

Why the characteristic zero proof of resolution of singularities fails in positive characteristic

HERWIG HAUSER

This is – for the time being – the last of a series of papers of the author on resolution of singularities. This series started with a collection of obstacles which make resolution in arbitrary dimension and characteristic difficult [Ha 1]. It was followed by a comprehensive study of Hironaka’s proposal for surface resolution in positive characteristic [Ha 2], in order to see whether this approach could be extended to higher dimensions. It turned out that its methods are too limited for the purpose, since they rely heavily on dimension two, with little chances of extension beyond.

The next step was to understand Hironaka’s original proof of resolution in characteristic zero for varieties of any dimension [Hi 4]. The algorithmic versions of this proof by Villamayor and Encinas-Villamayor [Vi 1, Vi 2, EV 1, EV 2], respectively by Bierstone-Milman [BM 1, BM 2, BM 3], were crucial for getting a clearer view on the reasoning. To axiomatize these proofs as much as possible seemed to be a prerequisite to locate the problems which pop up in characteristic p . Also, it was necessary in order to formulate a plausible and characteristic-free resolution procedure by specifying at each step a correct choice of the center of blowup (without, of course, proving that the process actually terminates).

This conceptualization and compactification of Hironaka’s proof was done together with Santiago Encinas in [EH], and is explained with much more motivations and background information in the monograph [Ha 3]. Special efforts are made in [EH] to construct the resolution invariant and the centers of blowup in a characteristic independent manner (as far as possible). Essentially, only the induction itself uses characteristic zero. This hypothesis ensures the existence of *osculating hypersurfaces* – these are hypersurfaces which are especially adapted to the singularity to be resolved and which need not exist in positive characteristic; they are defined by Tschirnhaus transformations in the sense of Abhyankar. Such hypersurfaces, if they exist, allow to descend in the embedding dimension, to apply induction on this dimension, and to show then that the resolution invariant drops under blowup.

In the paper [Ha 4], we developed techniques which allow to find a substitute for osculating hypersurfaces and which works in any characteristic and any dimension. The techniques are inspired by the work of Abhyankar. He proposes in [Ab 1] to consider as a significative resolution invariant of a plane curve singularity the maximum over all coordinate choices of the slope of a certain segment of the Newton polygon (related to the classical concept of the first characteristic pair of a curve). And indeed, the vector formed by the multiplicity and this slope forms a local invariant which drops lexicographically when blowing up the singular points of the curve. Together with Georg Regensburger we explain in [HR] the necessary ingredients from commutative algebra needed for this approach. It is shown explicitly when and why the invariant drops. Especially, the involved formal coordinate changes required to realize the maximum of the slope are studied with all details.

The paper [Ha 4] extends these methods to arbitrary dimension. They allow to construct a whole range of local invariants of singularities and to observe their behaviour under blowup. However, the definite decrease under blowup of any of these invariants can no longer be ensured ab initio.

With the above articles, we dispose nowadays of a very conceptual inductive proof for resolution in characteristic zero and arbitrary dimension (using osculating hypersurfaces), and all the necessary devices to replace osculating hypersurface by the more general concept of *hypersurfaces of weak maximal contact* (which, essentially, are hypersurfaces which maximize the order of the ideal generated by the coefficients of the defining polynomials of the singularity). To complete the picture, it remains to insert the second concept in the first proof and to search the circumstances where the induction fails in positive characteristic. This task shall be accomplished in the present paper.

So the intention is to look what happens when trying to carry out Hironaka's proof in a characteristic free manner and using weak maximal contact. The first observations in this direction go back to Narasimhan (a student of Abhyankar) in the papers [Na 1, Na 2] (see also Mulay's article [Mu]), and Moh (a student of Hironaka) in [Mo 1, Mo 2]. Narasimhan constructed an example of a variety in characteristic two, where the locus of points of highest multiplicity is not contained (even locally) in any regular hypersurface. This prohibits the existence of hypersurfaces of maximal contact (in the sense of Hironaka, which is stronger than weak maximal contact) and shows that the descending induction on the embedding dimension (which is instrumental in zero characteristic) cannot be applied directly in positive characteristic.

Moh shows that even when replacing osculating hypersurfaces by hypersurfaces of weak maximal contact there occur problems. Namely, he constructs an example of a variety in positive characteristic where Hironaka's resolution invariant (when adapted properly to the concrete situation) increases under blowup. In addition, Moh is able to bound the maximal increase of the invariant.

The present paper looks closely at this type of phenomena. We describe and study completely the cases in characteristic p where the arguments and conclusions of characteristic 0 fail. Very strange and subtle things seem to happen. The main observations we will make can be grouped in six items.

We shall always assume that the characteristic of the ground field is $p > 0$. For simplicity of the exposition, we restrict to hypersurface singularities (for which, in particular, the order of the defining equation at a point coincides with the local multiplicity).

- *Failure of maximal contact:* In a sequence of permissible blowups of a given ideal, the sequence of points where the order of the transforms of the ideal remains constant (equiconstant points) may leave eventually any regular hypersurface accompanying the process. This prohibits to apply induction on descending embedding dimension (as is done successfully in zero characteristic).
- *Uniqueness of blowups for failure of maximal contact:* The sequences of blowups where equiconstant points leave any given regular hypersurface are essentially unique. A particularity is that the equiconstant points must loose earlier on their way through the sequence of blowups at least two exceptional components passing originally through them.
- *Failure of induction on order:* The most popular resolution invariant is given by the lexicographic pair of numbers consisting of the order of the ideal and the order of its divided coefficient ideal (i.e., the order which is obtained by subtracting from the order of the coefficient ideal the exceptional multiplicity). In positive characteristic, it may increase.
- *Uniqueness of tangent cone for failure of induction:* For hypersurfaces, in order to have an increase of the above resolution invariant, the weighted tangent cone of the defining

equation must coincide with a unique universal polynomial. Such special polynomials will be called *hybrid*. They have prescribed coefficients (namely, certain binomial coefficients). In contrast, a sufficiently generic choice of the coefficients produces a non-increasing invariant. Thus only very special and explicitly known hypersurface singularities pose problems. But we do not know how to resolve these.

- *Failure of the Bernstein-Kushnirenko theorem modulo p -th powers:* We show that the Bernstein-Kushnirenko theorem equating the number of isolated zeros of a system of polynomial equations and the Minkowski mixed volume of the associated convex polytopes fails if the polynomials are replaced by their equivalence classes modulo p -th powers of the variables. It turns out that the counterexamples for this failure coincide with the weighted homogeneous hybrid polynomials where the induction on the resolution invariant falls short.
- *Estimates on the increase of the resolution invariant:* Following Moh, the increase of the invariant can be bounded. For surfaces, this suffices to show that it drops in the long run in a sequence of permissible blowups, though it may increase occasionally. This gives a new proof for the resolution of surfaces in characteristic p , using – in contrast to the existing proofs – the characteristic zero resolution invariant and the same sequence of blowups.

Some of the preceding circumstances have already been known for a long time, and appear – at least implicitly – in the work of Abhyankar, Giraud, Moh, Cossart and others. But it seems that they were never studied systematically. In particular, it is surprising that nobody observed that the obstruction can only occur in concrete series of polynomials.

The most striking fact in the above list is the coincidence that all three failures are related to the same type of equations, the hybrid ones. Let us therefore deviate briefly to have a look at these polynomials. A typical candidate of a hybrid polynomial in three variables looks like this (t being a constant in the ground field)

$$f = x^p + P(y, z) = x^p + y^r z^s \cdot \sum_{i=0}^k \binom{k+r}{i+r} \cdot y^i (tz - y)^{k-i}.$$

Here, r and s are positive integers not divisible by p , $r + s + k$ is a multiple of the characteristic p , and the residues \bar{r}^p and \bar{s}^p of r and s modulo p satisfy

$$\bar{r}^p + \bar{s}^p \leq p.$$

Such polynomials look quite harmless. Let us see three examples and their behaviour under the substitution $y \rightarrow y + tz$. For $p = 3$, $r = s = 1$, $k = 4$ and $t = 1$ we get

$$f = x^3 + yz \cdot (y^4 + y^3z + y^2z^2 - yz^3 - z^4),$$

which, by the coordinate change $y \rightarrow y + z$, transforms into

$$f^+ = x^3 + P^+(y, z) = x^3 + z \cdot (y^5 + z^5).$$

For $p = 3$, $r = s = 1$, $k = 7$ and $t = 1$ we get

$$f = x^3 - yz \cdot (y^7 + y^6z + y^5z^2 + y^4z^3 + y^3z^4 + y^2z^5 + yz^6 + z^7),$$

which, under $y \rightarrow y + z$, transforms into

$$f^+ = x^3 - z \cdot (y^8 - z^8).$$

For $p = 3$, $r = 2$, $s = 1$, $k = 3$ and $t = 1$ we get

$$f = x^3 + y^2z \cdot (-y^3 + yz^2 + z^3),$$

which, under $y \rightarrow y + z$, transforms into

$$f^+ = x^3 + z \cdot (-y^5 + y^4z + z^5).$$

What is the common feature of these examples? It has to do with the above coordinate change (which corresponds in the application to resolution problems to a translation in the exceptional divisor). In all cases we see that a pure z -power (z^6 in the first and last case, and z^9 in the second case) can be eliminated from the polynomial f^+ via the substitution $x \rightarrow x - z^2$ (respectively $x \rightarrow x - z^3$ for the second example). This elimination is the same as considering P^+ modulo p -th powers.

Now, after this substitution, the resulting polynomial (denoted again by P^+) has order at 0 with respect to the y -variable equal to $k + 1$, whereas the order at 0 of P minus $r + s$ (this subtraction corresponds to delete exceptional components from the total transform of the polynomial) equals k . Thus, modulo p -th powers, the order of P minus the exceptional multiplicity is smaller than the y -order of P^+ . This purely algebraic fact will be the clue in all observed phenomena. And there are very few polynomials where this “increase” can happen (we shall classify them completely in the case of two variables).

It is astonishing that the reason why Hironaka’s proof fails in positive characteristic has such a simple and dull appearance. It is simply the strange behaviour of homogeneous polynomials (in our case P) under linear coordinates changes when considered modulo p -th powers. The first example of a hybrid polynomial was given by Moh [Mo 1, Mo 2].

For each selection of p , r , s and k subject to the above conditions there is precisely one hybrid polynomial in three variables with the respective exponents and degrees (up to coordinate changes). For other values of the parameters there are no hybrid polynomials. In the course of the paper we shall also indicate how hybrid polynomials appear in the context of the Bernstein-Kushnirenko theorem.

It has to be added here that the hybrid polynomials represent only the weighted tangent cone (with respect to suitably chosen weights) of the singularity which has to be resolved, so that higher order terms may and will occur in general in the defining equation. If there are no such terms, the singularity can indeed be resolved by direct inspection. Otherwise, the higher order terms seem to prohibit the precise control of the singularities under blowup. One possible line of attack could consist in showing that the weighted tangent cone is always sufficiently dominant in the expansion of the polynomials so as to guarantee the existence of a resolution. This has not been achieved up to now.

Acknowledgements. While thinking on resolution of singularities in the last few years, the author has profited from many discussions with other mathematicians. In particular, we would like to mention O. Villamayor, G. Müller, S. Encinas, M.E. Alonso, A. Quirós, A. Bravo, M. Spivakovsky, F. Cano, J. Lipman, V. Cossart, J.-J. Risler, M. Vaquié, H. Maeda, H. Kawanoue, J. Schicho, G. Bodnár, F. Pauer, W. Förger. Thanks to all of them. The work on this paper has been supported by the Austrian Ministry of Science and Education.

CONTENTS

0. The ingredients of Hironaka’s proof

I. EXAMPLES

1. Regular hypersurfaces containing top locus
2. Failure of permanent contact
3. Increase of invariant under blowup
4. Estimating the decrease of invariant in earlier blowups

II. RESULTS

5. The main result
6. The main result in the case of surfaces

III. TECHNIQUES

7. Description of blowups in local coordinates
8. Height and volume of polynomials
9. Realizing weak maximal contact after blowup
10. Zwickels
11. Transformation matrices
12. Bounding the increase of invariant under blowup
13. Persistence of hypersurfaces of weak maximal contact

IV. OUTLOOK

14. Moh's upper bound for the increase of the invariant
15. Comparison with the Bernstein-Kushnirenko theorem
16. Description of tangent cone for increase of secondary order
17. Hybrid polynomials
18. Quings

References

0. The ingredients of Hironaka's proof

We now give a more detailed account on the situation and phenomena we will describe. For this it will be convenient to review briefly the main constructions developed and explained in [EH] and [Ha 3]. The reader is assumed to have some familiarity with blowups and the induced transforms of ideals. The necessary definitions can be found in the appendix to [Ha 3].

The basic invariant for the resolution of singular schemes is the order of ideals at a point. If the scheme X is defined in some regular ambient scheme W by the ideal K of the structure sheaf \mathcal{O}_W , then, for any point a of W , the order of K at a is defined as

$$\text{ord}_a K = \max \{k \in \mathbb{N}, K \subset m_a^k\},$$

where m_a denotes the maximal ideal of the local ring $\mathcal{O}_{W,a}$ of W at a . The same number is obtained when working in the completed local ring $\widehat{\mathcal{O}_{W,a}}$ of $\mathcal{O}_{W,a}$ (this often allows to simplify computations). If the embedding of X in W has minimal dimension, the order provides a rough measure how singular X is at a . It defines an upper semicontinuous function $\text{ord} : X \rightarrow \mathbb{N}$ on X , i.e., for any $c \in \mathbb{N}$, the subscheme $\text{top}(K, c)$ formed by the points of order $\geq c$ is closed in X . We thus get a stratification of X by locally closed subschemes along which the order of K is constant. The smallest stratum $\text{top}(K)$ consists of points of maximal order and is a closed reduced subscheme of X . It may be singular. We call it the *top locus* of K , respectively of X , in W .

A classical and easy to prove fact asserts that blowing up X in a regular center Z contained in a stratum along which the order of K is constant, the order of X does not increase, i.e., at any point a' of the exceptional divisor Y' of the blowup the order of X' is less or equal to the order of X at $a \in Z$. In particular, this holds if Z is contained in the top locus $\text{top}(K)$. Such centers are called *permissible*. Here, the order of the blowup X' of X is understood as the

order of the strict transform X^{st} of X in the blowup W' of W . In contrast, the order of the total transform X^* of X usually increases and thus does not serve for induction purposes.

Actually, the order of the strict transform of X decreases at most points of the exceptional divisor. At these points, the situation has improved, and induction on the order can be applied. At some special points, however, the order may remain constant. We call these points the *equiconstant points* of K or X in W' . They are also known in the literature as very infinitely near points. It is at these equiconstant points where we need some extra information on the singularity of X' in order to know that the situation has also improved there, though the improvement will occur in a less evident way than at the other points.

This information is usually exhibited by adding to the order of the ideal a second local invariant. The resulting pair of numbers will then be considered with respect to the lexicographic ordering: If the first component, the order of K , has decreased, we are done. If it has remained constant (recall that the order cannot increase if the center is permissible), the second component should have dropped. In particular, this second component becomes only relevant at the equiconstant points a' of K in W' . Now it is an exercise on blowups in local coordinates to see that the equiconstant points of K in the exceptional divisor Y' can be determined from the tangent cone of K at a , and that they lie necessarily in a regular hypersurface V' of W' . So they are relatively rare in Y' .

Let us assume for the moment that this hypersurface V' has a regular image V in W , so that V' is the strict transform V^{st} of V , and that V contains the center Z . It is a general fact that in this case V' coincides with the blowup of V in Z , yielding the commutative diagram

$$\begin{array}{ccc} V' & \subset & W' \\ & \downarrow & \downarrow \\ Z & \subset & V \subset W \end{array}$$

where the vertical arrows denote the blowups of V and W with center Z . Let now a be a point in $Z \subset V$, and let $a' \in Y' \cap V'$ be a point above a . To determine a suitable candidate for the second component of the induction invariant of K at a and its strict transform $K' = K^{st}$ at a' , it is then natural to use V and V' for its definition.

The idea is to associate to the ideals K in W and K' in W' ideals K_- in V and $(K')_-$ in V' which measure the improvement. At points where the order of K has remained constant, we should be able to compare K_- and $(K')_-$ in order to measure the improvement of the singularities when passing from X to X' . Best would be if $(K')_-$ would again be the strict transform of K_- in V' under the blowup of Z in V , analogously to K and K' . This would allow to control the change between K_- and $(K')_-$. In particular, if Z were contained in the top locus $\text{top}(K_-)$ of K_- , the order of $(K')_-$ at a' would automatically be less or equal to the order of K_- at a . Here, of course, it has to be shown that the order of K_- at a does not depend on the local choice of V . In the affirmative case the order of K_- in V would be appropriate to form the second component of the resolution invariant we are looking for.

It is mandatory here that $(K')_-$ is the transform of K_- , i.e., that the descent in dimension from W to V and W' to V' commutes with blowups

$$\begin{array}{ccc} K' & \rightsquigarrow & (K')_- = (K_-)' \\ & \downarrow & \downarrow \\ K & \rightsquigarrow & K_- \end{array}$$

We can then write simply K'_- for $(K')_- = (K_-)'$. As explained before, such a commutative diagram can only be expected at the equiconstant points of K in W' . Moreover, we have

to ensure that the center Z is contained in both $\text{top}(K)$ and $\text{top}(K_-)$. Therefore, when searching for a suitable ideal K_- , we have to be cautious so as to meet commutativity of the diagram and this inclusion simultaneously.

Even in case the construction of a suitable ideal K_- in V could be realized, there is no reason why, at an equiconstant point a' of K in W' , the order of K'_- should have dropped compared to the order of K_- . It may equally have remained constant, and the quandary of equiconstant points seems to repeat.

But now we are much better off, for we may apply induction on the dimension: the ideal K_- in V is defined in a lower dimensional ambient scheme, therefore, by induction, we may assume that we know how to associate to K_- a local invariant – it will consist of a lexicographically ordered vector of numbers given as the orders of a string of ideals – which decreases under blowup. We call this type of descent in the dimension *horizontal induction*, in contrast to the *vertical induction* implied by the decrease of the resolution invariant. Using horizontal induction we either arrive in some dimension at an ideal of order 0, in which case no further descent is possible, or at dimension 1. In the first case, the situation is sufficiently specific to allow a direct and combinatorial resolution argument, sometimes called the *monomial case*. In the second case one uses that the order of an ideal in a one dimensional regular scheme always drops under blowup to 0 when passing to its strict transform. This completes the induction argument.

Leaving aside (intricate) technical complications, the preceding reasoning represents the main outline of Hironaka's proof for resolution of singularities in characteristic zero, in the version developed by Villamayor, Bierstone-Milman, Encinas-Villamayor, Bodnár-Schicho, Encinas-Hauser and Bravo-Villamayor [Hi 4, AHV 1, AHV 2, V1, V2, BM 1, BM 2, BM 3, EV 1, EV 2, EV 3, BS, EH, BV]. You may consult [Ha 3] for an easily accessible introduction to the subject.

The most delicate part in this program is the adequate construction of the ideal K_- in V , starting from an ideal K in W . This works only locally on W , and depends on the choice of the local hypersurface V , which is by no means unique, nor patches on overlaps to give a global hypersurface. From V it is only required that, locally at each point a of W , V accompanies the resolution process of K as long as the order of K remains constant, i.e., that the successive transforms of V contain all points a' , a'' , etc. above a where the strict transforms of K have the same order as K at a . Such hypersurfaces will be said to have *permanent contact* with K . In characteristic 0, it is known that such hypersurfaces exist. They can even be chosen so as to contain locally $\text{top}(K)$ and are then known as *hypersurfaces of maximal contact*. There is a simple procedure through iterated derivatives how to find them at any point, see section 2.

Once some V is chosen at a , expand each element of K as a power series with respect to a local coordinate defining V in W . The resulting coefficients can be equilibrated by raising them to a suitable power and then generate an ideal in V , the *coefficient ideal* $\text{coeff}_V(K)$ of K in V . Do the same with K' and V' . Unfortunately, the coefficient ideal of the strict transform K' of K at equiconstant points a' is not the strict transform of the coefficient ideal of K at a . Thus the commutativity of the above diagram fails.

The inconvenience can be remedied by a suitable factorization of the coefficient ideals $\text{coeff}_V(K)$ into a product of a principal monomial ideal (supported on the exceptional divisor) and another ideal (the relevant part). With this factorization, commutativity with blowups can be established for the second factor. This second factor, which we call K_- for obvious reasons, will be the correct candidate for our descent in dimension (for details on the precise construction and transformation formulas, see again [EH] and [Ha 3]). The ideal K_- in V fulfills all the required properties so as to build on it the horizontal and vertical induction. But

it is essential here that the local hypersurface V in W has transform V' in W' which contains *all* equiconstant points of K in W' .

The present paper shows why and where the preceding procedure fails in positive characteristic. First, we exhibit an example where after a number of blowups any regular hypersurface containing the points where the order of the transforms of K has remained constant since the beginning has singular image in the initial ambient scheme W . Therefore there is no local regular hypersurface in W at a whose transforms contain permanently the points of constant order above a , and this until the moment where the order of K has dropped everywhere. This makes it necessary to replace occasionally the accompanying regular hypersurface so as to contain also in the subsequent blowup the points of constant order. The appropriate adjustment of the hypersurface can be explicitly determined. We can therefore try to apply again horizontal induction on the dimension by using these variable hypersurfaces.

In positive characteristic, the descent in dimension by constructing an ideal K_- in a suitable hypersurface V still works, provided one incorporates the necessary modifications. We thus dispose again of a string of local ideals in descending dimension, and, associated to it, a resolution invariant formed by the orders of these ideals. We wish to show that the invariant drops under blowup in the lexicographic order. If the accompanying hypersurface persists after blowup, i.e., if it has not to be replaced by a new one, the same argument as in characteristic 0 applies and shows that the invariant decreases. The occasional change of the hypersurface in a sequence of blowups would do no harm as long as the invariant of the ideal K_- decreases at these instances. This is, unfortunately, not the case, as was first observed by Moh [Mo 2]. We will see an example which shows that whenever the hypersurface V has to be adjusted, the invariant need not decrease under blowup, and, in certain circumstances, may even increase. This destroys the desired *vertical* induction.

The interesting fact here is that – as alluded to in the introduction – the increase only occurs in very specific situations, and for very special ideals. Among other things, we show that the tangent cone of the involved ideal must coincide, up to rescaling, with a uniquely given homogeneous polynomial. In particular, the algebraic relations between the coefficients of the tangent cone come into play. The “dangerous” homogeneous polynomials forming the tangent cone of the defining polynomial can be described explicitly. Moreover, the points in the blowup where the increase occurs lie in a codimension 2 subscheme. This, of course, suggests to investigate these bad cases further, and to profit of their well known internal structure in order to develop a separate induction argument for their resolution. As experience shows, the matter seems to be more delicate than to be solvable in a straightforward manner.

I. EXAMPLES

1. Regular hypersurfaces containing the top locus

For an ideal K in W we denote by $\text{top}(K)$ the closed reduced subscheme of W of points where the order of K in W is maximal. In characteristic 0, there always exists, locally at each point of W , a regular hypersurface V containing $\text{top}(K)$. This can be seen as follows. Let $a \in W$ be a given point, and set $c = \text{ord}_a K$. In characteristic 0, an element g of $\mathcal{O}_{W,a}$ belongs to m_a^k for some $k \geq 1$, if and only if $\partial_{x_i} g \in m_a^{k-1}$ for all first order partial derivatives ∂_{x_i} . Therefore $\text{top}(K)$ is given locally at a by the vanishing of all partial derivatives $\partial_x^\alpha f$ of elements of K up to order $|\alpha| < c$. At least one derivative $\partial_x^\alpha f$ with $|\alpha| = c - 1$ will have order 1 at a , and hence defines in a neighborhood of a a regular hypersurface V in W containing $\text{top}(V)$. It can be shown that any such hypersurface has permanent contact with

K at a : its transforms under blowup of W in a regular center $Z \subset \text{top}(K) \subset V$ contain the equiconstant points of K [Ab 2, p. 211, Hi 3, p. 106, Ha 3, p. 362].

The argument does not work in positive characteristic, take for K the ideal generated by $f = x^p + y^p$ in characteristic p . Here, all partial derivatives vanish up to arbitrary order, but K has order p at 0. Nevertheless, there still exists a regular hypersurface containing $\text{top}(K) = \text{top}(f)$ (namely, $x + y = 0$). Narasimhan showed that, in general, this is not the case.

Example 1. [Na 1, Na 2, Mu, Hi 1, ex. 4] Consider the polynomial $f = x^2 + yz^3 + zw^3 + y^7w$ in four variables over an algebraically closed field of characteristic 2. It has order 2 at $a = 0$. In a neighborhood of 0, its top locus $\text{top}(f)$ coincides with the singular locus and is given by the vanishing of f and its first order partial derivatives 0, $z^3 + y^6w$, $yz^2 + w^3$, $zw^2 + y^7$. It is verified that the monomial curve C in \mathbb{A}^4 parametrized by t^{32} , t^7 , t^{19} , t^{15} equals $\text{top}(f)$. Therefore, this locus cannot be embedded locally at 0 into a regular hypersurface in \mathbb{A}^4 .

Take now a regular hypersurface V passing through $a = 0$. We claim that for any sequence of point blowups whose first center is the origin, the sequence of equiconstant points of f above a will leave eventually the strict transforms of V . Indeed, as the point blowups keep $\text{top}(f)$ unchanged outside 0, the order of the transforms of f will remain constant equal to 2 at points above points of $\text{top}(f)$ outside 0. The strict transform of the curve $\text{top}(f)$ will therefore consist of points of order 2 for f , by the upper semicontinuity of the order. In particular, the points above 0 which lie in these strict transforms will all be equiconstant points above 0.

But, by a sequence of point blowups, the curve $\text{top}(f)$ will always be separated from the hypersurface V and its strict transforms (since it is not contained in V). Combining both observations we conclude that the equiconstant points above 0 will eventually leave the strict transforms of V .

Therefore, to have permanent contact it is necessary to have the entire top locus of K in W contained locally in a regular hypersurface V . It remains unclear in which situations this inclusion does not occur. Moreover, even if the top locus is contained locally in a regular hypersurface, this hypersurface may not have permanent contact with K . The examples of the next section will shed some light on this type of questions.

2. Failure of permanent contact

We give an example of a polynomial in three variables over a field of characteristic two which does not possess at the origin of \mathbb{A}^3 a local hypersurface of permanent contact, even though its top locus is contained in a regular hypersurface. For this, we indicate a sequence of blowups along which the equiconstant point of the polynomials leave eventually the transforms of any regular hypersurface chosen below.

The example is inspired by the observation of T.T. Moh on the possible increase of the order of coefficient ideals under blowup [Mo 1, Mo 2, ex. 3.2, Hi 1, ex. 5, Ha 1, ex. 16]. Recall that a local regular hypersurface V of W at a has maximal contact with the ideal K at a if it contains locally at a the locus $\text{top}(K)$ of points of maximal order of K in W , and if the consecutive strict transforms under permissible blowups contain the equiconstant points of K above a .

Our example is a polynomial f in characteristic 2 for which after six blowups the strict transform of any regular hypersurface at a does no longer contain the equiconstant points of f . Hence a new hypersurface must be chosen in the resolution process. This replacement

of V destroys the descent in dimension and the required commutativity for the passage to coefficient ideals.

Example 2. Consider a sequence of local blowups $W^6 \rightarrow \dots \rightarrow W^1 \rightarrow W$ at points a^i in W^i with $W = W^0$ a regular scheme of dimension three (e.g., affine space \mathbb{A}^3). All blowups are point or curve blowups. We choose the centers as follows. For given local coordinates x, y, z in W at $a = 0$, the first map is the monomial point blowup in the y -chart, the second the monomial point blowup in the z -chart. Hence, a^1 and a^2 will be the origins of the respective affine charts of W^1 and W^2 . Note that a^2 lies in the intersection of the two exceptional components having occurred so far. The coordinates x, y, z in W induce in a natural way coordinates in each W^i (which will be denoted again by x, y, z).

The third blowup is no longer monomial. Its center is the origin a^2 of the present chart of W^2 , but the blowup is considered in the z -chart of W^3 at the point a^3 with coordinates $(0, 1, 0)$. Said differently, this blowup is the composition of the monomial point blowup in the z -chart followed by the translation $y \rightarrow y + 1$. Hence a^3 belongs to the new exceptional component Y^3 in W^3 , but lies outside the strict transforms of the two exceptional components through a^2 . The fourth, fifth and sixth blowup are the monomial curve blowup with center a^2 . In figure 1 we see the evolution of the exceptional curves when restricting the first three blowups to the hypersurface $V : x = 0$ in W .

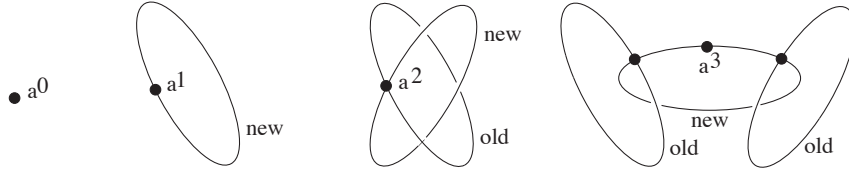


Figure 1. Configuration of exceptional components.

Take for K the ideal in W generated by $f = f^0 = x^2 + y^7 + yz^4$. The hypersurface V defined in W by $x = 0$ is regular and contains the top locus of f . The coefficient ideal of f in V is generated by the polynomial $y^7 + yz^4$. Under the above sequence of blowups, the strict transforms f^i of f and the coordinate changes are of the form

$$\begin{aligned}
 f^0 &= x^2 + 1 \cdot (y^7 + yz^4), & x, y, z, \\
 f^1 &= x^2 + y^3 \cdot (y^2 + z^4), & x, y, z \rightarrow xy, y, zy, \\
 f^2 &= x^2 + y^3 z^3 \cdot (y^2 + z^2), & x, y, z \rightarrow xz, yz, z, \\
 f^3 &= x^2 + z^6 \cdot (y + 1)^3 ((y + 1)^2 + 1), & x, y, z \rightarrow xz, yz + z, z, \\
 f^4 &= x^2 + z^4 \cdot (y + 1)^3 ((y + 1)^2 + 1), & x, y, z \rightarrow xz, y, z, \\
 f^5 &= x^2 + z^2 \cdot (y + 1)^3 ((y + 1)^2 + 1), & x, y, z \rightarrow xz, y, z, \\
 f^6 &= x^2 + 1 \cdot (y + 1)^3 ((y + 1)^2 + 1), & x, y, z \rightarrow xz, y, z.
 \end{aligned}$$

The strict transforms V^i of V are always given by $x = 0$. The monomial factors in front of the parentheses in f^i denote exceptional components of the restriction of f^i to V^i (more precisely, of the coefficient ideal of f^i in V^i). The order of f^i at a^i has remained constant equal to 2 for $i \leq 5$, and has dropped to 0 at a^6 . So a^6 is not an equiconstant point for f . But,

if the characteristic is 2, there is another equiconstant point in the exceptional divisor Y^6 of W^6 , namely the point $b^6 = (1, 0, 0)$. At this point, f^6 has again order 2, but b^6 does not lie in V^6 . The expansion of f^6 at b^6 is obtained by applying the translation $x \rightarrow x + 1$ to f^6 . It yields the polynomial

$$\tilde{f}^6 = x^2 + 1 \cdot (y^5 + y^3 + y^2)$$

of order 2 at the origin. This shows that the order of f^6 at b^6 is 2. Thus V is not a hypersurface of permanent contact for f at a . It has only *temporary* contact.

Of course, it is possible to choose at a^5 instead of V^5 a new local hypersurface U^5 whose strict transform U^6 in W^6 does contain b^6 . It suffices to take for U^5 the hypersurface defined by $x + z = 0$. Observe here that replacing x by $x + z$ in f^5 will eliminate the monomial z^2 from the expansion of f^5 , transforming the tangent cone into a monomial. But the image U of U^5 in W is singular at a . Therefore U^5 is not the strict transform of some regular hypersurface in W (see figure 2).

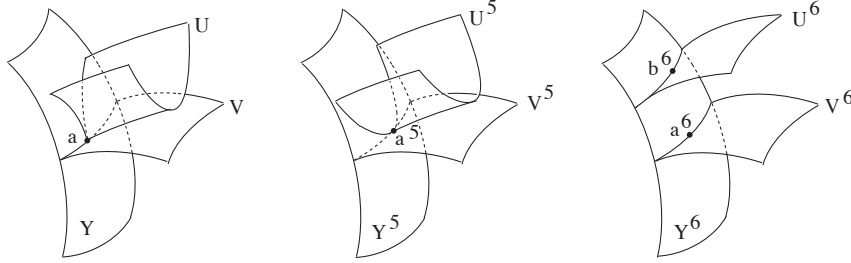


Figure 2. Failure of permanent contact.

The reason for the loss of contact in W^6 is that one step earlier, in W^5 , the hypersurface V^5 does not maximize the order of the coefficient ideal of f^5 in V^5 . Indeed, $\text{coeff}_{V^5}(f^5)$ has order 2, whereas the maximal order is 3 (take the coefficient ideal with respect to the hypersurface defined by $x + z(y + 1)^2 + z = 0$). It is easy to see that if a hypersurface U maximizes the order of the coefficient ideal of an ideal K , its strict transform U^{st} contains all equiconstant points of K in W' . Conversely, a hypersurface of permanent contact must maximize the order of the coefficient ideal at the beginning.

Maybe we can modify U^5 from before slightly to a hypersurface \tilde{U}^5 which maximizes the order of the coefficient ideal of f^5 in \tilde{U}^5 and which does stem from a regular hypersurface \tilde{U}^0 in W . A computation shows that the linear term of the equation of \tilde{U}^0 must be x (up to a constant factor). So let us write $g^0 = x + \sum g_{jk}y^jz^k$ for the equation of \tilde{U}^0 in W . We get

$$\begin{aligned} g^0 &= x + \sum g_{jk}y^jz^k, \\ g^1 &= x + \sum g_{jk}y^{j+k-1}z^k, \\ g^2 &= x + \sum g_{jk}y^{j+k-1}z^{j+2k-2}, \\ g^3 &= x + \sum g_{jk}(y+1)^{j+k-1}z^{2j+3k-4}, \\ g^4 &= x + \sum g_{jk}(y+1)^{j+k-1}z^{2j+3k-5}, \\ g^5 &= x + \sum g_{jk}(y+1)^{j+k-1}z^{2j+3k-6}. \end{aligned}$$

Substituting this polynomial in f^5 shall maximize the order of the coefficient ideal of f^5 , i.e., it must eliminate the term $z^2(y+1)^3y$ from f^5 . Therefore the coefficient g_{jk} of g^5 must be non-zero for $j+k-1=1$ and $2j+3k-6=1$. But there is no pair (j, k) of non-negative integers with $j+k=2$ and $2j+3k=7$. This shows that there is no regular hypersurface \tilde{U}^0 in W whose transform \tilde{U}^5 in W^5 maximizes the order of the coefficient ideal of f^5 .

Let us examine the recipe for constructing such type of examples. The first two monomial blowups in opposite charts are needed to produce two exceptional components and a point a^2 in their intersection. The third blowup is characterized by the “disappearance of the two exceptional components” when passing from a^2 to a^3 . The last three blowups are only used to exhibit the point b^6 outside V^6 by making the exceptional multiplicities drop until they equal 0.

It is between a^2 and a^3 , i.e., in the third blowup, where the key phenomenon occurs. It consists in the increase of the order of the coefficient ideal $y^2 + yz + z^2$ of f^2 in V^2 (after having factored from it the exceptional monomial yz). Applying the coordinate change $x \rightarrow x + z^3$ to f^3 eliminates the monomial z^6 and thus produces the (divided) coefficient ideal y^3 of f^3 in V^3 . It has order 3 at a^3 , whereas the (divided) coefficient ideal $y^2 + yz + z^2$ of f^2 has order 2 at a^2 . This increase of the order will be discussed separately in the next section.

It turns out that the above construction produces examples where permanent contact fails and where the order of the coefficient ideal increases if only if the exponents and the coefficients of f are chosen in a very specific manner. The conditions carry on the second transform f^2 of f , because the passage from f^2 to f^3 represents the substance of the phenomenon. The necessary (and sufficient) conditions on f^2 are as follows:

- The residues modulo p of the exceptional multiplicities, i.e., of the exponents of the monomial factors in front of the parentheses of f^2 , must satisfy a prescribed arithmetic inequality. For surfaces, both must be positive and their sum must not exceed p . The general inequality is given in section 5. In our case, the coefficient ideal of f^2 in V^2 is $y^3z^3 \cdot (y^2 + yz + z^2)$. The exceptional multiplicities are both 3 and satisfy $\overline{3}^2 + \overline{3}^2 = 1 + 1 = 2$.
- The order of the coefficient ideal of f^2 must be a multiple of the characteristic. In the example, the order is $3 + 3 + 2 = 8$.
- The coefficients of the weighted tangent cone of f^2 must be certain binomial coefficients (up to a rescaling of the coordinates). In the example, where all coefficients are 1, this condition is hidden by the fact that we are working in characteristic 2.

The given example is among the simplest ones with these properties. The actual values of the coefficients of the defining equations of the singularity seem to play a decisive role in positive characteristic. This is strikingly different from the case of zero characteristic, where the resolution never refers to the actual values of the coefficients.

Of course, due to the necessity of the three conditions for the failure of permanent contact, one might hope to find a strategy for the resolution of singularities in positive characteristic by distinguishing at each blowup two cases, a good and a bad one. The conditions show that the situations where the characteristic zero arguments fail are very special. One could then try to treat this critical case separately by a different ad hoc argument. This is done in a similar manner in characteristic zero in the case where the coefficient ideal is a monomial ideal supported by the exceptional divisor. There, a simple combinatorial argument saves the situation, cf. e.g. [Ha 3, section on shortcuts]. In positive characteristic, surprisingly enough, the special cases seem to be much more malicious than their outfit would suggest. Up to now, they resisted obstinately the various attempts of attack.

3. Increase of invariant under blowup

We shall now show that the resolution invariant proposed in [Hi 4, V 1, V 2, BM 1, BM 2, BM 3, EV 1, EV 2, BS, EH, BV] for characteristic 0 does not work in characteristic p in the special cases of the preceding section. First, we briefly recall its definition, restricting to surfaces in three-space and omitting some technical complications. For its precise definition, we refer to the literature.

We shall consider surfaces in $W = \mathbb{A}^3$ at $a = 0$ defined by a polynomial f . Let $c = \text{ord}_a f$ and assume given a local hypersurface V of equation $x = 0$ with respect to some local or affine coordinates x, y, z . Expand f with respect to x into $f(x, y, z) = \sum_i a_i(y, z)x^i$. After a generic linear coordinate change (assuming to have an infinite ground field) we may assume that $a_c(0, 0) \neq 0$, and thus, locally at a and after multiplication of f by an invertible power series, that $a_c = 1$ (this reduction is not obligatory but simplifies the notation). Thus

$$f = x^c + \sum_{i < c} a_i(y, z) \cdot x^i \quad \text{modulo } x^{c+1}.$$

The coefficient ideal of f with respect to V at a is defined as

$$\text{coeff}_V(f) = (a_i^{\frac{c}{c-i}}, i < c) \subset \mathcal{O}_{V,a}.$$

The rational exponents can be avoided by taking instead $\frac{c!}{c-i}$, but as all subsequent constructions commute with taking powers of ideals, we prefer to allow quotients $\frac{c}{c-i}$ in order to keep the notation simple. The coefficient ideal lives in the local ring $\mathcal{O}_{V,a}$ and allows to perform the descent in dimension. It depends on the choice of V .

Consider now all regular hypersurfaces V at a for which the order of the associated coefficient ideal is maximal. It can be shown that either $f = x^c$ modulo x^{c+1} and hence $\text{coeff}_V(f) = 0$, or the maximal order is finite. The first case being simple, let us restrict to the second. Let e be this maximal order, $e = \text{ord}_a(\text{coeff}_V(f))$. It is clear that e does not depend on any choices (of course it depends on the characteristic of the ground field). It will form (preliminarily) the second component of our local resolution invariant $i_a(f)$ of f at a . Thus

$$i_a(f) = (c, e, \dots).$$

Observe here that $c \leq e$. Let us now investigate the behaviour of coefficient ideals and of the invariant under blowup. Taking simply the coefficient ideal of f does not commute with blowup at points where the order of f remains constant: The coefficient ideal of the strict transform of f with respect to the transform V' of V at a point a' above a is not the strict transform of the coefficient ideal of f with respect to V at a . But we have the formula

$$\text{coeff}_{V'}(f^{st}) = I(Y')^{-c} \cdot (\text{coeff}_V(f))^*$$

where $I(Y')$ denotes the principal ideal defining the exceptional component Y' in W' and $(\text{coeff}_V(f))^*$ denotes the total transform (= inverse image) of $\text{coeff}_V(f)$ under the blowup of V in Z . This is easily checked by working in local coordinates for which the blowup is monomial (cf. section 7). The formula only holds if $Z \subset V$ and at points a' above a where $c' = \text{ord}_{a'}(f')$ equals c . For an ideal J and an integer c with $\text{ord}_Z J \geq c$ we define the controlled transform $J^!$ of J with respect to c as $J^! = I(Y')^{-c} \cdot J^*$. Thus we can phrase the commutativity of coefficient ideals with blowups as follows.

Blowing up a center $Z \subset V$, the coefficient ideal of the strict transform of f with respect to V' at a' equals the controlled transform with respect to $c = \text{ord}_Z f$ of the coefficient ideal of f with respect to V , at points a' of V' above a where $c' = c$.

Let us now see how this affects our invariant (c, e) . First we assume, as always, that the center Z of the blowup is chosen so that $Z \subset \text{top}(f)$ and $Z \subset V$ locally at a , in particular $c = \text{ord}_a f = \text{ord}_Z f$. This implies that $c' \leq c$. We wish to show that $e' \leq e$ whenever $c' = c$.

There are three obstructions to this. First, the order of the controlled transform of an ideal may increase under blowup. This can be overcome by factoring $(\text{coeff}_V(f))^!$ further into $(\text{coeff}_V(f))^! = I(Y')^{e-c} \cdot (\text{coeff}_V(f))^\vee$, where $K^\vee = I(Y')^{-\text{ord}_Z K} \cdot K^*$ denotes the weak transform of an ideal K . We have already mentioned that the order does not increase when passing to the weak transform (provided that the order of K is constant along the center). Thus we should take instead of $e = \text{ord}_a(\text{coeff}_V(f))$ the order o of the ideal obtained from $\text{coeff}_V(f)$ after factoring a suitable (and prescribed) power of the exceptional component. We call o the *secondary order* of f . The resulting *divided* coefficient ideal is defined in a way so that it passes to its weak transform under blowup, while the undivided coefficient ideal passes to its controlled transform (always considered at equiconstant points of f in the exceptional divisor), cf. [EH, Ha 3] for more details. Doing so we get for the pair (c, o) the inequality

$$(c', o') \leq (c, o)$$

lexicographically, where c' denotes the order of $f' = f^{st}$ at a' and o' the order of the divided coefficient ideal of f' at a' with respect to V' .

The second obstruction is that the transform V' of V with respect to which the coefficient ideal at a' is taken may not maximize the order of $\text{coeff}_{V'}(f')$. Thus o' need not be intrinsic. It may be necessary to choose a new local hypersurface V' at a' maximizing the order of $\text{coeff}_{V'}(f')$. It can be shown (in any characteristic) that a suitable choice of V (not just maximizing the order of $\text{coeff}_V(f)$, but subject to further conditions) yields a maximizing V' , cf. section 9. This question is studied extensively in [Ha 4]. But such a V may not admit the required factorization of $\text{coeff}_V(f)$, and, more essentially, its transforms need not maximize the order of the coefficient ideals through a given sequence of blowups. It may only work for one blowup. Therefore, the substitution of V' by a maximizing \tilde{V}' cannot be avoided.

The third obstruction is related to this, and relies on what we have seen in examples 1 and 2. After a finite number of blowups the strict transform V' of V may no longer contain all equiconstant points of f . Again it is necessary to adjust V from time to time. The reason for this is the same as before. To have the strict transforms of V maximize the order of the coefficient ideal throughout the sequence of points along which the order of f remains constant we must choose a new local hypersurface occasionally.

We will show in the example below that this adjustment may destroy the required inequality $(c', o') \leq (c, o)$, i.e., $c' = c$ and $o' > o$ may indeed occur, where now o' denotes the maximal order at a' of the divided coefficient ideal of f' , maximized over all choices of regular hypersurfaces V' at a' . We have seen a glance of this already in example 2.

The definition of the coefficient ideal shows that to understand the phenomena it is sufficient to consider polynomials of the form $f(x, y, z) = x^c + h(y, z)$, i.e., no other powers of x appear in the expansion of f . Here h generates the coefficient ideal of f with respect to $V = \{x = 0\}$ and c is the order of f at $a = 0$, with $e = \text{ord}_a h \geq c$. We assume further that e is maximal and that h comes with a factorization $h(y, z) = m(y, z) \cdot g(y, z)$ where m is a prescribed monomial in the exceptional components having been produced by the earlier blowups. Thus g is just the divided coefficient ideal of f in V . Take now

$$o = \text{ord}_a g$$

as the second component of the invariant of f at 0.

Let now $\pi : W' \rightarrow W$ be the blowup of $W = \mathbb{A}^3$ with center $Z = \{0\}$ and exceptional divisor $Y' = \pi^{-1}(Z) \subset W'$. Let a' be a point of Y' where the order c' of the strict transform f' of f has remained constant, $c = c'$. The transformation rules for f and g are as follows. Both f and g pass to their strict transforms f' and g' , and the exceptional monomial m transforms accordingly so as to yield again a decomposition

$$f'(x, y, z) = x^c + m'(y, z) \cdot g'(y, z).$$

The monomial x^c survives and $\text{ord}_{a'}(m'(y, z) \cdot g'(y, z)) \geq c$ because $c' = c$. We have

$$m'(y, z) = m^*(y, z) \cdot I(Y')^{o-c}$$

with $m^* = \pi^{-1}(m)$ the total transform of m . But in W' we may have to apply a coordinate change at a' in order to maximize again the order of g' , yielding the same m' but a new g' . The next example shows that in this setting the pair (c, o) may increase under blowup with respect to the lexicographic order. The phenomenon was first observed by Moh [Mo 2]. He gave in [Mo 1] a bound on the maximal increase, see section 14 for more details.

Example 3. Similarly as in example 2, consider the polynomial $f = f^0 = x^2 + y^7 + t^2yz^4$ where $t \in \mathbb{K}$ is a non-zero constant, and let V be the hypersurface of W defined by $x = 0$. Take the sequence of blowups $W^5 \rightarrow W$ as in example 2. The point a^5 will now have coordinates $(0, t, 0)$. Hence the third blowup is the composition of the monomial point blowup in the z -chart followed by the translation $y \rightarrow y + t$. The resulting sequence of strict transforms f^i of f is

$$\begin{aligned} f^0 &= x^2 + 1 \cdot (y^7 + t^2yz^4), \\ f^1 &= x^2 + y^3 \cdot (y^2 + t^2z^4), \\ f^2 &= x^2 + y^3z^3 \cdot (y^2 + t^2z^2), \\ f^3 &= x^2 + z^6 \cdot (y^3 + t^3). \end{aligned}$$

From f^0 to f^4 the strict transforms V^i of V maximized the order of the coefficient ideal of f^i with respect to V^i . This is no longer the case for f^3 . If t^3 is a square s^2 in the ground field we may apply the local coordinate change $x \rightarrow x + sz^3$ at a^3 to f^3 and get

$$\tilde{f}^3 = x^2 + z^2 \cdot y^3$$

with secondary order $\tilde{o}^3 = 3$. Hence the hypersurface $V^3 = \{x = 0\}$ at a^3 with $o^3 = 0$ did not maximize the order of the divided coefficient ideal of f^3 . But $\tilde{V}^3 = \{x + sz^3 = 0\}$ does maximize this order, and we get

$$(c^3, \tilde{o}^3) = (2, 3) >_{lex} (c^4, o^4) = (2, 2).$$

Actually, to be precise, the order o should always come with an index indicating the respective hypersurface.

We conclude that our invariant has increased when passing from a^2 to a^3 . Observe that the coordinates of a^3 are related to the coefficients of f , and that a^3 is the only point where the increase of (c, o) can happen. Moreover, changing the coefficients of f , the increase disappears. We will describe this fact with precision in later sections.

4. Estimating the decrease of the invariant in earlier blowups

We have seen before that the increase of the resolution invariant requires a special configuration of the exceptional divisor at the point in question. These exceptional components must have been produced by specific earlier blowups. It would be natural to expect that these preliminary blowups cause a drop in the invariant which is larger than the subsequent increase, so that in total a decrease would result. This works for surfaces – producing a new proof of surface resolution – but gets stuck in dimension 3 and higher. In the example of the last section, the order of g decreases from 5 to 2, then remains constant in the second blowup, and finally increases from 2 to 3. So in total, over all three blowups, the secondary order has dropped. For surfaces, this is a general fact, which will be proven in section 6.

In the next section, we will give an example of a hypersurface in four-space where after the whole sequence of blowups the secondary order has not decreased.

Let us call *kangaroo point* a point a' in a sequence of blowups where the jump of the resolution invariant may have occurred in the last local blowup $(W', a') \rightarrow (W, a)$. Necessarily, there pass less exceptional components through a' than through a , cf. Theorem 1 in section 5. The predecessor point a of a' is called the *antelope point* of a' . Let $(W, a) \rightarrow (W^\circ, a^\circ)$ be the shortest sequence of local blowups producing all the exceptional components in W which pass through a . We call a° the *oasis point* of a , for obvious reasons, cf. figure 3. Oasis points are the starting point of a sequence of blowups which gives rise to the described complications when passing from the antelope point a to the kangaroo point a' . It is then appropriate to compare the resolution invariant at a° with the one at a' .

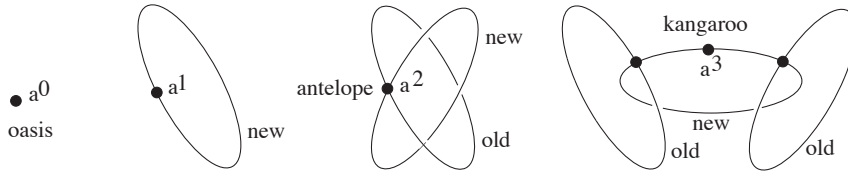


Figure 3. Oasis, antelope and kangaroo points.

For surfaces in three-space (and restricting to point blowups), the sequence of blowups between oasis a° and antelope point a can be characterized in suitable coordinates in W° as one monomial point blowup in the y -chart followed by an arbitrary number of monomial point blowups in the z -chart. Restricting to a hypersurface V° having permanent contact between a° and a we get a sequence of monomial point blowups in the plane with one change of charts after the first blowup. It is an amusing exercise to show that any polynomial of order o in two variables transforms under such a sequence into a polynomial of order $o' \leq o/2$ (taking always the strict transform of the polynomial).

If $o > 2$, this drop will make up with the increase by at most 1 in the blowup from a to a' . If $o = 2$, the polynomial has become regular at a . This gives a rough outline how to treat the case of surfaces in positive characteristic, using orders of ideals as invariants.

We next give an example where the order of the divided coefficient ideal of a polynomial f increases along a sequence of point blowups between oasis and kangaroo point. In the example, occasionally curves could be taken as permissible centers. However, transversality problems with still older exceptional components may prohibit to choose one-dimensional centers, thus forcing point blowups.

Example 4. Let W be a 4-dimensional regular ambient scheme, and let $a = 0$ be a point of W . Choose local coordinates x, y, z, w in W at a , and let V be the hypersurface in W defined by $x = 0$. We consider the sequence of local point blowups

$$(W^3, a^3) \rightarrow (W^2, a^2) \rightarrow (W^1, a^1) \rightarrow (W, a) \rightarrow (W^{-1}, a^{-1}) \rightarrow (W^{-2}, a^{-2})$$

given as follows: The first two blowups are monomial and of auxiliary nature, a^{-1} is the origin of the z -chart, a is the origin of the y -chart. We will be interested in the situation at a , the two prior blowups ensure to have already two exceptional components passing through a . The next two blowups are again monomial, a^1 is the origin of the y -chart, a^2 is the origin of the z -chart. The next blowup involves a translation: a^3 is the point with coordinates $(0, 1, 0)$ of the z -chart. Thus a^3 is a kangaroo point with antelope a^2 , and a is the oasis point of a^3 .

Take the hypersurface $f^{-2} = x^5 + 1 \cdot (yw^5 + y^2z^8)$ in $W^{-2} = \mathbb{A}^4$ at $a^{-2} = 0$, with coefficient ideal $h(y, z, w) = 1 \cdot (yw^5 + y^2z^8)$ in $V^{-2} : x = 0$. We get the following sequence of transforms and coordinate substitutions under the above sequence of point blowups.

$$\begin{aligned}
f^{-2} &= x^5 + 1 \cdot (yw^5 + y^2z^8), \\
f^{-1} &= x^5 + y \cdot (w^5 + y^4z^8), & x, y, z, w &\rightarrow xz, yz, z, wz, \\
f &= f^0 = x^5 + yz \cdot (w^5 + y^4z^7), & x, y, z, w &\rightarrow xy, y, zy, wy, \\
f^1 &= x^5 + yz^2 \cdot (w^5 + y^4z^6), & x, y, z, w &\rightarrow xy, y, zy, wy, \\
f^2 &= x^5 + y^3z^2 \cdot (w^5 + y^5z^6), & x, y, z, w &\rightarrow xz, yz, z, wz, \\
f^3 &= x^5 + z^5(y+1)^3 \cdot (w^5 + z^6(y+1)^5), & x, y, z, w &\rightarrow xz, yz + z, z, wz.
\end{aligned}$$

The order of f and of its transforms has remained constant at the successive points of the blowups, and V has permanent contact along the sequence. The order of the divided coefficient ideal of f^3 in these coordinates with respect to $V^3 : x = 0$ is 5. This order is not maximal, as is seen by applying the coordinate change $x \rightarrow x + zw$. After this substitution, the order has become 6 and is then maximal. The divided coefficient ideal of f at the oasis point a was $g = w^5 + y^4z^7$ of order 5. Thus the order of the divided coefficient ideal has increased between the oasis point and the kangaroo point (and not just between the antelope point and the kangaroo point). This seems to make also induction relying “on the long run” of the invariants obsolete.

II. RESULTS

5. The main result

Let W be a regular ambient scheme (excellent of finite type over an algebraically closed field), and let a be a point of W , $\dim_a W = n$. As all considerations are local, we may as well assume that W is n -dimensional affine space \mathbb{A}^n . Let D be a given normal crossings divisor in W at a (corresponding to the exceptional divisor of earlier blowups) and let K be an ideal in W . For simplicity of exposition and notation, we shall always restrict to *principal* ideals $K = (f)$. Let V be a hypersurface in W which has weak maximal contact with K at a relative to D . By this we mean that V maximizes the order of the coefficient ideal $J = \text{coeff}_V(K)$ of K in V at a , that V is transversal to D and that J allows a factorization $J = M \cdot I$ with $M = I_V(D \cap V)$ a principal monomial ideal and I an ideal in V at a . It can be shown that hypersurfaces of weak maximal contact always exist (at least as formal subschemes of W at a), see [Ha 4]. Although we shall use some of the constructions of [EH], this paper is not a prerequisite for understanding the results of the present paper. The references mainly serve to embed the used objects into a larger context so as to justify their consideration.

For Z a regular subscheme of W let $W' \rightarrow W$ denote the blowup of W in Z with exceptional component Y' . We assume that Z is contained in $\text{top}(K)$ and V and transversal to D and V . This is the case in the actual resolution process, see the section *Transversality* of [EH]. Note that $W' \rightarrow W$ induces by restriction a morphism $V^{st} \rightarrow V$ which coincides with the blowup of V with center Z . Let $a' \in Y'$ be a given point above a point a in Z . As all arguments and computations are local at a and a' , we will work with ideals in the local rings of W and W' at a and a' (or their completions).

Let $K' = K^\vee = K^* \cdot I(Y')^{-\text{ord}_Z K}$ denote the weak transform of K in W' . As K is assumed principal, K^\vee coincides with the strict transform K^{st} of K . From $Z \subset \text{top}(K)$ follows that the order c' of K' at a' is less or equal the order c of K at a . Since in case where

the inequality is strict induction on the order applies, we shall assume throughout that a' is an equiconstant point for K , say

$$c = \text{ord}_a K = \text{ord}_{a'} K' = c'.$$

Define $D' = D^* + (o - c) \cdot Y'$ with D^* the total transform of D and $o = \text{ord}_a I$ the order of the second factor I of $J = M \cdot I$ at a . As Z is transversal to D and $e = \text{ord}_a J \geq c$, D' is an effective normal crossings divisor in W' (all multiplicities of D' are non-negative). The strict transform $V^\diamond = V^{st}$ of V contains a' because V is assumed to have weak maximal contact with K (cf. [EH]). Moreover, V^\diamond is transversal to D' . A computation shows that the coefficient ideal $J^\diamond = \text{coeff}_{V^\diamond}(K')$ of K' with respect to V^\diamond admits a factorization $J^\diamond = M^\diamond \cdot I^\diamond$ with $M^\diamond = I_{V^\diamond}(D' \cap V^\diamond)$ and I^\diamond an ideal of V^\diamond at a' . It can be shown that I^\diamond is the weak transform of I in V^\diamond , cf. the section *Commutativity* of [EH]. Set $o^\diamond = \text{ord}_{a'} I^\diamond$.

In characteristic 0, the hypersurface V can be chosen so that V^\diamond has again weak maximal contact with K' at a' , i.e., o^\diamond is maximal over all choices of local hypersurfaces at a' . This is not the case in arbitrary characteristic, as we saw in the earlier examples. There may exist a regular hypersurface V' in W' at a' transversal to D' so that $J' = \text{coeff}_{V'}(K')$ factors into $J' = M' \cdot I'$ with $M' = I_{V'}(D' \cap V')$ and the order $o' = \text{ord}_{a'} I'$ of I' at a' exceeds o^\diamond . Any such hypersurface is the image of V^\diamond under a local automorphism of W' at a' . In the computations of the respective coefficient ideals, this will correspond to a coordinate change as already occurred in the examples when eliminating certain monomials from the expansion of f and f' . We set $e' = \text{ord}_{a'} J'$.

The resolution invariant of K and D at a point a of W is a vector $i_a(K)$ of numbers. Its first two components, which shall only interest us here, are the respective orders c and o as defined above. Our purpose is to observe the behaviour of (c, o) under blowup, i.e., to compare (c, o) with (c', o') . As we may (and will) assume that $c = c'$, we are left to compare $o = \text{ord}_a I$ with $o' = \text{ord}_{a'} I'$. For this, the general procedure will be to compute first $o^\diamond = \text{ord}_{a'} I^\diamond$ and then apply coordinate changes to maximize the order $o' = \text{ord}_{a'} I'$ of the resulting ideal I' .

The examples of section 3 have shown that $o' > o$ may occur. We shall classify in the sequel all cases where such an increase can happen.

For this, the multiplicities of the exceptional monomial factor $M = I_V(D \cap V)$ of the coefficient ideal J have to be taken into account. In reality, D is the exceptional divisor produced by earlier blowups, and $D \cap V$ is the exceptional divisor of the restriction of these blowups to hypersurfaces of weak maximal contact. We therefore call the components of $D \cap V$ the exceptional components of V at a . We may choose local coordinates (x, y_m, \dots, y_1) of W at a (with $m = n - 1$) so that V is given by $x = 0$ and M is generated by a monomial in the coordinates y_m, \dots, y_1 . Let $q \in \mathbb{N}^m$ be the vector of exponents of this monomial, i.e., the vector of exceptional multiplicities in V at a .

Let $V^\diamond = V^{st}$. Fix now a' in V^\diamond above a with $c' = c$, and consider the exceptional monomial $M^\diamond = M^* \cdot I_{V^\diamond}(Y' \cap V^\diamond)^{o-c}$ of $J^\diamond = \text{coeff}_{V^\diamond}(K')$ at a' , i.e., $J^\diamond = M^\diamond \cdot I^\diamond$. Here, M^* denotes the total transform of M under the induced blowup $V^\diamond \rightarrow V$. Observe that $M^\diamond = I_{V^\diamond}(D' \cap V^\diamond)$, by definition of D' .

We wish to describe the exponents q^\diamond of M^\diamond . Their detailed description using local coordinates is given in section 7. Exceptional components of the blowup $V^\diamond \rightarrow V$ at a' which are the strict transforms of components in V through a will have the same exponent as their image below. So for these, we will have $q_i^\diamond = q_i$. The new exceptional component at a' is $Y' \cap V^\diamond$. It has exponent $o - c$. The remaining exceptional components of D' will not pass through a' , so their exponent in M^\diamond is 0. Combining these observations, we can decompose q into $q = r + \ell$ where r and ℓ are obtained from $q \in \mathbb{N}^m$ by setting certain components q_i of

q equal to zero and leaving the others unchanged. The non-zero components of r correspond to components at a which *disappear* at a' , whereas the non-zero components of ℓ correspond to those which *persist* at a' . We call $r \in \mathbb{N}^m$ the *red* exceptional multiplicities (or exponents) of K in V , and $\ell \in \mathbb{N}^m$ the *yellow* exceptional multiplicities of K in V . The value of the red exponents will be of special interest for the phenomena to be studied. Of course, the decomposition of q depends on the choice of the point a' .

For an integral vector $r \in \mathbb{N}^m$, let $\phi_c(r)$ denote the number of components of r which are not divisible by c ,

$$\phi_c(r) = \#\{i, r_i \not\equiv 0 \pmod{c}\}.$$

For $r \in \mathbb{N}^m$ and $c \in \mathbb{N}$ define $\bar{r}^c = (\bar{r}_1^c, \dots, \bar{r}_m^c)$ as the vector of the remainders $0 \leq \bar{r}_i^c < c$ of the components of r modulo c . We set $|\bar{r}^c| = \bar{r}_1^c + \dots + \bar{r}_m^c$.

We say that an equiconstant point $a' \in W'$ above a of the ideal K in W is *tame* with respect to a given regular hypersurface V of W at a , if either the order c of K at a is not divisible by the characteristic of the ground field, or $e = \text{ord}_a J = \text{ord}_a(\text{coeff}_V(K))$ is not divisible by c (in both cases all points a' above a are tame), or e is a multiple of c and the residues \bar{r}_i^c modulo c of the red exceptional multiplicities r_i of K in V satisfy the arithmetic inequality

$$|\bar{r}^c| = \bar{r}_1^c + \dots + \bar{r}_m^c > (\phi_c(r) - 1) \cdot c.$$

A similar inequality appears in the work of Abhyankar on good points [Ab 2], but is used there with a completely different perspective. It is easy to see that the inequality is equivalent to

$$\lceil \frac{\bar{r}_1^c}{c} \rceil + \dots + \lceil \frac{\bar{r}_m^c}{c} \rceil = \lceil \frac{\bar{r}_1^c + \dots + \bar{r}_m^c}{c} \rceil,$$

where $\lceil u \rceil$ denotes the smallest integer $\geq u$. If none of the three conditions hold, a' is called *wild* above a . Hence, a' is wild above a with respect to K and V if and only if

- The characteristic p of the ground field divides $c = \text{ord}_a K$.
- The order $e = \text{ord}_a J$ of the coefficient ideal J of K in V at a is a multiple of c .
- The red exceptional multiplicities r_i of K satisfy

$$\bar{r}_1^c + \dots + \bar{r}_m^c \leq (\phi_c(r) - 1) \cdot c.$$

Note that if only one exceptional component is lost when passing from a to a' (or a lies in no exceptional component), then all r_i but one are 0 and $\phi_c(r)$ equals 1 or 0, so that $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ is satisfied. Therefore, at a wild point a' , at least two exceptional components have disappeared.

The main result of this paper is as follows.

Theorem 1. *Given are a principal ideal K in W at a with coefficient ideal $J = M \cdot I$ in a hypersurface V of weak maximal contact with K at a . Set $c = \text{ord}_a K$ and $o = \text{ord}_a I$. Let $(W', a') \rightarrow (W, a)$ be a local blowup with center $Z \subset \text{top}(K) \cap \text{top}(I)$ and exceptional component Y' . Set $c' = \text{ord}_{a'} K'$ with $K' = K^\vee$ the weak transform of K . Assume that $c' = c$. Let V' be a local hypersurface in W' at a' with weak maximal contact with K' , and let J' be the coefficient ideal of K' with respect to V' . Assume given the decomposition $J' = M' \cdot I'$ with $M' = I_{V'}(D' \cap V')$ and D' the normal crossings divisor $D' = D^* + (o - c) \cdot Y'$. Set $o' = \text{ord}_{a'} I'$.*

(a) *If a' is tame above a , then*

$$o' \leq o.$$

(b) If a' is tame above a , the strict transform V^{st} of V need not have weak maximal contact with K' .

(c) If a' is wild above a , then

$$o' > o,$$

may occur.

(d) If a' is wild above a and $o' > o$, the weighted tangent cone of (a generator of) K is uniquely determined, up to coordinate choices and multiplication with p -th powers, by o and the red exceptional multiplicities r_i of K in V .

In assertion (d), we understand by the weighted tangent cone of K the ideal in W generated by the initial weighted homogeneous forms of elements of K with respect to the weight $(w, 1, \dots, 1)$ where $w = e/c$ and $e = \text{ord}_a(\text{coeff}_V(K))$, cf. [AHV].

To illustrate assertion (d), we indicate the form of the tangent cone for surfaces in $W = \mathbb{A}^3$ defined by a polynomial of the special form

$$f = x^c + y_1^{r_1} y_2^{r_2} \cdot g(y_1, y_2).$$

Here, $a = 0$ is the origin, V is defined by $x = 0$, and the coefficient ideal of f in V is generated by $y_1^{r_1} y_2^{r_2} \cdot g(y_1, y_2)$ with exceptional monomial $y_1^{r_1} y_2^{r_2}$. The center Z is the origin of \mathbb{A}^3 and $(W', a') \rightarrow (W, a)$ is the local point blowup with a' a point of Y' outside the strict transforms of the two exceptional components $\{y_1 = 0\}$ and $\{y_2 = 0\}$ at a . Hence both exceptional multiplicities r_1 and r_2 of f in V are red (so, in the notation from above, $q = r$ and $\ell = 0$). We have $c = \text{ord}_a f \leq e = r_1 + r_2 + \text{ord}_a g$ and $o = \text{ord}_a g$. Assume that a' is wild above a . We may assume that a' lies in the y_2 -chart of W' and has coordinates $(0, t, 0)$ there, for some $t \neq 0$ in the ground field. For simplicity, we suppose that $c = p$ equals the characteristic. The arithmetic conditions for a' to be wild are

$$w = \frac{e}{c} \in \mathbb{N} \quad \text{and} \quad \bar{r}_1^c + \bar{r}_2^c \leq c.$$

Assume now that the order of g has increased at a' , say $o' > o$. Then g must have the following form

$$g(y_1, y_2) = \sum_{i=0}^{o'} \binom{o+r_1}{i+r_1} y_1^i (y_2 - t y_1)^{o-i}.$$

Similar but much more complicated formulas could be given for the tangent cone of arbitrary surfaces in \mathbb{A}^3 , cf. the computations of the next section.

If a' is wild above a it is possible to bound the increase of o' with respect to o , namely

$$o' \leq o + \mu$$

with $\mu = \sum_i \bar{r}_i^c - \max\{\bar{r}_j^c, j = 1, \dots, m\}$. A sharper bound is given by Moh [Mo 1]. Namely, if $c = p^k$ with p the characteristic and if the elements of the coefficient ideal J of K with respect to V are not pure p^i -th powers h^{p^i} for any $1 \leq i \leq k$, then

$$o' \leq o + p^{k-1}.$$

We are going to prove this inequality in section 14. The theorem is proven in sections 11 and 12. The form of the uniquely determined weighted tangent cone of (b) is described for three variables in sections 16 and 17 on hybrid polynomials. The non-persistence of weak maximal contact under tame blowups is discussed in section 13.

6. The theorem in the case of surfaces

The description of the observed phenomena in later sections and the proof of the main theorem will be somewhat technical. Therefore, to illustrate the underlying ideas, we first treat in detail a specific example. As before, we shall restrict to surfaces of the form

$$f(x, y, z) = x^c + y^r z^s \cdot g(y, z),$$

with $c = \text{ord}_a f$, $o = \text{ord}_a g$, $e = r + s + o = \text{ord}_a(y^r z^s \cdot g)$ and $a = 0$. We assume that c equals the characteristic p of the ground field, that e is divisible by c and that no monomial of $y^r z^s \cdot g(y, z)$ can be eliminated from f by a coordinate change in x , i.e., no monomial is a c -th power. Both exponents r and s will assumed to be red. In this section, we shall only consider the case where g is homogeneous of degree o . We set

$$P(y, z) = y^r z^s \cdot g(y, z) = \sum a_{ij} y^i z^j,$$

$$P^+(y, z) = P(y + z, z) = \sum b_{mn} y^m z^n,$$

where the sums range over i, j with $i + j = r + s + o$, $i \geq r$, $j \geq s$, respectively $m + n = r + s + o$, $m \geq 0$ and $n \geq s$. Of course, the indices j and n are determined by i and m and could be omitted. The *support* $\text{supp}(P)$ of a polynomial P is the set of exponents of the monomials of P with non-zero coefficients.

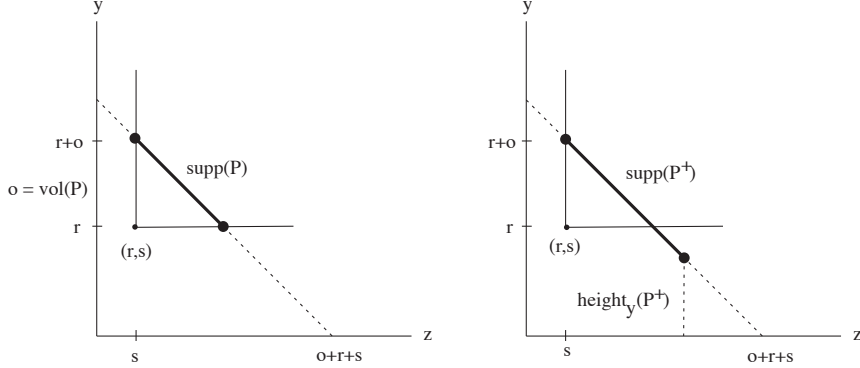


Figure 4. Height and volume of a polynomial.

Let us call *volume* $\text{vol}(P)$ of P the integer volume of the convex hull of the support of P in \mathbb{R}^2 , say the euclidean volume of this convex hull in the one-dimensional affine sublattice $\{(i, j) \in \mathbb{N}^2, i + j = r + s + o\}$ of \mathbb{N}^2 , setting the length of a generator of this lattice equal to 1. Thus

$$\text{vol}(P) = \max \{i, a_{ij} \neq 0\} - \min \{i, a_{ij} \neq 0\} \leq o.$$

We may assume that $y^r z^s$ is the maximal monomial which can be factored from P (using that P is homogenous). In this case we have $a_{r, s+o} \neq 0$ and $a_{r+o, s} \neq 0$, hence $\text{vol}(P) = o$. The *height* $\text{height}_y(P^+)$ of P^+ is the order of P^+ with respect to y , say

$$\text{height}_y(P^+) = \min \{m, b_{mn} \neq 0\}.$$

These two numbers can be easily read off from the Newton polygon of P and P^+ , see figure 4. In higher dimensions, the height will still be the order of P^+ with respect to some variables, but the volume of P will be defined differently as the order of g .

For later applications when observing the increase of the resolution invariant, we wish to bound $\text{height}_y(P^+)$ in terms of $\text{vol}(P)$. Let A denote the transformation matrix between the coefficients of P and P^+ ,

$$b_{mn} = \sum_{ij} A_{ij,mn} \cdot a_{ij}.$$

We will be particularly interested in the $(o+1)$ -square submatrix $A^\square = A^\square(o, r, s)$ of A of columns indexed by m, n with $m \leq o$, relating the coefficients of P with the coefficients of monomials of P^+ of y -degree $\leq o$. Computation gives by binomial expansion

$$A^\square = \begin{pmatrix} \binom{r}{0} & \cdots & \binom{r+o}{0} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \binom{r}{o} & \cdots & \binom{r+o}{o} \end{pmatrix}.$$

If P^+ would be obtained from P by the change $y \rightarrow y + tz$ the entries of this matrix would have to be multiplied with powers of t . Subtracting the m -th column from the $(m+1)$ -st column for every $m \geq 1$, and using the binomial identity $\binom{k}{\ell} + \binom{k+1}{\ell} = \binom{k+1}{\ell+1}$, we get a matrix with first row $(1, 0, \dots, 0)$ and whose submatrix obtained by deleting the first row and column is the matrix $A^\square(o-1, r, s)$ of size $o-1$

$$A^\square(o-1, r, s) = \begin{pmatrix} \binom{r}{0} & \cdots & \binom{r+o-1}{0} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \binom{r}{o-1} & \cdots & \binom{r+o-1}{o-1} \end{pmatrix}.$$

Induction on the size of the matrices shows that A^\square has determinant 1. This implies that the correspondence between the coefficients a_{ij} with $r \leq i \leq r+o$ and the coefficients b_{mn} with $0 \leq m \leq o$ is a linear bijection. In particular, if $\text{height}_y(P^+) > o$ and hence $b_{mn} = 0$ for all $0 \leq m \leq o$, then $P = 0$. Conversely, $P \neq 0$ implies that

$$\text{height}_y(P^+) \leq \text{vol}(P).$$

The equality $\text{height}_y(P^+) = \text{vol}(P) = o$ can only occur if $b_{mn} = 0$ for all $0 \leq m \leq o-1$, $b := b_{o, r+s} \neq 0$ and the coefficients a_{ij} of P are the b -th multiple of the last column of $(A^\square)^{-1}$. They are thus unique.

The inequality $\text{height}_y(P^+) \leq \text{vol}(P)$ is related to the Bernstein-Koushnirenko theorem [Be] on the comparison between the number of isolated zeroes of a system of n polynomial equations in $(\mathbb{C}^*)^n$ and the mixed volume of the convex polytopes given as the convex hulls of the supports of the polynomials (see section 15).

Let us now consider the surface

$$f = x^c + y^r z^s \cdot g(y, z) = x^c + P(y, z)$$

with g homogeneous of degree o . We set

$$f^+ = x^c + (y+z)^r z^s \cdot g(y+z, z) = x^c + P^+(y, z).$$

For general positions of the wild point a' in W' , we would have to consider coordinate changes of form $y \rightarrow y + tz$ with t in the ground field. The first two components of the resolution invariant of f at $a = 0$ are (c, o) . As we have seen in examples 3 and 4 of section 3, o may increase to $o+1$ under a point blowup when taking in W' the strict transform of f at a point

outside the two intersection points of the three exceptional components and the hyperplane $x = 0$. Our purpose here is to describe the circumstances where such an increase can happen.

It turns out that the order o' of the strict transform g' of g is bounded from above by the height $\text{height}_y P^+$ of P^+ . This is proved by a computation in local coordinates as given by the lemma of section 7. On the other hand, o bounds from above $\text{vol}(P)$. Thus

$$o' \leq \text{height}_y(P^+) \quad \text{and} \quad \text{vol}(P) \leq o.$$

Both inequalities are sharp, i.e., equality may hold. To compare o with o' , it is therefore plausible to investigate more closely the inequality $\text{height}_y(P^+) \leq \text{vol}(P)$ in the context of polynomials f . Observe here that the presence of x^c in f with c the characteristic of the ground field allows to eliminate monomials from P and P^+ which are c -th powers. Thus P and P^+ are only given modulo c -th powers. As $\text{vol}(P)$ and $\text{height}_y(P^+)$ are given by support conditions this will effect the validity of the inequality $\text{height}_y(P^+) \leq \text{vol}(P)$.

We shall assume throughout that the order o of g is maximal over all coordinate choices, i.e., that P is not a c -th power. However, some of its monomials could be c -th powers. Let us therefore define $\text{vol}^p(f)$ as the minimal volume $\text{vol}(P)$ over all polynomials $P(y, z)$ occurring after coordinate changes $x \rightarrow x + a(y, z)$ in f , and similarly $\text{height}_y^p(f^+)$ as the maximal height $\text{height}_y(P^+)$ over all polynomials obtained from $P(y + z, z)$ after elimination of c -th powers. We then still have

$$o' \leq \text{height}_y^p(f^+) \quad \text{and} \quad \text{vol}^p(f) \leq o,$$

and thus wish to compare $\text{height}_y^p(f^+)$ with $\text{vol}^p(f)$. Moh has shown in [Mo 1] that for $c = p$ one always has

$$\text{height}_y^p(f^+) \leq \text{vol}^p(f) + 1,$$

so that, by the above, $o' \leq o + 1$. To have equality $\text{height}_y^p(f^+) = \text{vol}^p(f) + 1$, some monomials of $P(y + z, z) = P^+(y, z)$ must be c -th powers, because of $\text{height}_y(P^+) \leq \text{vol}(P)$. A first necessary condition for $o' = o + 1$ is therefore that the degree $r + s + o$ of P and P^+ is divisible by c . Else no monomial of P or P^+ would be a c -th power. We shall prove that $o' = o + 1$ can only occur if y^r and z^s are red exceptional components and in addition

$$\bar{r}^c + \bar{s}^c \leq c,$$

where \bar{r}^c and \bar{s}^c denote the residues of r and s modulo c . Note that $\varphi_c(\bar{r}^c, \bar{s}^c) - 1 = 1$ if both \bar{r}^c and \bar{s}^c are positive. By prior auxiliary curve blowups with centers (x, y) or (x, z) one can always achieve that $r < c$ and $s < c$, in which case the inequality $\bar{r}^c + \bar{s}^c \leq c$ simply reads $r + s \leq c$. This reduction step is not a prerequisite.

Let us interpret the inequality $\bar{r}^c + \bar{s}^c \leq c$ geometrically. It is equivalent to

$$\left\lceil \frac{r}{c} \right\rceil + \left\lceil \frac{s}{c} \right\rceil > \left\lceil \frac{r+s}{c} \right\rceil.$$

Note that $\left\lceil \frac{r}{c} \right\rceil + \left\lceil \frac{s}{c} \right\rceil \geq \left\lceil \frac{r+s}{c} \right\rceil$ always holds. To see the equivalence, we may assume that both r and s are $< c$ and non-zero, so that the left hand side of $\left\lceil \frac{r}{c} \right\rceil + \left\lceil \frac{s}{c} \right\rceil > \left\lceil \frac{r+s}{c} \right\rceil$ equals 2. The right hand side is < 2 if and only if $r + s \leq c$.

The inequality $\left\lceil \frac{r}{c} \right\rceil + \left\lceil \frac{s}{c} \right\rceil \leq \left\lceil \frac{r+s}{c} \right\rceil$ signifies that the straight segment S in \mathbb{N}^2 connecting the points $(e - s, s)$ and $(r, e - r)$ contains as many c -multiples, i.e., points in $c \cdot \mathbb{N}^2$, as the segment U connecting $(o, e - o)$ with $(0, e)$ (see figure 5).

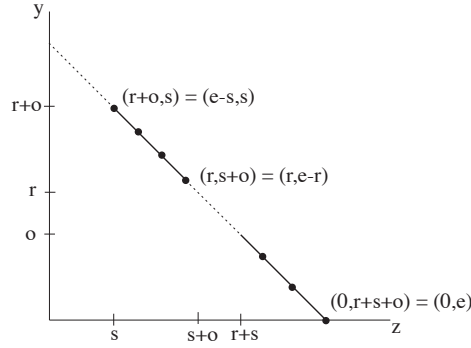


Figure 5. c -multiples on segments.

Note here that the first segment is just the maximal possible support of P , whereas the second corresponds to monomials whose coefficients in the expansion of P^+ must be zero in order to have $\text{height}_y P^+ > o$. And c -multiples in these segments correspond to monomials which can be eliminated from P or P^+ by coordinate changes in x applied to f and f^+ . It is clear that $S \setminus c \cdot \mathbb{N}^2$ has at most one element more than $U \setminus c \cdot \mathbb{N}^2$. Heuristically speaking, applying coordinate changes $x \rightarrow x + a(y, z)$ and $y \rightarrow y + z$ to f may produce at most o zero coefficients in P^+ , thus giving

$$\text{height}_y(f^+) \leq \text{vol}(f) + 1.$$

In addition to the preceding condition on the exceptional exponents, the occurrence of the equality $\text{height}_y^p(f^+) = \text{vol}^p(f) + 1$ implies for each value of o , r and s , that P is a uniquely determined polynomial (up to a rescaling of the coordinates). The uniqueness of P will be proven in Proposition 1 of section 11. We have assumed here that P is homogeneous, for arbitrary polynomials P only the tangent cone of P would be prescribed. The shape of P can be explicitly be determined. It is given as follows (cf. section 16). Consider the polynomial \mathbb{H}_r^o given as

$$\mathbb{H}_r^o(y, w) = \sum_{i=0}^o \binom{o+r}{i+r} y^i w^{o-i}.$$

We then have

$$P(y, z) = y^r z^s \cdot \mathbb{H}_r^o(y, z - y).$$

To prove the equality of the two polynomials it is sufficient, by the uniqueness of P , to show that $y^r z^s \cdot \mathbb{H}_r^o(y, z - y)$ has the same properties as P with respect to the substitution $y \rightarrow y + z$. This is easy to check, simply replace y by $y + z$ in $y^r z^s \cdot \mathbb{H}_r^o(y, z - y)$ and get by computation

$$(y + z)^r z^s \cdot \mathbb{H}_r^o(y + z, -y) = z^{o+r+s} - (-y)^{o+1} \cdot \mathbb{H}_{o+1}^{r-1}(-y, y + z).$$

Working modulo c -th powers, we may delete z^{o+r+s} from the sum on the right hand side so that the height with respect to y of this polynomial is $\geq o + 1$ (actually, it is equal to $o + 1$ since $\mathbb{H}_{o+1}^{r-1}(-y, y + z)$ has y -order 0). Observe that

$$y^r \cdot \mathbb{H}_r^o(y, z - y) = \sum_{i=0}^o \binom{o+r}{i+r} y^{i+r} (z - y)^{o-i} = \sum_{i=r}^{o+r} \binom{o+r}{i} y^i (z - y)^{o+r-i}$$

equals the terms of y -degree $\geq r$ of the binomial expansion of $z^{o+r} = (y + (z - y))^{o+r}$.

Let us return to $f = x^c + P(y, z)$. Assume that $\text{height}_y^p(f^+) = \text{vol}^p(f) + 1$ and that $f = x^c + P(y, z)$ with homogeneous polynomial $P(y, z) = y^r z^s \cdot \mathbb{H}_r^o(y, z - y)$. We see from the above that if $o + 1 \geq c$ then f has constant order c along the curve $(x - z^{(r+s+o)/c} + \dots, y - z)$, where the dots denote further monomials which eliminate c -th powers from P . Hence this curve is a permissible center of blowup for f (this is only

the case if P is homogeneous). Changing coordinates accordingly in x and y , f becomes $\tilde{f} = x^c - (-y)^{o+1} \cdot \mathbb{H}_{o+1}^{r-1}(-y, y+z)$. The permissible curve is defined now by (x, y) . Blowing it up decreases the exponent $o+1$ of y by c and leaves the rest of \tilde{f} unchanged. This can be repeated until the exponent of y is $< c$. As $\mathbb{H}_{o+1}^{r-1}(y, z-y)$ has y -order 0, a subsequent point blowup will make the order of f drop below c .

We may therefore assume $o+1 < c$ from the beginning. In this case, applying a point blowup to $f = x^c + y^r z^s \cdot \mathbb{H}_r^o(y, z-y)$ yields a strict transform $f' = x^c + z^{o+r+s} \cdot g'(y, z)$ with g' of order $\leq o+1 < c$. As $r+s+o$ is a power of c , we can now blow up the curve (x, z) several times to make the factor z^{r+s+o} equal to 1. This yields a strict transform $f'' = x^c + g''$ with $g'' = g'$ of order $< c$. Hence the order of f is $< c$, showing that it has dropped. Then induction on the order of f applies to resolve f .

This argument does not work if the polynomial P is not homogeneous, because then the curve $(x - z^{(r+s+o)/c} + \dots, y - z)$ need no longer be permissible for f .

III. TECHNIQUES

7. Description of blowups in local coordinates

Let W be a regular scheme of dimension n , and let Z be a closed regular subscheme of dimension d . Let $\pi : W' \rightarrow W$ be the induced blowup with center Z and exceptional component Y' , and let $(W', a') \rightarrow (W, a)$ denote the corresponding local blowup for some pair of points $a \in Z$ and $a' \in Y'$ above a . We shall assume that the ground field is algebraically closed. As the order of ideals is an upper semicontinuous function of the point in question, we may and will restrict to closed points a and a' . Let V be a local regular hypersurface of W at a containing Z locally.

Assume given an ideal K in W at a , with coefficient ideal $J = \text{coeff}_V(K)$ in V . Let D be a normal crossings divisor in W at a transversal to V , set $M = I_V(D \cap V)$ and assume that J factorizes into $J = M \cdot I$. Let $c = \text{ord}_a K$ and $c' = \text{ord}_{a'} K'$ with $K' = K^\vee$ the weak transform of K in W' . Assume that V has weak maximal contact with K (i.e., maximizes the order of J), and that Z is transversal to D . Moreover, we shall assume that a' is an equiconstant point for K , i.e., the order $c' = c$ has remained constant. In this situation, local coordinates can be chosen in W at a which make the description of the blowup and of the transforms of ideals particularly explicit.

Lemma. *There exist local coordinates $x = (x_n, \dots, x_1)$ of W at a , i.e., a regular system of parameters of $\mathcal{O}_{W,a}$, such that*

- (1) a has components $a = (0, \dots, 0)$ with respect to x .
- (2) V is defined in W by $x_n = 0$.
- (3) Z is defined in W by $x_n = \dots = x_{d+1} = 0$.
- (4) M is generated by the monomial $x_{n-1}^{q_{n-1}} \dots x_1^{q_1}$ for some $q \in \mathbb{N}^{n-1}$.
- (5) Let $\pi : (x_n, \dots, x_1) \rightarrow (x_n x_{n-1}, x_{n-1}, x_{n-2} x_{n-1}, \dots, x_{d+1} x_{n-1}, x_d, \dots, x_1)$ be the expression of the blowup $W' \rightarrow W$ in the x_{n-1} -chart. In the coordinates in W' induced by π , the point a' has components

$$a' = (0, 0, a'_{n-2}, \dots, a'_{j+1}, 0, \dots, 0)$$

for some $d \leq j \leq n-2$ and with $a'_{n-2}, \dots, a'_{j+1} \neq 0$. Here, $j-d$ is the number of components of D whose transforms pass through a' .

(6) Local coordinates in W' at a' are given by the monomial blowup π followed by the translation $x \rightarrow x+t$ with $t = (0, 0, t_{n-2}, \dots, t_{j+1}, 0, \dots, 0)$ and $t_i = a'_i$. Alternatively, they are given as the composition of the linear map $\lambda_t : x \rightarrow (x + tx_{n-1})$ in W at a , followed by the monomial blowup π of W . The map λ_t preserves Z and V and the factorization $J = M \cdot I$, but destroys the monomiality of M as in (4) with respect to the given coordinates.

(7) The decomposition $q = r + \ell$ of the exponent q of M in red and yellow components is given by $r = (q_{n-1}, \dots, q_{j+1}, 0, \dots, 0)$ and $\ell = (0, \dots, 0, q_j, \dots, q_1)$.

(8) The weak transform V' of V in W' is given in the induced coordinates at a' by $x_n = 0$.

(9) If condition (4) is not imposed, the coordinates x_n, \dots, x_1 can be chosen so that a' is the origin of the x_{n-1} -chart and so that $(W', a') \rightarrow (W, a)$ is the monomial blowup π from (5).

The assertions can be proven as follows. It is clear that (x_n, \dots, x_1) can be chosen satisfying (1) to (3), and (4) can be achieved because D and Z are transversal. As for (5), we know by (3) that the exceptional component Y' is covered by the charts corresponding to x_n, \dots, x_{d+1} . As $c' = c$ and x_n is supposed to appear in the tangent cone of K we conclude that a' cannot lie in the x_n -chart. Hence a' lies in the other charts and satisfies there $a'_n = 0$. A permutation of y_{n-1}, \dots, y_{d+1} allows to assume that a' lies in the x_{n-1} -chart. This permutation does not alter (2) and (3). As Y' is given in the x_{n-1} -chart by $x_{n-1} = 0$ and as $a' \in Y'$ we get $a'_{n-1} = 0$. From $a_d = \dots = a_1 = 0$ follows that $a'_d = \dots = a'_1 = 0$. After a permutation of x_{n-2}, \dots, x_{d+1} we may assume that $a'_i \neq 0$ for $n-2 \geq i \geq j+1$ and $a'_i = 0$ for $j \geq i \geq 1$ and $i = n-1$ with $n-2-j$ the number of non-zero components of a' . This establishes (5).

Assertions (6), (7) and (8) follow from (5) and direct computations in the coordinates. Finally, (9) is a consequence of the second part of (6).

In the sequel, we shall mostly assume that there are no yellow components in D (these are the simple ones), so that $q = r$ consists only of red components, i.e., all components of D at a have transforms in W' which do not pass through a' . This is not a substantial restriction, but simplifies the exposition considerably.

8. Height and volume of polynomials

The description of blowups in local coordinates as in the lemma of the last section allows to observe the behaviour of polynomials f in \mathbb{A}^n when passing to their strict transform. Assertion (6) shows that it suffices to apply to the polynomial the linear coordinate change $\lambda_t : (x_n, \dots, x_1) \rightarrow (x_n, \dots, x_1) + tx_{n-1}$ in \mathbb{A}^n at 0 with $t = (0, 0, t_{n-2}, \dots, t_{j+1}, 0, \dots, 0)$, and then the monomial substitution

$$\pi : (x_n, \dots, x_1) \rightarrow (x_n x_{n-1}, x_{n-1}, x_{n-2} x_{n-1}, \dots, x_{d+1} x_{n-1}, x_d, \dots, x_1).$$

The order of the coefficient ideal of f in $V : x_n = 0$ and of its strict transform can be estimated by two numbers, the height and the volume of f , associated to its Newton polyhedron. We first define them for polynomials in V , and then for polynomials in W by passage to their coefficient ideal in V .

To ease the notation, we shall write m for $n-1$ and y_m, \dots, y_1 for x_{n-1}, \dots, x_1 . Let $y = (y_m, \dots, y_1)$ and set $z = (y_{m-1}, \dots, y_1)$. Let $P(y)$ be a polynomial in y , and let y^r with $r \in \mathbb{N}^m$ be the largest monomial which can be factored from P ,

$$P(y) = y^r \cdot g(y),$$

with some polynomial $g(y)$. We write r instead of g because, as mentioned above, we shall assume that all components of the exponent are relevant, i.e., red. For a given vector $t = (0, t_{m-1}, \dots, t_1)$ of constants t_i in the ground field, we set

$$P^+(y) = Q(y) = P(y + ty_m).$$

We define the *volume* of P as

$$\text{vol}(P) = \text{ord } g = \text{ord } P - |r|,$$

i.e., as the order $\text{ord } g$ of g at 0. We shall often write o for $\text{ord } g$. The *height* of P with respect to z is defined as

$$\text{height}_z(P^+) = \text{ord}_z P^+,$$

where $\text{ord}_z P^+$ denotes the order of P^+ with respect to the variables $z = (y_{m-1}, \dots, y_1)$, i.e. $\text{ord}_z P^+ = \max \{k, P^+ \in \langle y_{m-1}, \dots, y_1 \rangle^k\}$. The height of an ideal is defined as the minimum of the heights of its elements. For a polynomial in two variables, the height and volume are illustrated in figure 6.

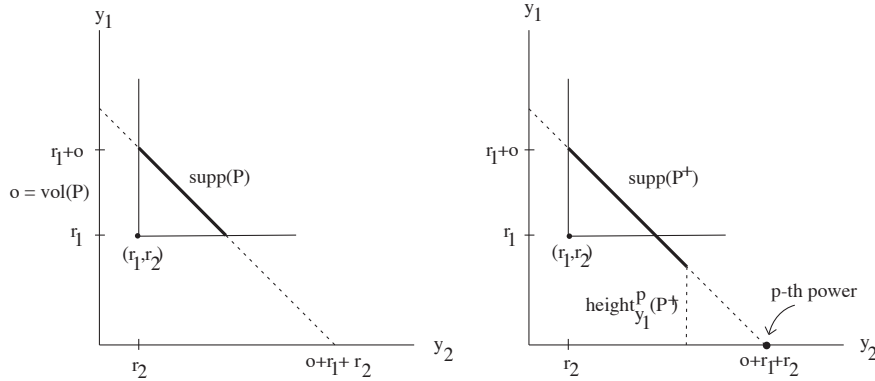


Figure 6. Height and volume of polynomials.

Let now x be another variable and let f be a weighted homogeneous polynomial in (x, y_m, \dots, y_1) of weighted degree e with respect to weights $(w, 1, \dots, 1)$ with $w \geq 1$. Let c be the order of f at 0. Write

$$f(x, y) = \sum a_{k\alpha} x^k y^\alpha,$$

$$f^+(x, y) = f(x, y + ty_m) = \sum b_{l\beta}(t) x^l y^\beta,$$

where $wk + |\alpha| = e$ and $wl + |\beta| = e$, with constants $a_{k\alpha}$ and polynomials $b_{l\beta}(t)$. Let V be the hypersurface $x = 0$ and let

$$\text{coeff}_V(f) = (a_i^{\frac{e}{c-i}}, i < c)$$

be the coefficient ideal of f in V (which we assume to be non-zero to avoid trivial cases). The order of $\text{coeff}_V(f)$ at 0 will be denoted by e . Let y^r be the maximal monomial which can be factored from $\text{coeff}_V(f)$, say $\text{coeff}_V(f) = y^r \cdot I$ for some ideal I . We set (see figure 7)

$$\text{vol}(f) = \text{ord } I = \text{ord}(\text{coeff}_V(f)) - |r| = e - |r|.$$

Similarly, we set

$$\text{height}_z(f^+) = \text{height}_z(\text{coeff}_V(f^+)).$$

In case where f has the form $f = x^c + P(y)$ with a homogeneous polynomial $P(y)$ of degree e (this case will be significant in the sequel) we simply get

$$\text{vol}(f) = \text{vol}(P)$$

and

$$\text{height}_z(f^+) = \text{height}_z(P^+).$$

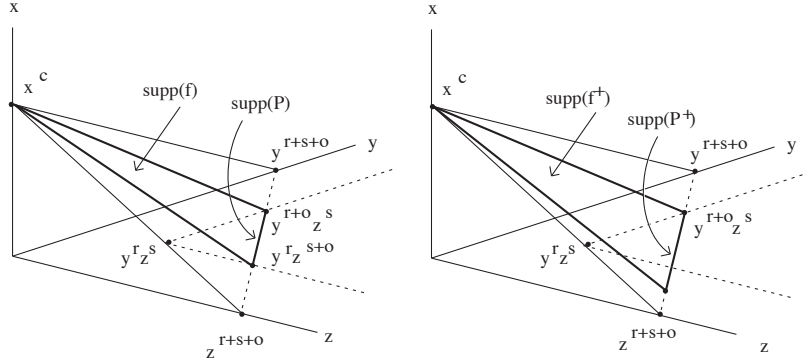


Figure 7. Height and volume of weighted homogeneous polynomials.

By definition, $\text{vol}(f)$ coincides with the order o at 0 of the divided coefficient ideal of f in V . We will see in the next section that $\text{height}_z(f^+)$ bounds from above the order o' at 0 of the divided coefficient ideal of the strict transform f' of f in V' under a blowup as in the lemma of the last section. As we wish to compare o with o' , we will be lead to compare $\text{vol}(f)$ with $\text{height}_z(f^+)$ (which can be treated as a question on the behaviour of polynomials under linear coordinate changes without involving blowups). To maximize the order of the divided coefficient ideals of f and f' we will have to eliminate p -th powers appearing in $\text{coeff}_V(f)$ and $\text{coeff}_{V'}(f')$ by coordinate changes in x . This can be best seen for $f = x^c + P(y)$ and characteristic $p = c$, where changes $x \rightarrow x + y^\alpha$ may cancel some monomials of P , say, of $\text{coeff}_V(f)$. Therefore we will also define volume and height of a polynomial modulo p -th powers, say

$$\text{vol}^p(f) = \text{ord}(\text{coeff}_V(f)/\text{modulo } p\text{-th powers}) - |r|.$$

and

$$\text{height}_z^p(f^+) = \text{height}_z(\text{coeff}_V(f^+)/\text{modulo } p\text{-th powers}),$$

where $/\text{modulo}$ denotes the ideal obtained after elimination of all monomials which are p -th powers in y .

9. Realizing weak maximal contact after blowup

We have seen in the examples that a hypersurface V which has weak maximal contact with an ideal K may transform under blowup into a hypersurface V^{st} which has no longer weak maximal contact with the transformed ideal K' of K . In this section we study the local isomorphisms of the blown up ambient scheme W' which map V^{st} into a hypersurface V' of weak maximal contact above. This will allow to read off the possible increase of the order of the corresponding coefficient ideal already below, and to relate it to the volume and height of

the weighted tangent cone of the ideal. For convenience we work in the completions of the local rings. In the sequel, “local” shall always refer to objects defined in the completions of the local rings.

Let $(W', a') \rightarrow (W, a)$ be a local blowup with exceptional component Y' , let K be an ideal in W and $K' = K^\vee$ its weak transform at a' . We assume that the order of K has remained constant at a' , i.e., that $c' = c$. Let already be chosen V in W at a with weak maximal contact with K relative to a normal crossings divisor D in W , i.e., $J = \text{coeff}_V K$ factorizes into $J = M \cdot I$ with $M = I_V(D \cap V)$, where V is transversal to D and $o = \text{ord}_a I$ is maximal among all such choices of V . Our objective is to find a hypersurface V' of weak maximal contact with K' at a' . In addition we wish to read off directly from K an upper bound for the order of the associated (divided) coefficient ideal I' of K' with respect to V' .

We place ourselves in the situation of the lemma of section 7, with local coordinates x_n, \dots, x_1 chosen at a so that the various conditions of this lemma are met. In particular, y_m, \dots, y_1 will denote x_{n-1}, \dots, x_1 . We assume for simplicity that the ideal K is principal. Let $V^\diamond = V^{st}$ be the strict transform of V at a' . The superscript \diamond will correspond to objects defined through V^\diamond , whereas a prime denotes objects in W' which play the same role as the corresponding objects without prime in W . As $c' = c$ and V has weak maximal contact with K at a we have by the lemma that $a' \in V^\diamond$. Let J^\diamond be the coefficient ideal of K' in V^\diamond . By the commutation of the passage to coefficient ideals with local blowups at equiconstant points (see the sections on commutativity in [EH] or [Ha 3]) we have that $J^\diamond = M^\diamond \cdot I^\diamond$ with $M^\diamond = I_{V^\diamond}(D' \cap V^\diamond)$ and $I^\diamond = I^\vee$, where $D' = D^* + (o - c) \cdot Y'$. However, as the examples 3 and 4 from section 3 show, V^\diamond need not maximize $o^\diamond = \text{ord}_{a'} I^\diamond$, i.e., V^\diamond need not have weak maximal contact with K' relative to D' .

Choose a local regular hypersurface V' in W' at a' which has weak maximal contact with K' relative to D' . There then exists a local automorphism ψ' of W' at a' which maps V^\diamond on V' . By the Gauss-Bruhat decomposition of the group of formal automorphisms with respect to the lexicographic order as described in [Ha 1] we may assume, up to permutations, that ψ' has in the induced coordinates at a' the form $\psi'(x_n, y) = (x_n + b'(y), y)$ with some formal power series $b'(y)$. In particular, ψ' preserves $D' \cap V^\diamond$ and hence the ideal M^\diamond .

A look at the Newton polyhedron of K shows, similarly as in [Hi 2, Ha 2, proof of Thm. 8.1], that ψ' is induced from an automorphism $\psi(x, y) = (x_n + b(y), y)$ of W at a for some formal power series b , i.e., the respective diagram is commutative

$$\begin{array}{ccc} (W', a') & \xrightarrow{\psi'} & (W', a') \\ \downarrow & & \downarrow \\ (W', a') & \xrightarrow{\psi} & (W', a') \end{array}$$

But ψ need not preserve V nor allow to factor the exceptional components from the coefficient ideal of K in $\psi(V)$ (cf. the examples in section 3). We denote by ψ^* the dual map of ψ between the local rings.

Let $f(x, y)$ be an element of the weighted tangent cone of K with respect to the given coordinates (x_n, y_m, \dots, y_1) at a and weights $(w, 1, \dots, 1) \in \mathbb{Q}^n$ with $w = e/c \geq 1$ and $e = \text{ord}_a J$. Then f has order c at 0. It is weighted homogeneous of weighted degree $e = |q| + o$, where $J = y^q \cdot I$ and $o = \text{ord} I$.

Set $\tilde{f}(x, y) = f(x + h(y), y + ty_m)$ with h the homogeneous tangent cone of b and $t = (t_m, \dots, t_1) = (0, t_{m-1}, \dots, t_{j+1}, 0, \dots, 0)$ with components t_i prescribed by the coordinates of a' . As $f(x + h(y), y)$ belongs to the weighted tangent cone of $\psi^*(K)$ and as the blowup $(W', a') \rightarrow (W, a)$ is the composition of $\lambda_t : (x, y) \rightarrow (x, y + ty_m)$ and the monomial blowup π of Z in the y_m -chart which maps (x_n, y_m, \dots, y_1) to $(x_n, y_m, y_{m-1}y_m, \dots, y_1y_m)$,

we see that the strict transform $(\tilde{f})^\vee$ of \tilde{f} under the monomial blowup π is an element of the weighted tangent cone of $(\psi')^*(K')$ whose coefficient ideal in V' belongs to the homogeneous tangent cone of J' .

As V' is assumed to maximize the order e' of J' in W' at a' , the order of $(\text{coeff}_{V'} \tilde{f})^\vee$ at a' thus bounds the order e' of J' , say

$$e' \leq \text{ord}_{a'}((\text{coeff}_{V'} \tilde{f})^\vee).$$

Let $q' \in \mathbb{N}^m$ denote the exponent of the monomial factor M' of $J' = M' \cdot I'$ and set $z = (y_{m-1}, \dots, y_1)$. The exponent q' stems from the exponent q of M via the formulas from the last section: q decomposes into $q = r + \ell$ with $r = (q_m, \dots, q_{j+1}, 0, \dots, 0)$ the red components and $\ell = (0, \dots, 0, q_j, \dots, q_1)$ the yellow components, and $q' = (q_m + o - c, 0, \dots, 0, q_j, \dots, q_1)$.

As the blowup π is monomial in the y_m -chart we can interpret the preceding inequality in terms of \tilde{f} before blowing up. A direct inspection of the Newton polyhedra yields the (slightly weaker) bound

$$e' \leq \text{ord}_z(\text{coeff}_V(\tilde{f})) + q'_m = \text{height}_z(\text{coeff}_V(\tilde{f})) + q'_m.$$

We could also take here the order with respect to z of $\text{coeff}_{\psi(V)}(f)$, but computationally it is easier to handle $\text{coeff}_V(\tilde{f})$. We obtain for $o' = \text{ord}_{a'} I'$ the inequality

$$o' = e' - |q'| \leq \text{height}_z(\tilde{f}) - |\ell|.$$

In order to show $o' \leq o$ it therefore suffices to show, using $o = e - |q| = \text{vol}(f)$, that

$$\text{height}_z(\tilde{f}) - |\ell| \leq \text{vol}(f).$$

In the particular case where no transforms of exceptional components through a persist at a' , say if $\ell = 0$ and $r = q$, we get the sufficient inequality

$$\text{height}_z(\tilde{f}) \leq \text{vol}(f).$$

If it holds, then $o' \leq o$ will follow. But, by the examples 3 and 4 of section 3, we know that $\text{height}_z(\tilde{f}) = \text{vol}(f) + 1$ may occur. The next section prepares the material to compare $\text{height}_z(\tilde{f})$ with $\text{vol}(f)$.

10. Zwickels

Zwickels are convex polytopes in \mathbb{N}^n which we shall use to prescribe the supports of our polynomials and to express conveniently the volume and the height of a weighted homogeneous polynomial.

Let be given $c \leq e$ in \mathbb{N} and write $cw = e$ with and $w \in \mathbb{Q}$. Let

$$L_c = \{(k, \alpha) \in \mathbb{N}^{1+m}, k < c\} \rightarrow \mathbb{Q}^m : (k, \alpha) \rightarrow \frac{c}{c-k} \cdot \alpha$$

be the map projecting elements (k, α) of the layer L_c in \mathbb{N}^{1+m} to elements of \mathbb{Q}^m . The center of the projection is the point $(c, 0, \dots, 0)$ (see figure 8).

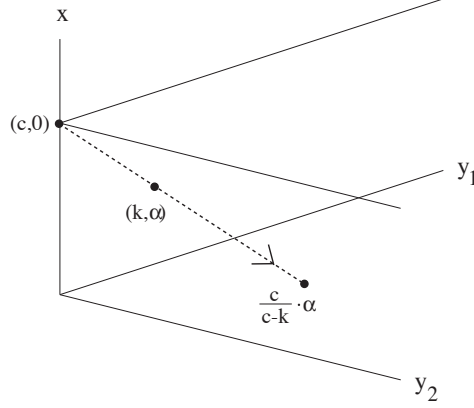


Figure 8. Projection of $L_c \subset \mathbb{N}^{1+m}$ to \mathbb{Q}^m with center $(c, 0, \dots, 0)$.

Let $q \in \mathbb{N}^m$ with $|q| = q_1 + \dots + q_m \leq e$ be fixed, and assume given a decomposition $q = r + \ell$ with $r = (q_m, \dots, q_{j+1}, 0, \dots, 0)$ and $\ell = (0, \dots, 0, q_j, \dots, q_1)$ for some j between $m - 1$ and $d \geq 0$. Define the *upper zwickel* $Z(q)$ in \mathbb{N}^{1+m} as the set of points (k, α) with $0 \leq k \leq c$, $wk + |\alpha| = e$ and projection $\frac{c-k}{c} \cdot \alpha \geq_{cp} q$, denoting by \geq_{cp} the componentwise order (see figure 9). Thus $Z(q)$ is given by

$$Z(q) : wk + |\alpha| = e \text{ and } \alpha \geq_{cp} \left[\frac{c-k}{c} \cdot (q_m, \dots, q_1) \right].$$

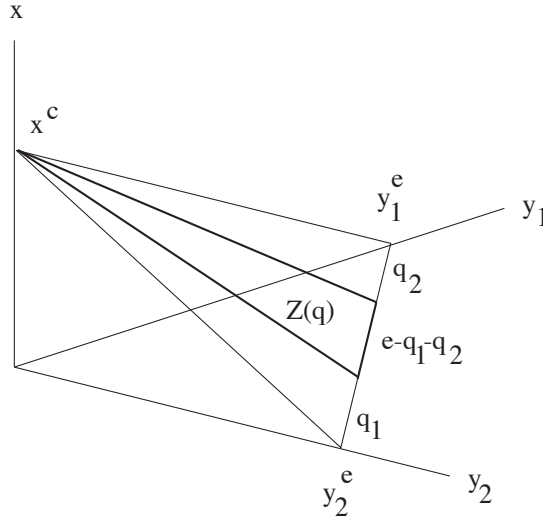


Figure 9. The upper zwickel $Z(q)$ in \mathbb{N}^3 .

Define the *lower zwickel* $Y(r, \ell)$ in \mathbb{N}^{1+m} as the set of points (k, β) in \mathbb{N}^{1+m} with $0 \leq k \leq c$, $wk + |\beta| = e$ and projection $\frac{c-k}{c} \cdot \beta \geq_{cp} (|r|, 0, \dots, 0, \ell)$ (see figure 10). Thus $Y(r, \ell)$ is given by

$$Y(r, \ell) : wk + |\beta| = e \text{ and } \beta \geq_{cp} \left[\left(\frac{c-k}{c} \cdot |r|, 0, \dots, 0, \frac{c-k}{c} \cdot q_j, \dots, \frac{c-k}{c} \cdot q_1 \right) \right].$$

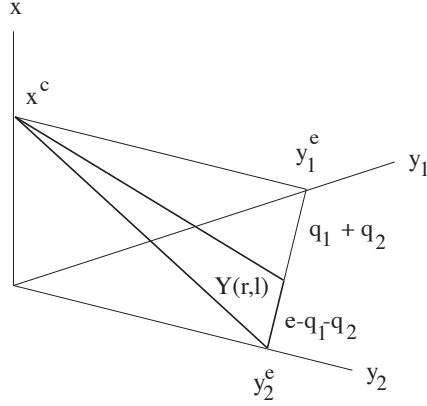


Figure 10. The lower zwickel $Y(r, \ell)$ in \mathbb{N}^3 .

For $j = m - 1$ and hence $r = (q_m, 0, \dots, 0)$ and $\ell = (0, q_{m-1}, \dots, q_1)$ we have $Z(q) = Y(r, \ell)$. In general, the two zwickels are different.

We will show first that for any r and ℓ and $0 \leq k \leq e/w = c$ the slice

$$Y(r, \ell)(k) = \{(k, \beta) \in Y(r, \ell)\} = Y(r, \ell) \cap (\{k\} \times \mathbb{N}^m)$$

has at least as many elements as the slice

$$Z(q)(k) = \{(k, \alpha) \in Z(q)\} = Z(q) \cap (\{k\} \times \mathbb{N}^m).$$

This holds for $k = 0$, by definition of $Z(q)$ and $Y(r, \ell)$. For arbitrary k , the inequality $\lceil \frac{c-k}{c} \cdot |r| \rceil \leq \lceil \frac{c-k}{c} \cdot r \rceil$ implies that the condition

$$wk + |\beta| = e \text{ and } \beta \geq_{cp} (\lceil \frac{c-k}{c} \cdot |r| \rceil, 0, \dots, 0, \lceil \frac{c-k}{c} \cdot q_j \rceil, \dots, \lceil \frac{c-k}{c} \cdot q_1 \rceil)$$

is more restrictive than the condition

$$wk + |\beta| = e \text{ and } \beta \geq_{cp} (\lceil \frac{c-k}{c} \cdot |r| \rceil, 0, \dots, 0, \lceil \frac{c-k}{c} \cdot q_j \rceil, \dots, \lceil \frac{c-k}{c} \cdot q_1 \rceil)$$

defining $Y(r, \ell)(k)$. For each k , the set of pairs k, β satisfying the first condition has as many elements as $Z(q)(k)$ because $|r| + q_j + \dots + q_1 = |q|$. The claim follows.

We now invoke the arithmetic inequality $\lceil \bar{r}^c \rceil > (\phi_c(r) - 1) \cdot c$ from the definition of tame and wild points in section 5. We will show that if it holds, the upper zwickel $Z(q)$ contains as many c -rays as the lower zwickel $Y(r, \ell)$. Here, a c -ray is the segment in $Z(q)$ between the point $(c, 0, \dots, 0) \in \mathbb{N}^{1+m}$ and a lattice point in $\{0\} \times c \cdot \mathbb{N}^m$.

For the proof, let $(0, c\alpha)$ be a point of $0 \times c \cdot \mathbb{N}^m$. It belongs to $Z(q)(0) \cap c \cdot \mathbb{N}^{m+1}$ if and only if $|c\alpha| = e$ and

$$c\alpha \geq_{cp} \lceil (q_m, \dots, q_1) \rceil = (\lceil q_m \rceil, \dots, \lceil q_1 \rceil).$$

As the components of α are integers, the second inequality is equivalent to

$$\alpha \geq_{cp} (\lceil \frac{q_m}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil).$$

Conversely, $(0, c\beta)$ in $0 \times \mathbb{N}^m$ belongs to $Y(r, \ell)(0) \cap c \cdot \mathbb{N}^{m+1}$ if $|c\beta| = e$ and

$$c\beta \geq_{cp} \lceil (|r|, 0, \dots, 0, q_j, \dots, q_1) \rceil = (\lceil |r| \rceil, 0, \dots, 0, \lceil q_j \rceil, \dots, \lceil q_1 \rceil),$$

say

$$\beta \geq_{cp} (\lceil \frac{|r|}{c} \rceil, 0, \dots, 0, \lceil \frac{q_j}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil).$$

The hypothesis $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ is equivalent to the equality

$$|\lceil \frac{\bar{r}^c}{c} \rceil| = \lceil \frac{|\bar{r}^c|}{c} \rceil$$

and hence also to

$$|\lceil \frac{r}{c} \rceil| = \lceil \frac{|r|}{c} \rceil.$$

This implies that the second condition on $(0, c\beta)$ can be written as

$$\beta \geq_{cp} (|\lceil \frac{r}{c} \rceil|, 0, \dots, 0, \lceil \frac{q_1}{c} \rceil, \dots, \lceil \frac{q_m}{c} \rceil).$$

Now the assertion follows from

$$|\lceil \frac{r}{c} \rceil| = \lceil \frac{q_m}{c} \rceil + \dots + \lceil \frac{q_{j+1}}{c} \rceil.$$

We have shown that the arithmetic inequality implies that the upper zwickel $Z(q)$ contains as many c -rays as the lower zwickel $Y(r, \ell)$.

The relation of zwickels with the concepts of height and volume of the last section is the following (we leave the verification as an exercise). If $f(x, y)$ is a weighted homogeneous polynomial of weighted degree e with respect to $(w, 1, \dots, 1)$ and if the hypersurface V is given by $x = 0$ then y^q is a factor of $\text{coeff}_V f$, i.e., $\text{coeff}_V f = y^q \cdot I$, if and only if f has support in $Z(q)$. If $\tilde{f}(x, y) = f(x + h(y), y + ty_m)$ is associated to $f(x, y)$ as in the last section then $\tilde{f}(x, y)$ satisfies the relevant inequality

$$\text{height}_z \tilde{f} - |\ell| > e - |q| = \text{vol}(f)$$

if and only if all coefficients of \tilde{f} in $Y(r, \ell)$ are zero except the coefficient of x^c . Therefore, to prove the inequality

$$\text{height}_z(\tilde{f}) - |\ell| \leq \text{vol}(f)$$

it is sufficient to show that some coefficient of \tilde{f} in $Y(r, \ell)$ different from the coefficient of x^c is non zero.

We will show in the next section that if all these coefficients of \tilde{f} are zero and if the arithmetic inequality $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ holds then f must be a c -th power, say $f = (x + a(y))^c$ for some series $a(y)$. This shows in turn that the order of $\text{coeff}_V(f)$ was not maximal over all coordinate choices, i.e. that the hypersurface V does not have weak maximal contact with f , contradictory to the assumption. Hence we may conclude that $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ implies $\text{height}_z(\tilde{f}) - |\ell| \leq \text{vol}(f)$ and hence, as seen earlier, $o' \leq o$. In case $|\bar{r}^c| \leq (\phi_c(r) - 1) \cdot c$ this need not follow, but we can show that at least $\text{height}_z(\tilde{f}) - |\ell| \leq \text{vol}(f) + 1$ holds. The polynomials f in three variables for which equality occurs here will be completely determined.

11. Transformation matrices

This section determines the relation between the coefficients of polynomials obtained from each other by specific coordinate changes. In the sequel let $f(x, y)$ and $\tilde{f}(x, y) = f(x + \sum_{\gamma} h_{\gamma} y^{\gamma}, y + ty_m)$ be weighted homogeneous polynomials of weighted degree e with respect to weights $(w, 1, \dots, 1)$ on $(x, y) = (x, y_m, \dots, y_1)$, where the sum $\sum_{\gamma} h_{\gamma} y^{\gamma}$ ranges over $\gamma \in \mathbb{N}^m$ with $|\gamma| = w$, and where h_{γ} and the components of $t = (0, t_{m-1}, \dots, t_1)$ belong to the ground field. Let $c = e/w$ be the order of f . Write

$$f(x, y) = \sum a_{k\alpha} x^k y^{\alpha} \quad \text{and} \quad \tilde{f}(x, y) = \sum b_{l\beta}(t) x^l y^{\beta}$$

with $wk + |\alpha| = wl + |\beta| = e$. We assume that $a_{c0} \neq 0$, i.e., that x^c appears with non-zero coefficient. This can be achieved for infinite ground fields by a generic linear coordinate change. For simplicity, we shall take $a_{c0} = 1$. Let V be the hypersurface in $W = \mathbb{A}^n$ defined

by $x = 0$. Let us fix the decomposition $q = r + \ell \in \mathbb{N}^m$ with $r = (q_m, \dots, q_{j+1}, 0, \dots, 0)$ and $\ell = (0, \dots, 0, q_j, \dots, q_1)$ for some index j between $m - 1$ and 0 (the center of blowup may still have dimension $d \geq 0$). Then y^q is a factor of $\text{coeff}_V f$ if and only if $f - x^c$ has support in the upper zwickel $Z = Z(q)$ with $q \in \mathbb{N}^m$, and $\text{coeff}_V \tilde{f}$ has order $> e - |r|$ in $z = (y_{m-1}, \dots, y_1)$, i.e., $\text{height}_z \tilde{f} - |\ell| > e - |q|$, if and only if all coefficients of $\tilde{f} - x^c$ in the lower zwickel $Y(r, \ell)$ are zero.

Write elements $\beta \in \mathbb{N}^m$ as (β_m, β^-) where $\beta^- = (\beta_{m-1}, \dots, \beta_1) \in \mathbb{N}^{m-1}$. Let $Y^*(r, \ell)$ be the subset of $Y(r, \ell)$ of elements $(k, \beta) \in \mathbb{N}^{1+m}$ given by

$$\begin{aligned} |\beta^-| &\leq e - wk - \lceil \frac{c-k}{c} \cdot |r| \rceil, \\ \beta^- &\geq_{cp} \lceil \frac{c-k}{c} \cdot (0, \dots, 0, q_j, \dots, q_1) \rceil. \end{aligned}$$

By definition, for each k , the slice $Y^*(r, \ell)(k)$ has the same cardinality as the slice $Z(q)(k)$ of the upper zwickel $Z(q)$. For α and δ in \mathbb{Z}^m set $\binom{\alpha}{\delta} = \prod_i \binom{\alpha_i}{\delta_i}$ where $\binom{\alpha_i}{\delta_i}$ is zero if $\alpha_i < \delta_i$ or $\delta_i < 0$. For Γ a subset of \mathbb{N}^m , define for $k \in \mathbb{N}$ and $\lambda = (\lambda_\gamma)_{\gamma \in \Gamma} \in \mathbb{N}^\Gamma$ the alternate binomial coefficient

$$\binom{k}{\lambda} = \prod_{\gamma \in \Gamma} \binom{k - |\lambda|^\gamma}{\lambda_\gamma} \quad \text{with } |\lambda|^\gamma = \sum_{\varepsilon \in \Gamma, \varepsilon < l_{\varepsilon x} \gamma} \lambda_\varepsilon.$$

Let $\Gamma \subset \mathbb{N}^m$ be the set of $\gamma \in \mathbb{N}^m$ with $|\gamma| = w$ and write $h = (h_\gamma)_{\gamma \in \Gamma} \in K^\Gamma$. Set $\lambda \cdot \Gamma = \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma \in \mathbb{N}^m$ and fix $t = (0, t_{m-1}, \dots, t_{j+1}, 0, \dots, 0)$. The transformation matrix between the coefficients $a_{k\alpha}$ and $b_{l\beta}(t)$ of f and \tilde{f} looks as follows.

Proposition 1. *Let $f(x, y) = \sum a_{k\alpha} x^k y^\alpha$ and $\tilde{f}(x, y) = f(x + \sum_{\gamma \in \Gamma} h_\gamma y^\gamma, y + ty_m) = \sum b_{l\beta}(t) x^l y^\beta$ be weighted homogeneous polynomials with respect to weights $(w, 1, \dots, 1)$ as above. Fix $q = r + \ell \in \mathbb{N}^m$ with zwickels $Z(q)$ and $Y^*(r, \ell) \subset Y(r, \ell)$.*

(1) *The transformation matrix $A = (A_{k\alpha, l\beta})$ from the coefficients $a_{k\alpha}$ of f to the coefficients $b_{l\beta}(t)$ of \tilde{f} is given by*

$$A_{k\alpha, l\beta} = \sum_{\lambda \in \mathbb{N}^\Gamma, |\lambda| = k-l} \binom{k}{l} \binom{k-l}{\lambda} \binom{\alpha}{\delta_{\alpha\beta\lambda}} \cdot h^\lambda \cdot t^{\alpha - \delta_{\alpha\beta\lambda}},$$

where $\delta_{\alpha\beta\lambda} = (\alpha_m, \beta^- - (\lambda \cdot \Gamma)^-) \in \mathbb{N}^m$ and $h^\lambda = \prod_\gamma h_\gamma^{\lambda_\gamma}$.

(2) *The quadratic submatrix $A^\square = (A_{k\alpha, l\beta})$ of A with $(k\alpha, l\beta)$ ranging in $Z(q) \times Y^*(r, \ell)$ has determinant $t^{\rho(Z, Y^*(r, \ell))}$ where $\rho(Z, Y^*(r, \ell))$ is a vector in \mathbb{N}^{m-1} independent of $h = (h_\gamma)_{\gamma \in \Gamma}$ with $\rho_m = 0$ and $\rho_j = \dots = \rho_1 = 0$.*

(3) *Assume that f has support in $Z(q)$. If t_{m-1}, \dots, t_{j+1} are non zero, the coefficients $b_{l\beta}$ of \tilde{f} in the lower zwickel $Y(r, \ell)$ determine all coefficients of f . In particular, there is at most one non-zero polynomial $f(x, y)$ with support in $Z(q)$ such that $\tilde{f}(x, y) - x^c$ has all coefficients in $Y(r, \ell)$ equal to zero.*

Proof. Multinomial expansion of $\tilde{f}(x, y) = f(x + \sum_\gamma h_\gamma y^\gamma, y + ty_m)$ gives for each $k\alpha \in \mathbb{N}^{1+m}$

$$\begin{aligned} &(x + \sum_{\gamma \in \Gamma} h_\gamma y^\gamma)^k (y + ty_m)^\alpha = \\ &= \sum_{l \in \mathbb{N}, l \leq k} \binom{k}{l} x^l (\sum_{\gamma \in \Gamma} h_\gamma y^\gamma)^{k-l} \sum_{\delta \in \mathbb{N}^m, \delta \leq_{cp} \alpha} \binom{\alpha}{\delta} y^\delta t^{\alpha - \delta} y_m^{|\alpha - \delta|} = \\ &= \sum_l \binom{k}{l} x^l \sum_{\lambda \in \mathbb{N}^\Gamma, |\lambda| = k-l} \prod_{\gamma \in \Gamma} \binom{k-l - |\lambda|^\gamma}{\lambda_\gamma} (h_\gamma y^\gamma)^{\lambda_\gamma} \cdot \sum_\delta \binom{\alpha}{\delta} y^\delta t^{\alpha - \delta} y_m^{|\alpha - \delta|} = \\ &= \sum_l \sum_\lambda \sum_\delta \binom{k}{l} \prod_\gamma \binom{k-l - |\lambda|^\gamma}{\lambda_\gamma} \binom{\alpha}{\delta} \cdot h^\lambda \cdot t^{\alpha - \delta} \cdot x^l \cdot y^{\sum_\gamma \lambda_\gamma} \cdot y^\delta \cdot y_m^{|\alpha - \delta|} = \\ &= \sum_l \sum_\lambda \sum_\delta \binom{k}{l} \binom{k-l}{\lambda} \binom{\alpha}{\delta} \cdot h^\lambda \cdot t^{\alpha - \delta} \cdot x^l \cdot y^{\lambda \cdot \Gamma + \delta} \cdot y_m^{|\alpha - \delta|}. \end{aligned}$$

As $\delta_{\alpha\beta\lambda} = ((\delta_{\alpha\beta\lambda})_m, \delta_{\alpha\beta\lambda}^-) = (\alpha_m, \beta^- - (\lambda \cdot \Gamma)^-)$ we can rewrite for given k, α and l a sum $\sum e_{\lambda\delta} \cdot y^{\lambda \cdot \Gamma + \delta} \cdot y_m^{|\alpha - \delta|}$ over $\lambda \in \mathbb{N}^\Gamma$ and $\delta \in \mathbb{N}^m$ with coefficients $e_{\lambda\delta}$ as

$$\begin{aligned}
& \sum_{\lambda \in \mathbb{N}^m, |\lambda|=k-l} \sum_{\delta \leq_{cp} \alpha} e_{\lambda\delta} \cdot y^{\lambda \cdot \Gamma + \delta} \cdot y_m^{|\alpha - \delta|} = \\
& = \sum_{|\lambda|=k-l} \sum_{\delta \leq_{cp} \alpha} e_{\lambda\delta} \cdot (y^-)^{(\lambda \cdot \Gamma)^- + \delta^-} \cdot y_m^{|\alpha - \delta| + (\lambda \cdot \Gamma + \delta)_m} = \\
& = \sum_{\beta \in \mathbb{N}^m} \sum_{|\lambda|=k-l} e_{\lambda\delta_{\alpha\beta\lambda}} \cdot (y^-)^{\beta^-} \cdot y_m^{\beta_m} = \\
& = \sum_{\beta \in \mathbb{N}^m} \left(\sum_{|\lambda|=k-l} e_{\lambda\delta_{\alpha\beta\lambda}} \right) \cdot y^\beta.
\end{aligned}$$

Here the coefficients $e_{\lambda\delta_{\alpha\beta\lambda}}$ of the last two sums are set equal to zero if $\delta_{\alpha\beta\lambda} \notin \mathbb{N}^m$ or $\delta_{\alpha\beta\lambda} \not\leq_{cp} \alpha$, say if $\binom{\alpha}{\delta_{\alpha\beta\lambda}} = 0$. Thus

$$\begin{aligned}
\tilde{f}(x, y) &= \sum_{k\alpha} a_{k\alpha} \cdot (x + \sum_{\gamma \in \Gamma} h_\gamma y^\gamma)^k \cdot (y + ty_m)^\alpha = \\
&= \sum_{k\alpha} \sum_{l\beta} \sum_{|\lambda|=k-l} a_{k\alpha} \binom{k}{l} \binom{\alpha}{\delta_{\alpha\beta\lambda}} \cdot h^\lambda \cdot t^{\alpha - \delta_{\alpha\beta\lambda}} \cdot x^l \cdot y^\beta = \\
&= \sum_{l\beta} \beta_{l\beta} \cdot x^l \cdot y^\beta.
\end{aligned}$$

This gives assertion (1). Observe here that we have used that $a_{c0} = 1$ and $b_{c0} = 1$.

For (2), note that $A_{k\alpha, l\beta} = 0$ if $k < l$. Hence the matrix A is block triangular with blocks $A(k) = (A_{k\alpha, k\beta})_{\alpha\beta}$ on the diagonal $k = l$. By the choice of $Y^*(r, s)$, the induced blocks $A^\square(k)$ of A^\square are square matrices. Hence A^\square is a square matrix. We get from assertion (1) that

$$A_{k\alpha, k\beta} = \sum_{|\lambda|=0} \binom{\alpha}{\delta_{\alpha\beta\lambda}} \cdot h^\lambda \cdot t^{\alpha - \delta_{\alpha\beta\lambda}} = \binom{\alpha}{\delta_{\alpha\beta 0}} \cdot t^{\alpha - \delta_{\alpha\beta 0}} = \binom{\alpha^-}{\delta_{\alpha\beta 0}^-} \cdot t^{\alpha^- - \delta_{\alpha\beta 0}^-},$$

with $\alpha = (\alpha_m, \alpha^-)$ and $\delta_{\alpha\beta 0} = ((\delta_{\alpha\beta 0})_m, \delta_{\alpha\beta 0}^-) = (\alpha_m, \beta^-)$. Recall that $k\alpha$ and $l\beta$ vary in $Z(q)$ and $Y(r, s)$ respectively so that

$$\begin{aligned}
wk + |\alpha| &= e \quad \text{and} \quad \alpha \geq_{cp} \lceil \frac{c-k}{c} \cdot (q_m, \dots, q_1) \rceil, \\
wl + |\beta| &= e \quad \text{and} \quad \beta \geq_{cp} \lceil \frac{c-l}{c} \cdot (|r|, 0, \dots, 0, q_j, \dots, q_1) \rceil.
\end{aligned}$$

Hence, as $k = l$, we have

$$\begin{aligned}
|\alpha^-| &= e - wk - \alpha_m \quad \text{and} \quad \alpha^- \geq_{cp} \lceil \frac{c-k}{c} \cdot (q_{m-1}, \dots, q_1) \rceil, \\
|\delta_{\alpha\beta 0}^-| &= |\beta^-| = e - wk - \beta_m \quad \text{and} \quad \delta_{\alpha\beta 0}^- \geq_{cp} \lceil \frac{c-k}{c} \cdot (0, \dots, 0, q_j, \dots, q_1) \rceil.
\end{aligned}$$

The determinant of $A^\square(k)$ is given by the lemma below, taking there $b = (c-k) \cdot w - \lceil \frac{c-k}{c} \cdot r \rceil$, $\mu = \lceil \frac{c-k}{c} \cdot (0, \dots, 0, q_j, \dots, q_1) \rceil$ and $\theta = \lceil \frac{c-k}{c} \cdot (q_{m-1}, \dots, q_{j+1}, 0, \dots, 0) \rceil$. Substituting there the variables t_{m-1}, \dots, t_1 by constants in the ground field with $t_{m-1}, \dots, t_{j+1} \neq 0$ the determinant is non zero. We conclude that all $A^\square(k)$ and hence A^\square are invertible. This proves (2).

Assertion (3) follows from (2) since the transformation matrix between the $k\alpha$ in $Z(q)$ and the $l\beta$ in $Y(r, s)$ has, by (2) and since t_{m-1}, \dots, t_{j+1} are non zero, maximal rank equal to the cardinality of $Z(q)$. This concludes the proof of the proposition.

Lemma. Let $b \in \mathbb{N}$, $\mu \in \mathbb{N}^{m-1}$ and $U = \{\delta \in \mathbb{N}^{m-1}, |\delta| \leq b, \delta \geq_{cp} \mu\}$. Set $g = \#U = \binom{m-1+b-|\mu|}{m-1}$. Let $\theta \in \mathbb{N}^{m-1}$ and let $t = (t_{m-1}, \dots, t_1)$ be a vector of variables. Then

$$\det((\binom{\gamma+\theta}{\delta} \cdot t^{\gamma+\theta-\delta})_{\gamma, \delta \in U}) = t^\rho$$

with $\rho = g \cdot \theta \in \mathbb{N}^{m-1}$ independent of t .

Proof. Write A^θ for the $(g \times g)$ -square matrix with entries $A_{\gamma\delta}^\theta = \binom{\gamma+\theta}{\delta} \cdot t^{\gamma+\theta-\delta}$. Observe that for $\theta = 0 \in \mathbb{N}^{m-1}$ we have $\det A^0 = 1$, since the matrix is upper triangular with 1's on the diagonal. From $\binom{j+1}{i} = \binom{j}{i} + \binom{j}{i-1}$ follows for any $\varepsilon \in \mathbb{N}^{m-1}$ with $|\varepsilon| = 1$ that

$$A_{\gamma\delta}^{\theta+\varepsilon} = t^\varepsilon \cdot A_{\gamma\delta}^\theta + A_{\gamma, \delta-\varepsilon}^\theta \quad \text{if } \delta \geq_{cp} \varepsilon,$$

$$A_{\gamma\delta}^{\theta+\varepsilon} = t^\varepsilon \cdot A_{\gamma\delta}^\theta \quad \text{else.}$$

Therefore the matrix $A^{\theta+\varepsilon}$ is obtained from A^θ by multiplying for $\delta \geq_{cp} \varepsilon$ the columns $A_{-\delta}^\theta$ by t^ε and then adding the column $A_{-\delta-\varepsilon}^\theta$. The other columns $A_{-\delta}^\theta$ are only multiplied with t^ε . This implies that

$$\det(A^{\theta+\varepsilon}) = t^{g \cdot \varepsilon} \cdot \det(A^\theta).$$

Now induction implies that $\det(A^\theta) = t^{g \cdot \theta} \cdot \det(A^0) = t^{g \cdot \theta}$.

12. Bounding the increase of o under blowup

Let again $f(x, y)$ be a weighted homogeneous polynomial of weighted degree e with respect to weights $(w, 1, \dots, 1)$ on $(x, y) = (x, y_m, \dots, y_1)$, and let $c = e/w$ be the order of f at 0. Set $\tilde{f}(x, y) = f(x + \sum_\gamma h_\gamma y^\gamma, y + ty_m)$ with $\gamma \in \mathbb{N}^m$, $|\gamma| = w$, where h_γ and the components of $t = (0, t_{m-1}, \dots, t_1)$ belong to the ground field. Write

$$f(x, y) = \sum a_{k\alpha} x^k y^\alpha \quad \text{and} \quad \tilde{f}(x, y) = \sum b_{l\beta}(t) x^l y^\beta$$

with indices $k\alpha$ and $l\beta$ subject to $wk + |\alpha| = wl + |\beta| = e$. We may assume that $a_{c0} = b_{c0} = 1$. Let V be the hypersurface defined by $x = 0$. Recall that $\text{height}_z(\tilde{f}) = \text{ord}_z(\text{coeff}_V \tilde{f})$ for $z = (y_{m-1}, \dots, y_1)$ and $\text{vol}(f) = e - q$ for $\text{coeff}_V(f) = y^q \cdot I$ with $q \in \mathbb{N}^m$.

Fix a decomposition $q = r + \ell = (q_m, \dots, q_{j+1}, 0, \dots, 0) + (0, \dots, 0, q_j, \dots, q_1)$ for some $m-1 \geq j \geq 0$ with induced zwickels $Z(q)$ and $Y(r, \ell)$ in \mathbb{N}^{1+m} . Let $\phi_c(r)$ denote the number of components of r not divisible by c .

Theorem 2. *Let $f(x, y) = \sum a_{k\alpha} x^k y^\alpha$ and $\tilde{f}(x, y) = f(x + \sum_\gamma h_\gamma y^\gamma, y + ty_m) = \sum b_{l\beta}(t) x^l y^\beta$ be as above, $t = (0, t_{m-1}, \dots, t_{j+1}, 0, \dots, 0)$ for some $0 \leq j \leq m-1$. Assume that t_{m-1}, \dots, t_{j+1} are non-zero. Let $q = r + \ell = (q_m, \dots, q_{j+1}, 0, \dots, 0) + (0, \dots, 0, q_j, \dots, q_1)$.*

(1) *Assume that $e/c \notin \mathbb{N}$ or $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ or $h = 0$. If f has support in $Z(q)$ and if $\tilde{f} - x^c$ has support outside $Y(r, \ell)$ then f is a c -th power.*

(2) *Assume that $e/c \notin \mathbb{N}$ or $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ or $h = 0$. If f has support in $Z(q)$ and is not a c -th power then \tilde{f} satisfies the inequality*

$$\text{ord}_z \text{coeff}(\tilde{f}) \leq e - |r|.$$

Equivalently,

$$\text{height}_z(\tilde{f}) - |\ell| \leq e - |q| = \text{vol}(f).$$

(3) *If $e/c \in \mathbb{N}$ and $|\bar{r}^c| \leq (\phi_c(r) - 1) \cdot c$ we have*

$$\text{ord}_z \text{coeff}(\tilde{f}) \leq e - |u|,$$

where $u \in \mathbb{N}^m$ is maximal with respect to the componentwise order satisfying $u \leq_{cp} r$ and $|\bar{u}^c| > (\phi_c(u) - 1) \cdot c$. In particular, we get the bound

$$\text{ord}_z \text{coeff}(\tilde{f}) \leq e - |r| + |\bar{r}^c| - \max_{m \geq i \geq j+1} \bar{r}_i^c,$$

or, equivalently,

$$\text{height}_z(\tilde{f}) - |\ell| \leq e - |q| + |\bar{r}^c| - \max_{m \geq i \geq j+1} \bar{r}_i^c.$$

As an immediate consequence of these estimates we obtain.

Corollary. *Let f be an element of the weighted tangent cone of a principal ideal K in W at a . Assume that $J = \text{coeff}_V(K)$ factors into $J = y^q \cdot I$ with $q \in \mathbb{N}^m$. Let $W' \rightarrow W$ be the blowup of W with regular center $Z \subset V$ and exceptional divisor Y' . Let $a' \in Y'$ be a point above $a \in Z$ such that $c' = c$ for the orders of K at a and its weak transform K' at a' . Let $o = e - |q| = \text{ord}_a I$, $o' = \text{ord}_{a'} I'$. If $e/c \notin \mathbb{N}$ or $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$, then*

$$o' \leq o.$$

Proof of the theorem. Let us prove assertion (1). From Proposition 1 of section 11 follows that f is uniquely determined by \tilde{f} . It therefore suffices to construct a c -th power f with support in $Z(q)$ whose associated polynomial \tilde{f} has support outside $Y(r, \ell)$.

If $w \notin \mathbb{N}$, there are no h_γ 's, say $h = 0$ and the coordinate change $x + h(y)$ is the identity. The proof is then similar to the proof in case $w \in \mathbb{N}$ by setting all $h_\gamma = 0$, but without using the arithmetic condition $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$. We will omit it.

So assume that $w \in \mathbb{N}$. The set T of γ 's in \mathbb{N}^m satisfying

$$|\gamma| = w \text{ and } \gamma \geq_{cp} (\lceil \frac{r}{c} \rceil, 0, \dots, 0, \lceil \frac{q_j}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil)$$

forms an equilateral $(m-1)$ -dimensional simplex in $\Gamma = \{\gamma \in \mathbb{N}^m, |\gamma| = w\} \subset \mathbb{N}^m$. Consider its projection T^- in \mathbb{N}^{m-1} obtained by omitting the first component γ_m . It consists of elements γ^- in \mathbb{N}^{m-1} subject to

$$T^- : |\gamma^-| \leq w - \lceil \frac{r}{c} \rceil \text{ and } \gamma^- \geq_{cp} (0, \dots, 0, \lceil \frac{q_j}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil).$$

Thus T^- forms an equilateral $(m-1)$ -dimensional simplex in \mathbb{N}^{m-1} with side length $w - \lceil \frac{r}{c} \rceil - \lceil \frac{\ell}{c} \rceil$ and $(m-1)$ -dimensional volume $\frac{1}{(m-1)!} \cdot (w - \lceil \frac{r}{c} \rceil - \lceil \frac{\ell}{c} \rceil)^{m-1}$. As $\gamma^- \in \mathbb{N}^{m-1}$ determines $\gamma \in \Gamma$ we may write h_{γ^-} for h_γ . Consider the system of equations

$$h_{\gamma^-} = - \sum_{\delta^- \geq_{cp} \gamma^-} \binom{\delta^-}{\gamma^-} t^{\delta^- - \gamma^-} g_{\delta^-}, \quad \gamma^- \in T^-,$$

with unknowns $g_{\delta^-} = g_\delta$ and indices δ^- ranging in the equilateral simplex S^- in \mathbb{N}^{m-1} given by

$$S^- : |\delta^-| \leq w - \lceil \frac{q_m}{c} \rceil \text{ and } \delta^- \geq_{cp} (\lceil \frac{q_{m-1}}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil).$$

Thus S^- has side length $w - \lceil \frac{q}{c} \rceil$ and hence $(m-1)$ -dimensional volume $\frac{1}{(m-1)!} \cdot (w - \lceil \frac{q}{c} \rceil)^{m-1}$. The assumption $|\bar{r}^c| > (\phi_c(r) - 1) \cdot c$ is equivalent to

$$\lceil \frac{r}{c} \rceil \leq \lceil \frac{r}{c} \rceil,$$

which in turn is equivalent to

$$\lceil \frac{q}{c} \rceil \leq \lceil \frac{r}{c} \rceil + \lceil \frac{\ell}{c} \rceil.$$

Hence $T^- = U^-$ with U^- as in the lemma of the last section, taking $b = w - \lceil \frac{r}{c} \rceil$ and $\mu = (0, \dots, 0, \lceil \frac{q_1}{c} \rceil, \dots, \lceil \frac{q_1}{c} \rceil)$. The lemma implies together with $t_{m-1}, \dots, t_{j+1} \neq 0$ that the system

$$h_{\gamma^-} = - \sum_{\delta^- \geq_{cp} \gamma^-} \binom{\delta^-}{\gamma^-} t^{\delta^- - \gamma^-} g_{\delta^-}, \quad \gamma^- \in T^-,$$

admits solutions g_{δ^-} with $\delta^- \in S^-$. Set $f(x, y) = (x + \sum_{\delta \in S} g_\delta y^\delta)^c$ with $\delta = (\delta_m, \delta^-)$ satisfying $\delta^- \in S^-$. This polynomial is a weighted homogeneous c -th power of weighted degree e and with support in $Z(q)$, by definition of S^- . Moreover, as $t_m = 0$,

$$f(x, y + ty_m) = (x + \sum_{\delta \in S} g_\delta \cdot (y + ty_m)^\delta)^c =$$

$$\begin{aligned}
&= (x + \sum_{\delta \in S} g_\delta \cdot y_m^{\delta_m} \cdot (y^- + t^- y_m)^{\delta^-})^c = \\
&= (x + \sum_{\delta^- \in S^-} g_\delta \cdot y_m^{w-|\delta^-|} \cdot \sum_{\gamma \in \Gamma, \gamma^- \leq_{cp} \delta^-} \binom{\delta^-}{\gamma^-} \cdot (y^-)^{\gamma^-} \cdot (t^-)^{\delta^- - \gamma^-} \cdot y_m^{|\delta^- - \gamma^-|})^c = \\
&= (x + \sum_{\gamma \in \Gamma} \sum_{\delta^- \in S^-, \delta^- \geq_{cp} \gamma^-} g_\delta \cdot y_m^{w-|\gamma^-|} \cdot \binom{\delta^-}{\gamma^-} \cdot (y^-)^{\gamma^-} \cdot (t^-)^{\delta^- - \gamma^-})^c = \\
&= (x + \sum_{\gamma \in \Gamma} y^\gamma \cdot \sum_{\delta^- \in S^-, \delta^- \geq_{cp} \gamma^-} g_\delta \cdot \binom{\delta^-}{\gamma^-} \cdot (t^-)^{\delta^- - \gamma^-})^c = \\
&= (x - \sum_{\gamma \in T} y^\gamma \cdot h_\gamma + \sum_{\gamma \in \Gamma \setminus T} y^\gamma \cdot (\dots))^c
\end{aligned}$$

with some unspecified sum (\dots) . Observe that if $h_\gamma = 0$ for all $\gamma \in T$, then all $g_\delta = 0$. The equalities imply that

$$\begin{aligned}
\tilde{f}(x, y) &= f(x + \sum_{\gamma \in \Gamma} h_\gamma y^\gamma, y + t y_m) = \\
&= f(x + \sum_{\gamma \in T} h_\gamma y^\gamma, y + t y_m) + R(x, y) = \\
&= x^c + R(x, y),
\end{aligned}$$

where R is a polynomial with support outside $Y(r, \ell)$, by definition of T . Thus $\tilde{f} - x^c$ has zero coefficients in $Y(r, \ell)$. This proves assertion (1). Assertion (2) follows from assertion (1) and the description of the order of $\text{coeff}_V \tilde{f}$ in terms of $Y(r, \ell)$ given in section 10 on zwickels. Assertion (3) holds by replacing in r all components r_i but the maximal component r_j by $u_i = r_i - \bar{r}_i^c$. This new u satisfies the arithmetic condition and hence the described inequalities.

13. Persistence of hypersurfaces of weak maximal contact

We have already seen that in specific circumstances, the strict transform of a hypersurface V of weak maximal contact with an ideal K at a need no longer have weak maximal contact with the weak transform K^\vee of K at a point a' above a . We show that if the order of the divided coefficient ideal does not increase, weak maximal contact may neither persist under blowup.

Example 5. Let $f = x^3 + yz^2 \cdot ((y-z)^2 + z^3)$ be given in characteristic $p = 3$ with coefficient ideal $yz^2 \cdot ((y-z)^2 + z^3)$ in the hypersurface of weak maximal contact $V = \{x = 0\}$. The divided coefficient ideal is $((y-z)^2 + z^3)$ of order 2 (this is the secondary order of f). Then $f^+(x, y, z) = f(x, y+z, z)$ equals $f^+(x, y, z) = x^3 + (y+z)z^2 \cdot (y^2 + z^3) = x^3 + z^2 \cdot (y^2(y+z) + z^3(y+z))$ (recall that in the examples, the role of y and z is exchanged with respect to the notation used for (y_m, \dots, y_1)). Its strict transform under the monomial point blowup in the z -chart is

$$f' = x^3 + z^2 \cdot (y^2(y+1) + z(y+1)).$$

The secondary order with respect to $V^\vee = \{x = 0\}$ is 1. It is not maximal since the coordinate change $x \rightarrow x - z$ yields

$$\tilde{f}' = x^3 + z^2 \cdot (y^2(y+1) + zy)$$

of secondary order 2. Thus the weak maximal contact of V with K does not persist at a' when passing to the transforms $V' = V^{st}$ and $K' = K^\vee$.

In figure 11, we see that in order to maximize o' the dotted points in the polygon on the right hand side have to be eliminated by a coordinate change in x . This may be also achieved before blowup by eliminating the dotted points in the polygon on the left hand side by a coordinate change in x , but this has no effect on the order o . Thus, though $V = \{x = 0\}$ has weak maximal contact, the maximality of o' after blowup is not necessarily realized by V^{st} .

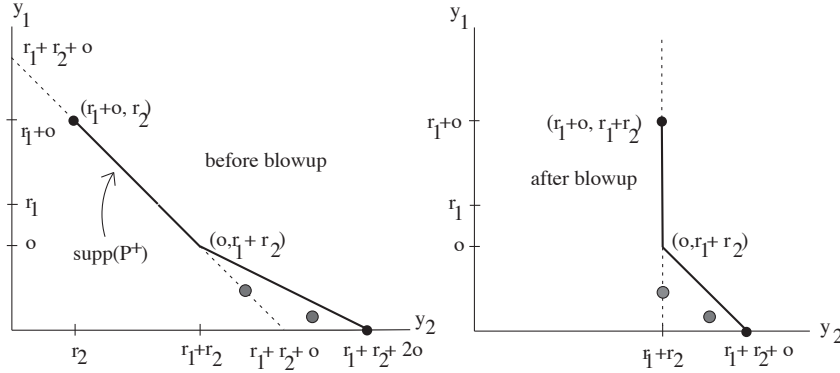


Figure 11. Newton polygon before and after blowup.

Observe that in the case where the secondary order drops, the persistence of V is not needed, since induction on the first two components of the invariant suffices. But if $o' = o$, further components of the invariant defined through subsequent coefficient ideals have to be compared, and this is only possible if the respective coefficient ideals are transforms of each other inside hypersurfaces of V and V^{st} .

IV. OUTLOOK

14. Moh's upper bound for the increase of the secondary order

We reproduce the proof of Moh's theorem from [Mo 1] showing that under blowup of a polynomial $f = x^c + y^q \cdot g$ of order $c = p^b$ the order o of g at any point of the exceptional divisor where c remains constant can increase at most by p^{b-1} .

Theorem 3. *Let x and $y = (y_m, \dots, y_1)$ be variables, and consider a polynomial $f = x^{p^b} + y^q \cdot g(y)$ modulo p^b -th powers in y , where p is the characteristic of the ground field, $b \geq 1$ and $q \in \mathbb{N}^m$. Let $\text{ord}_y^{p^b} g = \text{vol}^{p^b} f$ be the order of g after elimination of all p^b -th powers. Assume that $|q| + \text{ord}_y^{p^b} g \geq p^b$, and that $y^q \cdot g(y)$ is not a p^b -th power. Fix some vector $t = (0, t_{m-1}, \dots, t_1)$ with non-zero components in the ground field, and set $f^+ = x^{p^b} + (y + ty_m)^q \cdot g(y + ty_m)$, where $(y + ty_m)^q \cdot g(y + ty_m)$ is again considered modulo p^b -th powers. Set $z = (y_{m-1}, \dots, y_1)$. Then*

$$\text{height}_z^{p^b} f^+ \leq \text{vol}^{p^b} f + p^d,$$

where $d \leq b - 1$ is maximal such that the tangent cone of $y^q \cdot g(y)$ is a p^d -th power.

Thus the order o of g modulo p^b -th powers in y can increase under blowup at most by p^{b-1} at points in the exceptional divisor where the order of f remains constant, say

$$o' \leq o + p^{b-1}.$$

In particular, for $b = 1$, we have $o' \leq o + 1$. Here, the center of blowup is assumed to be regular and contained in the top locus of f , and coordinates are chosen as in the lemma on blowups in local coordinates of section 7.

Observe here that if the equality $\text{height}_z^{p^b} f^+ = \text{vol}^{p^b} f + p^{b-1}$ occurs, the tangent cone of $y^q \cdot g(y)$ is unique up to p^b -th powers, by Theorem 1 of section 5 (apply it first to the case $b = 1$ and take then p^{b-1} -st powers).

Proof. Let us treat first the case $b = 1$, say $d = 0$, in which case the tangent cone P of $y^q \cdot g(y)$ is not a p -th power. Set $P^+ = P(y + ty_m)$, $t^- = (t_l, \dots, t_1)$ and $q^- = (q_l, \dots, q_1)$ with $l = m - 1$. Let $o = \text{vol}^p f = \text{ord}_y P - |q| = \deg P - |q|$ and $u = \text{height}_z^p f^+ = \text{ord}_z P^+$, both orders taken up to p -th powers.

From P divisible by y^q follows that P^+ belongs to the ideal $y_m^{q_m} \cdot \langle z + t^- y_m \rangle^{|q^-|}$, which implies that

$$\partial_{z_i} P^+ \in y_m^{q_m} \cdot \langle z + t^- y_m \rangle^{|q^-|-1}$$

for all i between l and 1. From $u = \text{height}_z^p P^+$ follows that we can write P^+ up to p -th powers as

$$P^+(y_m, z) = \sum_{|\alpha|=u} a_\alpha(y_m, z) \cdot z^\alpha,$$

where for at least one $\alpha \in \mathbb{N}^l$ with $|\alpha| = u$ we have $a_\alpha(0) \neq 0$ and α is not a multiple of p . Choose such an α . There is an i between l and 1 such that α_i is not a multiple of p . Fix such an i , for example $i = l$. Then $\partial_{z_i} P^+ \neq 0$ and $\partial_{z_i} P^+ \in \langle z \rangle^{|\alpha|-1}$. Combining both inclusions we get

$$\partial_{z_i} P^+ \in y_m^{q_m} \cdot \langle z + t^- y_m \rangle^{|q^-|-1} \cap \langle z \rangle^{u-1} = y_m^{q_m} \cdot \langle z + t^- y_m \rangle^{|q^-|-1} \cdot \langle z \rangle^{u-1}.$$

The last equality holds because $t_i \neq 0$ for all i . The order of P^+ with respect to y is $|q| + o$, and as $\partial_{z_i} P^+ \neq 0$ we can conclude that

$$q_m + |q^-| - 1 + u - 1 \leq |q| + o - 1,$$

say

$$u \leq o + 1 \quad \text{or} \quad \text{height}_z^p f^+ \leq \text{vol}^p f + 1.$$

This settles the case $b = 1$. For arbitrary b , let $d \leq b - 1$ be maximal so that P and P^+ are p^d -th powers. The preceding argument applied to the p^d -th root Q^+ of P^+ yields $\text{height}_z^{p^b} Q^+ \leq \text{vol}^{p^b} Q + 1$, hence $\text{height}_z^{p^b} P^+ \leq \text{vol}^{p^b} P + p^d$, say $\text{height}_z^{p^b} f^+ \leq \text{vol}^{p^b} f + p^d$.

Remark 1. Let us assume $b = 1$ and $m = 2$, say $f = x^p + y^r z^s \cdot g(y, z)$ with $\text{ord } g = o$. Let g be homogeneous. Then, up to permutation of y and z , y^{r-1} and $(y - z)^o$ divide $\partial_y [y^r z^s \cdot g(y, z)]$, hence, by comparison of degrees and up to units

$$\partial_y [y^r z^s \cdot g(y, z)] = y^{r-1} z^s \cdot (y - z)^o.$$

This identity is in particular fulfilled by the examples in the table of section 17. Similar formulas should hold in several variables.

Remark 2. If the reasoning of the proof would go through considering $y^q \cdot g(y)$ instead of its tangent cone $P(y)$ we would get that

$$\partial_{z_i} (y + ty_m)^q \cdot g(y + ty_m) \in y_m^{q_m} \cdot \langle z + t^- y_m \rangle^{|q^-|-1} \cdot \langle z \rangle^{|\alpha|-1}.$$

In particular, the ideal of $\mathbb{K}[x, y]$ generated by $x + h(y)$ (with suitable p -th powers $h(y)$) and $\langle z - t^- y_m \rangle$ would give a permissible center for f as long as $u = o + 1$ and $o \geq \text{ord } f - 1 = c - 1$. Compare this with observations 1 and 2 from section 17.

15. Comparison with the Bernstein-Kushnirenko theorem

We show how the inequalities between $\text{height}_z^p(f^+)$ and $\text{vol}^p(f)$ are related to Bernstein-Kushnirenko theorem on the number of isolated zeros of systems of polynomial equations and the Minkowski mixed volume of the associated convex polytopes.

Let K_1, \dots, K_n be convex integer polytopes (= convex hull of finitely many points in \mathbb{N}^n) in \mathbb{R}^n . Assume given n complex polynomials P_1, \dots, P_n in n variables with support in K_1, \dots, K_n . For sufficiently generic coefficients the common zero-set of the polynomials is finite (we allow that some zeros appear with multiplicities). The following result was proven by Bernstein and Kushnirenko [Be, Kh1, Kh2, Kh3, Ku1, Ku2, HS, Ro and RW].

Theorem 4. *The number of isolated zeros of n complex polynomials P_1, \dots, P_n in the torus $(\mathbb{C}^*)^n$ is bounded from above by the mixed volume $\text{MV}(K_1, \dots, K_n)$ of K_1, \dots, K_n . Equality holds if all zeros are isolated, counted with multiplicity, and if the K_i are the convex hulls of the supports of the P_i .*

The mixed volume of K_1, \dots, K_n is defined as $\text{MV}(K_1, \dots, K_n) = \text{vol}_n(K_1 + \dots + K_n) - \sum_{i=1}^n \text{vol}_n(K_1 + \dots + \hat{K}_i + \dots + K_n) + \dots + (-1)^{n-1} \sum_{i=1}^n \text{vol}_n(K_i)$, where vol_n denotes the euclidean volume in \mathbb{R}^n (see figure 12).

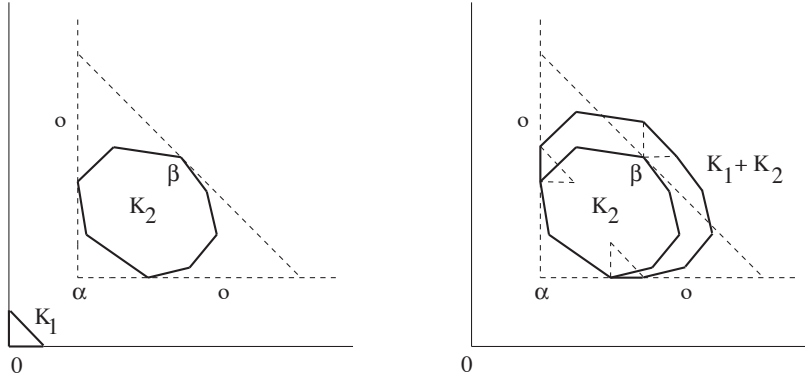


Figure 12. Mixed volume of two polytopes in \mathbb{R}^2 .

Example 6. In one variable, the number of non-zero roots of a polynomial is given by the difference of the degree of the polynomial and the order of its Taylor expansion at zero. This difference is just the volume of the convex hull of its support.

Example 7. The circle $P_1 : x^2 + y^2 = 25$ and the straight line $P_2 : 3x + 4y = 25$ meet tangentially in the point $(3, 4)$, which is a double point of the intersection. The associated polytopes are $K_1 = \text{conv}\{(0, 0), (2, 0), (0, 2)\}$ and $K_2 = \text{conv}\{(0, 0), (1, 0), (0, 1)\}$. We have $K_1 + K_2 = \text{conv}\{(0, 0), (3, 0), (0, 3)\}$. Therefore $\text{MV} = \text{vol}(K_1 + K_2) - \text{vol}(K_1) - \text{vol}(K_2) = 9/2 - 4/2 - 1/2 = 2$ as asserted.

Example 8. Let be given $n - 1$ quadratic homogeneus equations $P_i(x) = 0$ and one inhomogeneous linear equation $P_n(x) = 0$. If the common zeros are isolated, their number is 2^{n-1} . Indeed, the quadratic equations have identical polytopes $K = 2 \cdot S$ where S is the $(n - 1)$ -simplex spanned by the standard basis e_1, \dots, e_n of \mathbb{N}^n , and the polytope L of the linear polynomial P_n is spanned by the origin 0 of \mathbb{N}^n and e_1, \dots, e_n . The mixed volume $\text{MV}(K, \dots, K, L)$ equals by multilinearity the product $2^{n-1} \cdot \text{MV}(S, \dots, S, L)$. Using the Bernstein-Kushnirenko theorem, $\text{MV}(S, \dots, S, L)$ equals the number of isolated

zeros of one inhomogeneous linear and $n - 1$ homogeneous linear equations. This is 1, hence $MV(K, \dots, K, L) = 2^{n-1}$.

Example 9. Consider now a hypersurface X in \mathbb{C}^n given by the polynomial equation $P(x) = 0$. We wish to determine or bound from above the order of X at a point $a \in (\mathbb{C}^*)^n$. The first possibility is to compute the Taylor expansion of P at a by expanding $P(x + a)$ binomially and taking the order of the resulting series in x . This order is the order of X at a . Another way to determine the order of X at a is to take a sufficiently generic section of X with a straight line through a and to compute the multiplicity of the intersection point. For this we can use the Bernstein-Kushnirenko theorem. The straight line is given by $n - 1$ linearly independent equations P_1, \dots, P_{n-1} . They can be chosen non-homogeneous because $a \in (\mathbb{C}^*)^n$, and so that their polytopes K_1, \dots, K_{n-1} are spanned by $0 \in \mathbb{N}^n$ and e_1, \dots, e_n . It then suffices to compute the mixed volume of K_1, \dots, K_{n-1} and K_n , where K_n is the polytope associated to P .

Let's do this in two variables, $X : P(y, z) = 0$. Let K_1 be the 2-simplex spanned by $0, e_1$ and e_2 , and let $K_2 \subset \mathbb{N}^2$ be an arbitrary convex polytope. Let $Q \subset \mathbb{N}^2$ be the smallest quadrant $\alpha + \mathbb{N}^2$ containing K_2 , and set $o = \max_{\beta \in K_2} |\beta| - |\alpha|$. That the mixed volume of K_1 and K_2 equals o can be seen from figure 12.

Homogenizing the polynomial $P(y, z)$ we get a homogeneous polynomial $\bar{P}(y, z, w)$ of degree $|\alpha| + o$. The monomial $y^{\alpha_1} z^{\alpha_2}$ can be factored from \bar{P} , yielding a homogeneous polynomial of degree o . This is just the volume $\text{vol}(\bar{P})$ of \bar{P} as defined in earlier sections. Setting $\bar{P}^+(y, z, w) = \bar{P}(y, z + ty, w + t'y)$ with non-zero t, t' in the ground field, we conclude by the above that the (z, w) -order of \bar{P}^+ is bounded from above by o . Hence $\text{height}_{z, w} \bar{P}^+ \leq o$. This coincides with our results on transformation matrices (see Theorem 2 in section 12 in case $h = 0$).

It can be expected that similar assertions hold in higher dimensions. Already in three variables it is more complicated though feasible to compare the mixed volume $MV(K_1, K_1, K_2)$ – with K_1 spanned by 0 and e_1, \dots, e_3 in \mathbb{N}^3 and $K_2 \subset \mathbb{N}^3$ arbitrary – with the degree of the polynomial P minus the degree of the largest monomial which can be factored from P .

16. Description of tangent cone for increase of secondary order

We place ourselves again in dimension 3. Let $f(x, y, z) = x^c + y^r z^s \cdot g(y, z)$ with $c = p$ be a weighted homogeneous polynomial, with g homogeneous of degree k (the case $c = p^b$ with $b \geq 2$ should be treated analogously). Consider also $\tilde{f}(x, y, z) = f(x + h(y, z), y + tz, z)$ as earlier, for some given homogeneous polynomial $h(y, z)$ and a constant t . We assume that the ground field is perfect. Set $P(y, z) = y^r z^s \cdot g(y, z)$ and $P^+(y, z) = P(y + tz, z)$. We have by definition $\text{vol}(f) = \text{vol}(P)$ and $\text{height}_y(\tilde{f}) = \text{height}_y(P^+)$ provided all p -th powers were deleted from P and P^+ . Instead of eliminating these powers, we let again $\text{vol}^p(P)$ and $\text{height}_y^p(P^+)$ be the corresponding values after elimination, i.e., working in $\mathbb{K}[y, z]/\mathbb{K}[y^p, z^p]$. By Moh's result we then have

$$\text{height}_y^p(P^+) \leq \text{vol}^p(P) + 1.$$

We shall investigate in this section the special form P must have in order to produce the equality

$$\text{height}_y^p(P^+) = \text{vol}^p(P) + 1.$$

If the expansion of f has other x -powers of x -exponent not divisible by p these cannot be altered by coordinate changes in x due to characteristic p . Thus if such terms really

appear, $\text{height}_y^p(\tilde{f}) \leq \text{vol}^p(f)$ must hold. Therefore we may restrict to the case where $f = x^c + P(y, z)$. The case of higher dimension, though interesting, will be postponed.

The invertibility of the transformation matrix between the coefficients of f and \tilde{f} implies that $\text{height}_y^p(P^+) = \text{vol}^p(P) + 1$ can only hold if the following conditions are satisfied (cf. Theorem 2 of section 12)

- (1) the degree $r + s + k$ of P is a multiple of p ,
- (2) $\bar{r}^p + \bar{s}^p \leq p$ for the residues \bar{r}^p and \bar{s}^p of r and s modulo p .

In addition, whenever p does not divide $\binom{k+r}{k+1}$, the polynomial P is of the following form,

$$(3) P(y, z) = y^r z^s \cdot \mathbb{H}_r^k(y, tz - y) = y^r z^s \cdot \sum_{i=0}^k \binom{k+r}{i+r} y^i (tz - y)^{k-i},$$

modulo p -th powers and up to scalar multiplication of the variables, where $\mathbb{H}_r^k(y, w)$ is a *hybrid polynomial* defined as

$$\mathbb{H}_r^k(y, w) = \sum_{i=0}^k \binom{k+r}{i+r} y^i w^{k-i}.$$

Similar formulas should hold in higher dimension. In case where p divides $\binom{k+r}{k+1}$ the polynomial $y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ is a p -th power and thus does not give a candidate for $\text{height}_y^p(P^+) = \text{vol}^p(P) + 1$. We are indebted to R. Blanco for detecting at this place an inaccuracy in an earlier draft of this paper. For instance, taking $p = k = 2$ and $r = s = 3$ the polynomial P with $\text{height}_y^p(P^+) = \text{vol}^p(P) + 1$ equals $P = y^3 z^3 (y^2 + z^2)$ whereas $y^3 z^3 \cdot \mathbb{H}_3^2(y, z - y) = y^4 z^4$ yields a square.

There is an alternative description of polynomials P with $\text{height}_y^p(P^+) = \text{vol}^p(P) + 1$. The partial derivative $\partial_y P$ will eliminate all p -th powers (recall that $r + s + k$ is assumed to be divisible by p by (1)). As derivation with respect to y commutes with the coordinate change $(y, z) \rightarrow (y + tz, z)$, we have $(\partial_y P)^+ = \partial_y(P^+)$. Observe now that $\partial_y P$ has the same volume k as P , and that in the derivatives it is no longer necessary to neglect p -th powers. In order that $(\partial_y P)^+$ has height k it is then immediate that $\partial_y P$ must be of the form $y^{r-1} z^s (y - tz)^k$ (dehomogenize P , derive and use the fundamental theorem of algebra). Therefore, up to p -th powers and independently of whether p divides $\binom{k+r}{k+1}$ or not, P can be computed as the y -integral

$$(3') P(y, z) = \int y^{r-1} z^s (y - tz)^k dy.$$

To prove (3), recall that we have already seen in Theorem 1 of section 5 that for f as above and for given r, s, k and t the polynomial $P(y, z)$ is uniquely determined up to p -th powers by the equality $\text{height}_y^p(P^+) = \text{vol}^p(P) + 1$. To prove the formula from above it therefore suffices to show that $Q = y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ satisfies $\text{height}_y^p(Q^+) = \text{vol}^p(Q) + 1$ (note that $\text{height}_y^p(Q^+) \leq k + 1$ will imply that Q^+ and hence Q is not a p -th power). By definition, Q has volume $\leq k$. We will show that $Q^+(y, z) = (y + tz)^r z^s \cdot \mathbb{H}_r^k(y + tz, -y)$ has modulo p -th powers order $\geq k + 1$ with respect to y . Computation gives

$$\begin{aligned} Q^+(y, z) &= (y + tz)^r z^s \cdot \mathbb{H}_r^k(y + tz, -y) = \\ &= (y + tz)^r z^s \cdot \sum_{i=0}^k \binom{k+r}{i+r} (y + tz)^i (-y)^{k-i} = \\ &= z^s \cdot \sum_{i=0}^k \binom{k+r}{i+r} (y + tz)^{i+r} (-y)^{k-i} = \\ &= z^s \cdot \sum_{i=r}^{k+r} \binom{k+r}{i} (y + tz)^i (-y)^{k+r-i} = \\ &= z^s \cdot \sum_{i=0}^{k+r} \binom{k+r}{i} (y + tz)^i (-y)^{k+r-i} - z^s \cdot \sum_{i=0}^{r-1} \binom{k+r}{i} (y + tz)^i (-y)^{k+r-i} = \\ &= z^s (y + tz - y)^{k+r} - (-y)^{k+1} z^s \cdot \sum_{i=0}^{r-1} \binom{k+r}{i} (y + tz)^i (-y)^{r-1-i} = \\ &= z^s (y + tz - y)^{k+r} - (-y)^{k+1} z^s \cdot \sum_{i=0}^{r-1} \binom{k+r}{r-1-i} (-y)^i (y + tz)^{r-1-i} = \end{aligned}$$

$$\begin{aligned}
&= z^s (y + tz - y)^{k+r} - (-y)^{k+1} z^s \cdot \sum_{i=0}^{r-1} \binom{k+r}{i+k+1} (-y)^i (y + tz)^{r-1-i} = \\
&= t^{k+r} z^{r+s+k} - (-y)^{k+1} z^s \cdot \mathbb{H}_{k+1}^{r-1}(-y, y + tz).
\end{aligned}$$

As $\mathbb{H}_{k+1}^{r-1}(-y, y + tz)$ has y -order 0 if and only if its coefficient $\binom{k+r}{k+1}$ at z^k is not divisible by p we get assertion (3).

The above computations with hybrid polynomials give a certain duality between $(y + tz)^r z^s \cdot \mathbb{H}_r^k(y + tz, -y)$ and $(-y)^{k+1} z^s \cdot \mathbb{H}_{k+1}^{r-1}(-y, y + tz)$. A similar relation, exchanging y and z , should hold between $\mathbb{H}_s^k(-y, z + ty)$ and $\mathbb{H}_{k+1}^{s-1}(z + ty, -y)$.

To describe $P(y, z) = y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ more closely we consider for $k \in \mathbb{N}$ and variables y and w the binomial expansion of $(y + w)^{k+r}$, say

$$(y + w)^{k+r} = \sum_{i=0}^{k+r} \binom{k+r}{i} y^i w^{k+r-i} = \sum_{i=0}^{r-1} \binom{k+r}{i} y^i w^{k+r-i} + \sum_{i=r}^{k+r} \binom{k+r}{i} y^i w^{k+r-i}.$$

The second summand is divisible by y^r . Dividing it by y^r we get

$$\mathbb{H}_r^k(y, w) = \sum_{i=r}^{k+r} \binom{k+r}{i} y^{i-r} w^{k+r-i} = \sum_{i=0}^k \binom{k+r}{i+r} y^i w^{k-i}.$$

We call this sum the polynomial part of $y^{-r} \cdot (y + w)^{k+r}$. It is a homogeneous polynomial of degree k . Let $s \geq 0$ be such that $r + s + k$ is divisible by the characteristic p . We claim that if $\bar{r}^p + \bar{s}^p > p$ then $y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ is a p -th power. This shows why for $\bar{r}^p + \bar{s}^p > p$ the equality $\text{height}_y^p(f) = \text{vol}^p(f) + 1$ cannot occur. If the homogeneous part of the coefficient ideal is a p -th power, this order can be increased by a coordinate change in x . Moreover, we shall show that if $k = mp + \ell$ for some $m \geq 0$ and $\ell \geq 0$, we have modulo p -th powers the equality

$$(y + w)^r w^s \cdot \mathbb{H}_r^k(y, w) \equiv (y + w)^r w^s \cdot w^{mp} \cdot \mathbb{H}_r^\ell(y, w).$$

A direct proof of both assertions is pending. However we can use Proposition 1 of section 11: The non-existence of polynomials P modulo p -th powers with $\text{height}_y(P^+) = \text{vol}(P) + 1$ if $\bar{r}^p + \bar{s}^p > p$ implies that $y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ must be a p -th power. The uniqueness of P , and the fact that both $(y + w)^r w^s \cdot \mathbb{H}_r^k(y, w)$ and $(y + w)^r w^s \cdot w^{mp} \cdot \mathbb{H}_r^{\ell+r}(y, w)$ allow the increase of the secondary order when replacing w by $tz - y$ implies the above equality.

As g has degree k , the polynomial $P(y, z) = y^r z^s \cdot g(y, z)$ has volume $\leq k$ (i.e., the length of the convex hull of the support of g is k). After the substitution $y \rightarrow y + tz$ in $y^r z^s \cdot g(y, z)$ the preceding computation shows that if p divides $r + s + k$, then the y -order modulo p -th powers of the resulting polynomial $P^+(y, z) = P(y + tz, z)$ equals $k + 1$. Thus, modulo p -th powers, $\text{height}_y(P^+) = \text{vol}(P) + 1$.

Setting $k = mp + \ell$ with $m \geq 0$ and $0 \leq \ell < p$ we may also write modulo p -th powers

$$\begin{aligned}
y^r z^s \cdot \mathbb{H}_r^k(y, tz - y) &= y^r z^s \cdot (y - tz)^{mp} \cdot \mathbb{H}_r^{\ell+r}(y, tz - y) = \\
&= y^r z^s \cdot (y - tz)^{mp} \cdot \sum_{i=0}^{\ell+r} \binom{\ell+r}{i+r} y^i (tz - y)^{\ell-i} = \\
&= y^r z^s \cdot (y - tz)^{mp} \cdot \sum_{i=0}^{\ell} \sum_{j=0}^{\ell-i} \binom{\ell+r}{i+r} \binom{\ell-i}{j} y^{\ell-j} (tz)^j.
\end{aligned}$$

The expansion of $\mathbb{H}_r^k(y, tz - y)$ as a polynomial in y and z also equals

$$\mathbb{H}_r^k(y, tz - y) = \sum_{i=0}^k (-1)^i \binom{k+r}{k-i} \binom{i+r-1}{i} t^{k-i} y^i z^{k-i}.$$

This can be seen by expanding $\mathbb{H}_r^k(y, tz - y) = \sum_{i=0}^k \binom{k+r}{i+r} y^i (tz - y)^{k-i}$ binomially, and using the formulas

$$\begin{aligned}
\binom{k-i+j}{k-i} \binom{k+r}{k-i+j} &= \binom{k+r}{k-i} \binom{i+r}{j}, \\
\sum_{j=0}^i (-1)^j \binom{i+r}{j} &= (-1)^i \binom{i+r-1}{i}
\end{aligned}$$

from [Ri, p. 3, GKP, p. 168 and p. 165]. The coefficient of $t^{k-i} y^i z^{k-i}$ thus equals

$$\begin{aligned}
& \sum_{j=0}^i (-1)^j \binom{k-i+j}{k-i} \binom{k+r}{k-i+j} = \\
& = \sum_{j=0}^i (-1)^j \binom{k+r}{k-i} \binom{i+r}{j} = \\
& = \binom{k+r}{k-i} \sum_{j=0}^i (-1)^j \binom{i+r}{j} = \\
& = \binom{k+r}{k-i} (-1)^i \binom{i+r-1}{i} = \\
& = (-1)^i \binom{k+r}{k-i} \binom{i+r-1}{i}.
\end{aligned}$$

This leaves us with the question whether there is some intrinsic description of polynomials of form $\mathbb{H}_r^k(y, tz - y)$, for instance as a derivative of certain polynomials.

The computation also shows after dehomogenization by setting $z = 1$ that $\mathbb{H}_r^k(y + t, t) = \sum_{i=0}^k \binom{k+r}{i+r} (y+t)^i (-y)^{k-i}$ equals the expansion of $t^{k+r} (t+y)^{-r}$ truncated at degree k in y . This implies that

$$\mathbb{H}_r^k(y, w) = \sum_{i=0}^k \binom{-r}{i} (y+w)^{k-i} (-w)^i,$$

hence

$$P(y, z) = y^r z^s \cdot \mathbb{H}_r^k(y, tz - y) = y^r z^s \cdot \sum_{i=0}^k \binom{-r}{i} (y - tz)^i (tz)^{k-i}.$$

This description of P as a truncated inverse should be the starting point to determine P in the case of three variables (take the truncation of $R^{k+|r|}(t_1, t_2) \cdot (t_1 + y_1)^{-r_1} (t_2 + y_2)^{-r_2}$ at $y_1 y_2$ -degree k , with R homogeneous of degree $k + |r|$).

17. Examples of hybrid polynomials

We will now give some examples for hybrid polynomials in two variables. We shall always consider two prior monomial blowups in \mathbb{A}^3 in different charts producing the two exceptional components y^r and z^s appearing in P .

Example 10. Take $f = x^3 + y^7 - yt^9 z^9$ in characteristic 3 with secondary order $o = 7$. Two monomial point blowups, first in the z -chart, then in the y -chart, yield

$$\begin{aligned}
f &= x^3 + y^7 - yt^9 z^9, \\
f^1 &= x^3 + y^7 z^4 - yt^3 z^7 = x^3 + z^4 \cdot (y^7 - t^3 z^3), \\
f^2 &= x^3 + y^8 z^4 - y^5 t^3 z^7 = x^3 + y^5 z^4 \cdot (y^3 - t^3 z^3).
\end{aligned}$$

Two further curve blowups with centers (x, y) and (x, z) allow to reduce the exceptional components to

$$f^3 = x^3 + y^2 z \cdot (y - tz)^3.$$

The secondary order of f^3 is 3. Applying now the translation $y \rightarrow y + tz$ followed by the monomial point blowup in the z -chart and the change $x \rightarrow x - yzt^{2/3}$ we get

$$\begin{aligned}
f^3 &= x^3 + (y + tz)^2 z \cdot y^3, \\
f^4 &= x^3 + z^3 \cdot y^3 (y + t)^2, \\
f^4 &= x^3 + z^3 \cdot y^3 (y^2 + 2ty).
\end{aligned}$$

The secondary order of f^4 is 4. However, if there are no higher order terms in $g^4 = y^3(y^2 + 2ty)$ we may reduce it to 1 by a curve blowup with permissible center (x, y) . But if g^4 is not homogeneous, the order of f^4 will in general not be constant along the z -axis defined by (x, y) , and thus this center is not permissible.

Example 11. Let $p = 11$, $k = 12$ and $r = 6$, $s = 4$. A sequence of a monomial point blowups in the z -chart, a monomial curve blowup in the z -chart with center (x, z) and a monomial point blowup in the y -chart yield

$$\begin{aligned}
f &= x^{11} + 1 \cdot yz(6y^2 - 7tz^3)(y^2 - tz^3)^{11}, \\
f^1 &= x^{11} + z^{15} \cdot y(6y^2 - 7tz)(y^2 - tz)^{11}, \\
f^2 &= x^{11} + z^4 \cdot y(6y^2 - 7tz)(y^2 - tz)^{11}, \\
f^3 &= x^{11} + y^6 z^4 \cdot (6y - 7tz)(y - tz)^{11}.
\end{aligned}$$

The secondary order is 12. The translation $y \rightarrow y + tz$ applied to f^3 gives

$$f^3 = x^{11} + (y + tz)^6 z^4 \cdot (6y - tz)y^{11}.$$

We apply to f^3 the monomial point blowup in the z -chart followed by the coordinate change $x \rightarrow x - yzt^{7/11}$ and obtain

$$\begin{aligned}
f^3 &= x^{11} + z^{11} \cdot (y + t)^6 (6y - t)y^{11} = \\
&= x^{11} + z^{11} \cdot (6y^7 + \dots + 36y^2 t^5 - t^7)y^{11}, \\
f^3 &= x^{11} + z^{11} \cdot (6y^7 + \dots + 36y^2 t^5)y^{11},
\end{aligned}$$

This last polynomial has secondary order 13.

Example 12. We have proven earlier that the order cannot increase if $\bar{r}^p + \bar{s}^p > p = c$ or if r or s are divisible by p . The resulting \mathbb{H}_r^k is identically zero modulo p -th powers. Take for instance $p = 3$, $r = s = 2$ and $k = 2$. Then $\bar{r}^p + \bar{s}^p = 2 + 2 > p = 3$, and

$$\begin{aligned}
P(y, z) &= y^r z^s \cdot \mathbb{H}_r^k(y, tz - y) = \\
&= y^2 z^2 \left(\binom{k+r}{r} (tz - y)^2 + \binom{k+r}{1+r} y(tz - y) + \binom{k+r}{2+r} y^2 \right) = \\
&= y^2 z^2 \left(\binom{4}{2} (tz - y)^2 + \binom{4}{3} y(tz - y) + \binom{4}{4} y^2 \right) = \\
&= y^2 z^2 (0(tz - y)^2 + 4y(tz - y) + y^2) = ty^3 z^3 \equiv 0
\end{aligned}$$

is identically zero modulo third powers. We leave it as an exercise to check this for all r , s and k with $\bar{r}^p + \bar{s}^p > p$ and $r + s + k$ divisible by p .

Example 13. For $p = 5$, $r = 4$, $s = 4$ and $k = 2$ we have $\bar{r}^p + \bar{s}^p = 8 > p$ and $r + s + k = 10 = 2p$. We get

$$\begin{aligned}
P(y, z) &= y^4 z^4 \cdot \mathbb{H}_4^2(y, tz - y) = \\
&= y^4 z^4 \left(\binom{k+r}{r} (tz - y)^2 + \binom{k+r}{1+r} y(tz - y) + \binom{k+r}{2+r} y^2 \right) = \\
&= y^4 z^4 \left(\binom{6}{4} (tz - y)^2 + \binom{6}{5} y(tz - y) + \binom{6}{6} y^2 \right) = \\
&= y^4 z^4 (0(tz - y)^2 + y(tz - y) + y^2) = \\
&= y^4 z^4 (0y^2 + tyz + 0t^2 z^2) = ty^5 z^5 \equiv 0.
\end{aligned}$$

Example 14. If one of r or s is divisible by p we get the following examples. Taking $p = 3$, $r = 0$, $s = 2$ and $k = 1$ gives $g(y, z) = z^2(y + (tz - y)) = tz^3 \equiv 0$. Similarly, $p = 3$, $r = 0$, $s = 1$ and $k = 2$ gives

$$\begin{aligned}
P(y, z) &= z(y^2 + 2y(tz - y) + (tz - y)^2) = \\
&= z(y^2 - 2y^2 + 2tyz + t^2 z^2 - 2tyz + y^2) = z(0y^2 + 0tyz + t^2 z^2) = t^2 z^3 \equiv 0.
\end{aligned}$$

Example 15. In contrast, if $\bar{r}^p + \bar{s}^p \leq p$, the polynomial $P(y, z) = y^r z^s \cdot \mathbb{H}_r^k(y, tz - y)$ need not be a p -th power. We list the following examples, none of which is congruent 0 modulo p -th powers. For simplicity, we take always $t = 1$.

(a) $p = 3$, $r = s = k = 1$:

$$P(y, z) = -yz \cdot (y + z).$$

(b) $p = 3, r = s = 1, k = 4$:

$$P(y, z) = yz \cdot (y^4 + y^3z + y^2z^2 - yz^3 - z^4),$$

which, by the coordinate change $y \rightarrow y + z$, transforms into

$$P^+(y, z) = z \cdot (y^5 + z^5).$$

(c) $p = 3, r = s = 1, k = 7$:

$$\begin{aligned} P(y, z) &= -yz \cdot (y^7 + y^6z + y^5z^2 + y^4z^3 + y^3z^4 + y^2z^5 + yz^6 + z^7) = \\ &= -yz \cdot (y^8 - z^8)/(y - z), \end{aligned}$$

which, by the coordinate change $y \rightarrow y + z$, transforms into

$$P^+(y, z) = -z \cdot (y^8 - z^8).$$

(d) $p = 3, r = 2, s = 1, k = 3$:

$$P(y, z) = y^2z \cdot (-y^3 + yz^2 + z^3),$$

which, by the coordinate change $y \rightarrow y + z$, transforms into

$$P^+(y, z) = z \cdot (-y^5 + y^4z + z^5).$$

(e) $p = 5, r = 1, s = 2, k = 2$:

$$P(y, z) = yz^2 \cdot (y^2 + 2yz + 3z^2).$$

(f) $p = 5, r = 2, s = 1, k = 2$: symmetric to (e), say

$$P(y, z) = y^2z \cdot (y^2 - yz + 2z^2).$$

(g) $p = 5, r = 1, s = 1, k = 3$:

$$\begin{aligned} P(y, z) &= -yz \cdot (y^3 + y^2z + yz^2 + z^3) = \\ &= -yz \cdot (y + z)(y + 2z)(y + 3z). \end{aligned}$$

Observation 1. It turns out that we may assume that the secondary order $o = k$ of $f = x^c + y^r z^s \cdot g$ is $\geq c = p$ whenever an increase of o occurs. Indeed, after the increase the exceptional factor is z^{k+r+s} . Write $k + r + s = mp$. Then m curve blowups with center (x, z) make the exponent drop to 0 without changing the rest of the polynomial f . Now, if $\text{ord } g$ would be $< c - 1$, then the transform of $y^r z^s \cdot g$ under the first point blowup and the following sequence of curve blowups would have, using again Moh's result, order $o + 1 < c$, making the order of the transform of f drop below c . This shows that we may restrict to the case $o \geq c - 1$. But if $o = c - 1$, then $r + s + o$ cannot be divisible by c since $0 < r + s \leq c$, and no increase would occur. Thus we may even assume that $o \geq c$.

Observation 2. If $f = x^c + y^r z^s \cdot g$ were really weighted homogeneous, and hence g homogeneous with $g = \mathbb{H}_r^k(y, tz - y)$, we could perform the coordinate change $y \rightarrow y + tz$ getting modulo p -th powers the polynomial (setting again $k = o$)

$$x^c + (-y)^{k+1} z^s \cdot \mathbb{H}_{k+1}^{r-1}(y, y + tz).$$

Here the curve $x = y = 0$ is permissible since $k \geq c$, and blowing it up makes the exponent $k + 1$ drop to $k + 1 - p$. This can be repeated until the exponent of $(-y)$ is less than p , leaving the rest of the polynomial unchanged. But as $\mathbb{H}_{k+1}^{r-1}(y, y + tz)$ has degree $r - 1 < p - 1$, blowing up the origin followed by curve blowups makes as in observation 1 the order of f drop below c .

Thus only the higher order terms of g make troubles, since then $x = y = 0$ need no longer be permissible. Possibly it is appropriate to weaken the notion of permissibility (still ensuring

that the order of f does not increase). Compare this with remark 2 at the end of section 14. Also note that the Newton polygon of g is a simplex.

Observation 3. Let V be a finite dimensional vector space of polynomials in two variables, for example the space of polynomials of degree $\leq d$ for a certain d . Consider for $f = x^c + y^r z^s \cdot g(y, z)$ with $g \in V$ the sequence of blowups given by the resolution invariant as in [EH]. Assume that the process does not terminate. Then there exists a valuation along which the increase of the secondary order o of f happens infinitely many times. As we have seen in this section, each increase imposes linear relations between the coefficients of the polynomial we started with (no relations only occur if $o = 1$, which is a case excluded by observation 1). Thus the set of polynomials in V for which the resolution process may not terminate is Zariski-closed. If the linear relations imposed on the coefficients of g by each increase are sufficiently independent (e.g., define a regular sequence of polynomials in $\mathbb{K}[V]$), then their common solution set would be empty, thus showing that for any $g \in V$ the increase of the secondary order of $f = x^c + g$ can occur only finitely many times. This would prove the existence of resolutions in a non-constructive way.

The definition of $\mathbb{H}_r^k(y, w)$ shows that at each stage of the resolution process where the secondary order increases the tangent cone is specified by prescribed coefficients. Linear coordinate changes correspond to multiplying the vector of all coefficients of monomials of the tangent cone by an invertible matrix of binomial coefficients. Thus the equations appear on the orbits of the action of this matrix in each degree. A monomial blowup alters the tangent cones by making them weighted homogeneous. These structures had to be made more precise to show that the resulting linear equations on the coefficients at the initial stage of the process are linearly independent.

Observation 4. For surfaces, the point a' above a where the order may increase, is unique, and determined by the coefficients of the tangent cone at a . This suggests that in higher dimensions it is always contained in a regular codimension 2 scheme. It is however not clear how to profit of this descent in dimension.

Observation 5. As mentioned in section 14, the derivative of hybrid polynomials with respect to one variable is a product of three linear polynomials. For instance, in the examples above, we have

$$\partial_y [y^r z^s \cdot \mathbb{H}_r^k(y, z - y)] = y^{r-1} z^s \cdot (y - z)^k.$$

Exercise. Construct a polynomial in three variables and a sequence of blowups where the increase of the secondary order occurs twice (or k -times). Determine the shape of hybrid polynomials in several variables.

Table of hybrid polynomials $P = \mathbb{H}_r^k(y, z - y) = \sum_{i=0}^k \binom{k+r}{i+r} y^i (z - y)^{k-i}$ modulo p

$[p, r, s, k]$	$P(y, z) \pmod{p}$
$[2, 1, 1, 2]$	$y^2 + z^2$ (but $\mathbb{H}_r^k(y, z - y) = yz$ differs, because $\binom{k+r}{k+1}$ is even),
$[3, 1, 1, 1]$	$2z - y,$
$[3, 1, 1, 4]$	$2z^4 + 2z^3y + z^2y^2 + zy^3 + y^4$
$[3, 1, 1, 7]$	$2y^7 + 2z^6y + 2z^5y^2 + 2z^4y^3 + 2z^3y^4 + 2z^2y^5 + 2zy^6 + 2z^7,$
$[3, 1, 2, 3]$	$z^3 + zy^2 + 2y^3,$
$[3, 1, 2, 6]$	$y^6 + 2z^4y^2 + z^3y^3 + 2zy^5 + z^6,$
$[3, 2, 2, 2]$	$zy, (\bar{r} + \bar{s} > p, \text{ hence } y^r z^s P(y, z) \text{ is } p\text{-th power}),$

[3, 2, 2, 5]	$2z^4y + 2zy^4, (\bar{r} + \bar{s} > p, \text{ hence } y^r z^s P(y, z) \text{ is } p\text{-th power}),$
[5, 1, 1, 3]	$4z^3 + 4z^2y + 4zy^2 + 4y^3,$
[5, 1, 1, 8]	$y^8 + 4z^7y + 4z^6y^2 + 4z^5y^3 + z^4y^4 + z^3y^5 + z^2y^6 + zy^7 + 4z^8,$
[5, 1, 2, 2]	$3z^2 + 2zy + y^2,$
[5, 1, 2, 7]	$4y^7 + 3z^7 + 2z^6y + z^5y^2 + z^3y^4 + 2z^2y^5 + 3zy^6,$
[5, 1, 3, 1]	$2z - y,$
[5, 1, 3, 6]	$y^6 + 2z^6 + 4z^5y + z^2y^4 + 3zy^5,$
[5, 1, 4, 5]	$z^5 + zy^4 + 4y^5,$
[5, 2, 2, 1]	$3z + 3y,$
[5, 2, 2, 6]	$2y^6 + 3z^6 + 3z^5y + z^3y^3 + 2zy^5,$
[5, 2, 3, 5]	$z^5 + z^2y^3 + 4y^5,$
[5, 3, 3, 4]	$z^2y^2, (\bar{r} + \bar{s} > p, \text{ hence } y^r z^s P(y, z) \text{ is } p\text{-th power})$
[5, 4, 4, 2]	$zy, (\bar{r} + \bar{s} > p, \text{ hence } y^r z^s P(y, z) \text{ is } p\text{-th power}),$
[7, 1, 1, 5]	$6z^5 + 6z^4y + 6z^3y^2 + 6z^2y^3 + 6zy^4 + 6y^5,$
[7, 1, 2, 4]	$5z^4 + 4z^3y + 3z^2y^2 + 2zy^3 + y^4,$
[7, 1, 3, 3]	$4z^3 + z^2y + 4zy^2 + 6y^3,$
[7, 1, 4, 2]	$3z^2 + 4zy + y^2,$
[7, 1, 5, 1]	$2z + 6y,$
[7, 2, 2, 3]	$3z^3 + z^2y + zy^2 + 3y^3,$
[7, 2, 3, 2]	$6z^2 + 6zy + 3y^2,$
[7, 2, 4, 1]	$3z + 5y,$
[7, 3, 3, 1]	$4z + 4y,$

18. Quings

The failure of commutativity in characteristic $p > 0$ can be reformulated as an independent problem on equivalence classes of polynomials modulo p -th powers.

A *quing* is the quotient of a polynomial ring $Q = \mathbb{K}[x]_p := \mathbb{K}[x]/\mathbb{K}[x]^p$ in n variables $x = (x_1, \dots, x_n)$ over a field \mathbb{K} of characteristic p (setting $p = \infty$ if the characteristic is zero). Its elements will be called *qu'nomials*. We have $\mathbb{K}[x]/\mathbb{K}[x]^p = \mathbb{K}[x]/\mathbb{K}[x^p]$ where $\mathbb{K}[x^p]$ denotes the subring generated by all p -th powers of the variables. In this case, denoting by L the sublattice $p \cdot \mathbb{Z}^n$ of \mathbb{Z}^n , we have $Q = \mathbb{K}[x]_p = \mathbb{K}[\mathbb{Z} \setminus L]$, i.e., each qu'nomial in Q has support in $\mathbb{Z} \setminus L$.

Clearly, quings are abelian groups but not rings. A qu'hypersurface in affine space \mathbb{A}^n is defined as a non-zero qu'nomial. The first objective should be to prove the resolution of plane qu'curves. A plane qu'curve is given by a qu'nomial in two variables, i.e., the equivalence class in $\mathbb{K}[x, y]/\mathbb{K}[x^p, y^p]$ of a polynomial in $\mathbb{K}[x, y]$.

The main problem here is to develop a reasonable concept of the order of qu'nomials (more precisely, of products of monomials with qu'nomials). This order shall be intrinsic

(i.e., invariant under coordinate changes and under multiplication by invertible power series), upper semicontinuous and shall not increase under blowup in permissible centers.

We have seen that examples of type $f = y^r z^s \cdot g(y, z)$ of degree a power of p with g as before of order k are essentially the unique qu'nomials in two variables where the definition of the order causes problems. This is due to the fact that when applying to f the coordinate change $y \rightarrow y + z$ we get $\tilde{f} = z^{r+s+k} + h(y, z)$ where h consists of monomials of degree $\geq k + 1$ in y . As $r + s + k$ is a multiple of p , \tilde{f} is equivalent to h . Defining the order of f as k (the monomial factor is deduced as is done when passing from total to weak transforms under blowups), the Bernstein-Kushnirenko theorem would say in this case that the y -order of \tilde{f} is bounded from above by k . This does not hold here, since the y -order of h is $\geq k + 1$ (actually, equal to $k + 1$ by Moh's result or direct verification).

The example is rather special, and its coefficients are quite unique in order to allow the phenomenon to take place. To see this it suffices to take a homogeneous polynomial f of degree $r + s + k$ with unknown coefficients and monomial factor $y^r z^s$ and to apply a linear coordinate change $y \rightarrow y + tz$ such that the resulting \tilde{f} has y -order $\geq k + 1$ modulo p -th powers. You will fall on $f = y^r z^s \cdot g(y, z)$ of degree a multiple of p and with g of the special form from above.

The task for resolution of plane qu'curves in this vein would therefore be to detect these hybrid qu'nomials and either to treat them separately, or to define a new concept of order where these examples have order $\geq k + 1$, but the order still does not increase under blowup. Or, alternatively, to show that in an infinite sequence of point blowups, the homogeneous forms of lowest degrees of the transforms of the qu'nomial to be resolved can only be finitely many times hybrid.

A satisfactory theory of quings and a concept of order of qu'nomials cannot be anticipated yet.

References

- [Ab 1] Abhyankar, S.: Desingularization of plane curves. In: Summer Institute on Algebraic Geometry. Arcata 1981, Proc. Symp. Pure Appl. Math. **40** Amer. Math. Soc.
- [Ab 2] Abhyankar, S.: Good points of a hypersurface. Adv. Math. **68** (1988), 87-256.
- [AHV] Aroca, J.-M., Hironaka, H., Vicente, J.-L.: The theory of the maximal contact. Memorias Mat. Inst. Jorge Juan Madrid **29** (1975).
- [Be] Bernstein, D.N.: The number of roots of a system of equations. Funkt. Anal. Appl. **9** (1975), 183-185.
- [BM 1] Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. **128** (1998), 207-302.
- [BM 2] Bierstone, E., Milman, P.: Uniformization of analytic spaces. J. Amer. Math. Soc. **2** (1989), 801-836.
- [BM 3] Bierstone, E., Milman, P.: Semianalytic and subanalytic sets. Publ. Math. IHES **67** (1988), 5-42.
- [BS] Bodnár, G., Schicho, J.: Automated resolution of singularities for hypersurfaces. J. Symb. Comp. **30** (2000), 401-428.
- [BV] Bravo, A., Villamayor, O.: A strengthening of resolution of singularities in characteristic zero. To appear in Proc. London Math. Soc.

- [Co 1] Cossart, V.: Polyèdre caractéristique d'une singularité. Thèse d'Etat, Orsay 1987.
- [Co 2] Cossart, V.: Contact maximal en caractéristique positive et petite multiplicité. *Duke Math. J.* **63** (1991), 57-64.
- [EH] Encinas, S., Hauser, H.: Strong resolution of singularities in characteristic zero. *Comment. Math. Helv.* **77** (2002), 421-445.
- [EV 1] Encinas, S., Villamayor, O.: Good points and constructive resolution of singularities. *Acta Math.* **181** (1998), 109-158.
- [EV 2] Encinas, S., Villamayor, O.: A course on constructive desingularization and equivariance. In: *Resolution of Singularities* (ed. H. Hauser et al.), *Progress in Math.* **181**, Birkhäuser 2000.
- [Gi 1] Giraud, J.: Sur la théorie du contact maximal. *Math. Z.* **137** (1974), 285-310.
- [Gi 2] Giraud, J.: Contact maximal en caractéristique positive. *Ann. Scient. Ec. Norm. Sup. Paris* **8** (1975), 201-234.
- [GKP] Graham, R., Knuth, D., Patashnik, O.: *Concrete mathematics. A foundation for computer science.* Addison-Wesley 1994.
- [Ha 1] Hauser, H.: Seventeen obstacles for resolution of singularities. In: *Singularities* (ed.: V.I. Arnold, G.-M. Greuel, J. Steenbrink). *Progress in Math.* **162**, Birkhäuser 1998.
- [Ha 2] Hauser, H.: Excellent surfaces over a field and their taut resolution. In: *Resolution of Singularities* (ed. H. Hauser et al.), *Progress in Math.* **181**, Birkhäuser 2000.
- [Ha 3] Hauser, H.: The Hironaka Theorem on resolution of singularities. (Or: A proof that we always wanted to understand.) *Bull. Amer. Math. Soc.* **40** (2003), 323-403.
- [Ha 4] Hauser, H.: Three power series techniques. To appear in *Proc. London Math. Soc.*
- [HR] Hauser, H., Regensburger, G.: Explizite Auflösung von ebenen Kurvensingularitäten in beliebiger Charakteristik. Preprint Univ. Innsbruck.
- [Hi 1] Hironaka, H.: On the presentations of resolution data (Notes by T.T. Moh). In: *Algebraic Analysis, Geometry and Number Theory 1988* (ed. J.I. Igusa), The Johns Hopkins University Press 1989.
- [Hi 2] Hironaka, H.: Desingularization of excellent surfaces. Notes by B. Bennett at the Conference on Algebraic Geometry, Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: *Resolution of surface singularities. Lecture Notes in Math.* **1101**, Springer 1984.
- [Hi 3] Hironaka, H.: Idealistic exponents of singularity. In: *Algebraic Geometry, the Johns Hopkins Centennial Lectures.* Johns Hopkins Univ. Press 1977.
- [Hi 4] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. Math.* **79** (1964), 109-326.
- [HS] Huber, B., Sturmfels, B.: Bernstein's theorem in affine space. *Discrete Comput. Geom.* **17** (1997), 137-141.
- [Kh 1] Khovanskii, A.: Newton polyhedra, a new formula for mixed volume, product of roots of a system of equations. *The Arnoldfest*, 325-364, *Fields Inst. Commun.* **24**, Amer. Math. Soc. 1999.
- [Kh 2] Khovanskii, A.: Newton polyhedra, and the genus of complete intersections. (Russian) *Funkt. Anal. Appl.* **12** (1978), 51-61.
- [Kh 3] Khovanskii, A.: *Fewnomials.* Amer. Math. Soc., Providence 1991.
- [Ku 1] Kushnirenko, A.G.: Newton polyhedra and the Bézout Theorem. *Funct. Anal. Appl.* **10** (1976), 82-83.

- [Ku2] Kushnirenko, A. G.: Polyèdres de Newton et nombres de Milnor. *Invent. Math.* **32** (1976), 1-31.
- [Mo 1] Moh, T.-T.: On a stability theorem for local uniformization in characteristic p . *Publ. RIMS Kyoto Univ.* **23** (1987), 965-973.
- [Mo 2] Moh, T.-T.: On a Newton polygon approach to the uniformization of singularities of characteristic p . In: *Algebraic Geometry and Singularities* (eds. A. Campillo, L. Narváez). *Proc. Conf. on Singularities La Rábida*. Birkhäuser 1996.
- [Mu] Mulay, S.B.: Equimultiplicity and hyperplanarity. *Proc. Amer. Math. Soc.* **87** (1983), 407-413.
- [Na 1] Narasimhan, R.: Monomial equimultiple curves in positive characteristic. *Proc. Amer. Math. Soc.* **89** (1983), 402-413.
- [Na 2] Narasimhan, R.: Hyperplanarity of the equimultiple locus. *Proc. Amer. Math. Soc.* **87** (1983), 403-406.
- [Ri] Riordan, J.: *Combinatorial identities*. Krieger 1979.
- [Ro] Rojas, M.: A convex geometric approach to counting the roots of a polynomial system. *Selected papers of the Workshop on Continuous Algorithms and Complexity* (Barcelona, 1993). *Theoret. Comput. Sci.* **133** (1994), 105-140.
- [RW] Rojas, M., Wang, X.: Counting affine roots of polynomial systems via pointed Newton polytopes. *J. Complexity* **12** (1996), 116-133.
- [Vi 1] Villamayor, O.: Constructiveness of Hironaka's resolution. *Ann. Scient. Ec. Norm. Sup. Paris* **22** (1989), 1-32.
- [Vi 2] Villamayor, O.: Patching local uniformizations. *Ann. Scient. Ec. Norm. Sup. Paris* **25** (1992), 629-677.
- [Yo] Youssin, B.: Newton polyhedra without coordinates. *Memoirs Amer. Math. Soc.* **433** (1990), 1-74, 75-99.

Fakultät für Mathematik
 Universität Wien
 A-1090 Vienna, Austria
 herwig.hauser@univie.ac.at