Gabrielov's counterexample to nested linear Artin Approximation

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Abstract : Gabrielov's famous example for the failure of analytic Artin approximation in the presence of nested subring conditions is explained as a growth phenomenon in standard basis computations. This understanding, in turn, provides many more examples.

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In the Séminaire Henri Cartan of 1960/61 Grothendieck posed the question whether analytically independent analytic functions are also formally independent [Gr]. It came as a surprise when Gabrielov answered the question in 1971 in the negative. He constructed four analytic functions e, f, g, h in two variables admitting one formal relation but no analytic one [Gb1]. To our knowledge, this is essentially the only known counterexample to Grothendieck's question.[(2)] In a later paper, Gabrielov gave a sufficient condition for a positive answer in terms of the rank of the Jacobian matrix of the analytic functions [Gb2]. In a more general vein, Popescu proved in 1985 a difficult approximation theorem which contains as a particular case a positive answer whenever the analytic functions are $algebraic$ power series [Po1, Po2, Sp, Te]. Gabrielov's counterexample is based on an example of Osgood [Os] from 1916, complemented by a tricky construction and a cryptic calculation. The deeper reason for the existence of formal divergent relations between analytically independent analytic functions remained mysterious over the years.

In this note we explain the genesis of the phenomenon in Gabrielov's example and provide a systematic way to construct many more counterexamples: It turns out that the existence of formal but not analytic relations is caused by $accumulated\ growth\ occurrences$ in standard basis computations for echelons (an $echelon$ is a generalization of an ideal, see below). Such a growth behaviour is well known for standard bases of ideals of power series rings, but does not do any harm there due to the finiteness of the basis (which is ensured by noetherianity.) Standard bases of echelons need no longer be finite, and the iterated growth occurrence in their construction may indeed force divergence.[(3)]

The first step to understand the phenomenon, and this already appears in [Gb1], is to transcribe the existence of formal or analytic relations to a nested linear Artin approximation problem: Let $f_1(x), ..., f_m(x)$ be power series in variables $x = (x_1, ..., x_n)$ and let $r(y_1, ..., y_m)$ be a (formal or analytic) relation between them, say,

$$r(f_1(x), ..., f_m(x)) = 0.$$

This is equivalent to saying that $r(y)$ belongs to the ideal of the formal, respectively convergent, power series ring $\mathbb{C}[[x, y]]$, respectively $\mathbb{C}\{x, y\}$, generated by the series $y_i - f_i(x)$, for $i = 1, ..., m$. Therefore there exist power series $a_1(x, y), ..., a_m(x, y)$ such that

$$r(y) = \sum_{i=1}^{m} a_i(x, y) \cdot (y_i - f_i(x)).$$

Here, the series a_i are allowed to depend on both x and y, whereas the series r must be independent of x. This requirement is known in the context of Artin approximation as a "nested subring condition". Note that the unknown series r and a_i appear linearly in the equation. As an extension of Grothendieck's question one may then ask more generally whether linear nested Artin approximation holds for analytic functions: Given analytic functions e and $f_1, ..., f_m$ in n variables $x_1, ..., x_n$ such that the linear presentation

$$e(x) = \sum_{i=1}^{m} \widehat{a}_i(x) \cdot f_i(x)$$

holds with formal power series $\widehat{a}_i(x)$ depending only on the variables $x_1, ..., x_{n_i}$, for given $n_i \leq n$, does there exist a presentation

$$e(x) = \sum_{i=1}^{m} a_i(x) \cdot f_i(x)$$

with analytic functions $a_i(x)$ depending on the same sets of variables as $\widehat{a}_i(x)$?

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(2) Artin attributes in [Ar] the question to Abhyankar.

(3) For algebraic power series, the basis may still be infinite, but the echelon then has a finiteness property which seems to imply directly a positive answer to Grothendieck's question in this case.

Gabrielov also gives a counterexample to this case of linear nested analytic approximation: Consider the series $f = 1$, $g = z \cdot (e^x - 1)$, and $h = xy - z$ in three variables x, y, z. He then shows that the convergent series

$$e(x,z) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{i!}{(i+j)!} \cdot x^{j+1} z^i$$

admits a presentation

$$e = \widehat{a} \cdot f + \widehat{b} \cdot g + \widehat{c} \cdot h,$$

with formal series $\widehat{a}(y,z)$, $\widehat{b}(y,z)$, $\widehat{c}(x,y,z)$ but that there are no convergent series $a(y,z)$, $b(y,z)$, $c(x,y,z)$ representing e in this way. Setting

$$I = \mathbb{C}\{y,z\} \cdot f + \mathbb{C}\{y,z\} \cdot g + \mathbb{C}\{x,y,z\} \cdot h$$

with completion $\widehat{I} = I \cdot \mathbb{C}[[x,y,z]]$, one therefore has the strict inclusion of vector subspaces

$$I \subsetneq \widehat{I} \cap \mathbb{C}\{x,y,z\}.$$

We will explain in this note the deeper reason behind this fact. Subspaces of power series rings as I and $\widehat{I}$ will be called *echelons*: In n variables, these are finite sums

$$I = \sum_{i=1}^{k} \mathbb{C}\{x_1, ..., x_{n_i}\} \cdot f_i,$$

with series $f_i \in \mathbb{C}\{x_1, ..., x_n\}$ and integers $0 \leq n_i \leq n$ (analogous definition for formal power series.)

Note first that the sums in our chosen echelons $\widehat{I}$ and I are direct: If we had a non-trivial linear relation

$$a(y,z) \cdot 1 + b(y,z) \cdot z \cdot (e^x - 1) + c(x,y,z) \cdot (xy - z) = 0,$$

setting $z = 1$ and $x = \frac{1}{y}$ would express $e^{\frac{1}{y}}$ as a quotient of power series in y, which is impossible. Now order the monomials $x^i y^j z^k$ lexicographically by their exponents so that $x < y < z$. The *initial monomial* of a non-zero series $s(x,y,z)$ is the smallest monomial $\mathrm{in}(s) = x^i y^j z^k$ of its expansion. We have $\mathrm{in}(f) = 1$, $\mathrm{in}(g) = xz$, and $\mathrm{in}(h) = xy$. The $\mathbb{C}$-subspace $\mathrm{in}(I)$ of $\mathbb{C}\{x,y,z\}$ generated by all initial monomials of I turns out to be the infinite sum

$$\mathrm{in}(I) = \mathbb{C}\{y,z\} \cdot 1 + \sum_{k=1}^{\infty} \mathbb{C}\{z\} \cdot x^k z^k + \mathbb{C}\{x,y,z\} \cdot xy.$$

An *echelon standard basis* of I is obtained by adapting Buchberger's algorithm to the present situation. The basis is infinite, the first few elements after f, g, h are

$$\begin{aligned}
g_2 &= \tfrac{1}{12} \cdot [x^2z^2 + \tfrac{1}{2} \cdot x^3z^2 + \tfrac{3}{20} \cdot x^4z^2 + \tfrac{1}{30} \cdot x^5z^2 + \tfrac{1}{168} \cdot x^6z^2 + \ldots],\\
g_3 &= \tfrac{1}{720} \cdot [x^3z^3 + \tfrac{1}{2} \cdot x^4z^3 + \tfrac{1}{7} \cdot x^5z^3 + \tfrac{5}{168} \cdot x^6z^3 + \tfrac{5}{1008} \cdot x^7z^3 + \ldots],\\
g_4 &= \tfrac{1}{100800} \cdot [x^4z^4 + \tfrac{1}{2} \cdot x^5z^4 + \tfrac{5}{36} \cdot x^6z^4 + \tfrac{1}{36} \cdot x^7z^4 + \ldots],\\
g_5 &= \tfrac{1}{25401600} \cdot [x^5z^5 + \tfrac{1}{2} \cdot x^6z^5 + \tfrac{3}{22} \cdot x^7z^5 + \tfrac{7}{2461} \cdot x^8z^5 + \ldots],\\
g_6 &= \tfrac{1}{10059033600} \cdot [x^6z^6 + \tfrac{1}{2} \cdot x^7z^6 + \tfrac{7}{52} \cdot x^8z^6 + \tfrac{1}{39} \cdot x^9z^6 + \ldots].
\end{aligned}$$

The general formula is

$$g_k = z^k \cdot \sum_{i=k}^{\infty} q_{i,k} \cdot x^i,$$

with coefficients $q_{i,k}$ given by

$$q_{i,k} = \frac{(i-1)!}{4^{k-1} \cdot (i-k)! \cdot (i+k-1)! \cdot (\frac{1}{2})^{\overline{k-1}}},$$

where $(\frac{1}{2})^{\overline{k-1}}$ denotes $\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+k-1)$. The expansion of g_k results from the linear combination

$$s(g_{k-1}, h) = -y \cdot g_{k-1} + q_{k-1,k-1} \cdot x^{k-2}z^{k-1} \cdot h$$

of g_{k-1} and h given by the relation $(-y, x^{k-2}z^{k-1})$ between their initial monomials $x^{k-1}z^{k-1}$ and xy, taking into account the factor $q_{k-1,k-1}$ in front of g_{k-1}. Then g_k is obtained as the remainder of the division of $s(g_{k-1}, h)$ by the series f, g, h and $g_2, ..., g_{k-1}$. All g_k are convergent series.

The key observation here is that the coefficients $q_{k,k}$ of the initial monomials $x^k z^k$ of g_k *tend very fast* to 0. This is due to the rapidly decreasing coefficients $\frac{1}{k!}$ in $g = z \cdot (e^x - 1)$ (most other series g with this property would also produce counterexamples.)

Rewrite now the series g_k as linear combinations of the original generators f, g, h of I,

$$g_k = a_k \cdot f + b_k \cdot g + c_k \cdot h,$$

with uniquely defined convergent series $a_k, b_k \in \mathbb{C}\{y, z\}$ and $c_k \in \mathbb{C}\{x, y, z\}$. They are given by the recursions

$$\begin{aligned}
a_k &= -y \cdot a_{k-1} + \tfrac{1}{4(2k-3)(2k-5)} \cdot z^2 \cdot a_{k-2}, \\
b_k &= -y \cdot b_{k-1} + \tfrac{1}{4(2k-3)(2k-5)} \cdot z^2 \cdot b_{k-2}, \\
c_k &= -y \cdot c_{k-1} + \tfrac{1}{4(2k-3)(2k-5)} \cdot z^2 \cdot c_{k-2} - x^{-1} \cdot (a_{k-1} \cdot f + b_{k-1} \cdot g + c_{k-1} \cdot h),
\end{aligned}$$

with $a_1 = 0$, $a_2 = z^2$, $b_1 = 1$, $b_2 = -y + \frac{1}{2}z$, $c_1 = 0$, $c_2 = -x^{-1} \cdot z \cdot (1 - e^x)$. The preceding formulas imply that a_k, b_k are homogeneous polynomials in y and z of degree k, respectively $k-1$, while c_k is a polynomial in x, y, z, e^x, x^{-1} without poles. Note that in the expansions of a_k, b_k and c_k the monomials $y^{k-2}z^2$, y^{k-1} and $y^{k-2}z$, respectively, appear with coefficients ± 1.

The successive quotients

$$\frac{q_{k+1,k+1}}{q_{k,k}} = \frac{1}{4 \cdot (2k+1) \cdot (2k-1)}$$

of the coefficients of g_k tend quadratically towards 0, so that the $q_{k,k}$ themselves tend extremely fast to 0. As

$$\frac{q_{i,k}}{q_{k,k}} = \frac{k! \cdot (i-1)!}{(i-k)! \cdot (i+k-1)!} \leq 1$$

for $i \geq k$, all coefficients $q_{i,k}$ of the series g_k become very small as k increases. This then implies that *infinite* linear combinations of the series g_k with rapidly *increasing* coefficients may still produce *convergent* series. A typical example would be the convergent series

$$e(x, z) := \sum_{k=1}^{\infty} \frac{1}{q_{k,k}} \cdot g_k(x, z).$$

Various other combinations of the series g_k could be taken. By construction, the series e belongs to the intersection $\widehat{I} \cap \mathbb{C}\{x, y, z\}$. We show that it does not belong to I. By uniqueness of the presentation, it suffices to write e as a linear combination $e = a \cdot f + b \cdot g + c \cdot h$ of f, g, h with divergent series a, b, c. Set $r_k = \frac{1}{q_{k,k}}$ so that $e = \sum_{k=1}^{\infty} r_k \cdot g_k$ and $a = \sum r_k \cdot a_k$, $b = \sum r_k \cdot b_k$, $c = \sum r_k \cdot c_k$ with a_k, b_k and c_k as defined above. As we noted earlier, the monomials $y^{k-2}z^2$, y^{k-1} and $y^{k-2}z$ appear with coefficients ± 1 in the expansions of a_k, b_k and c_k, respectively. As the successive quotients r_{k+1}/r_k tend quadratically to infinity, it follows that the series a, b, c diverge.

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