifications of arrangements of subvarieties. Michigan Math. J. 58 (2009), 535 - 563.
Faber, E., Hauser, H.: Today's Menu: Geometry and resolution of singular algebraic surfaces. Bull. Amer. Math. Soc. 47 (2010), 373 - 417.

(11) Make first precise what could be meant by transversal (one possible option is that the union of the variety and the center is again mikado). The instability of mikado under blowup was observed by Li Li.

Reference: Li, L.: Wonderful compactifications of arrangements of subvarieties. Michigan Math. J. 58 (2009), 535 - 563.

(12) Compare this with the resolution of hyperplane arrangements, respectively wonderful models as proposed by Li Li. From a geometric viewpoint, mikado singularities are suitable final forms for the resolution of singular varieties, but algebraically, normal crossings are much easier to handle. The problem exhibits this difference.

Reference: Goward, G.: A simple algorithm for the principalization of monomial ideals. Trans. Amer. Math. Soc. 357 (2005), 4805 - 4812.
Li, L.: Wonderful compactifications of arrangements of subvarieties. Michigan Math. J. 58 (2009), 535 - 563.

(13) As an embedded resolution aims at transforming all singularities into normal crossings it is natural to expect that this can be achieved by blowups along centers which lie inside the non normal crossings locus. However, as accentuated by Kollár, there is a subtle difference between the algebraic and analytic setting.

Reference: Kollár, J.: Semi log resolutions. arXiv:0812.3592v1.

(14) The conjecture was proposed by Casas-Alvero. The common divisors may a priori be different for each derivative. A proof for polynomials of prime degree (and several more cases) was given by Graf von Bothmer, Labs, Schicho and van de Woestijne. For positive characteristic, there exist easy counterexamples.

References: Casas-Alvero, E.: Singularities of plane curves. London Math. Soc. Lect. Notes 276, Cambridge Univ. Press 2000.
Graf von Bothmer, H.C., Labs, O., Schicho, J., van de Woestijne, C.: The Casas-Alvero conjecture for infinitely many degrees. J. Algebra 316 (2007), 224 - 230.

(15) Following an idea of R. Bryant, you may try for (a) with a quasi-homogeneous equation f, using that f belongs to $m \cdot j(f)$, where m denotes the maximal ideal generated by the coordinates and $j(f)$ the Jacobian ideal.

Reference: Faber, E., Hauser, H.: Today's Menu: Geometry and resolution of singular algebraic surfaces. Bull. Amer. Math. Soc. 47 (2010), 373 - 417.

(16) By local symmetry of order 2 we understand that the singularity is locally at 0 analytically isomorphic to a hypersurface which is invariant under exchanging x and y. This type of singularities causes problems when trying to resolve them, because one first has to blow up the origin in order to separate the two local branches of the singular locus. Due to the symmetry, it would not be natural to select one branch of the locus as center. Moreover, as the center must be closed, the local choice of one branch would produce the whole node as center, with singularity at 0.
Allowing centers with mild singularities (e.g., normal crossings) could be convenient for resolution purposes. Rosenberg and Faber-Westra have described suitable non reduced ideals defining normal crossings and giving smooth blowups.

References: Hauser, H.: Excellent surfaces and their taut resolution. In: Resolution of Singularities, Progress in Math. 181, Birkhäuser 2000.
Rosenberg, J.: Blowing up non reduced subschemes of $\mathbb{A}^n$. Unpublished manuscript 1998.
Faber, E., Westra, D.: Blowups in tame monomial ideals. arXiv:0905.4511v1.

(17) Use the Theorem of Mather-Yau and Gaffney-Hauser.

References: Gaffney, T., Hauser, H.: Characterizing singularities of varieties and of mappings. Invent. Math. 79 (1985), 427 - 447 .
Mather, J., Yau, S. S.-T.: Classification of isolated hypersurface singularities by their moduli algebras. Invent. Math. 69 (1982), 243 - 251.

(18) A typical example is the cone $x^2 + y^2 = z^2$ whose resolution is the cylinder $x^2 + y^2 = 1$ induced by the affine map $\mathbb{A}^3_{\mathbb{R}} \to \mathbb{A}^3_{\mathbb{R}}$ sending (x, y, z) to (xz, yz, z). The respective smoothing

is given by the family $x^2+y^2=t+z^2$. A more interesting example is the A_n singularity $x^2-y^2=z^{n+1}$ with the "real smoothing" $x^2-y^2=z(z-t)(z-2t)\cdots(z-nt)$. The exceptional divisors are the compact components of the intersections with the planes $x=0$ and $y=0$.
A necessary criterion for a resolution to be affine real in the above sense is that the self intersection numbers of the real exceptional divisors are even. This follows from the fact that smooth hypersurfaces in $\mathbb{R}^3$ are orientable, so they cannot contain closed curves with an odd self intersection number. This also shows that any affine real resolution is real isomorphic to the minimal resolution.

Reference: Hacking, P., Prokhorov, Y.: Smoothable Del Pezzo surfaces with quotient singularities. Compos. Math. 146 (2010), 169 - 192.

(19) This is amusing and easy. However, it is not possible to specify the stage where the drop must occur. It is not clear whether the statement extends to higher dimension.

Reference: Hauser, H., Wagner, D.: Alternative invariants for the embedded resolution of surfaces in positive characteristic. Preprint 2009.

(20) If X and Y are smooth, just apply principalization to the ideal I_X+I_Y defining the intersection $X\cap Y$. In general, take a suitably weighted sum $I_X^a+I_Y^b$ of I_X and I_Y so that both powers of the ideals have the same maximal local multiplicity (suggestion of H. Hironaka and O. Villamayor).

References: Encinas, S., Hauser, H.: Strong resolution of singularities in characteristic zero. Comment. Math. Helv. 77 (2002), 421 - 445.
Hironaka, H.: Idealistic exponents of singularity. In: Algebraic Geometry, The Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977.
Villamayor, O.: Patching local uniformizations. Ann. Scient. Éc. Norm. Sup. Paris 25 (1992), 629 - 677.

(21) The first jet space is just the tangent bundle, and the resulting modification corresponds to the blowup with center the Jacobian ideal. For higher jet spaces, it is a priori not clear how to projectivize them suitably, i.e., how to take limits of jets.

References: Gonzalez-Springberg, G.: Désingularisation des surfaces par des modifications de Nash normalisées. Sém. Bourbaki 1985/86. Astérisque 145-146 (1987), 187 - 207.
Hironaka, H.: On Nash blowing-up. In: Arithmetic and Geometry II. Progr. Math. 36. Birkhäuser 1983, 103 - 111.
Moody, J.A.: On resolving singularities. J. London Math. Soc. 64 (2001), 548 - 564.
Nobile, A.: Some properties of the Nash blowing-up. Pacific J. Math. 60 (1975), 297 - 305.
Spivakovsky, M.: Sandwiched singularities and desingularization of surfaces by normalized Nash transformations. Ann. Math. 131 (1990), 411 - 491.
Yasuda, T.: Higher Nash blowups. Compos. Math. 143 (2007), 1493 - 1510.

(22) The classical descent via hypersurfaces of maximal contact is necessarily local, by the only local existence of these hypersurfaces. Włodarczyk showed how to make the local descents intrinsic up to analytic isomorphisms by using homogenized coefficient ideals. Using modules of derivations and differential operators there is a certain chance to construct globally defined objects which give a substitute for the local descent.

Reference: Hironaka, H.: Theory of infinitely near singular points. J. Korean Math. Soc. 40 (2003), 901 - 920.

(23) The first example was given by Narasimhan, of equation $x^2+y^3z+zw^3+wy^7=0$ over a field of characteristic 2. The equimultiple locus is the curve parametrized by $(t^{32}, t^7, t^{19}, t^{15})$. Note that this example also shows that there need not always exist locally at a singular point a smooth hypersurface whose transforms contain all points where the local multiplicity has remained constant. Namely, any such hypersurface through 0 would get separated from the above curve under a sequence of point blowups, while the multiplicity must remain the same for semicontinuity reasons (it remains constant along the curve).

$References$: Hauser, H.: Seventeen obstacles for resolution of singularities. In: The Brieskorn Anniversary Volume. Progress in Math. 162, Birkhäuser 1997.
Mulay, S.: Equimultiplicity and hyperplanarity. Proc. Amer. Math. Soc. 87 (1983), 407 - 413.
Narasimhan, R.: Monomial equimultiple curves in positve characteristic. Proc. Amer. Math. Soc. 89 (1983), 402 - 413.
Narasimhan, R.: Hyperplanarity of the equimultiple locus. Proc. Amer. Math. Soc. 87 (1983), 403 - 406.

(24) Taking the maximum order of a representative of elements in $K[[x]]/K[[x]]^p$ yields an invariant which is not upper semicontinuous (as observed, among others, by Hironaka).

$References$: Hauser, H., Wagner, D.: Alternative invariants for the embedded resolution of surfaces in positive characteristic. Preprint 2009.
Hironaka, H.: Program for resolution of singularities in characteristics $p > 0$. Notes from lectures at the Clay Mathematics Institute, September 2008.

(25) There have been important recent advances by Cutkosky and Abramovich, Karu, Matsuki and Włodarcyzk. In positive characteristic there are simple counterexamples.

$References$: Abramovich, D., Karu, K., Matsuki, M., Jaroslaw Włodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. 15 (2002), 531-572.
Cutkosky, D.: Local monomialization and factorization of morphisms. Astérisque 260 (1999).
Cutkosky, D.: Toroidalization of Dominant Morphisms of 3-Folds. Memoirs Amer. Math. Soc. 890 (2007).
Cutkosky, D.: Monomialization of Morphisms from 3-folds to surfaces, Lecture Notes in Math. 1786, Springer 2002.

(26) A formal power series is called algebraic or Nash if it is algebraic over the polynomial ring, i.e., satisfies a polynomial equation in one variable with polynomial coefficients. The simplest example is $\sqrt{1+x}$. For (a), you may want to find a suitable dominating series. For (b), normalize the variety defined by the minimal polynomial of h and use Zariski's Main Theorem.

$References$: Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Springer 1998.
Lafon, J.P.: Séries formelles algébriques. C. R. Acad. Sci. Paris 260 (1965), 3238 - 3241.
Raynaud, M.: Anneaux locaux henséliens. Lecture Notes in Math. 169. Springer 1970.
Ruiz, J.: The basic theory of power series. Vieweg 1993.

(27) A formal power series in one variable is called D-finite if it satisfies an ordinary differential equation with polynomial coefficients. For several variables, the notion is less studied.

$References$: Denef, J., Lipshitz. L.: Power series solutions of algebraic differential equations. Math. Ann. 267 (1984) 213 - 238.

Denef, J., Lipshitz. L.: Algebraic power series and diagonals. J. Number Theory 26 (1987), 46 - 67.
Stanley, R.: Differentiably finite power series, European J. Combin. 1 (1980), 175 - 188.

(28) Assertion (a) was proven by Hironaka and Rossi before Artin had proved his approximation theorem (which, obviously, gives (a)). Artin's proof relies on a multiple use of the Weierstrass Division Theorem. This involves in each application an analytic coordinate change which need not maintain the nested subring or the separatedness condition. Popescu's and Spivakovsky's proofs for systems of algebraic equations whose solutions satisfy the nested subring condition are very involved.

References: André, M.: Artin's theorem on the solution of analytic equations in positive characteristic. Manuscripta Math. 15 (1975), 314 - 348.
Artin, M.: On the solution of analytic equations. Invent. Math. 5 (1968), 277 - 291.
Artin, M.: Algebraic approximation of structures over complete local rings. Publ. math. IHES 36 (1969), 23 - 58.
Becker, J.: A counterexample to Artin approximation with respect to subrings. Math. Ann. 230 (1977), 195 - 196.
Gabrielov, A.: The formal relations between analytic functions. Funkts. Anal. Prilozh. 5 (1971), 64 - 65.
Greenberg, M.: Rational points in Henselian discrete valuation ring. Publ. Math. IHES 31 (1966), 59 - 64.
Hironaka, H., Rossi, H.: On the equivalence of imbeddings of exceptional complex spaces, Math. Ann. 156 (1964), 313 - 333.
Ogoma, T.: General Néron desingularization based on the idea of Popescu. J. Algebra 167 (1994), 57 - 84.
Popescu, D. : General Néron desingularization. Nagoya Math. J. 100 (1985), 97 - 126.
Popescu, D. : General Néron desingularization and approximation. Nagoya Math. J. 104 (1986), 85 - 115.
Popescu, D. : Polynomial rings and their projective modules. Nagoya Math. J. 113 (1989), 121 - 128.
Spivakovsky, M.: A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. J. Amer. Math. Soc. 12 (1999), 381 - 444.
Teissier, B.: Résultats récents sur l'approximation des morphisms en algèbre commutative (d'après Artin, Popescu, André, Spivakovsky). Sém. Bourbaki 784 (1994), 1 - 15.
Wavrik, J.: A theorem on solutions of analytic equations with applications to deformations of complex structures. Math. Ann. 216 (1975), 127 - 142.

(29) According to Hironaka, the example from (a) is, independently, due Gabber and Kashiwara. It yields a lacunary series as the remainder of the division which therefore is not algebraic. This and similar examples were studied in combinatorics by Bousquet-Mélou and Petkovšek as generating functions for the number of walks in lattices. Lafon proved (b), and Hironaka extended the division to ideals of algebraic series in the sense of the Grauert-Hironaka-Galligo Division Theorem, assuming that the initial ideal satisfies a natural generalization of x_n-regularity.

References: Bousquet-Mélou, M., Petkovšek, M.: Linear recurrences with constant coefficients: the multivariate case. Discrete Mathematics, 225 (2000), 51 - 75.

Hironaka, H.: Idealistic exponents of singularity. In: Algebraic Geometry, The Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977.
Lafon, J.P.: Séries formelles algébriques. C. R. Acad. Sci. Paris 260 (1965), 3238 - 3241.

(30) The assertion is known as Grothendieck's p-curvature conjecture and is still wide open. The name is due to the formulation of the conjecture in terms of the Galois group of the differential equation. In the discrete case, i.e., difference equations, there is a formulation and proof of the conjecture by Di Vizio.

References: Katz, N.: A conjecture in the arithmetic theory of differential equations. Bull. Soc. Math. France, 110 (1982), 203 - 239 and 347 - 348.
Di Vizio, L.: On the arithmetic theory of q-difference equations. The q-analogue of the Grothendieck-Katz's conjecture on p-curvatures. Invent. Math. 150 (2002), 517 - 578.

(31) The symmetries of finite order are those automorphisms one can "see". If the singularity X is embedded in $(\mathbb{C}^n, 0)$ one may also consider the subgroup of $\mathrm{Aut}(\mathbb{C}^n, 0)$ of ambient analytic automorphisms fixing X. This is an infinite dimensional Lie group whose Lie algebra consists of the germs of analytic vector fields on $(\mathbb{C}^n, 0)$ which are tangent to X. It was shown by Hauser and Müller that for $n \geq 3$ the group as well as its Lie algebra determine X up to isomorphism.
The automorphisms of finite order disappear when passing to the Lie-algebra and thus cannot de detected by the tangent vector fields.

References: Hauser, H., Müller, G.: A rank theorem for analytic maps between power series spaces. Publ. Math. IHES 80 (1994), 95 - 115.
Hauser, H., Müller, G.: Affine varieties and Lie algebras of vector fields. Manuscr. Math. 80 (1993), 309 - 337.
Hauser, H., Müller, G.: Automorphisms of direct products of algebroid spaces. In: Singularity Theory and its Applications. Warwick 1989, Part I Springer Lecture Notes in Math. 1462, 1991.

(32) The problem is one of the main challenges in Cauchy-Riemann geometry, and even for manifolds no complete answer is known.

References: Chern, S., Moser, J.: Real hypersurfaces in complex manifolds. Acta Math. 133 (1974), 219 - 271.
Lamel, B.; Mir, N.: Parametrization of local CR automorphims by finite jets and applications. J. Amer. Math. Soc. 20 (2007), 519 - 572.

(33) The equation $x^{2/3} + y^{2/3} = 1$ with rational exponents can be replaced by a polynomial equation by raising it twice to the third power. In three variables, it is much less obvious how to find the polynomial equation for $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

References: Fritz, A., Hauser, H.: Platonic Stars. Math. Intelligencer (2010), to appear.

(34) (a) is easy for isolated singularities. The general case of (a) is due to Shiota. Question: Is it possible to define an even more precise normal form, e.g., f having tail (= f minus initial monomial) in a direct monomial complement of its Jacobian ideal? Compare this with Arnold's classification of simple singularities.

References: Arnold, V.: Singularity Theory. Cambridge University Press.
De Jong, T., Pfister, G.: *Local Analytic Geometry.* Advanced Lectures Math., Vieweg 2000.

Shiota, M.: Equivalence of differentiable mappings and analytic mappings. Publ. Math. IHES 54 (1981), 37 - 122.

(35) Clearly, every plain variety is smooth and rational. The converse is true for curves and surfaces. In view of the Weak Factorization Theorem of Włodarczyk, it is sufficient to show that if the locus of points where the variety is not locally isomorphic to affine space has codimension ≥ 2, then the variety is plain.

References: Abramovich, D., Karu, K., Matsuki, M., Włodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. 15 (2002), 531 - 572.
Bodnár, G., Hauser, H., Schicho, J., Villamayor, O.: Plain varieties. Bull. London Math. Soc. 40 (2008), 965 - 971.
Włodarczyk, J.: Toroidal varieties and the weak factorization theorem. Invent. Math. 154 (2003), 223 - 331.

(36) The factorization by an arbitrary sequence of blowups and blowdowns, known as the Weak Factorization Theorem, was proven by Włodarczyk, extending ideas of Morelli. For surfaces, see Hartshorne V.5.4.

References: Abramovich, D., Karu, K., Matsuki, K., Włodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. 15 (2002), 531 - 572.
Cutkosky, D.: Local factorization of birational maps. Adv. Math. 132 (1997), 167 - 315.
Cutkosky, D.: Local monomialization and factorization of morphisms. Astérisque 260 (1999).
Hartshorne, R.: Algebraic Geometry. Springer 1977.
Karu, K.: Local strong factorization of birational maps. J. Alg. Geom. 14 (2005), 165 - 175.
Morelli, R., The birational geometry of toric varieties, J. Alg. Geometry 5 (1996), 751 - 782.
Włodarczyk, J., Toroidal varieties and the weak factorization theorem. Invent. Math. 154 (2003), 223 - 331.

(37) Let A be a finitely generated K-algebra which is a domain, and let A' be its integral closure in the quotient field of A. If A is the coordinate ring of a variety X, A' describes the normalization X' of X. It is well known (see e.g. Mumford) that the singular locus of X' has codimension ≥ 2 in X.
A positive answer to the problem would yield a one step resolution of surfaces, as the normalization does for curves. It is conceivable that such a concept had again to do with extension properties as is the case for weakly holomorphic functions on normal varieties. Note here that the extension or integration of differential forms in the context of the singular Frobenius Theorem often requires that the exceptional set has codimension ≥ 3.

References: De Jong, T., Pfister, G.: *Local Analytic Geometry.* Advanced Lectures Math., Vieweg 2000.
Malgrange, B.: Frobenius avec singularités. I. Codimension un. Publ. Math. IHES 46 (1976), 163 - 173.
Moussu, R.: Sur l'existence d'intégrales premières. Ann. Inst. Fourier 26 (1976), 171 - 220.
Mumford, D.: The Red Book of Varieties and Schemes. Lecture Notes Math. 1358, Springer 1999.
Zariski, O., Samuel, P.: Commutative Algebra. Graduate Texts Math. 28, Springer 1975.

(38) The inequality of (b) is due to Zariski. To prove it, reduce first to the case where R/p has dimension 1 and then use (a) and resolution of curves. You may also consult Hironaka, Thm. III. 3.1.

References: Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79 (1964), 109 - 326.

(39) A regular function f on a hypersurface X defined by $g = 0$ is an adjoint if the pullback of the differential form $\frac{f dx_1 \wedge \ldots \wedge dx_n}{\partial_{x_n} g}$ to a resolution of X is regular. This condition is independent of the choice of the resolution. The arithmetic and geometric genus of X can be read off from the Hilbert function of the sheaf $\mathcal{A}$ of adjoint functions. Similarily, the plurigenera of X are related to higher adjoint sheaves. The sections of the twisted adjoint sheaves can be used to define canonical maps and adjoint maps which play a prominent role in Mori theory.
For surfaces, there is an efficient algorithm for computing adjoints, available in the computer algebra system Magma. It uses an embedded resolution of the discriminant of X, which is a plane curve.

References: Beck, T., Schicho, J.: Adjoint computation for hypersurfaces using formal desingularizations. J. Algebra 320 (2008), 3984 - 3996.
Blass, P., Lipman, J.: Remarks on adjoints and arithmetic genera of algebraic varieties. Amer. J. Math. 101 (1979), 331 - 336.

(40) If the answer to (b) would be yes, then one could say something about a curve singularity based on the topogical behavior of the curve in distance $\epsilon > 0$, namely that there exists a curve with a certain link which is sufficiently close.

References: Hodorog, M. Schicho, J.: A symbolic-numeric algorithm for genus computation. Preprint, 2010.
Yamamoto, M.: Classification of isolated algebraic singularities by their Alexander polynomials. Topology 23 (1984), 277 - 287.

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