CYCLES OF SINGULARITIES APPEARING IN THE RESOLUTION PROBLEM IN POSITIVE CHARACTERISTIC

HERWIG HAUSER, STEFAN PERLEGA

ABSTRACT. We present a hypersurface singularity in positive characteristic which is defined by a purely inseparable power series, and a sequence of point-blowups so that, after applying the blowups to the singularity, the same type of singularity reappears after the last blowup, with just certain exponents of the defining power series shifted upwards. The construction hence yields a cycle. Iterating this cycle leads to an infinite increase of the residual order of the defining power series. This disproves a theorem claimed by Moh about the stability of the residual order under sequences of blowups. It is not a counter-example to the resolution in positive characteristic since larger centers are also permissible and prevent the phenomenon from happening.

1. INTRODUCTION

The Whitney-umbrella is the surface X in $\mathbb{A}^3$ defined by the equation

$$x^2 - y^2 z = 0.$$

Its singular locus is the z-axis, and 0 is the worst singularity; at the other points of the z-axis, the singularity has normal crossings and consists of two smooth transversal branches. A simple computation shows that blowing up $\mathbb{A}^3$ with center the origin 0 produces a strict transform X' of X which has at the origin of the z-chart the same singularity as X at 0. Thus a cycle occurs, and no improvement of the singularity is achieved. It turns out that the appropriate choice of center is the whole z-axis in $\mathbb{A}^3$: the respective blowup resolves the singularities of X in one stroke.

Consider more generally a surface in $\mathbb{A}^3$ of equation $x^a - y^b z^c = 0$ with $a < b, c$. Blowing up again the origin produces in the z-chart the equation $x^a - y^b z^{c+b-a} = 0$. As $c' = c + b - a > c$ the singularity has got worse, while the shape of the equation is the same. This type of repetition will be called a *cycle under blowup*, with *shifted* exponents (the z-exponent c transforms into c', the others remain constant).

In characteristic zero, the occurrence of cycles under blowup can always be avoided by the correct choice of the center: There is a systematic way via hypersurfaces of maximal contact and respective coefficient ideals to determine an appropriate (regular) center so that the induced blowup does improve the singularity [Hir64, Vil89, Vil92, BM97, Wło05, EH02]. This is the way to prove resolution of singularities in zero characteristic (up to a bunch of technicalities).

The situation is much more subtle in the case of positive characteristic $p > 0$. We will produce, for any p, examples where cycles of singularities appear after a sequence of blowups in a much more involved fashion. The examples exhibit a new difficulty which adds to the many already known problems one encounters in

MSC-2000: 14B05, 14E15, 12D10. Supported by project P-25652 of the Austrian Science Fund FWF.

positive characteristic. The blowups in the examples have always point centers. Taking larger centers would prevent the phenomenon to happen. We were not able to construct examples with cycles where the choice of point centers is forced (e.g., because the singularities are isolated). Of course, such an example would disprove the existence of resolution of singularities in positive characteristic.

The examples are of a relative simple form, namely purely inseparable power series of the form

$$f = z^{p^e} + x^r y^s w^t (w^d y^b + x^a v^c) + \dots,$$

up to higher order terms. The exponents and the sequence of blowups have to be chosen very carefully to produce a cycle. After all these blowups, the transformed equation has the same shape, with certain exponents shifted *upwards*. This prohibits to show that some improvement has occurred.

The rest of this section recalls the main ingredients of resolution proofs which are used in the discussion of the examples; it can be skipped by the expert reader.

Let X be a singular variety embedded in a regular ambient variety W. For every point a of X, one may choose locally at a a defining ideal J for X. The order $\operatorname{ord}_a X := \operatorname{ord}_a J$ of J at a is one of the main numerical ingredients for the construction of the resolution of the singularities of X: it stratifies X into a disjoint union of finitely many locally closed strata, the smallest one collecting the worst singularities of X (we assume here that the embedding of X in W is minimal). Blowing up regular centers inside this stratum, one then aims at lowering the maximum of the local orders by a suitable sequence of blowups. The key observation here is that this maximum cannot increase under such a *permissible* blowup. If it drops, one may apply induction to achieve a resolution. If it remains constant, an extra argument is necessary to proceed. Typically, one then tries to apply induction on the ambient dimension: Choosing suitable regular hypersurfaces V in W locally at the points of the worst stratum, one associates to the variety X ideals K in V whose complexity reflects the original singularity and whose resolution simplifies sufficiently X so as to allow likewise its resolution.

In characteristic zero, this program works due to the existence of hypersurfaces of *maximal contact*. They ensure that all local constructions patch and that the local descent in dimension to the ideals K commutes with blowup at those points of the exceptional divisor where the multiplicity of X has remained constant.

For the inductive argument it is necessary to factor the local ideals K into products $K = M \cdot I$, where M is a principal monomial ideal defining the exceptional divisor produced by the earlier blowups in the resolution process. The residual factor I carries the relevant information on the complexity of X locally at the respective point of the stratum. Its order $\operatorname{ord}_a I$ forms the second component of a local upper semicontinuous invariant $\operatorname{inv}_a X$ of X at a. This invariant consists of a string of orders in descending dimensions,

$$\operatorname{inv}_a X = (\operatorname{ord}_a X, \operatorname{ord}_a I, ...),$$

which then, taken lexicographically, serves as a resolution invariant for X (leaving aside technicalities): Namely, it refines the stratification of X by the order of J, yielding a smallest stratum which is regular, and, so that, when blowing up this stratum, the invariant drops at every point of the exceptional divisor. Then induction applies to establish resolution.

In positive characteristic, this approach meets serious obstructions: First, hypersurfaces of maximal contact need no longer exist. A substitute are regular hypersurfaces maximizing the order of the ideals K and I (which is also the case for hypersurfaces of maximal contact). They permit the definition of $\operatorname{ord}_a I$ as an intrinsic and significant second component of $\operatorname{inv}_a X$, the *residual order* of X at a. The next obstruction then is the fact that in positive characteristic the residual order may increase under blowup at points where $\operatorname{ord}_a X$ has remained constant. However, the increase is not too large, happens very rarely, and can explictly be bounded, as Moh showed [Moh87, Hau10, HP16]. This raised the hope that the invariant $\operatorname{inv}_a X$ may drop *in the long run* and thus serves again to establish resolution. This has been proven to work for surfaces [HW14, HP17] and is still open in higher dimensions. For arbitrary dimensions, Moh claimed in [Moh87] that the invariant cannot increase under permissible blowup beyond a certain bound. It turns out that this claim is false. In fact, we prove in the present note:

There exists a purely inseparable hypersurface singularity X and a sequence of permissible blowups along which the order of X remains constant but for which the orders of the residual ideals I tend to infinity.

The hypersurface and the sequence of blowups were constructed by the second named author (preceded by an earlier and less significant example of the first author). The centers are always points, chosen carefully inside the respective exceptional divisor. After a tricky composition of such blowups the defining equation of X will have transformed into one of exactly the same shape, but with certain exponents and the residual order $\operatorname{ord}_a I$ shifted upwards. The iteration of the blowups then makes the residual order go to infinity.

2. Setting

Let k be an algebraically closed field of characteristic $p > 0$. The kind of singularities that we consider in this article are defined by *purely inseparable equations* of the form

$$f(z, x) = z^{p^e} + F(x_1, \dots, x_n) = 0,$$

where e is a positive integer and $F \in k[[x_1, \dots, x_n]]$ is a power series of order $\operatorname{ord} F \geq p^e$.

The key observation here is that a change of parameters $z_1 = z - g$ with $g \in k[[x_1, \dots, x_n]]$ changes the expansion of f to

$$f = z_1^{p^e} + g(x_1, \dots, x_n)^{p^e} + F(x_1, \dots, x_n).$$

Hence, any p^e-th power that appears in the expansion of F can be eliminated via such a change. This suggests to consider all p^e-th powers in the power series expansion of F as artificial information. We then say that F is *clean* if no p^e-th powers appear in its expansion. Respectively, we refer to a change of parameters $z_1 = z - g$ which eliminates all p^e-th powers from F as *cleaning*.

To measure the improvement of the singularity defined by f under blowup, it is standard to make use of a normal crossings divisor E of the form $E = V(\prod_{i \in \Delta} x_i)$ for some subset $\Delta \subseteq \{1, \dots, n\}$. This divisor consists of the exceptional components produced by earlier blowups in the resolution process.

Consider now a purely inseparable power series $f = z^{p^e} + F(x_1, \ldots, x_n)$ where F is clean, together with $E = V(\prod_{i\in\Delta} x_i)$. Assume that F admits the factorization

$$F(x_1, \ldots, x_n) = \prod_{i\in\Delta} x_i^{r_i} \cdot G(x_1, \ldots, x_n)$$

where $G \in k[[x_1, \ldots, x_n]]$ is a power series and $r_i = \mathrm{ord}_{(x_i)} F$ denotes the order of F along the component $x_i = 0$. We call the order of this power series G the *residual order* of f with respect to the normal crossings divisor E,

$$\text{residual.order}_E\, f = \mathrm{ord}\, G.$$

This is a natural invariant which measures how far, up to multiplication by units in the power series ring, F is away from being a monomial in $x_1, \ldots, x_n$. The residual order and similar numerals as well as the cleaning process have appeared repeatedly, often only implicitly, in the literature on resolution of singularities in positive characteristic [Abh67, Moh87, CP08, CP09, Cut11, BV13, KM]. The terminology residual order was proposed by Hironaka in [Hir12].

Notice that the multiplicities r_i in the factorization $F = \prod_{i\in\Delta} x_i^{r_i} \cdot G$ only have an influence on the residual order of f up to their value modulo p^e. Hence, in the examples, these numbers will be understood as elements of $\mathbb{Z}/p^e\mathbb{Z}$. Consequently, also negative values for r_i will be used. If $r_i \equiv 0 \pmod{p^e}$, the term $x_i^{r_i}$ will be omitted altogether.

Notice further that, if $r_i \geq p^e$, we may blow up the center $z = x_i = 0$ which reduces r_i by p^e and leaves the power series F otherwise unchanged.

One can view the residual order as a generalization of a resolution invariant that is successfully used over fields of characteristic zero. We refer to [EH02, Hau10, HP16] for this interpretation.

An ideal P of the ring $k[[z, x_1, \ldots, x_n]]$ is said to define a *permissible center* of blowup for f and E as above if P is of the form $P = (z, x_i : i \in \Gamma)$ for some subset $\Gamma \subseteq \{1, \ldots, n\}$ and the following two conditions hold:

(1) $f \in P^{p^e}$.
(2) $G \in P^d$, where G is defined via the factorization $F = \prod_{i\in\Delta} x_i^{r_i} \cdot G$ as above and d is the residual order of f.

The maximal ideal $(z, x_1, \ldots, x_n)$ always defines a permissible center.

Now consider a blowup map $\pi : k[[z, x_1, \ldots, x_n]] \to k[[z, x_1, \ldots, x_n]]$ with center P as above and let f' denote the strict transform of f. It is well-known that the inequality

$$\mathrm{ord}\, f' \leq \mathrm{ord}\, f = p^e$$

always holds. If the order of f' is strictly smaller than p^e, the singularity has already significantly improved under this blowup . Hence, we only consider the case where equality holds. There then exists an index $j \in \Gamma$ and constants $t_i \in k$ such that the map π is of the form

$$\begin{aligned}
&\pi(z) = x_j z, \\
&\pi(x_j) = x_j, \\
&\pi(x_i) = x_j(x_i + t_i) \quad \text{for } i \in \Gamma \setminus \{j\}, \\
&\pi(x_l) = x_l \qquad\qquad\quad\, \text{for } l \notin \Gamma.
\end{aligned}$$

If P is the maximal ideal, we will refer to π as a *point-blowup*.

For the examples below, we adapt the following terminology for point-blowups: Only the parameter x_j (referred to as the x_j-chart) and the translations $x_i \mapsto x_i + t_i$ with non-zero constants t_i are mentioned. For instance, the expression

$$x_1\text{-chart},\ x_2 \mapsto x_2 + 1$$

encodes the map π given by $\pi(z) = x_1 z$, $\pi(x_1) = x_1$, $\pi(x_2) = x_1(x_2+1)$ and $\pi(x_i) = x_1 x_i$ for $i \geq 3$.

The strict transform of f under π has the form

$$f' = z^{p^e} + F'(x_1, \ldots, x_n)$$

where $F' = x_j^{-p^e} \pi(F)$. Notice that, even though F is clean, the power series F' is not necessarily clean again. After applying cleaning, we obtain a new expansion

$$f' = z^{p^e} + F'_{\text{clean}}(x_1, \ldots, x_n)$$

from which we can compute the residual order of f' with respect to the induced normal crossings divisor $E' = V(x_j \cdot \prod_{i \in \Delta \setminus B} x_i)$ where the set B is defined as $B = \{j\} \cup \{i : t_i \neq 0\}$.

3. Behavior of the residual order under blowup

Contrary to the behavior of the resolution invariants used in characteristic zero, it is known that the residual order of a purely inseparable power series $f = z^{p^e} + F(x_1, \ldots, x_n)$ may increase under permissible blowups. Moh was able to show in [Moh87] that the increase under a *single blowup* is bounded by p^{e-1}. A further analysis of the increase and necessary conditions that F has to fulfill for an increase to happen were given by the first named author in [Hau10], see also [HP16].

Moh claimed in [Moh87] that the following *Stability Theorem* holds:

Let be given a purely inseparable equation $f = z^{p^e} + F(x_1, \ldots, x_n) = 0$ with residual order d and a sequence of permissible blowups under which the order of f remains constant when passing to its strict transforms. Then the residual orders of the strict transforms of f cannot increase beyond the bound $d + p^{e-1}$.

In particular, this result would rule out the possibility of the residual order increasing indefinitely.

Below we will present examples of purely inseparable equations with $e = 3$ and infinite sequences of point-blowups under which the residual order tends to infinity. This disproves Moh's claim in the case $e \geq 3$ (it is known to be valid for $e = 1$).

Although the examples might seem discouraging for proving resolution of singularities in positive characteristic, we emphasize that they do not constitute a counterexample to the existence of resolutions in positive characteristic. The reason for this is that in the examples one could choose at various instances a larger center than a point and thus would end up with a different sequence of blowups for which the residual order need not tend to infinity.

The flaw in Moh's proof of the Stability Theorem is relatively subtle. On p. 970 in [Moh87], a non-negative integer r is defined as the maximal number with the property that the initial form of F is a p^r-th power. It is then claimed that one may assume without loss of generality that the residual order d is divisible by p^r. This leads to the false conclusion that, after blowup, the residual order is bounded by $d' \leq [d/p^r] \cdot p^r + p^r$. As the following example shows, this is not true. Consider the purely inseparable polynomial

$$f = z^4 + x^2y^2w^3(w(x+y)^4 + x^{13})$$

over a field of characteristic 2. The residual order of f is $d = 5$. Further, $p^r = 2$. Let f' be the strict transform of f under a point-blowup in the x-chart with translation $y \mapsto y + 1$. Then

$$\begin{aligned} f' &= z^4 + x^8w^3(wy^4 + wy^6 + x^8 + x^8y^2) \\ &= z_1^4 + x^8w^3(wy^6 + x^8 + x^8y^2) \end{aligned}$$

after applying cleaning. Hence, the residual order of f' is $d' = 7$. This exceeds the bound $[d/p^r] \cdot p^r + p^r = 6$.

4. Examples for cycles and the indefinite increase of the residual order

We provide two types of examples, the first covering characteristic 2 and the second covering all odd characteristics: After a well chosen sequence of point blowups, the defining equation has the same shape as at the beginning, but some exponents have increased. Also, the residual order has increased.

In the examples, the letters A and B will be used to denote unspecified units $A, B \in k[[x_1, \dots, x_n]]^*$, and these may change from line to line without further notice. Similarly, λ will denote unspecified non-zero constants $\lambda \in k^*$ and Q unspecified power series $Q \in k[[x_1, \dots, x_n]]$. These unspecified objects can be chosen arbitrarily at the beginning of the sequence and are from then on prescribed by the transformation rules dictated by the blowups.

By abuse of notation, the consecutive strict transforms of the purely inseparable powers series f under the sequence of blowups will all be denoted again by f.

First example: Field of characteristic $p = 2$, $n = 5$, $\operatorname{ord} f = 8$.

The parameters $x_1, \dots, x_5$ will be denoted by u, v, w, x, y. Choose for d any even positive integer. All blowups are point-blowups.

(0) Starting equation:
$f = z^8 + x^4y^4w^{-d}(w^d(\lambda + u^{2d+6} \cdot Q) + x^{d+1}u^{2d+6} \cdot A)$.
The residual order equals d.

(1) x-chart, $y \mapsto y + 1$, $u \mapsto u + 1$:
$f = z^8 + w^{-d}(\lambda \cdot w^d + w^dy^4 \cdot B + w^dx^{2d+6} \cdot Q + x^{2d+7} \cdot A)$
$= z_1^8 + w^{-d}(w^dy^4 \cdot B + w^dx^{2d+6} \cdot Q + x^{2d+7} \cdot A)$,
after applying cleaning.
At this point, the residual order has increased to $d + 4$.

(2) x-chart:
$f = z^8 + x^4w^{-d}(w^dy^4 \cdot B + w^dx^{2d+2} \cdot Q + x^{d+3} \cdot A)$
$= z^8 + x^4w^{-d}(w^dy^4 \cdot B + x^{d+3} \cdot A)$.
The residual order has decreased to $d + 3$.

(3) u-chart:
$f = z^8 + u^7x^4w^{-d}(w^dy^4u \cdot B + x^{d+3} \cdot A)$.

(4) $\frac{d}{2}$ times v-chart:
$f = z^8 + v^{3d}u^7x^4w^{-d}(w^dy^4uv^d \cdot B + x^{d+3} \cdot A)$.

(5) w-chart, $v \mapsto v+1$, $y \mapsto y+1$:
$f = z^8 + u^7 x^4 w^{-(d+2)}(w^{d+2}u \cdot B + x^{d+3} \cdot A)$.

(6) u-chart:
$f = z^8 + u^4 x^4 w^{-(d+2)}(w^{d+2} \cdot B + x^{d+3} \cdot A)$.
The residual order has decreased to $d+2$.

(7) $(2d+10)$ times y-chart:
$f = z^8 + u^4 x^4 w^{-(d+2)}(w^{d+2}(\lambda + y^{2(d+2)+6} \cdot Q) + x^{d+3} y^{2(d+2)+6} \cdot A)$.
The residual order equals $d+2$.

At this point, f has again the same form as the starting equation (exchanging y with u), with the exception of d being raised to $d+2$. Effectively, the residual order has increased by 2 during the above sequence of blowups. Since this sequence can be iterated indefinitely, the residual order also increases indefinitely.

The sequence of blowups has a relatively simple structure: The residual order increases only under blowup (1). The remaining blowups in the sequence are necessary to rebuild the starting equation with increased d. After blowup (2), the residual order decreases by 1 and the initial form of G changes to x^{d+3}. The blowup sequences (3) and (4) leave the initial form unchanged, but add powers of u and v to the term $w^d y^4$. These powers are necessary to create the new term $w^{d+2}u \cdot B$ with blowup (5). Under blowup (6) the residual order decreases again by 1 and w^{d+2} becomes the new initial form of G. At this point, the initial form already has the same form as in the starting equation. The last sequence of blowups adds powers of y to x^{d+3} until it is of the same form as in the starting equation. It also has the effect of transforming the unit V into a constant λ plus a higher order term that is divisible by $y^{2(d+2)+6}$.

Second example: Field of characteristic $p \geq 3$, $n = 4$, $\operatorname{ord} f = p^3$.

The parameters $x_1, \ldots, x_4$ will be denoted by v, w, x, y. Choose for d a positive integer that is divisible by $\frac{p-1}{2}p$. Set $d' = d + \frac{p-1}{2}p$, $m = \frac{2d}{p-1} + p - 1$, and $q = \frac{p+1}{2}(d + p^2 - 1)$, $q' = \frac{p+1}{2}(d' + p^2 - 1)$. All blowups are point blowups.

(0) Starting equation:
$$f = z^{p^3} + x^{\frac{p-1}{2}p^2} y^{\frac{p^3+1}{2}} w^{-d}(w^d y^{\frac{p^2-1}{2}}(\lambda + v^q \cdot Q) + x^{d+\frac{p^2+1}{2}} v^q \cdot A).$$
The residual order equals $d + \frac{p^2-1}{2}$.

(1) x-chart, $y \mapsto y+1$, $v \mapsto v+1$:
$$f = z^{p^3} + w^{-d}(\lambda \cdot w^d + w^d y^{p^2} \cdot B + w^d x^q \cdot Q + x^{q+1} \cdot A)$$
$$= z_1^{p^3} + w^{-d}(w^d y^{p^2} \cdot B + w^d x^q \cdot Q + x^{q+1} \cdot A),$$
after applying cleaning.
At this point, the residual order has increased by $\frac{p^2+1}{2}$ to $d + p^2$.

(2) $\frac{p-1}{2}$ times x-chart:
$$f = z^{p^3} + x^{\frac{p-1}{2}p^2} w^{-d}(w^d y^{p^2} \cdot B + w^d x^{\frac{p+1}{2}(d-1)+p^2} \cdot Q + x^{d+p^2-\frac{p-1}{2}} \cdot A)$$
$$= z^{p^3} + x^{\frac{p-1}{2}p^2} w^{-d}(w^d y^{p^2} \cdot B + x^{d'+\frac{p^2+1}{2}} \cdot A).$$
The residual order has decreased to $d' + \frac{p^2+1}{2} = d + p^2 - \frac{p-1}{2}$.

(3) m times v-chart:
$$f = z^{p^3} + v^{m(p^2+\frac{p-1}{2}(p^2-1))} x^{\frac{p-1}{2}p^2} w^{-d}(w^d y^{p^2} v^{m\frac{p-1}{2}} \cdot B + x^{d'+\frac{p^2+1}{2}} \cdot A).$$

(4) w-chart, $v \mapsto v+1$:
$$f = z^{p^3} + x^{\frac{p-1}{2}p^2} w^{-d'}(w^{d'} y^{p^2} \cdot B + x^{d'+\frac{p^2+1}{2}} \cdot A).$$

(5) y-chart:

$$f = z^{p^3} + x^{\frac{p-1}{2}p^2} y^{\frac{p^3+1}{2}} w^{-d'} (w^{d'} y^{\frac{p^2-1}{2}} \cdot B + x^{d'+\frac{p^2+1}{2}} \cdot A).$$

The residual order has decreased to $d' + \frac{p^2-1}{2} = d + p^2 - \frac{p+1}{2}$.

(6) q' times v-chart:

$$f = z^{p^3} + x^{\frac{p-1}{2}p^2} y^{\frac{p^3+1}{2}} w^{-d'} (w^{d'} y^{\frac{p^2-1}{2}} (\lambda + v^{q'} \cdot Q) + x^{d'+\frac{p^2+1}{2}} v^{q'} \cdot A).$$

The residual order equals $d' + \frac{p^2-1}{2}$.

At this point, f has again the same form as the starting equation, with the exception of d being raised to d'. The residual order has increased along the sequence by $\frac{p-1}{2}p$ from $d + \frac{p^2-1}{2}$ to $d' + \frac{p^2-1}{2}$. Since the process can be iterated indefinitely, the residual order also increases indefinitely.

This example follows the same pattern as the first example. Again, the residual order increases only under blowup (1) and the remaining blowups are performed to rebuild the starting equation. Under the last blowup of sequence (2), the residual order decreases by $\frac{p-1}{2}$ and the initial form of G changes to $x^{d'+\frac{p^2+1}{2}}$. The blowup sequences (3) and (4) leave the initial form unchanged, but bring the first term into the form $w^{d'} y^{p^2} \cdot B$. Under blowup (5) the residual order decreases by 1 and $w^{d'} y^{\frac{p^2-1}{2}}$ becomes the new initial form. Again, the last sequence of blowups is only needed to bring the terms of higher degree into the same form as in the original equation.

Actually, the exponents of both minimal monomials of F increase. The relevant edges of the Newton polyhedron of F connecting the two minimal monomials $x^r y^s w^t \cdot w^d y^b$ with $x^r y^s w^t \cdot x^a v^c$ under blowup are illustrated schematically in Figure 1 by the red segments. The yellow regions represent the xv-, respectively yw-planes. The segments become longer and move from left to right with each cycle.

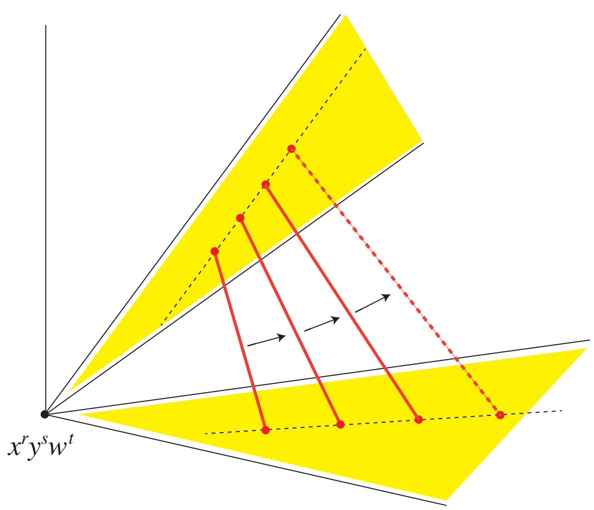

Figure 1: Transformation of the significant edge of the Newton polyhedron.

References

[Abh67] S. Abhyankar, *Nonsplitting of valuations in extensions of two dimensional regular local domains*, Math. Ann. **170** (1967), 87–144.

[BM97] E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[BV13] A. Benito and O. Villamayor, *Monoidal transforms and invariants of singularities in positive characteristic*, Compos. Math. **149** (2013), no. 8, 1267–1311.

[CP08] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra **320** (2008), no. 3, 1051–1082.

[CP09] ———, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** (2009), no. 7, 1836–1976.

[Cut11] S. D. Cutkosky, *A skeleton key to Abhyankar's proof of embedded resolution of characteristic p surfaces*, Asian J. Math. **15** (2011), no. 3, 369–416.

[EH02] S. Encinas and H. Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. **77** (2002), no. 4, 821–845.

[Hau10] H. Hauser, *On the problem of resolution of singularities in positive characteristic (or: a proof we are still waiting for)*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 1, 1–30.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[Hir12] ———, *Resolution of singularities*, Manuscript distributed at the CMI Summer School 2012, 138 pp.

[HP16] H. Hauser and S. Perlega, *Characterizing the increase of the residual order under blowup in positive characteristic*, Manuscript, 2016.

[HP17] ———, *A new proof for the embedded resolution of surface singularities*, Manuscript, 2017.

[HW14] H. Hauser and D. Wagner, *Alternative invariants for the embedded resolution of purely inseparable surface singularities*, Enseign. Math. **60** (2014), no. 1-2, 177–224.

[KM] H. Kawanoue and K. Matsuki, *Resolution of singularities of an idealistic filtration in dimension 3 after Benito-Villamayor*, to appear in Adv. Stud. Pure Math.

[Moh87] T.-T. Moh, *On a stability theorem for local uniformization in characteristic p*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 6, 965–973.

[Vil89] O. Villamayor, *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32.

[Vil92] ———, *Patching local uniformizations*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677.

[Wło05] J. Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822 (electronic).

Faculty of Mathematics
University of Vienna, Austria
herwig.hauser@univie.ac.at
stefan.perlega@univie.ac.at