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The cancellation property for direct products of analytic space germs

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Dedicated to Hans Grauert on the occasion of his 60th birthday

We propose:

Theorem 1 (Cancellation property for analytic space germs). *Let X, Y, and Z be three germs of complex analytic spaces. If $X \times Z \simeq Y \times Z$ then $X \simeq Y$.*

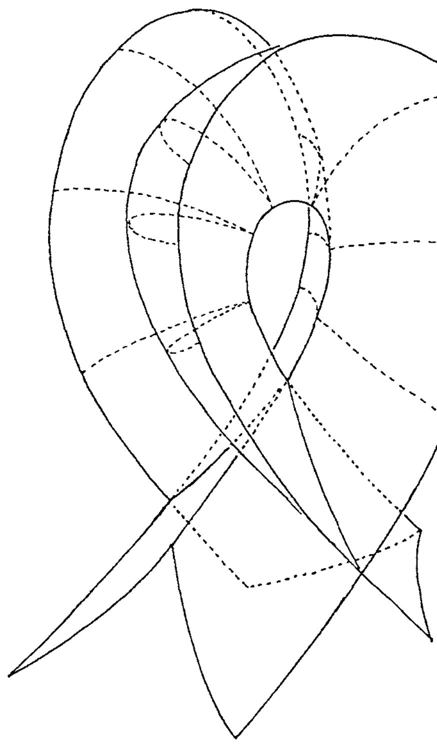

Fig. 1. Direct product of cusp and node

Cancellation properties have been studied in various situations. For algebraic varieties and biregular equivalence the problem is unsolved to a large extent. The most prominent result is due to Fujita (see [K]): If X is an affine surface (over an algebraically closed field of characteristic zero) such that $X \times Z \simeq \mathbb{A}^2 \times Z$ for some variety Z then X is the affine plane. If biregular is replaced by birational equivalence the cancellation property does not hold. In fact, Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [BCSS] exhibited (over every algebraically closed field of characteristic different from two) a non-rational variety X of dimension three such that the product $X \times \mathbb{P}^3$ is rational.

In global complex analytic geometry Horst [H, H 1] proved a cancellation theorem which combined with a result of Parigi [P] gives a complete and explicit description of those compact (not necessarily reduced) analytic spaces which cannot be cancelled. In local complex analytic geometry the cancellation property was proven by Ephraim [E] and Spallek [S] for reduced X, Y, Z as well as for arbitrary X, Y provided Z is smooth. Horst [H, H 2] established the case where one of the three germs is a fat point.

In the present article it will be proven that the cancellation property holds for all complex analytic space germs. By Artin's approximation theorem it is equivalent to the cancellation property for algebroid spaces (defined by ideals of formal power series). This in turn is an immediate consequence of the following structure theorem for algebroid spaces. Call an algebroid space Z decomposable if there are non-trivial algebroid spaces Z_1 and Z_2 (i.e., different from the reduced point) such that $Z \simeq Z_1 \times Z_2$.

Theorem 2 (Unique factorization property of algebroid spaces). *For any non-trivial algebroid space Z there are a unique integer p and non-trivial indecomposable algebroid spaces $Z_1, \ldots, Z_p$ unique up to permutation and isomorphism such that $Z \simeq Z_1 \times \ldots \times Z_p$.*

The analytic version of this statement (although proven for reduced germs by Ephraim [E] and Spallek [S]) remains open in general. However, it does hold if the germ is algebraic (= Nash analytic), i.e., if it can be defined by power series which in suitable coordinates are algebraic over the ring of polynomials:

Theorem 3 (Unique factorization property of algebraic analytic space germs). *For any non-trivial algebraic analytic space germ Z there are a unique integer p and non-trivial indecomposable analytic space germs $Z_1, \ldots, Z_p$ unique up to permutation and isomorphism such that $Z \simeq Z_1 \times \ldots \times Z_p$. The factors Z_i are algebraic. Moreover, passing to completions, $\hat{Z} \simeq \hat{Z}_1 \times \ldots \times \hat{Z}_p$ is the factorization of $\hat{Z}$ into indecomposable algebroid spaces.*

Note that this contains Horst's factorization theorem for fat points [H, H 2]. Theorem 3 is deduced from Theorem 2 by applying the Popescu-Rotthaus approximation theorem with nested subring condition [Po, R]. This is why the germ Z has to be assumed algebraic. And, by Gabriélov's example [G] one knows that this approximation theorem may fail in the non-algebraic situation.

The whole story becomes considerably simpler if one deals with local analytic families instead of individual germs:

Theorem 4. *Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two local analytic families of analytic space germs parametrized by some reduced germ T. Assume that $X_t\times Y_t\simeq X_0\times Y_0$ for all $t\in T$. Then $X_t\simeq X_0$ and $Y_t\simeq Y_0$ for all $t\in T$.*

This result has a nice consequence:

Corollary. *Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two local analytic families of analytic space germs with T reduced. If $X_t\simeq Y_0$ and $Y_t\simeq X_0$ for all $t\in T$ different from 0 then $X_0\simeq Y_0$.*

We thank D. Popescu for helpful suggestions concerning approximation arguments.

1. Proof of Theorem 1

We shall denote by $\hat{X}$ the completion of some analytic space germ X. Note that $X\simeq Y$ if and only if $\hat{X}\simeq\hat{Y}$. This can be seen by choosing defining equations of X and Y in smooth spaces and applying Artin's approximation theorem [A, Theorem 1.2]. So assume $X\times Z\simeq Y\times Z$. Then $\hat{X}\times\hat{Z}\simeq\hat{Y}\times\hat{Z}$. Factorize $\hat{X},\hat{Y},\hat{Z}$ into indecomposable algebroid spaces. The uniqueness of the factorization given by Theorem 2 yields $\hat{X}\simeq\hat{Y}$ and hence $X\simeq Y$.

2. Proof of Theorem 3

The deduction of the unique factorization property of algebraic analytic space germs from the corresponding formal property uses the Popescu-Rotthaus approximation theorem with nested subring condition (cf. [Po, Theorem 1.4], resp. [R, Theorem 4.2] in conjunction with [BDL, Remark 1.5]). We shall need an extension of this result to systems of algebraic equations instead of polynomial ones. This generalization goes along the lines of [B–M, Theorem 12.6].

Let $\mathcal{O}_n=\mathbb{C}\{x\}$ denote the algebra of convergent power series in n variables, $\mathbb{C}\langle x\rangle$ the subalgebra of algebraic power series, i.e., of those f in $\mathbb{C}\{x\}$ or, equivalently, in $\mathbb{C}[[x]]$ which are algebraic over the ring $\mathbb{C}[x]$ of polynomials. We then have:

Approximation theorem (with nested subring condition, Popescu, Rotthaus). *If a system of algebraic equations $f\in\mathbb{C}\langle x,y,u,v\rangle^p$ has a formal solution*

$$f(x,y,\bar{u}(x),\bar{v}(x,y))=0$$

with $\bar{u}\in\mathbb{C}[[x]]^r$, $\bar{v}\in\mathbb{C}[[x,y]]^s$, both without constant term, then there exists for any $k\in\mathbb{N}$ an algebraic solution

$$f(x,y,u(x),v(x,y))=0$$

with $u\in\mathbb{C}\langle x\rangle^r$ and $v\in\mathbb{C}\langle x,y\rangle^s$ approximating $\bar{u}$ and $\bar{v}$ up to order k:

$$u\equiv\bar{u},\quad v\equiv\bar{v}\mod(x,y)^{k+1}.$$

Proof. By [A 1, pp. 4, 5] one can choose an $\dot{f}\in\mathbb{C}\langle x,y,u,v\rangle^q$ without constant term such that there exists a polynomial $P\in\mathbb{C}[x,y,z,\dot{z},u,v]^{p+q}$ satisfying

$$P(x,y,f,\dot{f},u,v)=0\quad\text{and}\quad\det(\partial_z P,\partial_{\dot{z}}P)(0)\neq 0.$$

The implicit function theorem yields a $p\times(p+q)$-matrix $\bar{A}$ with entries in $\mathbb{C}[[x, y, z, \dot{z}, u, v]]$ such that

$$\dot{z}-f=\bar{A}\cdot P.$$

Consider then the polynomial $g=z-w\cdot P$ where w denotes some system of coordinates on $M_{p,p+q}(\mathbb{C})=\mathbb{C}^{p(p+q)}$. If we replace in g the variable w by $\bar{w}=\bar{A}(x,y,z,\dot{z},\bar{u}(x),\bar{v}(x,y))$ we obtain zero:

$$g(x,y,z,\dot{z},\bar{u}(x),\bar{v}(x,y),\bar{w}(x,y,z,\dot{z}))=0.$$

Since g is a polynomial, the original Popescu-Rotthaus theorem [loc. cit.] applies to this equation. We thus obtain an algebraic solution

$$g(x,y,z,\dot{z},u(x),v(x,y),w(x,y,z,\dot{z}))=0$$

approximating the formal one up to order k. Replace now in this last equation z by $f(x,y,u(x),v(x,y))$ and $\dot{z}$ by $\dot{f}(x,y,u(x),v(x,y))$; using $P(x,y,f,\dot{f},u,v)=0$ one obtains

$$f(x,y,u(x),v(x,y))=0$$

proving the assertion.

We apply this theorem to prove first:

Proposition. *Let Z be an algebraic analytic space germ. If its completion $\hat{Z}$ is formally decomposable then Z is analytically decomposable into algebraic factors.*

Proof. Let Z be defined in $(\mathbb{C}^n,0)$ by an ideal I of $\mathbb{C}\{x\}$ generated by the components of some vector $f\in\mathbb{C}\langle x\rangle^p$. By the implicit function theorem for algebraic map germs (cf. [L–T, Sect. 2, Teorema 1]) we may assume that the embedding of Z is minimal, i.e., that $I\subset(x)^2$. Let $\mathbb{A}^n$ denote the affine algebroid space of local ring $\mathscr{F}_n=\mathbb{C}[[x]]$. If $\hat{Z}$ is formally decomposable there exist an automorphism $\bar{\phi}$ of $\mathbb{A}^n$ and a partition $x=(x_1,x_2)$ such that

$$J\bar{\phi}(x)=J\bar{\phi}(x_1,0)+J\bar{\phi}(0,x_2),$$

where $J=I\cdot\mathbb{C}[[x]]$ and $J\bar{\phi}(x)=\bar{\phi}^*(J)\subset\mathbb{C}[[x]]$. We may assume that $\bar{\phi}$ is tangent to the identity. The preceding equation is equivalent to saying that there exists a $2p\times p$-matrix $\bar{A}=\begin{pmatrix}\bar{A}_1\\ \bar{A}_2\end{pmatrix}$ with entries in $\mathbb{C}[[x]]$ and rank p at 0 such that

$$\bar{A}_1(x)\cdot(f\circ\bar{\phi})(x)=(f\circ\bar{\phi})(x_1,0),$$
$$\bar{A}_2(x)\cdot(f\circ\bar{\phi})(x)=(f\circ\bar{\phi})(0,x_2).$$

We shall treat these two systems of equations separately: The approximation theorem yields

(i) $p\times p$-matrices A_1, A_2 with entries in $\mathbb{C}\langle x\rangle$ such that $A=\begin{pmatrix}A_1\\ A_2\end{pmatrix}$ has rank p at 0;

(ii) algebraic coordinate changes $\phi, \dot{\phi}$ of $(\mathbb{C}^n,0)$ tangent to the identity;

(iii) algebraic map germs $\alpha_i\in\mathbb{C}\langle x_i\rangle^n$, $i=1,2$, with

$$A_1\cdot(f\circ\phi)=f\circ\alpha_1 \quad\text{and}\quad A_2\cdot(f\circ\dot{\phi})=f\circ\alpha_2.$$

Let $\psi=(\psi_1,\psi_2)$ and $\dot\psi=(\dot\psi_1,\dot\psi_2)$ be the inverses of ϕ and $\dot\phi$ with components according to the partition $x=(x_1,x_2)$. Then

$$(A_1\circ\psi)\cdot f=f\circ\alpha_1\circ\psi_1 \quad\text{and}\quad (A_2\circ\dot\psi)\cdot f=f\circ\alpha_2\circ\dot\psi_2.$$

As ψ and $\dot\psi$ are tangent to the identity the pair $(\psi_1,\dot\psi_2)$ is an automorphism of $(\mathbb{C}^n,0)$. Let χ be its inverse. Then

$$(A_1\circ\psi\circ\chi)\cdot(f\circ\chi)=f\circ\alpha_1 \quad\text{and}\quad (A_2\circ\dot\psi\circ\chi)\cdot(f\circ\chi)=f\circ\alpha_2.$$

Since $f\circ\alpha_i\in\mathbb{C}\langle x_i\rangle^p$ (cf. [L–T, Sect. 2, Lemma 2]) and $I\subset(x)^2$, X is decomposable into non-trivial algebraic factors.

Proof of Theorem 3. Let $Z\simeq Z_1\times\ldots\times Z_q$ be a factorization of Z into non-trivial indecomposable analytic factors. Using the preceding proposition we can moreover choose a factorization $Z\simeq Y_1\times\ldots\times Y_p$ into non-trivial indecomposable algebraic factors such that p equals the number of indecomposable formal factors of the completion $\hat Z$ of Z. The algebroid spaces $\hat Y_i$ are thus indecomposable. Theorem 2 now implies that $\hat Z_1$ is isomorphic to a product of some $\hat Y_i$'s say $\hat Z_1\simeq\hat Y_1\times\ldots\times\hat Y_m$. Artin's approximation theorem (cf. Sect. 1) gives $Z_1\simeq Y_1\times\ldots\times Y_m$. As Z_1 is indecomposable, m must equal one, say $Z_1\simeq Y_1$. By Theorem 1 we obtain $Z_2\times\ldots\times Z_q\simeq Y_2\times\ldots\times Y_p$. Now induction gives the assertion of Theorem 3.

Remark. We could prove the unique factorization property for arbitrary analytic space germs if the following statement could be proven: An analytic space germ Z is analytically decomposable provided that $\hat Z$ is formally decomposable.

However this is not clear. We want to remark that the statement is true for hypersurfaces (and consequently the unique factorization property does hold for products of hypersurfaces.) In fact, let $Z\subset(\mathbb{C}^n,0)$ be a hypersurface defined by $f\in\mathbb{C}\{x\}$. If $\hat Z$ is decomposable then necessarily one of the factors is smooth. Hence there is a formal vector field $\bar\xi$ vanishing at 0 and a formal $\bar a\in\mathbb{C}[[x]]$ such that $\bar\xi f=\bar a\cdot f$. Since $\mathbb{C}[[x]]$ is flat over $\mathbb{C}\{x\}$ we obtain an analytic vector field ξ not vanishing at 0 and an analytic $a\in\mathbb{C}\{x\}$ such that $\xi f=a\cdot f$. By the analytic version of Lemma 1, Sect. 5, (see [F, pp. 92, 93]) Z is decomposable.

3. Proof of Theorem 2

The existence of a factorization into indecomposable factors is readily established using induction on the embedding dimension of Z. The hard part is the proof of uniqueness. Let

$$Z\simeq Z_1\times\ldots\times Z_p\simeq\dot Z_1\times\ldots\times\dot Z_q$$

be two factorizations into non-trivial indecomposable factors. Combining Lemma 1 and 2, Sect. 5, with the argument given in the proof of [E, Reduction 2 on p. 368] we may assume

$$Z\not\simeq\tilde Z\times\mathbb{A}^k$$

for any algebroid space $\tilde Z$ and any positive dimensional affine space $\mathbb{A}^k$. We shall proceed by induction on p, the case $p=1$ being obvious. So assume $p>1$. This implies $q>1$. We set:

$$X=Z_1, \quad Y=Z_2\times\ldots\times Z_p,$$
$$U=\grave{Z}_1, \quad V=\grave{Z}_2\times\ldots\times\grave{Z}_q.$$

We shall eventually show that after a suitable permutation of the factors Z_i we have $X\simeq U$ and $Y\simeq V$. This will establish the induction step.

(0) As we shall work with defining ideals we choose minimal embeddings $X\subset\mathbb{A}^m$, $Y\subset\mathbb{A}^n$, $U\subset\mathbb{A}^r$, $V\subset\mathbb{A}^s$. Then $m+n=r+s=:d$ and $X\times Y=(X\times\mathbb{A}^n)\cap(\mathbb{A}^m\times Y)\subset\mathbb{A}^d$. Let $M, N, R, S\subset\mathscr{P}_d$ denote the defining ideals of $X\times\mathbb{A}^n$, $\mathbb{A}^m\times Y$, $U\times\mathbb{A}^s$, $\mathbb{A}^r\times V$ in $\mathbb{A}^d$ such that $M+N$ defines $X\times Y$ etc. The given isomorphism between $X\times Y$ and $U\times V$ can be extended to an automorphism of $\mathbb{A}^d$ which we denote by (ϱ,σ) according to $\mathbb{A}^d=\mathbb{A}^r\times\mathbb{A}^s$. We then have:

$$M+N=(\varrho,\sigma)^*(R+S),$$

which we shall write for short:

$$M+N=R\varrho+S\sigma,$$

where $R\varrho$ denotes the ideal $\tilde{\varrho}^*(R)$ of $\mathscr{P}_d$ with $\tilde{\varrho}=(\varrho,0):\mathbb{A}^d\to\mathbb{A}^r\times\mathbb{A}^s=\mathbb{A}^d$. Moreover choose systems of coordinates x, y, u, v on $\mathbb{A}^m$, $\mathbb{A}^n$, $\mathbb{A}^r$, $\mathbb{A}^s$. After a suitable permutation of the factors Z_i we may assume that

$$\partial_x\varrho(0)\neq 0. \qquad (\square)$$

The subsequent proof relies on a detailed study of possible forms of the automorphism (ϱ,σ) of $\mathbb{A}^d$. We first treat a particularly simple case. Later we shall show that actually this is the only case which can occur:

(1) The components ϱ,σ of (ϱ,σ) are both tangent to the identity, i.e., $m=r$ and $n=s$ and (ϱ,σ) has linear term (x,y). We shall deduce that then $X\simeq U$ and $Y\simeq V$.

Let $(\mu,\nu):\mathbb{A}^d\to\mathbb{A}^m\times\mathbb{A}^n=\mathbb{A}^d$ denote the inverse of (ϱ,σ). With the notation fixed above we then have:

$$M+N=R\varrho+S\sigma$$

and

$$R+S=M\mu+N\nu.$$

For an ideal I of $\mathscr{P}_d$ let $I(x,0)$ be the ideal of $\mathscr{P}_d$ generated by all $f(x,0)$ with $f\in I$. As $M(x,0)=M$ and $N(x,0)=0$ we get:

$$M\supset R\varrho(x,0), \quad N\supset S\sigma(0,y),$$
$$R\supset M\mu(u,0), \quad S\supset N\nu(0,v).$$

Substitution gives:

$$M\supset M\mu(\varrho(x,0),0), \quad N\supset N\nu(0,\sigma(0,y)).$$

As $\mu(\varrho(x,0),0)$ has linear term x it defines an automorphism of $\mathbb{A}^m$. Because of $M=M(x,0)$ Lemma 6, Sect. 5, applies to the above inclusions and yields equalities:

$$M=M\mu(\varrho(x,0),0), \quad N=N\nu(0,\sigma(0,y)).$$

These in turn imply

$$M = R\varrho(x,0), \qquad N = S\sigma(0,y).$$

As $\varrho(x,0)$ and $\sigma(0,y)$ define automorphisms of $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively, we have proven that $X \simeq U$ and $Y \simeq V$.

(2) Reduction to the case where (ϱ, σ) has linear term (x_1, y_1, x_2, y_2). Here (x_1, x_2) and (y_1, y_2) denote partitions of the coordinates x and y according to factorizations $\mathbb{A}^m = \mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ and $\mathbb{A}^n = \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$ with $m_1 + n_1 = r$ and $m_2 + n_2 = s$.

Indeed, in case the jacobian matrices of ϱ and σ satisfy

$$rk\,\partial_{xy}\varrho(0) = rk\,\partial_x\varrho(0) + rk\,\partial_y\varrho(0), \qquad (\wedge)$$

$$rk\,\partial_{xy}\sigma(0) = rk\,\partial_x\sigma(0) + rk\,\partial_y\sigma(0), \qquad (\vee)$$

Lemma 4, Sect. 5, gives the stated normal form. If inequality "<" holds in either $(\wedge)$ or $(\vee)$ we shall deduce that $Z \simeq \tilde{Z} \times \mathbb{A}^k$ for some $k > 0$ and contradiction. So assume that "<" holds in $(\wedge)$, the reasoning for $(\vee)$ being symmetric. By Lemma 3, Sect. 5, we may suppose that

$$\partial_{xy}\varrho(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} m_1 = r_1 \\ m_2 = r_2 \\ m_3 \\ n_1 \\ n_2 = r_2 \\ n_3 = r_3 \end{matrix}$$
$$\qquad\qquad r_1 \;\; r_2 \;\; r_3$$

with $m_2 = n_2 = r_2$ strictly positive. Let Φ be the automorphism of $\mathscr{P}_d = \mathbb{C}[[x,y]] = \mathbb{C}[[u,v]]$ associated to $(\mu, v) = (\varrho, \sigma)^{-1}$, and ∂_x, ∂_y the column vectors of partial derivatives. The conjugate $\delta = \Phi\partial_y\Phi^{-1}$ of ∂_y induces by putting $v = 0$ a column vector $D = A \cdot \partial_u$ of derivations of $\mathbb{C}[[u]]$ with some $(n \times r)$-matrix A of entries in $\mathbb{C}[[u]]$. Recall now that $M + N = R\varrho + S\sigma$ and therefore $\Phi(M) + \Phi(N) = R + S$. From $\partial_y M \subset M$ we deduce $\delta(\Phi(M)) \subset \Phi(M) \subset R + S$ and then

$$\delta(\Phi(M))_{|v=0} \subset R_{|v=0} = R \cap \mathbb{C}[[u]]. \qquad (\equiv)$$

Since R is generated by $R \cap \mathbb{C}[[u]]$ and $S \subset (v)^2 \cdot \mathbb{C}[[u,v]]$ we have for $f \in R$ and $g \in S$:

$$\delta(f)_{|v=0} \equiv D(f_{|v=0}) \bmod R \cap \mathbb{C}[[u]] \quad \text{and} \quad \delta(g)_{|v=0} = 0 = D(g_{|v=0}).$$

Applying this to the elements of $\Phi(M) \subset R + S$ we conclude from $(\equiv)$ that

$$D(\Phi(M)_{|v=0}) \subset R \cap \mathbb{C}[[u]], \qquad (+)$$

and by symmetry:

$$E(\Phi(N)_{|v=0}) \subset R \cap \mathbb{C}[[u]], \qquad (-)$$

where $E = B \cdot \partial_u$ is induced from $\Phi\partial_x\Phi^{-1}$. Let A_i and B_i $(i = 1, 2, 3)$ denote the $(n_i \times r)$-, resp. $(m_i \times r)$-blocks of A and B. As $A(0) = \partial_y\varrho(0)$ and $B(0) = \partial_x\varrho(0)$ the $(r \times r)$-

matrix K of blocks B_1, B_2, and A_3 will be invertible. Hence there is an $(r_2 \times r)$-matrix $L=(L_1\ L_2\ L_3)$ such that $L \cdot K = A_2$. Clearly $L(0)=(0\ 1\ 0)$ and $L_1B_1+L_2B_2 = A_2 - L_3A_3 =: C$. Inclusions $(+)$ and $(-)$ thus imply that both $C \cdot \partial_u(\Phi(M)_{|v=0})$ and $C \cdot \partial_u(\Phi(N)_{|v=0})$ lie in $R \cap \mathbb{C}[[u]]$, where $C \cdot \partial_u$ is now an r_2-column of derivations of $\mathbb{C}[[u]]$. As $\Phi(M)_{|v=0} + \Phi(N)_{|v=0} = R_{|v=0} = R \cap \mathbb{C}[[u]]$ we get finally the inclusion

$$C \cdot \partial_u(R \cap \mathbb{C}[[u]]) \subset R \cap \mathbb{C}[[u]].$$

Since $C(0)=(0\ 1\ 0)$ and $r_2>0$ Lemma 1, Sect. 5, implies that $U \subset \mathbb{A}^r$ defined by $R \cap \mathbb{C}[[u]]$ and consequently Z can be factorized involving a smooth factor. This proves the reduction.

(3) The automorphism (ϱ, σ) has linear term (x_1, y_1, x_2, y_2) according to $\mathbb{A}^m = \mathbb{A}^{m_1} \times \mathbb{A}^{m_2}$ and $\mathbb{A}^n = \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$. In this case we shall show that $m_2 = n_1 = 0$, i.e., (ϱ, σ) is tangent to the identity as in (1). The proof of this implication is somewhat involved and goes in several steps:

(i) We start with the two equations

$$M+N=R\varrho+S\sigma \quad \text{and} \quad R+S=M\mu+N\nu.$$

We use $M(x,0)=M$ and $N(x,0)=0$ to get:

$$M=R\varrho(x,0)+S\sigma(x,0)$$

and

$$R=M\mu(u,0)+N\nu(u,0),$$

$$S=M\mu(0,v)+N\nu(0,v).$$

We substitute:

$$\begin{aligned} M &= M\mu(\varrho(x,0),0)+N\nu(\varrho(x,0),0) \\ &\quad + M\mu(0,\sigma(x,0))+N\nu(0,\sigma(x,0)) \\ &\supset M\mu(\varrho(x,0),0)+M\mu(0,\sigma(x,0)). \end{aligned}$$

We define:

$$\alpha, \beta: \mathbb{A}^m \to \mathbb{A}^m, \qquad \alpha(x)=\mu(\varrho(x,0),0), \qquad \beta(x)=\mu(0,\sigma(x,0)).$$

Thus:

$$M \supset M\alpha + M\beta,$$

where $M\alpha$ and $M\beta \subset \mathscr{P}_d$ are defined in the natural way via the inclusion $\mathbb{A}^m \subset \mathbb{A}^d$. Iteration gives:

$$M \supset M\alpha^k + M\beta^k, \qquad k \in \mathbb{N}, \tag{$*$}$$

(ii) We describe α^k and β^k: The normal form of (ϱ, σ) implies that α and β have linear terms $(x_1, 0)$ and $(0, x_2)$ respectively. Applying Lemma 5, Sect. 5, we can assume that

$$\alpha(x)=(\alpha_1(x_1), \alpha_2(x)) \quad \text{and} \quad \beta(x)=(\beta_1(x), \beta_2(x_2))$$

with α_2, $\beta_1 \equiv 0 \bmod (x)^2$. This implies:

$$\alpha^k(x) \equiv \alpha^k(x_1, 0) \bmod (x)^{k+1},$$
$$\beta^k(x) \equiv \beta^k(0, x_2) \bmod (x)^{k+1}.$$

(iii) We define

$$\gamma_k : \mathbb{A}^m \to \mathbb{A}^m, \qquad \gamma_k(x) = \alpha^k(x_1, 0) + \beta^k(0, x_2).$$

By substitution in (*):

$$M + (x)^{k+1} \supset M\gamma_k(x_1, 0) + M\gamma_k(0, x_2) \tag{$\triangle$}$$

and symmetrically:

$$N + (y)^{k+1} \supset N\delta_k(y_1, 0) + N\delta_k(0, y_2). \tag{$\triangledown$}$$

We shall show that these inclusions are actually equalities $\bmod (x, y)^{k+1}$: Fixing $k \in \mathbb{N}$, we may omit indices and write γ instead of γ_k in this section. Consider again

$$M + N = R\varrho + S\sigma.$$

Then

$$(M+N)(\gamma \times \delta) = M\gamma + N\delta = R\varrho(\gamma \times \delta) + S\sigma(\gamma \times \delta)$$

with $\gamma \times \delta : \mathbb{A}^{m+n} \to \mathbb{A}^{m+n}$. Restriction to $\mathbb{A}^{m_1} \times 0 \times \mathbb{A}^{n_1} \times 0$ yields:

$$M\gamma(x_1, 0) + N\delta(y_1, 0) \supset R\varrho(\gamma(x_1, 0), \delta(y_1, 0))$$

and symmetrically

$$M\gamma(0, x_2) + N\delta(0, y_2) \supset S\sigma(\gamma(0, x_2), \delta(0, y_2)).$$

Combining this with ($\triangle$) and ($\triangledown$) gives:

$$M + N + (x, y)^{k+1} \supset R\phi(x_1, y_1) + S\psi(x_2, y_2),$$

where

$$\phi(x_1, y_1) := \varrho(\gamma(x_1, 0), \delta(y_1, 0)),$$
$$\psi(x_2, y_2) := \sigma(\gamma(0, x_2), \delta(0, y_2)).$$

As

$$R + S = (M+N)(\mu, \nu)$$

we finally obtain

$$M + N + (x, y)^{k+1} \supset (M+N)(\mu, \nu) \circ \chi,$$

where χ is defined by $\chi(x, y) = (\phi(x_1, y_1), \psi(x_2, y_2))$. As χ has linear term (x_1, y_1, x_2, y_2) the composite $(\mu, \nu) \circ \chi$ is an automorphism of $\mathbb{A}^d$. Hence we can apply Lemma 6, Sect. 5, to establish the congruence

$$M + N \equiv (M+N)(\mu, \nu) \circ \chi \bmod (x, y)^{k+1}.$$

This is only possible if the inclusions ($\triangle$) and ($\triangledown$) already were equalities $\bmod (x, y)^{k+1}$.

(iv) We have shown for arbitrary $k \in \mathbb{N}$:

$$M \equiv M\gamma_k(x_1, 0) + M\gamma_k(0, x_2) \bmod (x)^{k+1}.$$

We restrict again:

$$M(x_1, 0) \equiv M\gamma_k(x_1, 0) \quad \bmod (x)^{k+1},$$
$$M(0, x_2) \equiv M\gamma_k(0, x_2) \quad \bmod (x)^{k+1}$$

and substitute:

$$M \equiv M(x_1, 0) + M(0, x_2) \bmod (x)^{k+1}.$$

We apply Krull's intersection theorem and obtain:

$$M = M(x_1, 0) + M(0, x_2).$$

By (□) we know $m_1 > 0$. If m_2 were positive then the last equality would imply that X is decomposable, a contradiction. Symmetrically $n_1 = 0$ since U is indecomposable. This concludes the proof of step (3) and of Theorem 2.

4. Proof of Theorem 4

Let us first give some specifications concerning the assertion of Theorem 4: Let $\pi: X \to T$ be a morphism of analytic space germs with section $\sigma: T \to X$. For $t \in T$ denote by X_t the germ in $\sigma(t)$ of the fiber of π over t. We then say that $\{X_t\}_{t \in T}$ is a local analytic family of analytic space germs. Such a family is called trivial – or π is called trivial along σ – if there is an isomorphism $X \simeq X_0 \times T$ over T mapping $\sigma(T)$ onto $0 \times T$. For a trivial family we clearly have $X_t \simeq X_0$ for all $t \in T$. Conversely it was shown in [H–M] that if T is reduced and $X_t \simeq X_0$ for all $t \in T$ then the family $\{X_t\}_{t \in T}$ is trivial.

Let us now prove Theorem 4. First consider the special case $T = (\mathbb{C}, 0)$. Choose embeddings $X \subset (\mathbb{C}^m, 0) \times T$ and $Y \subset (\mathbb{C}^n, 0) \times T$ over T. We may assume that the sections are given by $t \to (0, t)$. Choose $f \in \mathbb{C}\{x, t\}^p$ and $g \in \mathbb{C}\{y, t\}^q$ defining X and Y. Then both together define an analytic space germ Z in $(\mathbb{C}^{m+n}, 0) \times T$. The projection $(x, y, t) \to t$ with section $t \to (0, t)$ gives a local analytic family $\{Z_t\}_{t \in T}$ of analytic space germs with $Z_t \simeq X_t \times Y_t$. Hence our assumption implies via [H–M, Theorem 1′] that the family $\{Z_t\}_{t \in T}$ is trivial. Now apply the triviality criterion [G–H, Proposition 2]: There are a $(p+q) \times (p+q)$-matrix $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ and an $(m+n)$-column $\begin{pmatrix} a \\ b \end{pmatrix}$ both with entries in $\mathbb{C}\{x, y, t\}$ such that $\begin{pmatrix} a \\ b \end{pmatrix}(0, t) = 0$ and

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = M \cdot \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} \partial_x f & 0 \\ 0 & \partial_y g \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

We obtain:

$$\partial_t f = M_{1|y=0} \cdot f + \partial_x f \cdot a_{|y=0},$$

and similarly for g. Applying again [G–H, Proposition 2] the families $\{X_t\}_{t \in T}$ and $\{Y_t\}_{t \in T}$ will be trivial.

Now let T be an arbitrary reduced germ. By [H–M, Theorem 1'] there is a reduced analytic germ $R \subset T$, called the trivial locus of the family $\{X_t\}_{t\in T}$, with the following property: For any base change $\alpha: T' \to T$ with T' reduced the induced morphism $\pi': X' = X \times_T T' \to T'$ is trivial along the induced section $\sigma': T' \to X'$ if and only if α maps into R.

So let R and S denote the trivial loci of $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$. We have to prove that $R \cap S = T$. Assume that $R \cap S$ is strictly contained in T. We can clearly reduce to the case that T is irreducible. Hence $\dim R \cap S < \dim T$. Embed T in some smooth space $(\mathbb{C}^k, 0)$. If $d = \dim R \cap S$ and H is a generic $(k-d)$-plane in $(\mathbb{C}^k, 0)$ then $R \cap S \cap H = 0$ but $T \cap H$ has positive dimension. Hence there is an irreducible curve $C \subset T$ which is not contained in $R \cap S$. Take the normalization $(\mathbb{C}, 0) \to C \subset T$. By the special case considered before and the very definition of the trivial loci we must have $C \subset R \cap S$, and contradiction. This proves Theorem 4.

Remark. One might hope that Theorem 4 combined with Mather's path-method (cf. [G–H, proof of Theorem, Sect. 2]) gives the following result: If X, Y and X', Y' are indecomposable analytic space germs with $X \times Y \simeq X' \times Y'$ then $X \simeq X'$ up to permutation. This would yield in particular Theorem 3 without any assumption on algebraicity.

However, one encounters a serious obstacle in this approach: It looks difficult to find analytic families $\{X_t\}_{t\in\mathbb{C}}$ and $\{Y_t\}_{t\in\mathbb{C}}$ with $X_0 = X$, $X_1 = X'$, $Y_0 = Y$, $Y_1 = Y'$ and such that $X_t \times Y_t \simeq X_0 \times Y_0$ for all t in a domain of $\mathbb{C}$ containing 0 and 1. Once this could be done the assertion would be immediate from Theorem 4.

5. Some auxiliary results

Let $\mathscr{P}_n = \mathbb{C}[[x]]$, $x = (x^1, \ldots, x^n)$, be the algebra of formal power series and $\mathbb{A}^n$ the affine algebroid space of local ring $\mathscr{P}_n$.

Lemma 1 (a) [Z, Lemma 4]. *Let $D \in \operatorname{Der}\mathscr{P}_n$ be a formal vector field not vanishing at 0. Then there is a formal change of coordinates $\Phi \in \operatorname{Aut}\mathscr{P}_n$ such that $\Phi D \Phi^{-1} = \partial_{x^1}$.*
(b) [F, p. 93] *Let $I \subset \mathscr{P}_n$ be an ideal such that $\partial_{x^1} I \subset I$. Then $I = I(0, x^2, \ldots, x^n)$.*

Combining these assertions with the proof of [E, Lemma 1.5] gives:

Lemma 2. *Algebroid spaces X and Y satisfying $X \times \mathbb{A}^n \simeq Y \times \mathbb{A}^n$ for some n are isomorphic.*

Next we prove two results about normal forms of certain matrices.

Lemma 3. *Let $\begin{pmatrix} A \\ B \end{pmatrix}$ be an $(m+n) \times r$-matrix of* rank r with entries in $\mathbb{C}$. *Then there are invertible matrices $K \in GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \subset GL(m+n, \mathbb{C})$ and $L \in GL(r, \mathbb{C})$ such that*

$$K \cdot \begin{pmatrix} A \\ B \end{pmatrix} \cdot L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} m_1 \\ m_2 \\ m_3 \\ n_1 \\ n_2 \\ n_3 \end{matrix}$$
$$\qquad\qquad\quad m_1 \;\; m_2 \;\; n_3$$

The integers m_i, n_i are uniquely determined by the conditions $\operatorname{rk} A = m_1 + m_2$, $\operatorname{rk} B = n_2 + n_3$, $\operatorname{rk}\begin{pmatrix} A \\ B \end{pmatrix} = r = m_1 + m_2 + n_3$ *and* $m = \sum m_i$, $n = \sum n_i$.

Proof. Uniqueness is clear. Existence goes by induction on r. If $\operatorname{rk} A = r = \operatorname{rk} B$ then $K_1 \cdot A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $K_2 \cdot B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for suitable $(K_1, K_2) \in GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$. If $\operatorname{rk} A < r$ then after multiplication with an element from $GL(r, \mathbb{C})$ we may assume that the last column of A is zero. As $\operatorname{rk}\begin{pmatrix} A \\ B \end{pmatrix} = r$ the last column of B is then non-zero. After multiplication with an element from $GL(n, \mathbb{C})$ we may assume that the last column of B equals $(0 \ldots 01)^t$. Multiplication with a suitable element from $GL(r, \mathbb{C})$ transforms the last row of B into $(0 \ldots 01)$ leaving A invariant. Now apply the induction hypothesis to the $(m+n-1) \times (r-1)$-matrix obtained from $\begin{pmatrix} A \\ B \end{pmatrix}$ by deleting the last row and column.

Lemma 4. *Let* $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ *be an invertible* $(m+n) \times (r+s)$*-matrix with entries in* $\mathbb{C}$. *Assume that*

$$\operatorname{rk}\begin{pmatrix} A_1 \\ A_3 \end{pmatrix} = \operatorname{rk} A_1 + \operatorname{rk} A_3$$

and

$$\operatorname{rk}\begin{pmatrix} A_2 \\ A_4 \end{pmatrix} = \operatorname{rk} A_2 + \operatorname{rk} A_4 .$$

Then there are invertible matrices $K \in GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \subset GL(m+n, \mathbb{C})$ *and* $L \in GL(r, \mathbb{C}) \times GL(s, \mathbb{C}) \subset GL(r+s, \mathbb{C})$ *such that*

$$K \cdot A \cdot L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{matrix}$$
$$\qquad\qquad m_1 \; n_2 \; m_2 \; n_1$$

The integers m_i, n_i are uniquely determined.

Proof. Uniqueness is clear. Using Lemma 3 and the first rank condition we may assume that

$$\begin{pmatrix} A_1 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{matrix}$$
$$\qquad m_1 \; n_2$$

Now write

$$\begin{pmatrix} A_2 \\ A_4 \end{pmatrix} = \underset{m_2+n_1}{\begin{pmatrix} * \\ C \\ * \end{pmatrix}} \begin{matrix} m_1 \\ m_2+n_1 \\ n_2 \end{matrix}.$$

As A is invertible, C is. Multiplication of A with $(1\ C^{-1}) \in GL(r,\mathbb{C}) \times GL(m_2+n_1,\mathbb{C})$ leaves $\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$ invariant and transforms C into 1. We now have

$$\begin{pmatrix} A_2 \\ A_4 \end{pmatrix} = \underset{m_2\ n_1}{\begin{pmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix}} \begin{matrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{matrix}.$$

Actually by the second rank condition the upper and lower block are of form $(*\ 0)$ and $(0\ *)$ respectively. Multiplication with a suitable element from $GL(m,\mathbb{C}) \times GL(n,\mathbb{C})$ transforms $\begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$ into the desired form leaving $\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$ invariant.

Also we shall need a result on normal forms of formal endomorphisms of $\mathbb{A}^n$. If this statement could be proven in the analytic category one could also prove the unique factorization property for all analytic space germs without assumption on algebraicity.

Lemma 5. *Let $x=(x_1,x_2)$ be a partition of the coordinates on $\mathbb{A}^n=\mathbb{A}^{n_1}\times\mathbb{A}^{n_2}$. Consider formal morphisms $\alpha,\beta:\mathbb{A}^n\to\mathbb{A}^n$ satisfying*

$$\alpha=(\alpha_1,\alpha_2)\equiv(x_1,0)\operatorname{mod}(x)^2,$$

$$\beta=(\beta_1,\beta_2)\equiv(0,x_2)\operatorname{mod}(x)^2.$$

Then there is a formal automorphism ϕ of $\mathbb{A}^n$, tangent to the identity, such that the conjugates $\grave{\alpha}=\phi\alpha\phi^{-1}$ and $\grave{\beta}=\phi\beta\phi^{-1}$ satisfy:

$$\begin{aligned} \grave{\alpha}(x)&=(\grave{\alpha}_1(x_1),\grave{\alpha}_2(x)),\\ \grave{\beta}(x)&=(\grave{\beta}_1(x),\grave{\beta}_2(x_2)). \end{aligned} \qquad (:)$$

Proof (cf. [E 1, pp. 21–23]). Let $\phi_1=\mathrm{Id}$. We shall prove by induction on k: For every $k\geqq 2$ there is an automorphism ϕ_k of $\mathbb{A}^n$ such that $\phi_k\equiv\phi_{k-1}\operatorname{mod}(x)^k$ and the conjugates of α and β with respect to ϕ_k satisfy $(:)$ modulo $(x)^{k+1}$. The assertion of the lemma then follows by taking ϕ to be the projective limit of the ϕ_k's.

For the induction step we may assume that $\phi_{k-1}=\mathrm{Id}$ and

$$\alpha_1(x)\equiv x_1+\lambda(x_1)+\sigma(x)\operatorname{mod}(x)^{k+1},$$

$$\alpha_2(x)\equiv 0,\quad \beta_1(x)\equiv 0 \quad \operatorname{mod}(x)^2,$$

$$\beta_2(x)\equiv x_2+\mu(x_2)+\tau(x)\operatorname{mod}(x)^{k+1},$$

where the components of λ and μ are polynomials of order $\geqq 2$ and degree $\leqq k$, and the components of σ (resp. τ) are homogeneous polynomials of degree k contained in the ideal $(x_2)\cdot\mathbb{C}[[x]]$ (resp. $(x_1)\cdot\mathbb{C}[[x]]$). Define

$$\phi_k(x)=(x_1+\sigma(x), x_2+\tau(x))=:(\xi_k(x),\zeta_k(x)).$$

Then we have the following congruences modulo $(x)^{k+1}$:

$$\begin{aligned}\xi_k(\alpha(x)) &\equiv x_1+\lambda(x_1)+\sigma(x)+\sigma(\alpha(x))\\ &\equiv x_1+\sigma(x)+\lambda(x_1)\\ &\equiv \xi_k(x)+\lambda(\xi_k(x))\end{aligned}$$

and

$$\zeta_x(\beta(x))\equiv\zeta_k(x)+\mu(\zeta_k(x)),$$

hence:

$$\begin{aligned}(\xi_k\alpha\phi_k^{-1})(x)&\equiv x_1+\lambda(x_1),\\ (\zeta_k\beta\phi_k^{-1})(x)&\equiv x_2+\mu(x_2).\end{aligned}$$

This proves the induction step.

We conclude with a simple but useful fact:

Lemma 6. *Let $I\subset\mathscr{P}_n$ be an ideal and α an automorphism of $\mathbb{A}^n$. If $I\subset I\alpha:=\alpha^*(I)$ then $I=I\alpha$.*

Proof. We have by iteration: $I\subset I\alpha\subset I\alpha^2\subset\ldots$ and the sequence becomes stationary since $\mathscr{P}_n$ is Noetherian. Hence $I\alpha^k=I\alpha^{k+1}$ for some k and then $I=I\alpha$.

References

[A] Artin, M.: On the solutions of analytic equations. Invent. Math. **5**, 277–291 (1968)

[A1] Artin, M.: Algebraic spaces. New Haven London: Yale University Press 1971

[BCSS] Beauville, A., Colliot-Thélène, J.-L., Sansuc, J.-J., Swinnerton-Dyer, P.: Variétés stablement rationelles non rationelles. Ann. Math. **121**, 283–318 (1985)

[BDL] Becker, J., Denef, J., Lipshitz, L.: The approximation property for some 5-dimensional Henselian rings. Trans. Am. Math. Soc. **276**, 301–309 (1983)

[B-M] Bierstone, E., Milman, P.: Relations among analytic functions. II. Ann. Inst. Fourier **37**, 49–77 (1987)

[E] Ephraim, R.: Isosingular loci and the cartesian product structure of complex analytic singularities. Trans. Am. Math. Soc. **241**, 357–371 (1978)

[E1] Ephraim, R.: C^∞ and analytic equivalence of singularities. Rice Univ. Stud. **59**, 11–32 (1973)

[F] Fischer, G.: Complex analytic geometry (Lect. Notes Math., Vol. 538) Berlin Heidelberg New York: Springer 1976

[G] Gabriélov, A.M.: Formal relations between analytic functions. Funct. Anal. Appl. **5**, 318–319 (1971)

[G-H] Gaffney, T., Hauser, H.: Characterizing singularities of varieties and of mappings. Invent. Math. **81**, 427–447 (1985)

[H] Horst, C.: On product decompositions of complex spaces. Habilitationsschrift, Universität München

[H 1] Horst, C.: Decomposition of compact complex varieties and the cancellation problem. Math. Ann. **271**, 467–477 (1985)

[H 2] Horst, C.: A cancellation theorem for Artinian local algebras. Math. Ann. **276**, 657–662 (1987)

[H–M] Hauser, H., Müller, G.: Analytic curves in power series rings. Compos. Math. (to appear)

[K] Kambayashi, T.: On Fujita's strong cancellation theorem for the affine plane. J. Fac. Sci. Univ. Tokyo, Sect. I A Math. **27**, 535–548 (1980)

[L–T] Lazzeri, F., Tognoli, A.: Alcune proprietà degli spazi algebrici. Ann. Sc. Norm. Super. Pisa Cl. Sci., III. Ser. **24**, 597–632 (1970)

[P] Parigi, G.: Caractérisation des variétés compactes simplifiables et applications aux surfaces algébriques. Math. Z. **190**, 371–378 (1985)

[Po] Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. **104**, 85–115 (1986)

[R] Rotthaus, C.: On the approximation property for excellent rings. Invent. Math. **88**, 39–63 (1987)

[S] Spallek, K.: Produktzerlegung und Äquivalenz von Raumkeimen I. In: C. Andreian Cazacu et al. (eds.) Complex analysis Fifth Romanian-Finnish Seminar Part 2. Proceedings, Bucharest 1981 (Lect. Notes Math., Vol. 1014, pp. 78–100). Berlin Heidelberg New York: Springer 1983

[Z] Zariski, O.: Studies in equisingularity. I. Equivalent singularities of plane algebroid curves. Am. J. Math. **87**, 507–536 (1965)

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