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Characterizing singularities of varieties and of mappings

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Introduction

In the two parts of this paper we propose to study singularities of complex-analytic varieties and mappings. The first part investigates singularities of germs of complex-analytic varieties in $(\mathbb{C}^n, 0)$ under analytic isomorphism and their relation to the singular subspace of the variety. The second part presents a characterization of analytic mappings under $\mathscr{A}$-equivalence, i.e., coordinate changes in source and target.

Part I

Let $(X, 0)$ and $(Y, 0)$ be two germs of complex-analytic varieties in $(\mathbb{C}^n, 0)$. Assume they are smooth; then they are isomorphic, if and only if they have the same dimension. Assume they are singular; then they are isomorphic, if and only if ...!

The purpose of Part I is to provide the dots with a syntactic, and the singular locus of $(X, 0)$ with an analytic structure, such that the first characterizes via the second the initial structure of the variety $(X, 0)$.

For an embedded hypersurface $(X, 0) \subseteq (\mathbb{C}^n, 0)$ of equation $f(x)=0$, let $\mathrm{Sing}(X, 0)$ denote the singular subspace of $(X, 0)$ defined by the ideal $(f)+j(f)$ of $\mathcal{O}_n$ ($j(f)=$jacobian ideal of f). $\mathrm{Sing}(X, 0)$ being an analytic invariant of the variety, it is natural to ask to what extent it determines the variety $(X, 0)$, i.e., whether $\mathrm{Sing}(X, 0) \cong \mathrm{Sing}(Y, 0)$ implies $(X, 0) \cong (Y, 0)$?

Answers to this question have been found in special cases: Benson and independently Shoshitaishvili could affirm the question for homogeneous (resp. quasihomogeneous) hypersurfaces with isolated singularity [B], [Sh]. Then Mather and Yau extended these results to arbitrary isolated hypersurface singularities [M-Y].

In Part I we propose the study and complete answer of the question for arbitrary varieties $(X, 0)$ in $(\mathbb{C}^n, 0)$.

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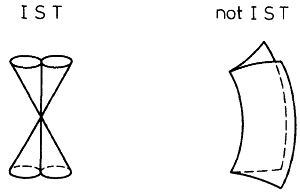
The answer is "Yes" for varieties of isolated singularity type. By definition, $(X,0)$ has isolated singularity type (I.S.T.), if $\operatorname{Sing}(X,0) \not\cong \operatorname{Sing}(X,a)$ for $a \neq 0$ varying in a sufficiently small neighborhood of $0 \in \mathbb{C}^n$.



The answer is "No" for varieties which are not of isolated singularity type: we shall give the example of a family of hypersurfaces $(X_t, 0)$, $t \in (\mathbb{C}, 0)$, with constant singular subspace $\operatorname{Sing}(X_t, 0) = \operatorname{Sing}(X_0, 0)$ but varying analytic type (§4). This shows in particular that even the continuous version of the question, i.e., $(X,0)$ varying in a local family, has a negative answer.

However, if one replaces $\operatorname{Sing}(X,0)$ by a slightly bigger subspace of $(X,0)$, the answer turns again to the affirmative: more precisely, let S be a submanifold of $\mathbb{C}^n$ and $Z \subseteq \mathbb{C}^n$ a manifold whose coordinates form a set of equations for S. Let $j_S(f)$ and $j_Z(f)$ denote the ideal of partial derivatives in direction of S and Z, and $\operatorname{Sing}_S(X,0)$ the subvariety of $(X,0)$ defined by $(f) + \mathfrak{m}_n \cdot j_S(f) + j_Z(f)$ ($\mathfrak{m}_n$ = maximal ideal of $\mathcal{O}_n$)

Theorem. *For all manifolds $S \subseteq \mathbb{C}^n$ containing the stratum $\Sigma \subseteq \mathbb{C}^n$ along which* $\operatorname{Sing}_S(X,0)$ *is trivial (i.e., a product), the analytic type of $(X,0)$ is determined by* $\operatorname{Sing}_S(X,0)$.

By the example of §4, the assertion is sharp. As a special case one obtains, that $\operatorname{Sing}_{\mathbb{C}^n}(X,0)$ defined by $TK(f) = (f) + \mathfrak{m}_n \cdot j(f)$ always characterizes $(X,0)$.

The Theorem will be generalized to the non hypersurface case defining $\operatorname{Sing}_S(X,0)$ by the analytic module

$$\mathcal{O}_n^p/(f) \cdot \mathcal{O}_n^p + \mathfrak{m}_n \cdot j_S(f) + j_Z(f)$$

($f = (f_1, \ldots, f_p) \in \mathcal{O}_n^p$ defining equations of $(X,0)$) and introducing a category of generalized analytic varieties defined by analytic modules.

Yau has informed us, that he and Mather have found that $TK(f)$ determines $(X,0)$ for $(X,0)$ complete intersection with isolated singularity. In the same case, Martin [Ma 1] shows that $\operatorname{Sing}_0(X,0)$ determines $(X,0)$. On the other hand, Dimca proved an analogous result using the ideal of minors of the jacobian matrix in the case of 0-dimensional or homogeneous complete intersections [D].

The first part concludes with the mentioned example and, as an application of the techniques used, an extension of Saito's result [Sa] characterizing quasi-homogeneous polynomials.

Part II

In Part II we prove analogues of the main theorems of Part I for map germs. Two map germs $f, g \in \mathcal{O}_n^p$ are $\mathcal{A}$-equivalent if there are coordinate changes in source and target, say r and l such that $l \circ f \circ r = g$. We let $\mathscr{A}$ denote the group formed from pairs of coordinate changes on source and target. The object that plays the part of the tangent space to the $\mathscr{A}$-orbit of f at f is $m_n j_{\mathbb{C}^n}(f) + f^*(m_p \mathcal{O}_p^p)$, denoted $T\mathscr{A}(f)$. Given Part I, it is natural to conjecture that $T\mathscr{A}(f) \cong T\mathscr{A}(g) \Rightarrow f \underset{\mathscr{A}}{\approx} g$.

If f has finite singularity type (i.e., $V(f)$ is a complete intersection with isolated singularity or $V(f)$ is an isolated point) then this conjecture is true, and is proved in Part B of the Theorem.

Just as there is a geometric interpretation of $\mathcal{O}_n^p/TK(f)$ in Part I, so there is also a geometric interpretation of $\mathcal{O}_n^p/T\mathscr{A}(f)$. This $\mathcal{O}_p$-module (via f) can be viewed as the stalk of a sheaf on the target $\mathbb{C}^p$ (the construction of a related sheaf is given by the proof of Part A of the theorem). The support of this sheaf is just the locus of instability in the target. This consists of all $y \in \mathbb{C}^p$ such that the germ of f at $S = f^{-1}(y)$ is unstable. Since two stable germs are $\mathscr{A}$-equivalent if they are K-equivalent, it is precisely over the locus of instability that the more complicated behavior of the map takes place.

If the behavior of a map germ at the origin is different from its behavior at nearby points, following Part I it is natural to conjecture that if $T\mathscr{A}_e(f) = j_{\mathbb{C}^n}(f) + f^*(\mathcal{O}_p^p) = T\mathscr{A}_e(g)$, then f and g are $\mathscr{A}$-equivalent. We prove this in Part A of the Theorem with the additional hypothesis that f is finitely determined i.e., $T\mathscr{A}_e(f) \supseteq m_n^k \mathcal{O}_n^p$ for some k, but f not stable.

Unfortunately, it is not clear how to prove the precise analogue of the corresponding theorem of Part I, because the sheaves in the $\mathscr{A}$ case are more difficult to work with (they involve direct images). In actually classifying germs, the finitely determined case is the most important and the one in which our theorem will be most useful.

The proofs of our theorems also illustrate some of the advances in technique of recent years. In their paper, Mather and Yau used the notion of finite determinacy to reduce to lie groups acting on jet spaces. The advantage of this method is that algebraic problems are reduced to ones involving finite dimensional vector spaces.

However, this limits the generality of the results that can be obtained. This paper shows that sufficient algebraic techniques exist to work with germs which are not finitely determined so that jets can be dispensed with. Thus, we can handle arbitrary varieties and map germs which are of finite singularity type. It even turns out that the algebra is strong enough to reduce everything to questions about finite dimensional vector spaces and linear maps between them again.

Lemma 3 is of independent interest to singularists for it shows that the module structure of $T\mathscr{A}(f)$ is "convex" in some sense. It implies that if $T\mathscr{A}(f) = T\mathscr{A}(g)$ then $T\mathscr{A}(f)$ is an $f_t^*\mathcal{O}_p$-module for all t, where $f_t = f + t(g - f)$.

1. Deformations

Let K denote the group $K = Gl_p(\mathcal{O}_n) \times \mathrm{Aut}(\mathbb{C}^n, 0)$ acting on $\mathcal{O}_n^p$ via $(L, \phi) \cdot f = L \cdot (f \circ \phi)$. For $f \in \mathcal{O}_n^p$ let (f) denote the ideal of $\mathcal{O}_n$ generated by the components f_i of f and $(X, 0)$ the complex-analytic variety in $(\mathbb{C}^n, 0)$ of local ring $\mathcal{O}_{X,0} = \mathcal{O}_n/(f)$.

Lemma 1. ([M], p. 136) *The fibers of the map*

$$\pi : \mathcal{O}_n^p \to \{\textit{isomorphism classes of analytic varieties}\}$$
$$f \to [(X,0)]$$

(respectively $\tau : \mathcal{O}_n^p \to \{\textit{analytic varieties}\}$, $f \to (X,0)$) are the K-orbits (respectively $Gl_p(\mathcal{O}_n)$-orbits) in $\mathcal{O}_n^p$. In particular, the fibers of τ contain with any two elements $f, g \in \mathcal{O}_n^p$ all elements except a finite number of the complex line $l = f + \mathbb{C}(f-g) \subseteqq \mathcal{O}_n^p$ through f and g. As the fibers of π are connected, any two isomorphic varieties can be joined by a family of isomorphic varieties.

Definition 1. A map from an analytic manifold T to a free $\mathcal{O}_n$-module, $f:(T,0) \to (\mathcal{O}_n^p, f_0)$, is called analytic at a point $0 \in T$, if the induced map-germ $F:(T \times \mathbb{C}^n, 0) \to \mathbb{C}^p$ defined by $F(t,x) = f(t)(x)$ is analytic. For given $f_0 \in \mathcal{O}_n^p$, $f:(T,0) \to (\mathcal{O}_n^p, f_0)$ is called a deformation of f_0; f is K-trivial, if its images $f_t := f(t)$ lie for t close to $0 \in T$ in the K-orbit of f_0, i.e., for any $t \in (T,0)$ there exists $(L_t, \phi_t) \in K$ with $L_t \cdot (f_t \circ \phi_t) = f_0$. Geometrically speaking, a deformation $h:(X,0) \to (T,0)$ of $(X_0,0) = (h^{-1}(0),0)$ is trivial, if all its fibers $(X_t,0)$ are isomorphic to $(X_0,0)$.

Proposition 1. (Ephraim, [E 2]) *If $f:(T,0) \to (\mathcal{O}_n^p, f_0)$ is trivial, there exists an analytic map $(L,\phi):(T,0) \to (K,e)$, $e = (1, \mathrm{Id}_{\mathbb{C}^n}) \in K$, such that $L_t \cdot (f_t \circ \phi_t) = f_0$. Equivalently, a trivial deformation $h:(X,0) \to (T,0)$ of $(X_0,0)$ is a product*

$$\begin{tikzcd}
(X,0) \arrow[rr, "\sim"] \arrow[dr] & & (X_0 \times T, 0) \arrow[dl] \\
& (T,0) &
\end{tikzcd}$$

Proposition 2. *Let $f:(T,0) \to (\mathcal{O}_n^p, f_0)$ be a deformation.*

(a) *f is K-trivial, if and only if for any coordinate s on $(T,0)$ and any point $t \in (T,0)$ the tangent-vector $\partial_s f_t := \dfrac{\partial f}{\partial s}(t) \in \mathcal{O}_n^p$ to the path f_t belongs analytically to the tangent-space $TK(f_t) := (f_t) \cdot \mathcal{O}_n^p + \mathfrak{m}_n \cdot j(f_t)$ of the K-orbit of f_t, i.e., by definition, there exist analytic mappings $M:(T,0) \to (M_p(\mathcal{O}_n), M_0)$ and $\psi:(T,0) \to (\mathcal{O}_n^n, \psi_0)$, $\psi_t(0) = 0 \in \mathbb{C}^n$, with*

$$\partial_s f_t = M_t \cdot f_t + \psi_t \cdot \partial_x f_t \qquad (x = (x_1, \ldots, x_n) \textit{ coordinates}).$$

(b) *Let G denote the group $G = Gl_p(\mathcal{O}_n)$; then f is G-trivial, if and only if $\partial_s f_t$ belongs analytically to $(f_t) \cdot \mathcal{O}_n^p$ for $t \in (T,0)$.*

Proof. To simplify notation, we assume $(T,0) = (\mathbb{C},0)$. For given deformations $L:(T,0) \to (Gl_p(\mathcal{O}_n),1)$ and $\phi:(T,0) \to (\mathrm{Aut}(\mathbb{C}^n,0), \mathrm{Id}_{\mathbb{C}^n})$ the equation $f_0 = L_t \cdot (f_t \circ \phi_t)$ is equivalent to $0 = \partial_s(L_t \cdot (f_t \circ \phi_t))$ for some coordinate s on T. By chain and product-rule we obtain

$$0 = \partial_s L_t \cdot (f_t \circ \phi_t) + L_t \cdot \partial_s \phi_t \cdot (\partial_x f_t \circ \phi_t) + L_t \cdot (\partial_s f_t \circ \phi_t) \in \mathcal{O}_n^p. \qquad (*)$$

Multiplication from the left by L_t^{-1} and composition from the right with ϕ_t^{-1} give

$$\partial_s f_t \in (f_t)\cdot \mathcal{O}_n^p + m_n \cdot j(f_t) = TK(f_t) \qquad (**)$$

analytically in $t\in(T,0)$. Conversely, assume (**): then

$$\partial_s f_t = M_t \cdot f_t + \psi_t \cdot \partial_x f_t \in TK(f_t) \qquad (***)$$

with deformations $M:(T,0)\to(M_p(\mathcal{O}_n), M_0)$ and $\psi:(T,0)\to(\mathcal{O}_n^n, \psi_0)$ with $\psi_t(0) = 0\in\mathbb{C}^n$. The differential equations

$$\begin{aligned} \partial_s\phi_t &= \psi_t\circ\phi_t \qquad & \phi_0 &= \mathrm{Id}_{\mathbb{C}^n} \\ \partial_s L_t &= L_t\cdot(M_t\circ\phi_t) \qquad & L_0 &= 1 \end{aligned}$$

admit analytic solutions $\phi:(T,0)\to(\mathrm{Aut}(\mathbb{C}^n,0),\mathrm{Id}_{\mathbb{C}^n})$ and $L:(T,0)\to(Gl_p(\mathcal{O}_n),1)$ by integration of vectorfields. By substitution in (***) we get (*) and hence the K-triviality of f. This proves (a) and (b) follows similarly.

Remark. Note that in the above proof, $\psi_t(0)=0$ implies $\phi_t(0)=0$. This assures, that the varieties X_t and X_0 defined by f_t and f_0 are actually isomorphic at the origin, $(X_t,0)\cong(X_0,0)$. The weaker assumption

$$\partial_s f_t \in (f_t)\cdot\mathcal{O}_n^p + j(f_t)$$

might allow $\phi_t(0)=a_t \neq 0\in\mathbb{C}^n$ and would only imply $(X_t,a_t)\cong(X_0,0)$ but not necessarily $(X_t,0)\cong(X_0,0)$.

The following result shows that in certain cases, the hypothesis of Proposition 2, $\partial_s f_t\in TK(f_t)$ analytically in $t\in T$ can be weakened to $\partial_s f_t\in TK(f_t)$ pointwise:

Lemma 2. *Let $g_t:(T,0)\to((\mathcal{O}_n^p)^q, g_0)$ be an analytic map and M_t be the submodule of $\mathcal{O}_n^p$ generated by the q components of g_t.*

If $\{M_t\}_{t\in(T,0)}$ is a flat family over $(T,0)$, then for any analytic map $h_t:(T,0)\to(\mathcal{O}_n^p, h_0)$

$$h_t\in M_t \quad \text{pointwise for} \quad t\in(T,0)$$

implies

$$h_t\in M_t \quad \text{analytically in} \quad t\in(T,0).$$

This holds in particular for families $M_t = M_0$ constant.

Proof. By [Ha], Th. 1, ch. I.2, (cf. also [Ga] and [Hi]), $\{M_t\}_{t\in(T,0)}$ is flat if and only if

$$M_t\oplus\Delta(M_0)=\mathcal{O}_n^p \quad \text{for} \quad t\in(T,0), \qquad (*)$$

where $\Delta(M_0)$ is a complement of M_0 in $\mathcal{O}_n^p$ given by the Division Theorem for modules:

$$M_0\oplus\Delta(M_0)=\mathcal{O}_n^p.$$

Note that the equality (*) holds analytically in t (loc. cit. Th. 2, A II). Division of h_t by M_t gives

$$h_t \equiv \bar{h}_t \operatorname{mod} M_t$$

analytically in t with $\bar{h}_t \in \Delta(M_0)$. As $h_t \in M_t$ pointwise, $\bar{h}_t \in M_t \cap \Delta(M_0) = 0$, i.e., $\bar{h}_t = 0$ and therefore $h_t \in M_t$ analytically in t.

The singular subspace of a variety

Definition 2. For a manifold S in $\mathbb{C}^n$ and $f \in \mathcal{O}_n^p$ we denote $j_S(f)$ the submodule of $\mathcal{O}_n^p$ generated by the partial derivatives of f in direction S:

$$j_S(f) = (\partial_{s_1} f, \ldots, \partial_{s_k} f) \subseteqq \mathcal{O}_n^p$$

for a coordinate system $s = (s_1, \ldots, s_k)$ on S. Note that $j_S(f)$ does not depend on the choice of coordinates. We then define

$$T_S f = (f) \cdot \mathcal{O}_n^p + m_n \cdot j_S(f) + j_Z(f) \subseteqq \mathcal{O}_n^p$$

where $Z \subseteqq \mathbb{C}^n$ denotes any complement of S in $\mathbb{C}^n$, $S \times Z \cong \mathbb{C}^n$, whose coordinates form a set of equations of S. The module $T_S f$ does not depend on the choice of Z. In the special cases $S = \mathbb{C}^n$ and $S = 0$ we get $T_{\mathbb{C}^n} f = TK(f)$ and $T_0 f = (f) \cdot \mathcal{O}_n^p + j(f)$. Let $(X, 0)$ be the variety defined by f. We denote by $\operatorname{Sing}_S(X, 0)$ and call the singular subspace in direction S of $(X, 0)$ the set of singular points of $(X, 0)$ provided with the analytic structure of the module $\mathcal{O}_n^p / T_S f$. Even though $\operatorname{Sing}_S(X, 0)$ is not a variety in the strict sense, if $(X, 0)$ is not a hypersurface, we consider it as a subvariety of $(X, 0)$. In the appendix we shall indicate how one could extend the notion of analytic varieties and morphisms between them to varieties defined by analytic modules instead of local rings.

For two varieties $(X, 0)$ and $(Y, 0)$ in $(\mathbb{C}^n, 0)$ and two manifolds $S, S' \subseteqq \mathbb{C}^n$ we say that $\operatorname{Sing}_S(X, 0)$ and $\operatorname{Sing}_{S'}(Y, 0)$ are isomorphic, if there exist a coordinate change $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending S to S' and an $\mathcal{O}_n$-linear isomorphism $L : \mathcal{O}_n^p \to \mathcal{O}_n^p$ such that

$$T_{S'} g = T_{\phi(S)}(L \cdot (f \circ \phi^{-1})) \subseteqq \mathcal{O}_n^p.$$

where f and $g \in \mathcal{O}_n^p$ denote defining equations for $(X, 0)$ and $(Y, 0)$.

Lemma 3. *The singular subspace* $\operatorname{Sing}_S(X, 0)$ *is an analytic invariant of* $(X, 0)$. *More explicitely, if* $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *is an isomorphism sending* $(X, 0)$ *to* $(Y, 0)$, *then* $\operatorname{Sing}_S(X, 0)$ *and* $\operatorname{Sing}_{\phi(S)}(X, 0)$ *are isomorphic in the above sense.*

The proof follows immediately from the definitions.

Remarks. For $S = 0 \subseteqq \mathbb{C}^n$ we mostly write $\operatorname{Sing}(X, 0)$ instead of $\operatorname{Sing}_0(X, 0)$. If $(X, 0)$ is a hypersurface, $\operatorname{Sing}(X, 0)$ is the usual singular subspace defined by the ideal $(f) + j(f)$. If $(X, 0)$ is not a hypersurface, the singular subspace defined by the minors of the jacobian matrix of f and the singular subspace defined here are actually different. As it turns out, our proof needs the module structure of $\mathcal{O}_n^p / T_S f$ on the singular locus of $(X, 0)$ to be able to recover $(X, 0)$ from it.

Note that for $S \subseteqq S'$ one has $\operatorname{Sing}_S(X, 0) \subseteqq \operatorname{Sing}_{S'}(X, 0)$.

The analytic and the singular stratum of a variety

Definition 3. The analytic stratum of a variety $(X,0)$ in $(\mathbb{C}^n,0)$ is the set-germ A of points in $(\mathbb{C}^n,0)$ along which a representative X of $(X,0)$ is trivial:

$$A=\{a\in(\mathbb{C}^n,0),(X,a)\cong(X,0)\}\subseteqq(\mathbb{C}^n,0).$$

We say that $(X,0)$ has isolated analytic type, if $A=0$, i.e., $(X,a)\ncong(X,0)$ for a in a neighborhood of $0\in\mathbb{C}^n$.

Theorem. (Ephraim, [E 2]) *Let $(X,0)$ be a variety in $(\mathbb{C}^n,0)$.*

(1) *The analytic stratum A of $(X,0)$ is the germ of a submanifold of $(\mathbb{C}^n,0)$.*

(2) *The variety $(X,0)$ is a product along A:*

$$(X,0)\cong(X_0,0)\times A,$$

for some variety $(X_0,0)$ in $(\mathbb{C}^n,0)$.

Definition 4. For a manifold $S\subseteqq\mathbb{C}^n$ we define the singular stratum in S-direction of $(X,0)$ as the stratum $\Sigma=\Sigma_S$ in $(\mathbb{C}^n,0)$ along which $\mathrm{Sing}_S(X,0)$ is trivial. This is again a manifold as Ephraim's Theorem extends to analytic modules (cf. appendix). We say that $(X,0)$ has isolated singularity type, if $\Sigma_0=0$, i.e.,

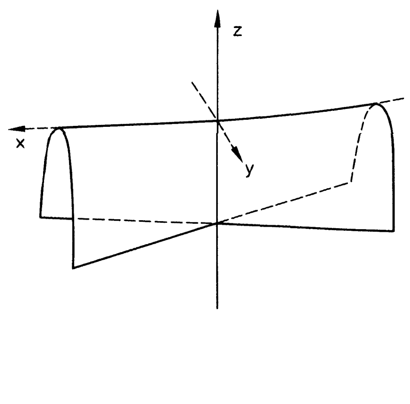
$$\mathrm{Sing}(X,a)\ncong\mathrm{Sing}(X,0)$$

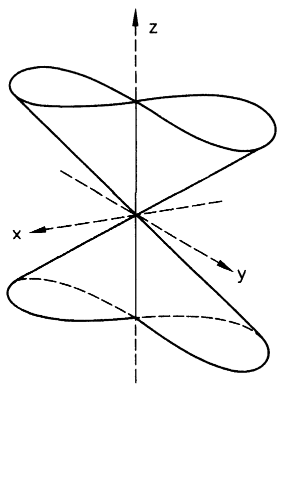
for a in a small neighborhood of $0\in\mathbb{C}^n$. Obviously, $(X,0)$ being of isolated singularity type implies $(X,0)$ of isolated analytic type. The converse is not true in general: in §4 we shall construct a variety $(X,0)$ in $(\mathbb{C}^n\times\mathbb{C},0)$ with constant singular type along $(0\times\mathbb{C},0)$ but varying analytic type.

Examples.

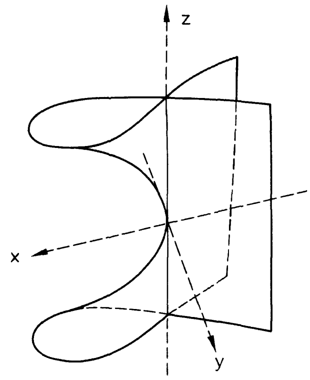
(1) Whitney Umbrella: $(X,0)\subseteqq(\mathbb{C}^3,0):x^2z-y^2=0$.
$\mathrm{Sing}(X,0):(x^2,xz,y)\subseteqq\mathcal{O}_3$, I.S.T.

(2) Cone over the Tacnode: $(X,0)\subseteqq(\mathbb{C}^3,0):x^4+y^4-xyz^2=0$.
$\mathrm{Sing}(X,0):(4x^3-yz^2,4y^3-xz^2,xyz)\subseteqq\mathcal{O}_3$, I.S.T.



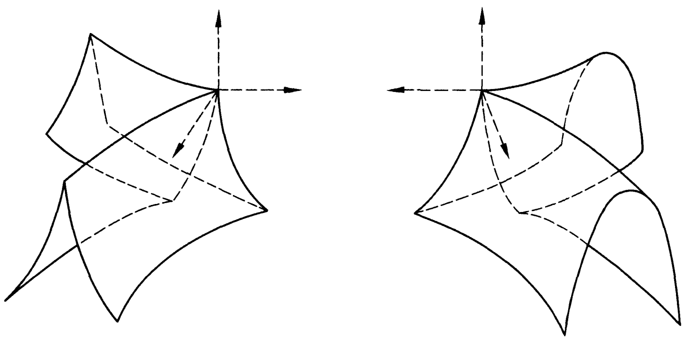


(3) Vanishing node: $x^3+x^2z^2-y^2=0$.
$\mathrm{Sing}(X,0):(3x^2+2xz^2, y, x^2z)$, I.S.T.

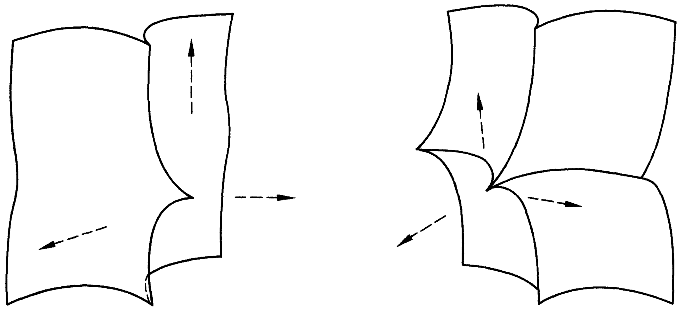


Here some other examples of varieties with isolated singularity type:

(4) The product of two cusps with perpendicular tangentlines at 0.
(5) Cusp deforming along a cusp to a parabola.



(6) The Peak: Cusp walking along cusp with same tangentline, stretching as 0 approaches.
(7) The Thorn: Product of two cusps with same tangentline at 0:
$x=s^2+t^2$, $y=s^3$, $z=t^3$.



Remark. For different $S \subseteqq \mathbb{C}^n$ the stratum Σ_S along which $\mathrm{Sing}_S(X,0)$ is trivial, might vary. However, one should expect that the stratum becomes smaller as S increases:

Proposition 3. *Let $S \subseteqq S'$ be two manifolds in $\mathbb{C}^n$ and Σ, Σ' the corresponding singular strata of $(X,0)$. Assume $(X,0)$ is a hypersurface. If $\Sigma \supseteqq S'$, then $\Sigma \supseteqq \Sigma'$.*

Remark. We don't know, if $\Sigma \supseteqq \Sigma'$ in general. For hypersurfaces, setting $S=0$ and $S'=\Sigma$, it follows that $S' \supseteqq \Sigma'$. This will be useful in applications of the theorem of the next section.

Proof. Let $g=(g_1,\dots,g_q)$ and $g'=(g'_1,\dots,g'_q)$ denote systems of defining equations of $\mathrm{Sing}_S(X,0)$ and $\mathrm{Sing}_{S'}(X,0)$. Choosing a decomposition $S' \times Z \cong \mathbb{C}^n$ and coordinates x of $\mathbb{C}^n$, s of S, (s,s') of S' and z of Z we can write

$$g=(\bar{f}, x\partial_s f, x\partial_{s'} f, \partial_z f, \partial_{s'} f) \quad \text{and} \quad g'=(\bar{f}, x\partial_s f, x\partial_{s'} f, \partial_z f, 0)$$

with the obvious notation. For $t \in \mathbb{C}^n$, we set $h_t(x)=g(x+t)$, $h'_t(x)=g'(x+t)$. After a suitable coordinate change in $\mathbb{C}^n$, we can assume by Ephraim's Theorem that $\mathrm{Sing}_S(X,0)$ is constant along Σ. As $S' \subseteqq \Sigma$, Proposition 2(b) gives

$$j_{S'}(h_t) \subseteqq (h_t) \cdot \mathcal{O}_n^q \tag{1}$$

analytically in $t \in S'$. By part (a) of Proposition 2:

$$j_{\Sigma'}(h'_t) \subseteqq (h'_t) \cdot \mathcal{O}_n^q + \mathfrak{m}_n \cdot j_{\mathbb{C}^n}(h'_t) \tag{2}$$

analytically in $t \in \Sigma'$. We have to show that $\mathrm{Sing}_S(X,0)$ is trivial along Σ', i.e.,

$$j_{\Sigma'}(h_t) \subseteqq (h_t) \cdot \mathcal{O}_n^q + \mathfrak{m}_n \cdot j_{\mathbb{C}^n}(h_t) \tag{3}$$

analytically in $t \in \Sigma'$. This will follow from (2) if one can prove that

$$j_V(h_t) \equiv j_V(h'_t) \bmod (h_t) \cdot \mathcal{O}_n^q \tag{4}$$

analytically in $t \in \Sigma'$ and for any submanifold V of $\mathbb{C}^n$.

Let v be a coordinate on V. Comparing h_t and h'_t componentwise, it becomes clear that it is sufficient to show

$$\partial_v \partial_{s'} f_t \equiv 0 \bmod (h_t)$$

analytically in $t \in \Sigma'$, where $f_t(x)=f(x+t)$. For $v=s$ or $v=s'$, this follows directly from (1). For $v=z$, write $\partial_z \partial_{s'} f = \partial_{s'} \partial_z f$ and again (1) applies. This proves the Proposition.

2. Characterization of varieties by their singular subspace

Theorem. *Let $(X,0)$ and $(Y,0)$ be two germs of complex-analytic varieties in $(\mathbb{C}^n,0)$. Let $S \subseteqq \mathbb{C}^n$ be a manifold containing the stratum $\Sigma \subseteqq \mathbb{C}^n$ along which the singular subspace* $\mathrm{Sing}_S(X,0)$ *of $(X,0)$ is trivial.*

The varieties $(X,0)$ *and* $(Y,0)$ *are isomorphic if and only if their singular subspaces* $\mathrm{Sing}_S(X,0)$ *and* $\mathrm{Sing}_{\phi(S)}(Y,0)$ *are isomorphic via an isomorphism* $h=(L,\phi)$.

Remarks. (1) The theorem fails over $\mathbb{R}$ or a field of positive characteristic. It also fails if the manifold S does not contain the stratum Σ (cf. the example, § 4).

(2) The assertion becomes stronger the smaller a manifold S satisfying $S \supseteq \Sigma$ can be chosen.

(3) For practical computations, the most interesting case occurs, when $(X,0)$ and $(Y,0)$ are reduced hypersurfaces: the singular subspaces are then varieties of smaller dimension and have usually simpler equations.

Example. The Whitney umbrella $(X,0) \subseteq (\mathbb{C}^3,0)$ defined by $x^2-y^2z=0$ is of isolated singularity type and hence determined by $\mathrm{Sing}(X,0)=\mathrm{Sing}_0(X,0)$, i.e., the ideal $(x,yz,y^2) \subseteq \mathcal{O}_3$. Therefore in order to check that a variety $(Y,0) \subseteq (\mathbb{C}^3,0)$ is isomorphic to $(X,0)$, it is sufficient to show that the ideal defining $\mathrm{Sing}(Y,0)$ is isomorphic to (x,yz,y^2).

Particular cases. (1) Take $S=\mathbb{C}^n$: The analytic type of $(X,0)$ is always determined by the largest singular subspace $\mathrm{Sing}_{\mathbb{C}^n}(X,0)$.

(2) Assume $S \subseteq \mathbb{C}^n$ contains both the strata along which $\mathrm{Sing}_S(X,0)$ and $\mathrm{Sing}_S(Y,0)$ are trivial: then $(X,0) \cong (Y,0)$ if and only if $\mathrm{Sing}_S(X,0) \cong \mathrm{Sing}_S(Y,0)$.

(3) The geometrically most interesting case occurs perhaps when $(X,0)$ and $(Y,0)$ are of isolated singularity type (def. 4): $(X,0) \cong (Y,0)$ if and only if $\mathrm{Sing}(X,0) \cong \mathrm{Sing}(Y,0)$ $(S=0)$.

(4) Let S be now the stratum of $\mathrm{Sing}(X,0)$-triviality of a hypersurface $(X,0)$. Then by Proposition 3 and the remark following it, S contains the stratum of $\mathrm{Sing}_S(X,0)$-triviality. Let S' denote the corresponding stratum for the hypersurface $(Y,0)$, then $(X,0) \cong (Y,0)$ if and only if $\mathrm{Sing}_S(X,0) \cong \mathrm{Sing}_{S'}(Y,0)$. This provides a way to find a manifold S strictly smaller than $\mathbb{C}^n$ such that $\mathrm{Sing}_S(X,0)$ characterizes $(X,0)$.

(5) Suppose $(X,0)$ is of isolated singularity type. If $(X,0)$ is also of finite singularity type ($\dim_{\mathbb{C}}(\mathcal{O}_n^p/T_0 f<\infty$), then the analytic type of $(X,0)$ is determined by the finite-dimensional vectorspace $\mathcal{O}_{\mathrm{Sing}(X,0)}=\mathcal{O}_n^p/T_0 f$ and its $\mathcal{O}_n$-modulestructure. Example: $(X,0)$ is a complete intersection with isolated singularity.

(6) Let $(X,0)$ be a hypersurface. As $\mathrm{Sing}(X,0)$ is then defined by an ideal, we can consider $\mathrm{Sing}(\mathrm{Sing}(X,0))$ (cf. appendix for the non hypersurface case and the iteration of passing to the singular subspace). If for instance $\mathrm{Sing}(X,0)$ has isolated singularity type (and consequently $(X,0)$ as well), $(X,0)$ is characterized by $\mathrm{Sing}(\mathrm{Sing}(X,0))$, which for $\mathrm{Sing}(X,0)$ finite singularity type, would be again finite dimensional.

(7) Let $(X,0)$ be arbitrary and $S \subseteq \mathbb{C}^n$ containing the $\mathrm{Sing}_S(X,0)$-stratum of triviality. Then $(X,0)$ is a product $(Y\times Z,0)$ with $(Z,0)$ smooth, if and only if $\mathrm{Sing}_S(X,0)$ is a product along $(Z,0)$.

(8) The theorem also provides an extension of Shoshitaishvili's result on right-equivalence [Sh]: Let f and $g \in \mathcal{O}_n$ define hypersurfaces of isolated singularity type. Assume f weakly quasihomogeneous (cf. Prop. 4, § 4). Then f and g are right equivalent if and only if $\mathcal{O}_n/j(f)$ and $\mathcal{O}_n/j(g)$ are isomorphic as $\mathbb{C}$-algebras.

Remark. Let $i: \mathcal{O}_n \to \mathcal{O}_n^q$ be a $\mathbb{C}$-linear map such that the subvariety $C(X,0)$ of $(X,0) = (f^{-1}(0),0)$, $f \in \mathcal{O}_n$, defined by the components of $i(f)$ is an analytic invariant of $(X,0)$. Then if $\mathrm{Sing}_S(X,0) \subsetneqq C(X,0) \subsetneqq (X,0)$ with $S \supseteqq \Sigma_S = \mathrm{Sing}_S(X,0)$-stratum, $C(X,0)$ determines the analytic type of $(X,0)$. This combined with the example of § 4 illustrates, that in general $\mathrm{Sing}_S(X,0)$ with $S \supseteqq \Sigma_S$ is the "smallest" subvariety of the hypersurface $(X,0)$ which characterizes the analytic type of $(X,0)$.

Proof of the Theorem. If $(X,0) \cong (Y,0)$, then $\mathrm{Sing}_S(X,0) \cong \mathrm{Sing}_{\phi(S)}(Y,0)$ by Lemma 3. To prove the inverse implication, we shall proceed in several steps:

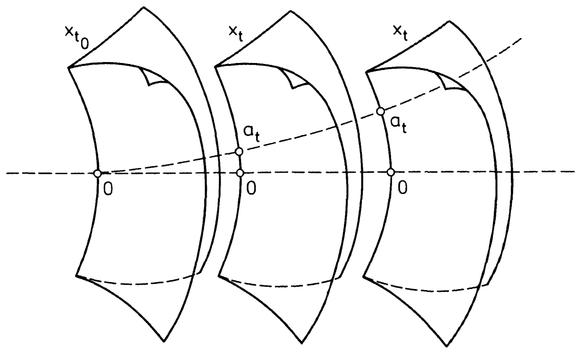
(1) Reduction to the case $\mathrm{Sing}_S(X,0) = \mathrm{Sing}_S(Y,0)$: The isomorphism $h = (L, \phi)$ between $\mathrm{Sing}_S(X,0)$ and $\mathrm{Sing}_{\phi(S)}(Y,0)$ is induced from $\phi \in \mathrm{Aut}(\mathbb{C}^n,0)$ and $L \in Gl_p(\mathcal{O}_n)$. Replacing $(Y,0)$ by $\phi^{-1}(Y,0)$ and the system of equations $g \in \mathcal{O}_n^p$ of $\phi^{-1}(Y,0)$ by $(L^{-1} \circ \phi) \cdot g$, one checks immediately that $\mathrm{Sing}_S(X,0) = \mathrm{Sing}_S(Y,0)$.

(2) Set $f_t = f + t(g-f) \in \mathcal{O}_n^p$, $t \in \mathbb{C}$, $f, g \in \mathcal{O}_n^p$ equations for $(X,0)$ and $(Y,0)$: Then the family $\{(X_t,0)\}_{t \in \mathbb{C}}$ defined by f_t describes a deformation from $(X,0)$ to $(Y,0)$.

(3) $\mathrm{Sing}_S(X_t,0) = \mathrm{Sing}_S(X_0,0)$ for $t \in T \subsetneqq \mathbb{C}$ Zariski-open: Let h and $k \in (\mathcal{O}_n^p)^q$ be defining equations for $\mathrm{Sing}_S(X,0)$ and $\mathrm{Sing}_S(Y,0)$ derived from f and g in the obvious way. Then $h_t := h + t(k-h)$ defines $\mathrm{Sing}_S(X_t,0)$ and Lemma 1 proves the equality (cf. also Lemma 1' of appendix).

(4) Reduction to the assertion: For any $t_0 \in T$, the deformation $\{(X_t,0)\}_{t \in (T,t_0)}$ of $(X_{t_0},0)$ is trivial: If $\{(X_t,0)\}$ is locally trivial at any $t_0 \in T$, it will be globally trivial over any compact subset of the connected set T, hence $(X_0,0) = (X,0)$ and $(X_1,0) = (Y,0)$ will be isomorphic. (This argument fails over $\mathbb{R}$ or for char > 0, as then T need not to be connected.)

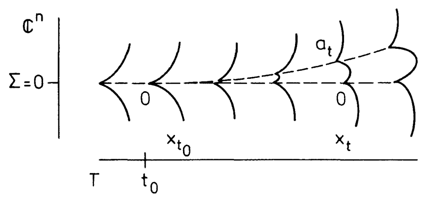
(5) The deformation $\{(X_t,0)\}_{t \in (T,t_0)}$ of $(X_{t_0},0)$ is trivial for any $t_0 \in T$: By (3), $\partial_t f_t = g - f \in T_S f_t = T_S f = T_S g$ for any $t \in T$. By Lemma 2, this inclusion is analytic in t. We can therefore apply Proposition 2 with $TK(f_t)$ replaced by $T_S f_t$ to the deformation f_t and obtain isomorphisms $\phi_t: (\mathbb{C}^n,0) \to (\mathbb{C}^n, a_t)$, $a_t = \phi_t(0)$, sending (X_t, a_t) to $(X_{t_0},0)$. As mentioned in the remark following Proposition 2, ϕ_t may not send origin to origin:



We claim that the initial hypothesis on the manifold S, namely that S contains the $\mathrm{Sing}_S(X,0)$-triviality stratum, will insure that ϕ_t actually does preserve the origin. Hence $a_t = 0 \in \mathbb{C}^n$ and $(X_t,0) \cong (X_{t_0},0)$ for all $t \in (T,t_0)$.

Notice first that from the way $T_S f_t$ is defined and enters in the proof of Proposition 2, the S-coordinates of a_t must be 0. Therefore, if a_t moves with t, it must move off S.

Second, the established isomorphism between (X_t, a_t) and $(X_{t_0}, 0)$ implies $\mathrm{Sing}_S(X_t, a_t) \cong \mathrm{Sing}_S(X_{t_0}, 0)$ and by (3): $\mathrm{Sing}_S(X_t, a_t) \cong \mathrm{Sing}_S(X_t, 0)$. This suggests that a_t belongs to the $\mathrm{Sing}_S(X_t, 0)$-stratum $\Sigma_t = \Sigma$. As $\Sigma \subsetneqq S$ by assumption, $a_t = 0$ would follow. However, one might imagine the following phenomena:



Here, $\mathrm{Sing}_S(X_t, a_t) \cong \mathrm{Sing}_S(X_t, 0)$ does not imply $a_t \in \Sigma$. In order to show that this type of coalescing cannot arise for the family $\{(X_t, 0)\}$, represent the situation on a small open neighborhood U of $0 \in \mathbb{C}^n$. By Lemma 1′(b) of the appendix, there exists $D \subsetneqq T$ open, $t_0 \in D$, such that for U sufficiently small and representatives X_t of $(X_t, 0)$ on U, $\mathrm{Sing}_S X_t = \mathrm{Sing}_S X_{t_0}$ holds on U for $t \in D$. This implies that the strata Σ_t coincide on U, $\Sigma_t = \Sigma_{t_0}$ for $t \in D$. Furthermore, reducing eventually U again, one can assume that $\Sigma_t \subsetneqq S$ on U. But then $\mathrm{Sing}_S(X_t, a_t) \cong \mathrm{Sing}_S(X_t, 0)$ implies for $a_t \in U$ and $t \in D$ that $a_t \in \Sigma_t \subsetneqq S$. As a_t was not allowed to move inside of S, $a_t = 0$, and $(X_t, 0) \cong (X_{t_0}, 0)$. This establishes (5) and concludes the proof of the theorem.

3. Appendix: Generalized analytic varieties

In this section we describe briefly a category of generalized analytic varieties in which the singular subspace of a variety as defined in § 1 is again a variety and an extension of the theorem of § 2 holds.

Definition 1. A germ of a (generalized) analytic variety $(X, 0)$ in $(\mathbb{C}^n, 0)$ is a couple

$$(X, 0) = ((X, 0), \mathcal{O}_{X,0})$$

such that

(1) $\mathcal{O}_{X,0} = \mathcal{O}_n^m / I$ is a quotient module of $\mathcal{O}_n^m := \mathcal{O}_n^{m_1} \otimes_{\mathcal{O}_n} \dots \otimes_{\mathcal{O}_n} \mathcal{O}_n^{m_d} = \mathcal{O}_n^{m_1 \cdots m_d}$ for some d-tuple $m = (m_1, \dots, m_d) \in N^d$.

(2) $(X, 0) = \{(X_r, 0)\}_{r \in N}$ is a collection of germs of subsets $(X_r, 0)$ of $(\mathbb{C}^n, 0)$ defined by

$$(X_r, 0) = \{x \in (\mathbb{C}^n, 0), rk(f(x)) \leqq r\}$$

for some presentation

$$\mathcal{O}_n^p \xrightarrow{f} \mathcal{O}_n^m \to \mathcal{O}_{X,0} \to 0 \quad (p \in N)$$

of $\mathcal{O}_{X,0}$. The rank of an element $A=(A_1, \ldots, A_p) \in (\mathbb{C}^m)^p$ is the maximal number of $\mathbb{C}$-linearly independent components A_i of A. The germs $(X_r, 0)$ do not depend on the choice of the presentation f.

Remark. As $\mathcal{O}_n^m \cong \mathcal{O}_n^m \otimes \mathcal{O}_n \otimes \ldots \otimes \mathcal{O}_n$, one can identify varieties with analytic modules $\mathcal{O}_n^m/I$ and $\mathcal{O}_n^{\bar{m}}/\bar{I}$, where $\bar{m}$ is obtained from m by adding 1's, $\bar{m}=(m_1, \ldots, m_d, 1, \ldots, 1)$, and $\bar{I}$ is the image of I in $\mathcal{O}_n^{\bar{m}}$.

On the other hand, for $m, m' \in N^d$ with $m \leqq m'$ component by component, we shall identify $\mathcal{O}_n^m/I$ and $\mathcal{O}_n^{m'}/I \times 0$, where $I \times 0$ denotes the image of I under the canonical injection $\mathcal{O}_n^m \hookrightarrow \mathcal{O}_n^{m'}$.

This signifies that we won't distinguish between varieties, whose modules differ only by free components (we are interested in the image of the presentation f but not in the cokernel). Therefore, whenever we compare two varieties $(X, 0)$ and $(X', 0)$, $\mathcal{O}_{X,0}=\mathcal{O}_n^m/I$, $\mathcal{O}_{X',0}=\mathcal{O}_n^{m'}/I'$, we may tacitly assume that $m=m' \in N^d$. Obviously, the underlying sets $(X_r, 0)$ are not affected by these identifications.

Morphisms. Any analytic map-germ $\phi:(\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ induces for arbitrary $m \in N^d$ a morphism

$$\phi^*: \mathcal{O}_n^m \to \mathcal{O}_n^m$$

sending $g \in \mathcal{O}_n^m$ to $\phi^*(g)=g \circ \phi$. Second, any matrix $L \in M_m(\mathcal{O}_n) := M_{m_1}(\mathcal{O}_n) \times \ldots \times M_{m_d}(\mathcal{O}_n)$ defines an $\mathcal{O}_n$-linear map

$$L: \mathcal{O}_n^m \to \mathcal{O}_n^m$$

multiplying component by component.

Definition 2. A morphism $h: \mathcal{O}_n^m \to \mathcal{O}_n^m$ is a couple $h=(L, \phi)$ with underlying map $L \circ \phi^*: \mathcal{O}_n^m \to \mathcal{O}_n^m$. As $\phi^* \circ M = \phi^*(M) \circ \phi^*$ for any $M \in M_m(\mathcal{O}_n)$, the composition of morphisms is well defined.

Let $(X, 0)$ and $(Y, 0)$ be two varieties in $(\mathbb{C}^n, 0)$ with $\mathcal{O}_{X,0}=\mathcal{O}_n^m/I$ and $\mathcal{O}_{Y,0}=\mathcal{O}_n^{m'}/J$, $m \in N^d$, $m' \in N^{d'}$. As described above, we may assume $m=m' \in N^d$. We say that two morphisms $h: \mathcal{O}_n^l \to \mathcal{O}_n^l$ and $h': \mathcal{O}_n^{l'} \to \mathcal{O}_n^{l'} (l, l' \in N^d)$ are equivalent on $(X, 0)$, if for any $k \in N^d$ with $k \geqq l, l', m$ (by components) the image of the induced map

$$h-h': \mathcal{O}_n^k \to \mathcal{O}_n^k$$

lies in $I \subseteqq \mathcal{O}_n^k$ (more precisely, if $\operatorname{Im}((h,0)-(h',0)) \subseteqq I \times 0 \subseteqq \mathcal{O}_n^k$). A morphism $h:(X,0) \to (Y,0)$ is the equivalence class over $(X, 0)$ of a morphism $H: \mathcal{O}_n^k \to \mathcal{O}_n^k$ $(k \geqq m)$ satisfying

$$H(J) \subseteqq I \subseteqq \mathcal{O}_n^k.$$

In particular, h induces a map $h: \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$.

The composition of two morphisms is well defined – we can talk of isomorphic analytic varieties: $(X,0) \cong (Y,0)$ if and only if there exist $\phi:(\mathbb{C}^n,0) \xrightarrow{\sim} (\mathbb{C}^n,0)$ and $L: \mathcal{O}_n^m \xrightarrow{\sim} \mathcal{O}_n^m$, $L \in Gl_m(\mathcal{O}_n)$, such that $h=L \circ \phi^*: \mathcal{O}_n^m \to \mathcal{O}_n^m$ induces a bijection $h: \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$.

Definition 3. Let $(X, 0)$ and $(X', 0)$ be two varieties in $(\mathbb{C}^n, 0)$, $\mathcal{O}_{X,0}=\mathcal{O}_n^m/I$ and $\mathcal{O}_{X',0} = \mathcal{O}_n^{m'}/I'$. We may assume $m=m'$. We say that $(X', 0)$ is a subvariety of $(X, 0)$ if $\mathcal{O}_{X',0}$ is a quotient of $\mathcal{O}_{X,0}$, say , $I \subseteqq I'$.

A morphism $h:(X,0)\to(Y,0)$ is an embedding, if it induces an isomorphism from $(X,0)$ onto a subvariety of $(Y,0)$.

In the so defined category one may carry out the usual operations, in particular, show that the fiber-product of two varieties over a third exists.

We shall now indicate, how one has to modify the statements and proofs of the preceeding sections in order to obtain the analogue results for generalized analytic varieties.

For $m\in N^d$ and $p\in N$, let K denote the group $K=Gl_{mp}(\mathcal{O}_n)\times \mathrm{Aut}(\mathbb{C}^n,0)$ with $Gl_{mp}(\mathcal{O}_n)=Gl_m(\mathcal{O}_n)\times Gl_p(\mathcal{O}_n)=Gl_{m_1}(\mathcal{O}_n)\times \ldots \times Gl_{m_d}(\mathcal{O}_n)\times Gl_p(\mathcal{O}_n)$, acting on $\mathcal{O}_n^{mp}:=\mathcal{O}_n^m\otimes_{\mathcal{O}_n}\mathcal{O}_n^p$ component by component. We have:

Lemma 1′. (a) *The fibers of the map*

$$\pi:\mathcal{O}_n^{mp}\to\{\text{isomorphism classes of analytic varieties}\}$$
$$f\to[(X,0)]$$

(respectively $\tau:\mathcal{O}_n^{mp}\to\{\text{analytic varieties}\}$, $f\to(X,0)$*) given by*

$$\mathcal{O}_n^p\xrightarrow{f}\mathcal{O}_n^m\to\mathcal{O}_{X,0}\to 0$$

are the K-orbits (respectively $Gl_p(\mathcal{O}_n)$*-orbits) in* $\mathcal{O}_n^{mp}$.

(b) *For any family* $\{(X_t,0)\}_{t\in(T,0)}$ *of varieties such that* $(X_t,0)\cong(X_0,0)$ *(respectively* $(X_t,0)=(X_0,0)$*), there exist open neighborhoods D of* $0\in T$ *and U of* $0\in\mathbb{C}^n$ *with representatives* X_t *of* $(X_t,0)$ *on U such that* $X_t\cong X_0$ (respectively $X_t=X_0$) *on U for* $t\in D$.

Proposition 1 holds as it stands.

Proposition 2′. *Let* $f:(T,0)\to(\mathcal{O}_n^{mp},f_0)$ *be a deformation.*

(a) *f is K-trivial, if and only if for any coordinate s on* $(T,0)$, $\partial_s f_t$ *belongs analytically in* $t\in(T,0)$ *to*

$$TK(f_t)=\sum M_{m_i}(\mathcal{O}_n)\cdot f_t+M_p(\mathcal{O}_n)\cdot f_t+\mathfrak{m}_n\cdot j(f_t)\subseteqq\mathcal{O}_n^{mp}.$$

(b) *f is G-trivial* $(G=Gl_p(\mathcal{O}_n))$, *if and only if* $\partial_s f_t\in M_p(\mathcal{O}_n)\cdot f_t\subseteqq\mathcal{O}_n^{mp}$ *analytically.*

Definition 2′. Set $T_S f=\sum M_{m_i}(\mathcal{O}_n)\cdot f+M_p(\mathcal{O}_n)\cdot f+\mathfrak{m}_n\cdot j_S(f)+j_Z(f)\subseteqq\mathcal{O}_n^{mp}$ and $\mathrm{Sing}_S(X,0)$ = the singular subspace of $(X,0)$ defined by $T_S f$. It is obvious from the definitions, that $\mathrm{Sing}_S(X,0)$ is a generalized analytic variety and a subvariety of $(X,0)$.

Lemmas 2 and 3, Definitions 3 and 4, and Ephraim's Theorem remain valid in the generalized context as they are. We don't know if Proposition 3 extends.

Theorem. *Let* $(X,0)$ *and* $(Y,0)$ *be two germs of generalized complex-analytic varieties in* $(\mathbb{C}^n,0)$. *Let* $S\subseteqq\mathbb{C}^n$ *be a manifold containing the stratum* $\Sigma\subseteqq\mathbb{C}^n$ *along which the singular subspace* $\mathrm{Sing}_S(X,0)$ *of* $(X,0)$ *is trivial.*

The varieties $(X,0)$ *and* $(Y,0)$ *are isomorphic if and only if the singular subspaces* $\mathrm{Sing}_S(X,0)$ *and* $\mathrm{Sing}_{\phi(S)}(Y,0)$ *of* $(X,0)$ *and* $(Y,0)$ *are isomorphic via an isomorphism* $h=(L,\phi)$.

The proof of the theorem can be translated word by word from the proof of the theorem of § 2.

Remarks. The procedure of passing to the singular subspace can now be iterated, looking at $\mathrm{Sing}_S(X,0)$, $\mathrm{Sing}_{S'}(\mathrm{Sing}_S(X,0))$ etc. It might happen that after a certain number of steps, the corresponding singular subspace is defined by a finite dimensional vectorspace, which, in turn, will therefore determine together with its $\mathcal{O}_n$-module structure the analytic type of the original variety $(X,0)$.

Example. $\mathrm{Sing}(X,0)$ is a complete intersection with isolated singularity.

4. Example

We shall construct a family of hypersurfaces $(X_t,0)$, $t\in\mathbb{C}$, with constant singular subspace $\mathrm{Sing}(X_t,0):=\mathrm{Sing}_0(X_t,0)=\mathrm{Sing}_0(X_0,0)$ (for $t\neq\pm 1$) but varying analytic type $(X_t,0)\not\cong(X_0,0)$.

This shows that the singular subspace $\mathrm{Sing}_0(X_0,0)$ need not determine the analytic type of the variety, which, in turn, illustrates, that the assumptions made in the theorem, § 2, are actually necessary.

Let $h:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be any function satisfying $h\notin j(h)\subseteq\mathcal{O}_n$. Define a family $f_t:(\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C},0)\to(\mathbb{C},0)$ by $f_t(x,y,z)=h(x)+(1+z+t)\cdot h(y)$, and let $(X_t,0)\subseteq(\mathbb{C}^{2n+1},0)$ be the hypersurface defined by f_t.

We claim that $\mathrm{Sing}_0(X_t,0)=\mathrm{Sing}_0(X_0,0)$ for $t\neq\pm 1$: Computation gives

$$j(f_t)=[j(h(x))+j(h(y))+(h(y))]\cdot\mathcal{O}_{2n+1}.$$

Hence

$$T_0 f_t=[(h(x))+(h(y))+j(h(x))+j(h(y))]\cdot\mathcal{O}_{2n+1}.$$

On the other hand, the family $\{(X_t,0)\}_{t\in\mathbb{C}}$ is not trivial: For, if $\{f_t\}_{t\in\mathbb{C}}$ were trivial, we would have by Proposition 2, § 1,

$$\begin{aligned}\partial_t f_t=h(y)&\in(f_t)+\mathfrak{m}_{2n+1}\cdot j(f_t)\\&=(f_t)+\mathfrak{m}_{2n+1}\cdot j(h(x))+\mathfrak{m}_{2n+1}\cdot j(h(y))+\mathfrak{m}_{2n+1}\cdot(h(y)).\end{aligned}$$

Solving for $h(y)$ implies either $h(y)\in j(h(y))$ or $h(x)\in j(h(x))$ contradicting the assumption on h.

Proposition 4. *(cf.* [Sa] *for notations and the isolated singularity case). Let $f\in\mathcal{O}_n$ define a hypersurface $(X,0)$ which is of isolated analytic type* (def. 3). *Then $f\in j(f)$ $=j_{\mathbb{C}^n}(f)$ if and only if f is K-equivalent to a weakly quasihomogeneous $g\in\mathcal{O}_n$.*

Proof. If f is weakly quasihomogeneous with respect to weights $d_1,\dots,d_n\in Z$ on coordinates $x_1,\dots,x_n$ on $\mathbb{C}^n$, Euler's formula says

$$\deg(f)\cdot f(x)=\sum_{i=1}^{n} d_i\cdot x_i\cdot\frac{\partial f}{\partial x_i}(x)\in j(f).$$

Assume conversely $f\in j(f)$. Note that the proof of 4.1 in [Sa] is valid for any $f\in\mathcal{O}_n$, provided $f\in j(f)$ implies $f\in\mathfrak{m}_n\cdot j(f)$ by means of 3.3. It will be therefore sufficient to show that f isolated analytic type implies $f\in m_n\cdot j(f)$. Write $f=\sum a_i\cdot\partial_{x_i}f$ with

$a_i \in \mathcal{O}_n$. If for at least one i, $a_i = c_i - b_i$ with $b_i \in m_n$ and $c_i \in \mathbb{C}$, $c_i \neq 0$, the vector $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ would be non-zero. By a $\mathbb{C}$-linear coordinate change, one can achieve $c = (0, 1, 0, \dots, 0)$, i.e.,

$$\partial_{x_2} f = f + \sum b_i \cdot \partial_{x_i} f .$$

By Proposition 2(a) it follows that $(X, 0)$ is trivial along the x_2-axis and hence $(X, 0)$ not of isolated analytic type.

5. Part II: $\mathscr{A}$-equivalence theorems

One of the principal tools for our results on $\mathscr{A}$-equivalence will be C^k-level preserving maps. These are maps of the form $F: \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^p \times \mathbb{C}^k$, $F(0) = 0$, $F(x, u) = (\bar{f}(x, u), u)$. If $f = \bar{f}(x, 0)$ has particular importance, we may call F an unfolding of f. We first define some objects for $\mathbb{C}$-level preserving maps, analogous to the $\mathscr{A}$-tangent spaces for map germs.

Definition 1. Let $F: (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ be a $\mathbb{C}$-level preserving map germ. Then $j_{\mathbb{C}^n}(F)$ is the $\mathcal{O}_{n+1}$-module generated by $\dfrac{\partial \bar{f}}{\partial x_i}$,

a) $\bar{T}\mathscr{A}(F) = m_n \cdot j_{\mathbb{C}^n}(F) + F^*(m_p \mathcal{O}^p_{p+1}) \subseteqq \mathcal{O}^p_{n+1}$

b) $\bar{T}\mathscr{A}_e(F) = j_{\mathbb{C}^n}(F) + F^*(\mathcal{O}^p_{p+1}) \subseteqq \mathcal{O}^p_{n+1}$.

In this section the exponent of an ideal indicates a power of the ideal, while the exponent on $\mathcal{O}_{p+1}$ indicates Cartesian product. The ideal generated by the $p \times p$ minors of the jacobian matrix of f is denoted $J(f)$.

We say a level preserving map germ is trivial in a region $B \subseteqq \mathbb{C}$ if there exist $\mathbb{C}$-level preserving biholomorphic germs H of $(\mathbb{C}^n \times \mathbb{C},\ 0 \times B)$ and K of $(\mathbb{C}^p \times \mathbb{C}, 0 \times B)$ such that

$$K \circ F \circ H = f \times 1 .$$

We now give some well known criteria for triviality.

Proposition 1. (Thom-Levine) a) *Suppose B is a region of $\mathbb{C}$. A $\mathbb{C}$-level preserving map germ $F: (\mathbb{C}^n \times \mathbb{C}, 0 \times B) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times B)$ is trivial iff $\partial_t F^a \in \bar{T}\mathscr{A} F^a$ for all $a \in B$. (Here F^a denotes the germ of F at $(0, a)$.)*

b) *Suppose $B = 0$, then F is trivial iff $\partial_t F^0 \in \bar{T}\mathscr{A}_e F^0$* (cf. [duP] p. 113 for a) *and* [M] p. 8 for b)).

Throughout this section $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ will be a complex analytic germ of finite singularity type. If $g: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is any other holomorphic germ, let $G(x, t) = (f(x) + t(g(x) - f(x)), t)$ be the level preserving germ linking f and g. We wish to show under various hypotheses on f and g that G is trivial, using Proposition 1. Our method of attack is to compare $\bar{T}\mathscr{A}(G)$ with $\bar{T}\mathscr{A}(F)$, where $F(x, t) = (f(x), t)$. The key technique in this comparison is to study the various module structures of $\bar{T}\mathscr{A}(F)$ and of $\bar{T}\mathscr{A}_e(F)$. We begin with a lemma which shows that the module structure of $\bar{T}\mathscr{A}(F)$ has a certain convexity property.

In what follows, let $H(x, t) = (f(x), g(x), t) \in \mathcal{O}^{2p+1}_{n+1}$.

Lemma 1. a) *Suppose $T\mathscr{A}(f)=T\mathscr{A}(g)$, then $\bar{T}\mathscr{A}(F)\supseteqq H^*(\mathcal{O}^p_{2p+1})$ and $\bar{T}\mathscr{A}(F)$ is an $H^*(\mathcal{O}_{2p+1})$-module.*

b) *Suppose $T\mathscr{A}_e(f)=T\mathscr{A}_e(g)$ then $\bar{T}\mathscr{A}_e(F)$ is an $H^*(\mathcal{O}_{2p+1})$-module.*

Proof. a) It is obvious that $j_{\mathbb{C}^n}(F)$ is closed under multiplication by $H^*(\mathcal{O}_{2p+1})$. It suffices to consider $H^*(\mathfrak{m}_{2p+1})\cdot F^*(\mathfrak{m}_p\mathcal{O}^p_{p+1})$. Our method of proof will also show $H^*(\mathfrak{m}_{2p}\cdot\mathcal{O}^p_{2p+1})\subseteqq\bar{T}\mathscr{A}(F)$.

Note that $T\mathscr{A}(f)\subseteqq\bar{T}\mathscr{A}(F)$, and that this implies $(F^*\mathcal{O}_{p+1})\cdot(g^*(\mathfrak{m}_p\mathcal{O}^p_p))\subseteqq\bar{T}\mathscr{A}(F)$. In particular, $g_k\cdot e_i=a_{k,i}\circ F+r_{k,i}\,\partial_x F$, where e_i is a canonical basis of $\mathbb{C}^p$. Hence, $A_{ki}(x,y)=x_k e_i-a_{ki}(y)$ has the property that $A_{k,i}(g,f)\in\mathfrak{m}_n\cdot j_{\mathbb{C}^n}F$.

Suppose $B(x,y,t)\in\mathfrak{m}_{2p+1}$. Consider $(B\circ H)\cdot(D\circ F)=\sum(D\circ F)(B\circ H)e_l$, $D\in O^p_{p+1}$. By the Division Theorem for modules we can write $B(x,y,t)e_l=\sum_{k,i}Q_{k,i,l}(x,y,t)\cdot A_{ki}(x,y)+C_l(y,t)$. Then $(B\circ H)e_l\equiv C_l\circ F\bmod\mathfrak{m}_n\cdot j_{\mathbb{C}^n}F$. Hence, $\sum(D_l\circ F)\cdot(B\circ H)e_l\in\bar{T}\mathscr{A}(F)$.

If $B\in\mathfrak{m}_{2p}$, then $C_l(y,t)\in\mathfrak{m}_p\cdot\mathcal{O}_{p+1}$, thus $(B\circ H)e_l\in\bar{T}\mathscr{A}(F)$ also. This implies $H^*(\mathfrak{m}_{2p}\cdot\mathcal{O}^p_{2p+1})\subseteqq\bar{T}\mathscr{A}(F)$. The proof of b) is similar.

Remark. The above proof gives the following useful fact. Suppose $h\in\mathcal{O}_n$ such that $h\cdot e_i\in T\mathscr{A}(f)$ or $T\mathscr{A}_e(f)$ then, $T\mathscr{A}(f)$ (resp. $T\mathscr{A}_e(f)$) is an $\mathcal{O}_{p+1}$-module via (h,f).

A key assumption in what follows is that all our germs have finite singularity type, or F.S.T. (i.e. $T\mathscr{K}(f)\supseteqq\mathfrak{m}_n^k\cdot\mathcal{O}^p_n$ for some k). It is obvious that if $T\mathscr{A}(f)=T\mathscr{A}(g)$, and f has F.S.T. then so does g. Does the equality $T\mathscr{A}_e(f)=T\mathscr{A}_e(g)$ imply the same result?

Proposition 2. *Suppose $T\mathscr{A}_e(f)=T\mathscr{A}_e(g)$ and f has F.S.T. Then g has F.S.T. also.*

Proof. Let R be the ring of all functions h such that $h\cdot T\mathscr{A}_e(g)\subset T\mathscr{A}_e(g)$. Let M be the ideal of all such functions vanishing at zero. Then if $h\in M$, $\frac{1}{1+h}$ is in R also. For $\frac{1}{1+h}\cdot T\mathscr{A}_e(g)$ lies in $h^*(\mathcal{O}_1)\cdot T\mathscr{A}_e(g)$, and the above remark shows that if h is in R, $T\mathscr{A}_e(g)$ is an $h^*\mathcal{O}_1$-module, so $h^*\mathcal{O}_1\cdot T\mathscr{A}_e(g)$ is in $T\mathscr{A}_e(g)$. Since $h\in M$ implies $1+h$ is a unit in R, Nakayama's lemma can be applied to R-modules. We have that $\frac{T\mathscr{A}_e(g)}{\mathfrak{m}_n j(g)}$ is a finitely generated R-module, for R contains $g^*\mathcal{O}_p$. Further, it is easy to check that $\frac{T\mathscr{A}(g)}{\mathfrak{m}_n j(g)}$ is a R-submodule of $\frac{T\mathscr{A}_e(g)}{\mathfrak{m}_n j(g)}$. Since $\dim\frac{T\mathscr{A}_e(g)}{T\mathscr{A}(g)}\leqq n+p$ it follows by a variant of Nakayama's lemma that $M^{n+p+1}\cdot T\mathscr{A}_e(g)\subset T\mathscr{A}(g)$. This implies that $T\mathscr{K}(g)\supset T\mathscr{A}(g)\cdot\mathcal{O}_n\supset M^{n+p+1}\cdot\mathcal{O}^p_n$. We have

$$M^{n+p+1}\cdot\mathcal{O}^p_n\supset(f,g,J(f))^{n+p+1}\cdot\mathcal{O}^p_n\supset\mathfrak{m}_n^k\cdot\mathcal{O}^p_n$$

for some k, since f has F.S.T., and M contains f_i, g_j, and the generators of $J(f)$. Thus, $T\mathscr{K}(g)\supset\mathfrak{m}_n^k\cdot\mathcal{O}^p_n$ and g has F.S.T.

The next result shows that we can lift the hypotheses of Lemma 1 concerning the equality of tangent spaces from the level of germs to the level of unfoldings. We compare $\bar{T}\mathscr{A}(G^a)$ which may vary with a, to $\bar{T}\mathscr{A}(F^a)$ which is constant with a. This will allow us to show that for most a values $\bar{T}\mathscr{A}(G^a)$ is constant, and will imply that $\partial_t G^a \in \bar{T}\mathscr{A}(G^a)$ which we need to apply Proposition 1.

Proposition 3. *Suppose f has finite singularity type.*

a) *If $T\mathscr{A}(f)=T\mathscr{A}(g)$, then $\bar{T}\mathscr{A}(G^a)=\bar{T}\mathscr{A}(F^a)$ for all $a\in\mathbb{C}$ except a finite number of values.*

b) *If $T\mathscr{A}_e(f)=T\mathscr{A}_e(g)$, then $\bar{T}\mathscr{A}_e(G^a)=\bar{T}\mathscr{A}_e(F^a)$ for all except a finite number of values.*

Proof. a) As noted before $T\mathscr{A}(f)=T\mathscr{A}(g)\Rightarrow T\mathscr{K}(f)=T\mathscr{K}(g)$. Thus, g has F.S.T., and by our results on $\mathscr{K}$-equivalence, G^a has F.S.T. for all except at most a finite number of values of a.

Further, it follows that if $J(f)+f^*(m_p)\mathcal{O}_n \supseteq m_n^k$ and $J(f)+f^*(m_p)\mathcal{O}_n \not\supseteq m_n^{k-1}$, then the analogous statements are true for G^a except for the finite number of a values.

For we know the G^a are $\mathscr{K}$-equivalent except for the odd a values; hence the ideals $J(G^a)+G^{a*}(m_{p+1})\mathcal{O}_{n+1}$ are right equivalent for all values of a, apart from the above finite number.

It is easy to see that $\bar{T}\mathscr{A}(G^a)\subseteq \bar{T}\mathscr{A}(F)$, for by Lemma 1, $G^{a*}(m_p\mathcal{O}^p_{p+1})\subseteq H^*(m_{2p}\mathcal{O}^p_{2p+1})\subseteq \bar{T}\mathscr{A}(F)$. We also have that $m_n\cdot j_{\mathbb{C}^n}(G^a)\subseteq \bar{T}\mathscr{A}(F)$ for $m_n\cdot j_{\mathbb{C}^n}(g)\subseteq T\mathscr{A}(f)\subseteq \bar{T}\mathscr{A}(F)$.

We now must show that $\bar{T}\mathscr{A}(G^a)=\bar{T}\mathscr{A}(F)$ for all except a finite number of values of a. The next lemma gives us an easy to apply criterion for equality.

Lemma 2. *Suppose $\bar{T}\mathscr{A}(G^a)+H^*(m_{2p+1})\,\bar{T}\mathscr{A}(F)=\bar{T}\mathscr{A}(F)$, then $\bar{T}\mathscr{A}(G^a)=\bar{T}\mathscr{A}(F)$.*

Proof. We first show that $\bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F)=\bar{T}\mathscr{A}(F)$. Consider

$$\frac{\bar{T}\mathscr{A}(F)}{\bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F)} \tag{$*$}$$

this has the structure of an $H^*(\mathcal{O}_{2p+1})$-module. For,

$$\begin{aligned}
H^*(\mathcal{O}_{2p+1})&(m_n\cdot j_{\mathbb{C}^n}G^a+G^{a*}(m_p\mathcal{O}^p_{p+1})+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F))\\
&\subseteq \bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F)+H^*(\mathcal{O}_{2p+1})\,(G^{a*}(m_p\mathcal{O}^p_{p+1}))\\
&\subseteq \bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F)+H^*(m_{2p})\,(G^{a*}(m_p\mathcal{O}^p_{p+1})\\
&\subseteq \bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F),\quad \text{since}\quad H^*(m_{2p})e_i\subseteq \bar{T}\mathscr{A}(F).
\end{aligned}$$

Further $\bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\ \bar{T}\mathscr{A}(F)$ contains $m_n(J(G^a)+G^{a*}(m_{p+1}))j_{\mathbb{C}^n}(F)$, and except for a finite number of a values, this contains $m_n(m^k_{n+1})j_{\mathbb{C}^n}F$.

This is important, for $F^*(m_p\mathcal{O}^p_{p+1})$ is always finitely generated as an $H^*(\mathcal{O}_{2p+1})$ module, but $j_{\mathbb{C}^n}(F)$ may not be. The above inclusion shows that all but a finite number of the elements of $j_{\mathbb{C}^n}F$ are already contained in the denominator of $(*)$. Thus $(*)$ is a f.g. $H^*(\mathcal{O}_{2p+1})$-module, so by Nakayama's lemma $(*)$ is zero.

Given that $\bar{T}\mathscr{A}(F)=\bar{T}\mathscr{A}(G^a)+G^{a*}(m_{p+1})\,\bar{T}\mathscr{A}(F)$, we now want to show that $\dfrac{\bar{T}\mathscr{A}(F)}{\bar{T}\mathscr{A}(G^a)}$ is a finitely generated $G^{a*}(\mathcal{O}_{p+1})$-module. For good values of a, $\dfrac{\mathcal{O}_{n+1}}{m_n j_{\mathbb{C}^n}(G^a)}$

is f.g. as a $G^{a*}(\mathcal{O}_{p+1})$-module, hence $\frac{\bar{T}\mathscr{A}(F)}{m_n j_{\mathbb{C}^n}(G^a)}$ is finitely generated also since $G^{a*}(\mathcal{O}_{p+1})$ is Noetherian. Consequently, $\frac{\bar{T}\mathscr{A}(F)}{\bar{T}\mathscr{A}(G^a)}$ is finitely generated, and Nakayama's lemma again applies.

Returning to the proof of Proposition 3, we now show that checking the hypothesis of Lemma 2 is a finite, algebraic problem. Let

$$A=\frac{\bar{T}\mathscr{A}(F)}{m_{n+1}^k(m_n)\cdot j_{\mathbb{C}^n}(F)+H^*(m_{2p+1})\,\bar{T}\mathscr{A}(F)} \quad \text{where} \quad m_n^k \subseteq J(f)+f^*(m_p)\mathcal{O}_n.$$

Notice that $\mathscr{A}$ is finite dimensional as a $\mathbb{C}$-vector space, and $\bar{T}\mathscr{A}(G^a)+H^*(m_{p+1})\,\bar{T}\mathscr{A}(F)\subseteq m_{n+1}^k(m_n j_{\mathbb{C}^n}(F))$.

$$\begin{aligned}(\text{For } m_{n+1}^k(m_n j_{\mathbb{C}^n}(F)) &\subseteq (G^{a*}(m_{p+1})+J(G^a))\,m_n j_{\mathbb{C}^n}(F)\\ &\subseteq m_n\cdot j_{\mathbb{C}^n}(G^a)+G^{a*}\bar{T}\mathscr{A}(F)).\end{aligned}$$

Further, $m_{n+1}^k(m_n j_{\mathbb{C}^n}(F))+H^*(m_{2p+1})\,\bar{T}\mathscr{A}(F)$ contains $m_{n+1}^{2k}(m_n\cdot j_{\mathbb{C}^n}(G^a))+G^{a*}(m_{p+1}(m_p\mathcal{O}_{n+1}^p))$.

Consider the polynomial map of finite dimensional vector spaces induced by $D_x(G^a)+G^{a*}$, which maps the direct sum $\frac{m_n\mathcal{O}_{n+1}}{m_{n+1}^{2k}(m_n\mathcal{O}_{n+1}^n)}\oplus\frac{m_p\mathcal{O}_{p+1}^p}{m_{p+1}(m_p\mathcal{O}_{p+1}^p)}$ into A. The above remarks ensure this mapping is well defined. The map is linear for fixed a, and surjective at a iff the hypothesis of Lemma 2 is satisfied. It is easy to check that the map is surjective at $a=0,1$. By the lemma $\bar{T}\mathscr{A}(G^a)=\bar{T}\mathscr{A}(F^a)$ for all except a finite number of a values.

The proof of Part b) is similar. As before we have that $\bar{T}\mathscr{A}_e(F)\supseteq\bar{T}\mathscr{A}_e(G^a)$ and that $T\mathscr{A}_e(F)$ is an $H^*(\mathcal{O}_{2p+1})$-module.

Claim. Suppose $\bar{T}\mathscr{A}_e(G^a)+(G^{a*}(m_{p+1})+H^*(m_{2p+1}^{p+1}))\,\bar{T}\mathscr{A}_eF=\bar{T}\mathscr{A}_eF$ then $\bar{T}\mathscr{A}_e(G^a)=\bar{T}\mathscr{A}_e(F)$.

The proof is similar to the proof of 3.1 in [G 1]:

The map that we consider in this case is induced by $D_x(G^a)+G^{a*}$ with domain $\frac{\mathcal{O}_{n+1}^n}{m_{n+1}^{k+l}\mathcal{O}_{n+1}^n}\oplus\frac{\mathcal{O}_{p+1}^p}{m_{p+1}^{p+1}\mathcal{O}_{p+1}^p}$ and target $\frac{\bar{T}\mathscr{A}_e(F)}{m_{n+1}^l\cdot j_{\mathbb{C}^n}(F)+H^*(m_{2p+1}^{p+1})\,\bar{T}\mathscr{A}_e(F)}$. Here l refers to the power of m_{n+1} which lies in $(J(G^a)+G^{a*}m_{p+1})\mathcal{O}_{n+1}$ for generic a. (We can ensure 1 is in this generic set.)

It is easy to check that $D_x(G^a)+G^{a*}$ surjects iff the hypothesis of the claim holds, and that it surjects for $a=0,1$. Hence, except for a finite number of bad values, $\bar{T}\mathscr{A}_e(G^a)=\bar{T}\mathscr{A}_e(F)$.

We now develop the notion of morphism for the $\mathscr{A}$-case; in what follows we call $h\in\mathcal{O}_n^p$ a generating map for $T\mathscr{A}(f)$ if $T\mathscr{A}(h)=T\mathscr{A}(f)$. (Similarly for $T\mathscr{A}_e(f)$.)

Our notion of morphism must be modeled on the change that $T\mathscr{A}(f)$ undergoes as we pass from $T\mathscr{A}(f)$ to $T\mathscr{A}(l\circ f\circ r)$. At the same time, the notion of morphism ought to be independent of the mapping which gives rise to $T\mathscr{A}(f)$.

Definition 2. Suppose $M=\frac{\mathcal{O}_n^p}{T\mathscr{A}(f)}$, $M_e=\frac{\mathcal{O}_n^p}{T\mathscr{A}_e(f)}$.

A morphism with domain M (resp. M_e) is given by a triple (r, L, h) where $r \in \mathrm{Aut}(\mathbb{C}^n, 0)$, L is a $p \times p$ invertible matrix with entries in $\mathcal{O}_p$ such that $L = Dl$ for some $l \in \mathrm{Aut}(\mathbb{C}^p, 0)$, and h is a generating map of $T\mathscr{A}(f)$ (resp. $T\mathscr{A}_e(f)$). The morphism is given by $\xi \to (L \circ h \circ r)(\xi \circ r)$, where $\xi \in M$. It is easy to see that this gives a well defined mixed module homorphism between M and $\frac{\mathcal{O}_n^p}{T\mathscr{A}(l \circ h \circ r)}$ $\left(\text{resp. } M_e \text{ and } \frac{\mathcal{O}_n^p}{T\mathscr{A}_e(l \circ h \circ r)}\right)$ which is an isomorphism of mixed modules.

Before we give our main theorem, recall that f is a stable germ if $T\mathscr{A}_e(f) = \mathcal{O}_n^p$, and f is finitely determined if $T\mathscr{A}_e(f) \supseteq m_n^k \mathcal{O}_n^p$ for some k.

Theorem. *Suppose f and g are complex analytic map germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $g: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Suppose f has F.S.T.*

A) *If f is finitely determined, but not stable, then $\frac{\mathcal{O}_n^p}{T\mathscr{A}_e(f)} \cong \frac{\mathcal{O}_n^p}{T\mathscr{A}_e(g)}$ iff $f \underset{\mathscr{A}}{\sim} g$.*

B) $\frac{\mathcal{O}_n^p}{T\mathscr{A}(f)} \cong \frac{\mathcal{O}_n^p}{T\mathscr{A}(g)}$ *iff $f \underset{\mathscr{A}}{\sim} g$.*

Proof of B). If $\frac{\mathcal{O}_n^p}{T\mathscr{A}(f)} \cong \frac{\mathcal{O}_n^p}{T\mathscr{A}(g)}$ we can find h such that $T\mathscr{A}(f) = T\mathscr{A}(h)$, $T\mathscr{A}(g) = T\mathscr{A}(l \circ h \circ r)$.

It follows from the Propositions 1 and 3 that we can choose a path connecting $l \circ h \circ r$ with g and a path connecting h and f such that the family of mappings given by these paths are $\mathscr{A}$-equivalent to h and $l \circ h \circ r$. Hence f is $\mathscr{A}$-equivalent to g.

Proof of A). By a preliminary coordinate change we may assume $T\mathscr{A}_e(f) = T\mathscr{A}_e(g)$. Thus $\bar{T}\mathscr{A}_e(G^a) = \bar{T}\mathscr{A}_e(F)$ for all a values off some Z-closed subset of $\mathbb{C}$.

By Proposition 1 and the compactness of our path connecting f and g, we can cover this path by a finite set of open intervals I_i with distinguished points x_i such that the unfolding G given by (I_i, x_i) is trivial. We must show that the families of trivializing diffeomorphisms that occur are origin preserving in source and target.

Since f is finitely determined, $T\mathscr{A}_e(f) \supseteq m_n^k \mathcal{O}_n^p$ for some k, hence $\bar{T}\mathscr{A}_e(F) \supseteq m_n^k \mathcal{O}_{n+1}^p$. Thus $\bar{T}\mathscr{A}_e(G^a) \supseteq m_n^k \mathcal{O}_{n+1}^p$ for all a values along our path. This implies that the target sheaf $G_* \frac{O_{n+1}^p}{j_{\mathbb{C}^n} G} \Big/ G^* \mathcal{O}_{p+1}^p$ has support only along the t-taxis. The assumption that f is not stable implies it does have support there. This implies that the sheaves $g_{t*} \frac{\mathcal{O}_n^p}{j(g_t)} \Big/ g_t^* \mathcal{O}_p^p$ on $\mathbb{C}^p$ have support only at the origin, for all t values in a neighborhood of our curve in $\mathbb{C}$. This forces the trivializing diffeomorphisms to be origin preserving so f and g are equivalent.

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