ON THE FORMAL NEIGHBORHOOD OF DEGENERATE ARCS

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ABSTRACT. For any k-point x of a variety X and its image $\overline{x}$ in the arc space $\mathcal{L}(X)$ we prove a structure theorem for the formal neighborhood $\mathcal{L}(X)_{\overline{x}}$ in the case $\mathrm{char}(k) = 0$. In particular, our result implies that, for x singular, there does not exist a decomposition for $\mathcal{L}(X)_{\overline{x}}$ as in the theorem of Grinberg, Kazhdan and Drinfeld.

1. Introduction

Let X be a scheme of finite type over a field k and let $\mathcal{L}(X)$ be the arc space of X. Any k-point of $\mathcal{L}(X)$ corresponds to an arc $\mathrm{Spec}(k[[t]]) \to X$ on X. Call $\gamma \in \mathcal{L}(X)$ *non-degenerate* if the image of γ is not entirely contained in $\mathrm{Sing}(X)$. The following result - proven first by Grinberg and Kazhdan for characteristic 0 and later by Drinfeld in general - is an important step in understanding the singularities of $\mathcal{L}(X)$.

Theorem 1.1 ([7], [5]). *Let $\gamma \in \mathcal{L}(X)(k)$ be non-degenerate and write $\mathcal{L}(X)_\gamma$ for the formal neighborhood of γ. Then there exists a scheme Y of finite type over k and $y \in Y$ such that $\mathcal{L}(X)_\gamma \cong Y_y \times \mathrm{D}^\infty$, where D^∞ is the product of countably many copies of* $\mathrm{Spf}(k[[t]])$.

For more recent work in this direction see [3] and [4].

In the case where γ is degenerate, Bourqui and Sebag [2] found an example of an irreducible curve X over a non-algebraically closed field such that the conclusion of Theorem 1.1 fails. In the same paper, they ask whether the same holds for the constant arc $0 \in V(y^2 - x^3) \subset \mathbb{A}^2$. Our main result clarifies the situation for degenerate constant arcs on arbitrary varities in characteristic 0. We say that the formal neighborhood X_x *has a smooth factor* if there exists an isomorphism $X_x \cong Z \times \mathrm{D}$ with Z a formal scheme.

Theorem 1.2. *Let k be a field of characteristic 0. Let X a scheme locally of finite type over k and $x \in X(k)$. Assume that $X_x \cong Y_y \times D^r$ and Y_y has no smooth factor. Denote by $\overline{x}$, $\overline{y}$ the constant arcs in x resp. y. Then $\mathcal{L}(X)_{\overline{x}} = \mathcal{L}(Y)_{\overline{y}} \times \mathrm{D}^\infty$ and $\mathcal{L}(Y)_{\overline{y}}$ admits no smooth factor.*

In particular, the theorem implies that for any $x \in \mathrm{Sing}(X)$ there exists no decomposition for $\mathcal{L}(X)_{\overline{x}}$ as in Theorem 1.1, see Corollary 3.5. The proof of Theorem 1.2

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uses a version of the criterion for triviality of germs of analytic varieties, which we are going to discuss in the next section.

Conventions. Our definition of a formal k-scheme is in agreement with [8]. A *test-ring* is a local k-algebra A with nilpotent maximal ideal $\mathfrak{m}$ and such that $A/\mathfrak{m} \cong k$. In particular, any test-ring is a discrete topological ring with respect to the $\mathfrak{m}$-adic topology. As in [5], for any k-scheme Y and $y \in Y(k)$ the formal neighborhood $Y_y = \operatorname{Spf}(\widehat{\mathcal{O}_{Y,y}})$ is given by its functor of points from the category of test-rings to sets.

2. Derivations and the Triviality criterion

Throughout this paper let k be a field of characteristic 0. Let A be any index set and denote by $k[x^A]$ the polynomial ring in the variables x_α, $\alpha \in A$. By $k[[x^A]]$ we denote the completion of $k[x^A]$ with respect to the (x^A)-adic topology. Its elements are formal power series of the form $f = \sum_{n \geq 0} f_n$, where each f_n is a homogeneous polynomial in $k[x^A]$ of degree n. Observe that $k[[x^A]]$ is local with maximal ideal $\mathfrak{m}$ consisting of all power series of order ≥ 1. Furthermore, when endowed with the inverse limit topology $k[[x^A]]$ becomes an admissible topological ring (see [8, 0;(7.1.2)]) with a system of neighborhoods for 0 given by

$$\overline{\mathfrak{m}^n} = \ker(k[[x^A]] \to k[x^A]/(x^A)^n).$$

Remark 2.1. If A is infinite, then $k[[x^A]]$ is not Noetherian and the inverse limit topology is no longer equivalent to the $\mathfrak{m}$-adic one. Moreover, $k[[x^A]]$ is not flat as a module over $k[x^A]$ and we have $\mathfrak{m} \supsetneq (x^A) \cdot k[[x^A]]$. For more details, see [10, Example 1.8].

Let us remark that the following version of the inverse function theorem holds independent of the size of the index set A. The proof is the same as in the finite-dimensional case.

Lemma 2.2 (Formal inverse function theorem)**.** *Let $\varphi : k[[x^A]] \to k[[x^A]]$ be a (continuous) endomorphism. Then φ is an isomorphism if and only if the induced map $\mathfrak{m}/\overline{\mathfrak{m}^2} \to \mathfrak{m}/\overline{\mathfrak{m}^2}$ is an isomorphism of k-vector spaces.*

For any topological k-algebra R define $\operatorname{Der}_k^{\text{cont}}(R)$ to be the module of continuous k-derivations $R \to R$. Observe that $\operatorname{Der}_k^{\text{cont}}(k[[x^A]])$ is isomorphic to the topological module of sequences $k[[x^A]]^A$ via $d \mapsto (d(x_\alpha))_{\alpha \in A}$. We write each $d \in \operatorname{Der}_k^{\text{cont}}(k[[x^A]])$ as a (possibly infinite) sum

$$d = \sum_{\alpha \in A} f_\alpha \partial_\alpha,$$

where $f_\alpha = d(x_\alpha)$ and ∂_α the derivation given by $\partial_\alpha(x_\beta) = \delta_{\alpha\beta}$. Such a derivation d is called *regular* if $d \not\equiv 0 \mod \mathfrak{m}$, or equivalently, if there exists $\alpha \in A$ with $f_\alpha \in k[[x^A]]^*$.

Let $R = k[[x^A]]/I$ be some quotient, then for any $d \in \mathrm{Der}_k^{\mathrm{cont}}(R)$ there exists a derivation $d' \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ with $d'(I) \subset I$ making the following diagram commute:

$$\begin{array}{ccc} k[[x^A]] & \xrightarrow{d'} & k[[x^A]] \\ \downarrow & & \downarrow \\ k[[x^A]]/I & \xrightarrow{d} & k[[x^A]]/I. \end{array}$$

We say that d is regular if any lifting of it is.

If $\varphi : R \to S$ is an isomorphism of quotients of power series rings, then φ induces a map

$$\varphi_* : \mathrm{Der}_k^{\mathrm{cont}}(S) \to \mathrm{Der}_k^{\mathrm{cont}}(R),\ d \mapsto \varphi^{-1} \circ d \circ \varphi\,.$$

It is obvious that $\varphi_*(d)$ is regular if and only if d is regular.

Let Z be any formal k-scheme. Recall that Z has a smooth factor if there exists some formal k-scheme Z' and an isomorphism $Z \cong Z' \times \mathrm{D}$, where $\mathrm{D} = \mathrm{Spf}(k[[t]])$ denotes the formal disk. Now let Y be a scheme over k, $y \in Y(k)$ and Y_y be the formal neighborhood of y. The following proposition is well-known in the case of Y being locally of finite type (see for example [6, p. 91]). In the general case the proof is exactly the same and we shall reproduce it here.

Proposition 2.3 (Triviality criterion)**.** *Y_y has a smooth factor if and only if there exists a regular derivation $d \in \mathrm{Der}_k^{\mathrm{cont}}(\widehat{\mathcal{O}_{Y,y}})$.*

Proof. Write $S = \widehat{\mathcal{O}_{Y,y}}$. There exists an index set A such that $S \cong k[[x^A]]/I$ (as topological rings), for some closed ideal I.

Assume now that $Y_y \cong Z \times \mathrm{D}$, then there exists an isomorphism $\varphi : S \to S' \widehat{\otimes} k[[t]]$ with S' an admissible local k-algebra. Observe that $d' = \mathrm{id}_{S'} \widehat{\otimes} \partial_t$ gives a regular derivation of $S' \widehat{\otimes} k[[t]]$. Then $d := \varphi_*(d) \in \mathrm{Der}_k^{\mathrm{cont}}(S)$ is regular by definition and we are done.

Now assume that there exists $d = \sum f_\alpha \partial_\alpha \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ with $d(I) \subset I$ and $\beta \in A$ with $f_\beta \in k[[x^A]]^*$. Applying Lemma 2.4 to the system

$$\frac{\partial \phi_\alpha}{\partial x_\beta} = f_\alpha(\underline{\phi}), \phi_\alpha|_{x_\beta = 0} = (1 - \delta_{\alpha\beta}) x_\alpha,$$

shows that there exists an endomorphism $\phi : k[[x^A]] \to k[[x^A]]$ which is an isomorphism by Lemma 2.2 and such that $\phi_*(\partial_\beta) = d$. In particular, for any $g \in I$ we have that $\partial_\beta(\phi(g)) = \phi(d(g)) \in I$. Taking Taylor expansion with respect to x_β we see that every $\phi(g)$ can be expressed as a power series in x_β with coefficients of the form $\phi(\tilde{g})|_{x_\beta=0}$ with $\tilde{g} \in I$. Thus

$$\phi(I) \subset \overline{(\phi(g)|_{x_\beta = 0}, g \in I)}.$$

The converse inclusion follows from the formula

$$h|_{x_\beta=0} = \sum_n \frac{(-1)^n}{n!}\frac{\partial^n h}{\partial x_\beta^n} x_\beta^n.$$

So we can assume that I is the closure of an ideal generated by elements $g \in k[[x^A]]$ which do not depend on the variable x_β. Consider the image I' of I under the map $k[[x^A]] \to k[[x^{A-\{\beta\}}]]$, it follows that

$$k[[x^A]]/I \cong k[[x^{A-\{\beta\}}]]/I' \widehat{\otimes} k[[x_\beta]].$$

Since $S' = k[[x^{A-\{\beta\}}]]/I'$ is again admissible we can define $Z = \mathrm{Spf}(S')$.

Now all that is left to prove is the following lemma:

Lemma 2.4. *Let $g_\alpha \in k[[x^A]]$ for $\alpha \in A$ and write $\mathfrak{m}$ for the maximal ideal in $k[[x^A]]$. For any collection of elements $c_\alpha \in \mathfrak{m}$ there exists a solution $\phi = (\phi_\alpha)_{\alpha\in A}$ in $k[[x^A]]\widehat{\otimes}k[[t]]$ of the following initial value problem:*

$$\frac{\partial \phi_\alpha}{\partial t} = g_\alpha(\underline{\phi}),\ \phi_\alpha(0) = c_\alpha.$$

Note that the composition $g_\alpha(\underline{\phi})$ is well-defined since $\phi_\alpha(0) \in \mathfrak{m}$.

Proof. Write $\phi_\alpha = \sum_{d\geq 0} \phi_{\alpha,d} t^d$, then the above gives a triangular system of equations for the coefficients $\phi_{\alpha,d}$. Namely, assume we have constructed $\phi_\alpha^n = \sum_{d=0}^n \phi_{\alpha,d} t^d$ such that

$$\frac{\partial \phi_\alpha^n}{\partial t} = g_\alpha(\underline{\phi^n}) \bmod (t)^n$$

holds. Now write

$$g_\alpha(\underline{\phi^n}) - \frac{\partial \phi_\alpha}{\partial t} = \widetilde{g_\alpha} t^n + t^{n+1}(\ldots)$$

and observe that $\widetilde{g_\alpha} \in R$ is polynomial in $\phi_{\alpha,d}$, $d \leq n$. Set $\phi_{\alpha,n+1} := \frac{1}{n+1}\widetilde{g_\alpha}$ and $\phi_\alpha^{n+1} := \sum_{d=0}^{n+1} \phi_{\alpha,d} t^d$ and proceed by induction. □

This finishes the proof of the proposition.

□

An easy application of the triviality criterion yields the following lemma, which appears in a weaker version in [1, Lemma 9.4]. It was proven there for Y, Y' locally of finite type, using [8, 0;(7.2.4)] (which turns out to be wrong in the full generality in which it is stated).

Lemma 2.5. *Let Y, Y' be two k-schemes and $y \in Y(k)$, $y' \in Y'(k)$. Suppose there exist index sets A, B such that*

$$Y_y \times \mathrm{D}^A \cong Y'_{y'} \times \mathrm{D}^B\,.$$

If Y_y and $Y'_{y'}$ admit no smooth factors, then the above isomorphism induces an isomorphism of k-vector spaces

$$\mathfrak{m}_y/\overline{\mathfrak{m}_y^2} \cong \mathfrak{m}_{y'}/\overline{\mathfrak{m}_{y'}^2},$$

where $\mathfrak{m}_y$ *and* $\mathfrak{m}_{y'}$ *denote the maximal ideals of* $\widehat{\mathcal{O}_{Y,y}}$ *resp.* $\widehat{\mathcal{O}_{Y',y'}}$.

Proof. By assumption we have an isomorphism of complete topological rings

$$\varphi : \widehat{\mathcal{O}_{Y,y}}\widehat{\otimes}_k k[[t^A]] \to \widehat{\mathcal{O}_{Y',y'}}\widehat{\otimes}_k k[[u^B]].$$

Now assume there exists $f \in \mathfrak{m}_y$ with $\varphi(f) = \ldots + cu_\beta + \ldots$, with $c \in k^*$, $\beta \in B$. Then $d = \varphi_*(\partial_{u_\beta})$ is a regular derivation since $d(f) = c \bmod \overline{\mathfrak{m}_y^2}$. But Proposition 2.3 yields a contradiction, hence the module $\mathfrak{m}_y/\overline{\mathfrak{m}_y^2}$ maps into $\mathfrak{m}_{y'}/\overline{\mathfrak{m}_{y'}^2}$ via φ. Applying the same argument in the opposite direction yields the desired statement. □

The next result is proven here only in the finite-dimensional case and, as of now, it is unclear to the authors whether or not it can be generalized to arbitrary schemes Y.

Proposition 2.6. *Let* Y *be a scheme locally of finite type over* k *and* $y \in Y(k)$. *Then there exists* $r \in \mathbb{N}$, *a scheme* Y' *locally of finite type over* k *and* $y' \in Y'(k)$ *such that* $Y_y \cong Y'_{y'} \times \mathrm{D}^r$ *and* $Y'_{y'}$ *has no smooth factor. Furthermore, this* $Y'_{y'}$ *is unique in the following sense: suppose* Z *is a* k*-scheme and* $z \in Z(k)$ *such that* Z_z *admits no smooth factor. Any isomorphism* $Y_y \cong Z_z \times \mathrm{D}^s$ *induces an isomorphism* $Y'_{y'} \cong Z_z$ *via composition with* $Y'_{y'} \to Y_y$ *and* $Z_z \times \mathrm{D}^s \to Z_z$.

Proof. We may assume that $Y = V(I) \subset \mathbb{A}^N$ and $y = 0$. If Y_y has a smooth factor, then, by 2.3, there exists an automorphism φ of $k[[x_1, \ldots, x_N]]$ such that $\varphi^{-1}(I)$ has a system of generators not depending on x_j. Take any polynomial $f \in k[x_1, \ldots, x_N]$ with $f \equiv \varphi(x_j) \mod (x_1, \ldots, x_N)^2$ and define an automorphism φ' via

$$\varphi'(x_i) = \begin{cases} \varphi(x_i), & i \neq j \\ f, & i = j. \end{cases}$$

Then φ' induces an isomorphism $Y_0 = Y'_0 \times D$ with $Y' = V(I + (f))$. If Y'_0 has no smooth factor then we are done; otherwise we repeat the above argument.
For the second assertion it is sufficient to prove that the isomorphism

$$\varphi : \widehat{\mathcal{O}_{Z,z}}\widehat{\otimes}_k k[[u_1, \ldots, u_s]] \to \widehat{\mathcal{O}_{Y',y'}}\widehat{\otimes}_k k[[t_1, \ldots, t_r]]$$

identifies $R = \widehat{\mathcal{O}_{Y',y'}}$ and $S = \widehat{\mathcal{O}_{Z,z}}$. From φ we obtain maps $R \to S$ and $S \to R$; it is sufficient to prove that their composition σ is an automorphism of R. By Lemma 2.5 it follows that σ induces an isomorphism of k-vector spaces

$$\bar{\sigma} : \mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2 \to \mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2.$$

Now applying a variant of the inverse function theorem finishes the proof. Namely, we get induced maps on the homogeneous parts

$$\bar{\sigma}_n : \mathfrak{m}_{y'}^n/\mathfrak{m}_{y'}^{n+1} \to \mathfrak{m}_{y'}^n/\mathfrak{m}_{y'}^{n+1},$$

which are surjective by assumption and thus already bijective. Hence σ is an automorphism. □

3. Application to the formal neighborhoods of arcs

From now on let X be a scheme locally of finite type over k. Then, for any arc $\gamma \in \mathcal{L}(X)(k)$, the formal neighborhood $\mathcal{L}(X)_\gamma$ depends only on the formal structure at $\gamma(0) \in X$. More precisely, for every test-ring A there is a natural bijection between the set of A-deformations of γ and the set

$$\{\widetilde{\gamma} : \mathrm{Spf}(A[[t]]) \to X_{\gamma(0)}, \widetilde{\gamma} \equiv \gamma \mod \mathfrak{m}_A\}.$$

In particular, we see that for an isomorphism $X_{\gamma(0)} \cong Y_y$ of formal neighborhoods, the arc γ induces an arc $\gamma' \in \mathcal{L}(Y)(k)$ with $\gamma'(0) = y$. With these observations the following lemma is immediate:

Lemma 3.1. *Let $\gamma \in \mathcal{L}(X)(k)$ and suppose there exist an isomorphism of formal neighborhoods $X_{\gamma(0)} \cong Y_y \times Y'_{y'}$. Let (γ_1, γ_2) denote the arc on $Y \times_k Y'$ corresponding to γ. Then $\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_{\gamma_1} \times \mathcal{L}(Y')_{\gamma_2}$.*

We will now use the previous section to prove Theorem 1.2. The main step is the following result:

Proposition 3.2. *Let $x \in X(k)$ and write $\overline{x}$ for the corresponding constant arc. Then X_x has a smooth factor if and only if $\mathcal{L}(X)_{\overline{x}}$ does so.*

Proof. Suppose that X_x has a smooth factor. By Lemma 3.1 we may assume that X_x is isomorphic to D^r, i.e. x is a smooth point of X. But then $\mathcal{L}(X)_{\overline{x}}$ is isomorphic to D^∞, so we are done.

For the converse, we first assume $X = V(I) \subset \mathbb{A}^N$ is affine and $x = 0$. Then $X_x = \mathrm{Spf}(R_0)$, where

$$R_0 = k[[x_1, \dots, x_N]]/I_0,$$

with $I_0 = I \cdot k[[x_1, \dots, x_N]]$. Recall that $\mathcal{L}(X)$ can be constructed as follows: let $f \in I$ and consider the power series expansion

$$f(\sum_{n\geq 0} x_{1,n}t^n, \dots, \sum_{n\geq 0} x_{N,n}t^n) = \sum_{n\geq 0} F_n t^n,$$

with $F_n \in k[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]$. Then $\mathcal{L}(X) = \mathrm{Spec}(k[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]/I_\infty)$, where $I_\infty = (F_n, f \in I)$.

Remark 3.3. Let $f = \sum_{\alpha\in\mathbb{N}^N} c_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ be a polynomial in $k[x_1, \dots, x_N]$. Then F_n is of the form

$$F_n = \sum_{\alpha\in\mathbb{N}^N} c_\alpha \sum_{\substack{\lambda_i \in \mathbb{N}^{\alpha_i}, \\ \sum_{i\in[N]} |\lambda_i| = n}} \prod_{i\in[N]} x_{i,\lambda_i^1} \cdots x_{i,\lambda_i^{\alpha_i}}.$$

Note that there occurs no cancellation of monomials in F_n, since k has characteristic 0.

Finally, we have that $\mathcal{L}(X)_{\overline{x}} = \mathrm{Spf}(R_\infty)$ with

$$R_\infty = k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]/I_{\infty,0},$$

where $I_{\infty,0} = \overline{I_\infty}$ denotes the topological closure of I_∞. Note that we have maps

$$k[[x_1, \dots, x_N]] \xrightarrow{\iota} k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]] \xrightarrow{\sigma} k[[x_1, \dots, x_N]] \ ,$$

with ι being the inclusion $x_i \mapsto x_{i,0}$ and σ its retraction given by $x_{ij} = 0$ for $j \geq 1$. Observe that $\sigma(I_{\infty,0}) = I_0$.

Now assume $\mathcal{L}(X)_0$ has a smooth factor. By Proposition 2.3 there exists a regular derivation d of $k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]$ with $d(I_{\infty,0}) \subset I_{\infty,0}$. Write

$$d = \sum_{i \in [N], j \in \mathbb{N}} g_{i,j} \partial_{x_{i,j}},\ g_{i,j} \in k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]].$$

The regularity of d means that there exists an index $l \in \mathbb{N}$ such that some $g_{i,l}$ is invertible. Set $g_i := \sigma(g_{i,l}) \in k[[x_1, \dots, x_N]]$. Then we define a derivation $d_0 \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x_1, \dots, x_N]])$ by

$$d_0 = \sum_{i=1}^{N} g_i \partial_{x_i}.$$

It is regular by construction. If we prove that $d_0(I) \subset I_0$, then we can use Proposition 2.3 together with Remark 2.6 and we are done. Let $f \in I$. Then

$$d(F_l) = \sum_{i \in [N], j \in [l]} g_{i,j} \partial_{x_{i,j}}(F_l) = \sum_{i \in [N]} g_{i,l} \partial_{x_{i,l}}(F_l) + \widetilde{d}(F_l) \in I_{\infty,0}$$

From Remark 3.3 we see that the only terms in F_n involving $x_{i,n}$ are of the form

$$c_\alpha \alpha_i x_{1,0}^{\alpha_1} \cdots x_{i,0}^{\alpha_i - 1} x_{i,n} \cdots x_{N,0}^{\alpha_N},$$

with $c_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ a term of f. Hence we have $\partial_{x_{i,l}}(F_l) = \partial_{x_{i,0}}(F_0)$ for all i. Furthermore, by a similar argument, we see that $\sigma(\widetilde{d}(F_l)) = 0$. Applying σ to $d(F_l)$ we get that

$$\sum_{i \in [N]} \sigma(g_{i,l}) \sigma(\partial_{x_{i,0}}(F_0)) = \sum_{i \in [N]} g_i \partial_{x_i}(f) = d_0(f) \in \sigma(I_{\infty,0}) = I_0.$$

□

Example 3.4. The proposition does not hold true for non-constant arcs, in the following sense: let X be as in the proof, take any arc γ on X and write it as

$$\gamma = (\sum_n \gamma_1^n t^n, \dots, \sum_n \gamma_N^n t^n).$$

After applying the automorphism $x_i^n \mapsto x_i^n + \gamma_i^n$ we can assume

$$\widehat{\mathcal{O}_{\mathcal{L}(X),\gamma}} = k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]/I.$$

Then it is not hard to verify that the derivation

$$d = \sum_{i,n} (n+1)(x_i^{n+1} - \gamma_i^{n+1})\partial_{x_i^n} \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x_1^{\mathbb{N}}, \ldots, x_N^{\mathbb{N}}]])$$

fulfills $d(I) \subset I$ (in fact d is just the pushforward of the universal derivation coming from the differential algebra description of $\mathcal{L}(X)$, see [9, 3.3.1]). Observe that the arc γ being nonconstant is equivalent to d being regular. Hence, by Proposition 2.3 the formal neighborhood $\mathcal{L}(X)_\gamma$ has a smooth factor, irrespective of the structure of $X_{\gamma(0)}$.

Proof of Theorem 1.2. Let $X_x \cong Y_y \times \mathrm{D}^r$ with Y_y having no smooth factor. We will prove a stronger assertion here: let $\gamma \in \mathcal{L}(X)(k)$ with $\gamma(0) = x$ and γ *constant on* Y, i.e. γ factors as $(\bar{y}, \gamma_2)$ on $Y \times \mathbb{A}^r$, where $\bar{y}$ denotes the constant arc. By Lemma 3.1 we have $\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_{\bar{y}} \times \mathrm{D}^\infty$ and by Proposition 3.2 $\mathcal{L}(Y)_{\bar{y}}$ admits no smooth factor. □

Corollary 3.5. *Let $x \in X(k)$ and $X_x \cong Y_y \times \mathrm{D}^r$, with Y having no smooth factor. For any arc $\gamma \in \mathcal{L}(X)(k)$ which is constant on Y, the formal neighborhood $\mathcal{L}(X)_\gamma$ does not have a decomposition as $\mathcal{L}(X)_\gamma \cong Z_z \times \mathrm{D}^\infty$ with Z of finite type.*

Proof. By Theorem 1.2, $\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_{\bar{y}} \times \mathrm{D}^\infty$ and $\mathcal{L}(Y)_{\bar{y}}$ admits no smooth factor. Assume that $\mathcal{L}(X)_\gamma$ has a decomposition as above, then

$$\mathcal{L}(Y)_{\bar{y}} \times \mathrm{D}^\infty \cong Z_z \times \mathrm{D}^\infty\,.$$

Using Lemma 2.5 yields a contradiction to Z_z being of finite dimension. □

4. Further questions

In this final section we would like to summarize some problems which are related to this paper and whose solutions are, to our knowledge, still open.

(1) As mentioned before, Proposition 2.6 does not extend immediately to the case where the scheme Y is no longer locally of finite type. An additional argument seems to be necessary to avoid using the dimension of Y.

(2) Corollary 3.5 is still open for general degenerate arcs. Consider the arc $\gamma = (0, 0, t)$ on $X = V(y^2 - x^2 z) \subset \mathbb{A}^3$ with $\gamma(0) = 0$. Then computing explicitly as in Example 3.4 shows that $\mathcal{L}(X)_\gamma$ is isomorphic to $\mathrm{Spf}(k[[x^{\mathbb{N}}, y^{\mathbb{N}}, z^{\mathbb{N}}]]/I)$, with I the closure of the ideal generated by the equations

$$\begin{aligned}
G_0 &= y_0^2,\\
G_1 &= 2y_0y_1 - x_0^2(z_1 + 1),\\
G_2 &= y_1^2 + 2y_0y_2 - 2x_0x_1(z_1 + 1) - x_0^2 z_2,\\
&\ \ldots
\end{aligned}$$

More precisely, G_n is obtained from $f = y^2 - x^2 z$ by considering the image of F_n under the transformation $z_1 \mapsto z_1 + 1$ and then setting $z_0 = 0$. In particular, we have $\mathcal{L}(X)_\gamma = Z \times \mathrm{Spf}(k[[z_0]])$. It is not known to the authors whether the formal scheme Z has smooth factors.

(3) Finally, all the arguments used in this paper have been developed in the case of $\mathrm{char}(k) = 0$ and have yet to be extended to positive characteristics.

References

[1] D. Bourqui and J. Sebag. The Drinfeld-Grinberg-Kazhdan theorem for formal schemes and singularity theory. *In pre-print*, 2015.

[2] D. Bourqui and J. Sebag. The Drinfeld-Grinberg-Kazhdan theorem is false for singular arcs. *Journal of the Institute of Mathematics of Jussieu*, 2015.

[3] A. Bouthier, N. B. Chau, and Y. Sakellaridis. On the formal arc space of a reductive monoid. *American Journal of Mathematics*, 138(1):81–108, 2014.

[4] A. Bouthier and D. Kazhdan. Faisceaux pervers sur les espaces d'arcs I: Le cas d'égales caractéristiques. arXiv:1509.02203, 2015.

[5] V. Drinfeld. On the Grinberg-Kazhdan formal arc theorem. arXiv:math/0203263, 2002.

[6] G. Fischer. *Complex Analytic Geometry. Lecture Notes in Mathematics, Vol. 538.* Springer-Verlag, 1976.

[7] M. Grinberg and D. Kazhdan. Versal deformations of formal arcs. *Geometric & Functional Analysis*, 10(3):543–555, 2000.

[8] A. Grothendieck. *Éléments de géométrie algébrique: I. Le langage des schémas.* Publications Mathématiques de l'IHÉS, 1960.

[9] J. Sebag. Arc schemes, derivations and Lipman's theorem. *Journal of Algebra*, 347:173–183, 2011.

[10] A. Yekutieli. On Flatness and Completion for Infinitely Generated Modules over Noetherian Rings. arXiv:0902.4378v4, 2010.