THREE POWER SERIES TECHNIQUES

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In this paper we shall describe three techniques which allow to work efficiently with power series in various problems of singularity theory, especially in the context of resolution of singularities. The key problem in resolution of singularities is the construction of local invariants of singular schemes and their control under blowup and localization. Our methods provide a general frame to produce such invariants and to observe their transformation laws. As all constructions are characteristic independent, they can be used in particular to investigate the still unsolved characteristic p case.

The techniques are the following.

Local flags in regular schemes. Given a point a on a regular analytic or algebroid ambient scheme, a *flag* at a is a chain of local regular subschemes passing through the point. In dimension two, a flag is just a regular curve, in dimension three, the flag consists of a regular curve contained in a regular surface. Fixing a flag at a allows to define subordinate coordinates, i.e., coordinates whose first k components induce coordinates on the k-th member of the flag. These in turn allow to construct local invariants of singular schemes.

For example, certain portions of the Newton polyhedron of the singular scheme at a are the same for all choices of subordinate coordinates, and thus only depend on the chosen flag. They are automatically invariants of the singularity, i.e., remain the same when applying a local automorphism which preserves the flag. Thus it suffices to choose any subordinate coordinates and to construct the invariant in these specific coordinates. This, of course, makes things much more explicit and allows to perform effective calculations.

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To study the behaviour of the invariants under transformations of the scheme it is necessary to associate also local flags to the transformed scheme. We shall prove that flags persist in a natural way under blowup and localization. In fact, to each point of the blown up scheme a local flag can be attached, and this flag is canonically related to the flag before blowup. Choosing then local coordinates which are subordinate to the flags before and after the transformation, the invariant is realized in the respective coordinates and the change it undergoes by the transformation of the scheme can be directly read off.

The Gauss-Bruhat decomposition of formal automorphisms. The classical Gauss-decomposition BUS of $\mathrm{GL}_n(\mathbf{K})$ says that any linear automorphism ℓ of $\mathbf{K}^n$ can be written as a product $\ell = bus$ where b is an upper triangular matrix, u is a unipotent lower triangular matrix and s is a permutation. The stronger Bruhat-decomposition $\mathrm{GL}_n(\mathbf{K}) = BSU$ asserts that ℓ can even be written as $\ell = bsu$.

We shall prove the analog of the Gauss-decomposition for the group $\mathbf{G}$ of automorphisms of the formal (or convergent) power series ring in n variables, say the group of local automorphisms of affine space $\mathbf{A}^n$: Any automorphism φ in $\mathbf{G}$ is a product $\varphi = bus$ where b belongs to the "Borel subgroup" $\mathbf{B}$ of upper triangular automorphisms (say $x_i \to x_i + b_i(x_1, \ldots, x_n)$ with b_i in the ideal generated by $x_i, \ldots, x_n$), u belongs to the "opposite unipotent subgroup" $\mathbf{U}$ of lower unipotent Jonquière automorphisms (say $x_i \to x_i + u_i(x_1, \ldots, x_{i-1})$), and s is a permutation. In contrast to the linear case, a factorization of type $\varphi = bsu$ may not be possible, as is seen by a simple example. Thus $\mathbf{B}S\mathbf{U}$ is in general strictly contained in $\mathbf{G}$.

Obviously, given a local flag in $\mathbf{A}^n$ and subordinate local coordinates $x_1, \ldots, x_n$, the Borel automorphism b are just those automorphisms of $\mathbf{A}^n$ which stabilize the flag. In this way the Gauss-decomposition of $\mathbf{G}$ allows to separate the part us of an automorphism φ which may alter a flag invariant. Or, said conversely, given any local coordinates, not necessarily subordinate, the automorphisms of type us suffice to construct subordinate coordinates and to realize the associated flag invariants.

As we will see, the possibility of restricting to automorphisms us when constructing invariants becomes the decisive clue for controlling a flag invariant under blowup or localization.

Coordinate-free initial ideals. Initial ideals are monomial ideals associated to ideals of power series (or polynomials) by choosing a monomial order $<$ on $\mathbb{N}^n$ and taking for each element f of the ideal its initial monomial $\text{in}(f)$, say the monomial of its expansion which has minimal exponent with respect to $<$ (maximal in case of polynomials). Such ideals have turned out to be very useful when investigating singularities, for they contain precise information on the geometry.

Their defect is that they require to work with fixed coordinates and are therefore not invariants of the singularity. This has been remedied by considering the generic initial ideal, which is the initial ideal appearing for a generic choice of coordinates [15, 19, 13, 14, 9, 30].

In this paper, we propose another view point by ordering monomial ideals through the lexicographic order taken on their minimal monomial generator systems. This produces a well-order on the set of monomial ideals in n variables. In particular, the set of initial ideals of a given ideal with respect to all coordinate choices is naturally ordered. It is easily seen that the *minimal initial ideal* coincides with the generic initial ideal. Much more interesting is the *maximal initial ideal*. Its existence is not obvious. It is realized in very specific coordinates and contains much more precise information on the singularity than the generic initial ideal. By definition, both the minimal and the maximal initial ideal are coordinate independent, and thus invariants of the singularity.

Given any coordinates, the first task is to determine a coordinate change so that the new coordinates realize the maximal initial ideal. This will be solved by using the Gauss decomposition of $\mathbf{G}$. First, the decomposition will be defined and proven for subgroups of $\mathbf{G}$ related to monomial orders on $\mathbb{N}^n$, extending the notion of Jonquière automorphisms. If the monomial order is graded, even the Bruhat-decomposition $\mathbf{G} = \mathbf{B}S\mathbf{U}$ holds for $\mathbf{G}$. It is then shown that automorphisms from the Borel factor $\mathbf{B}$ (associated to the monomial order used for the respective initial ideal) fix the initial ideal in the given coordinates. Therefore automorphisms of type us suffice to realize maximal initial ideals (of type su if the order is graded).

This in turn allows to observe the behaviour of minimal and maximal initial ideals under blowup and localization. As an immediate consequence one gets a quick proof for Bennett's famous theorem on the upper semicontinuity of the Hilbert-Samuel function of a singularity: The Hilbert-Samuel function does not increase under blowup with permissible centers and under localization.

These few remarks already indicate that the three concepts *flags*, *Gauss-Bruhat* *decomposition* and *initial ideals* are strongly related. Their original interest was to use them for studying resolution of singularities. Certainly, they can also be useful for many other problems in singularity theory.

Let us briefly explain how local flags appear in resolution problems, see [22] for more background on this. Given a singular subscheme of a regular ambient scheme, consider its defining ideal. To build up an induction on the local embedding dimension, one associates to the ideal locally at each point an ideal in one variable less. It will be defined in the hypersurface which is the largest member of the flag. This is Hironaka's approach in his proof of resolution of singularities in characteristic zero. He chooses the hypersurface by requiring maximal contact with the ideal, and the ideal in smaller dimension will be a certain ideal of coefficients with respect to this hypersurface. By induction on the embedding dimension, a local resolution invariant can be defined for this "smaller" ideal. From this the appropriate resolution invariant is built up for the original ideal, viz the singular scheme.

This construction is known to fail in positive characteristic, mainly because hypersurfaces of maximal contact need no longer exist. And choosing an arbitrary local hypersurface does not work since the resulting invariant may depend on the choice and would therefore be non-intrinsic. In particular, after blowup a new choice would be necessary, prohibiting the comparison of the invariant before and after blowup. This basic obstruction can be circumvented to a certain extent with the introduction of flags. Once a flag is chosen locally at each point of the initial ambient scheme, it accompanies the whole resolution process: there is a natural way to associate to each point of the blown up scheme a local flag. Moreover, the blowup admits at each point a coordinate description in local charts so that the local flags below and above are related by a *monomial* substitution of the variables. Again, this is substantial to control the resolution invariant along the process.

As an application of our methods, we show that all standard resolution invariants are in their substance special cases of constructions carrying on local flags and coordinate free initial ideals. They can therefore be treated in a unified way, yielding simpler proofs in characteristic 0. It is known that these invariants may increase in characteristic p, which requires to search for additional and refined invariants.

The methods produce a whole range of new characteristic free invariants. They form a potential testing ground for approaching resolution of singularities in positive characteristic, cf. [22, 23, 31]. It should be clear from the complexity of the subject that the correct resolution invariant will not be easily detected. We show explicitly up to where certain maximal initial ideals can serve to this end, and where they hit obstructions which still have to be overcome. We do not claim that the three techniques already suffice to prove resolution in positive characteristic. But they provide a powerful and flexible machinery for the study of these obstructions.

We now describe in more detail the content of the paper. Essentially, two types of singularity invariants will be constructed. The first class are flag invariants. A flag invariant is a singularity invariant which depends on the flag but not on the regular system of local parameters chosen to construct it. The flag persists at each closed point of the exceptional divisor of the blown up ambient scheme. Hence, the invariant is also defined there and can be compared with the invariant below. This is exhibited for polyhedral invariants of hypersurface singularities. Hironaka's invariant β used in [17] and [21] is thus interpreted as a flag invariant and can then be extended to arbitrary dimension (though it is not sufficient for induction purposes in embedding dimension ≥ 4).

The second class are coordinate free initial ideals. Let $\mathbf{R}$ be the completion of a local ring of the ambient scheme and let $\mathbf{I}$ in $\mathbf{R}$ be the completed stalk of the ideal sheaf of the singular scheme to be resolved. Monomial ideals are ordered lexicographically via their ordered minimal monomial generator system. We prove that for any monomial order ε, the associated initial ideal $\mathrm{in}_{x\varepsilon}\mathbf{I}$ of $\mathbf{I}$ possesses a maximum $\max_\varepsilon \mathbf{I}$ when x ranges over all regular systems of parameters of $\mathbf{R}$. The minimum $\min_\varepsilon \mathbf{I}$ exists since the set of monomial ideals is shown to be well ordered; it equals for infinite ground fields the generic initial ideal. For every ε, $\max_\varepsilon \mathbf{I}$ is an invariant of $\mathbf{I}$ which is much more subtle than $\min_\varepsilon \mathbf{I}$ since it is realized in most specific coordinates. We show how the Hilbert-Samuel function, Hironaka's ν^* and τ, Abhyankar's magnitude, the weighted order and the local embedding dimension are refined or recovered by this class of invariants. Coordinates which maximize initial ideals present a characteristic free substitute for hypersurfaces of maximal contact and for Tschirnhaus transformations [22, 23, 10, 31].

The changes the invariants undergo by permissible blowups are made explicit by choosing coordinates which are subordinate to the flag or maximize the initial ideal and for which the blowup is monomial. This is possible due to the Gauss-Bruhat-decomposition of the automorphism group $\mathbf{G}$ of $\mathbf{R}$.

In the first three sections we prove the persistence of flags under blowup, the Gauss decomposition of automorphisms and the existence of maximal initial ideals. Section four embeds the classical invariants into this setting and introduces new resolution invariants. The subsequent section relates automorphisms, flag invariants, initial ideals and blowup. These results are used in section six to determine the behaviour of the invariants under blowup. We show that the minimal and maximal initial ideals with respect to a graded monomial order do not increase under blowup, and determine the cases where they decrease. The same question is treated for elimination orders and flag invariants. Here the difficulties of resolution of singularities in arbitrary characteristic become apparent. In the last section, it is shown that $\min_\varepsilon \mathbf{I}$ is upper semicontinuous with respect to localization if the monomial order is graded.

In this paper, we restrict to the algebroid situation. Throughout, $\mathbf{R}$ denotes an equicharacteristic complete local Noetherian regular ring of dimension n with maximal ideal $\mathbf{M}$ and residue field $\mathbf{K}$. Most results are valid without completeness assumption or for convergent power series rings.

Some special cases of the constructions of this paper, mostly appearing in implicit form, can be found in the literature, see e.g. [1, 2, 5, 6, 8, 11,12, 10, 16, 17, 18, 21, 26, 33, 35, 36, 37]. A conceptual and stringent treatment of resolution invariants and their properties is necessary to be able to attack the main open resolution problems. We are indebted to Orlando Villamayor, Adolfo Quirós, Vincent Cossart, Mariemi Alonso, Mark Spivakovsky, Gerd Müller, Luis Narváez and Franz Pauer for very valuable conversations. The accurate suggestions of the referee helped to eliminate several ambiguities in a preliminary draft of the paper. The graphics were done by Sebastian Gann using the design-program POV-Ray.

Flag varieties

A flag in $\mathbf{R}$ is a sequence $\mathcal{F} : \mathbf{M}_n = 0 \subseteq \ldots \subseteq \mathbf{M}_0 = \mathbf{M}$ of regular ideals $\mathbf{M}_i$ of height $n - i$ of $\mathbf{R}$. Set $F_i = \text{Spec } \mathbf{R}/\mathbf{M}_i$ and $W = F_n = \text{Spec } \mathbf{R}$. A regular system of parameters $x_1, \ldots, x_n$ of $\mathbf{R}$ is called subordinate to $\mathcal{F}$ if $\mathbf{M}_i$ is generated by the last $n - i$ parameters for all i.

Let $\mathbf{S} = \oplus_{i \geq 0} \mathbf{P}^i$ be the blowup (or Rees) algebra of $\mathbf{R}$ with center a regular ideal $\mathbf{P}$ of $\mathbf{R}$. Localizing and completing $\mathrm{Proj}(\mathbf{S})$ at a maximal ideal $\mathbf{Q}'$ containing $\mathbf{M}$, the resulting complete local ring $\mathbf{R}'$ with inclusion $\varphi : \mathbf{R} \to \mathbf{R}'$ will be called the local blowup of $\mathbf{R}$ in $\mathbf{P}$ at $\mathbf{Q}'$. For detailed information on blowups we refer to [16, III.2, 5, 22, appendix]. Let $\pi : W' \to W$ be the corresponding morphism of local schemes with $W' = \mathrm{Spec}\, \mathbf{R}'$, center $Z = \mathrm{Spec}\, \mathbf{R}/\mathbf{P}$ and exceptional divisor $E = \pi^{-1}Z$. The normal cone $\mathrm{N}_W Z$ of Z in W is given by $\mathbf{P}/\mathbf{P}^2$, and its fibre $\mathrm{N}_{W,0}Z$ at the closed point 0 of W corresponds to $\mathbf{P}/\mathbf{MP}$. Choose any ideal $\mathbf{Q}$ in $\mathbf{R}$ whose class in $\mathbf{P}/\mathbf{MP}$ corresponds to $\mathbf{Q}'$, and denote by L the associated linear subspace in $\mathrm{N}_W Z$ of dimension the dimension of Z plus 1.

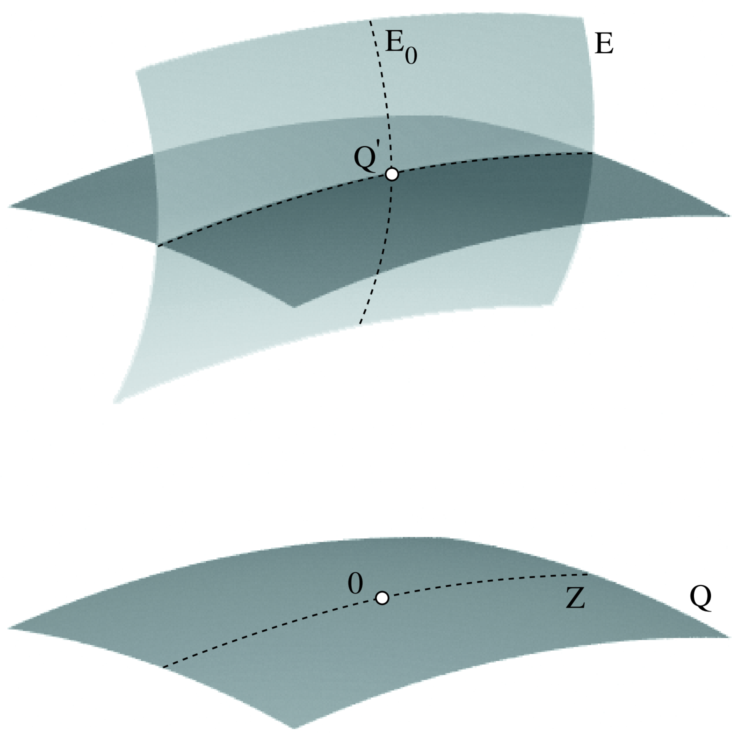

Figure 1: Blowup of $\mathbb{A}^3$ with center a regular curve.

If $\mathbf{P} = \mathbf{M}$, the center Z is the closed point, say 0, of W, and $W' \to W$ is the corresponding point blowup, with L a line in the tangent space $\mathrm{T}_0 W$ of W at 0. If

$\mathbf{P} \subsetneq \mathbf{M}$, then $W' \to W$ is the cartesian product of a point blowup with the identity on Z: Choose V in W regular and transversal to Z of complementary dimension, so that $W \cong Z \times V$. Let $\tau : V' \to V$ denote the point blowup of V in 0, yielding a commutative diagram

$$\begin{array}{ccc} W' & \cong & Z \times V' \\ \downarrow \pi & & \downarrow \mathrm{Id}_Z \times \tau \\ W & \cong & Z \times V \end{array}$$

The choice of $\mathbf{Q}$ in $\mathbf{R}$ determines a local section of π over Z. The class of $\mathbf{Q}$ in $\mathbf{P}/\mathbf{MP}$ specifies the intersection point $\mathbf{Q}'$ of this section with the exceptional fibre $E_0 = \pi^{-1}(0)$ of π over 0 (figure 1).

We say that Z is transversal to $\mathcal{F}$ if $Z \cap F_i$ is (scheme-theoretically) regular for all i. There then exist coordinates $x_1, \ldots, x_n$ in $\mathbf{R}$ subordinate to $\mathcal{F}$ and such that Z is defined by the ideal $<x_i,\, i \in J>$ for some $J \subset \{1, \ldots, n\}$.

Our first result concerns the persistence of local flags under blowup.

Theorem 1. *Any flag $\mathcal{F}$ in $\mathbf{R}$ induces canonically a flag $\mathcal{G}$ in every residually rational local blowup $\mathbf{R}'$ of $\mathbf{R}$ with center Z transversal to $\mathcal{F}$.*

Canonical means that for any two flags in $\mathbf{R}$ which are mapped onto each other by an automorphism of $\mathbf{R}$ preserving $\mathbf{P}$ and $\mathbf{Q} + \mathbf{MP}$ the induced flags are isomorphic via the induced automorphism of $\mathbf{R}'$.

For $n = 3$, a flag at a point consists of a regular curve F_1 contained in a regular surface F_2. Blow up the point. At the intersection point p_1 of the strict transform $C_1 = F_1'$ of F_1 with E, the induced flag is $F_1' \subseteq F_2'$. Along the intersection C_2 of F_2' with E, the flag at each point p_2 different from p_1 is $C_2 \subseteq F_2'$. At any point p_3 not on C_2 the flag is $C_3 \subseteq E$, where C_3 is the projective line in E through p_1 and p_3. The discontinuity of $\mathcal{G}$ when p moves in E is on purpose. See figure 2.

Blowing up a regular curve Z transversal to $\mathcal{F}$ there occur six configurations of Z with respect to $\mathcal{F}$. These are: (1) $Z = F_1$ and $L = \mathrm{T}_0 F_2$, (2) $Z = F_1$ and $L \neq \mathrm{T}_0 F_2$, (3) $Z \neq F_1$, $Z \subseteq F_2$ and $L = \mathrm{T}_0 F_2$, (4) $Z \neq F_1$, $Z \subseteq F_2$ and $L \neq \mathrm{T}_0 F_2$, (5) $Z \not\subseteq F_2$ and $\mathrm{T}_0 F_1 \subseteq L$, (6) $Z \not\subseteq F_2$ and $\mathrm{T}_0 F_1 \not\subseteq L$. The induced flags are easily determined by the formulas given below.

Proof. Assume first that Z is a point, say 0, so that L is a line in the tangent space $\mathrm{T}_0 W$ of W. Let $k \leq n$ be the minimal index for which $\mathrm{T}_0 F_k$ contains L. For $i < k$, choose a regular $(i+1)$-dimensional subscheme H_i of W with tangent space $L + \mathrm{T}_0 F_i$. In particular, $\mathrm{T}_0 H_{k-1} = \mathrm{T}_0 F_k$. Let H_i' be the strict transform of H_i in W. Then set

$$\begin{aligned} G_i &= E \cap H_i' \qquad && \text{for } i < k, \\ G_i &= F_i' && \text{for } i \geq k. \end{aligned}$$

This defines a flag $\mathcal{G}$ in W' which does not depend on the choice of H_i. It is immediate that $\mathcal{G}$ is canonical.

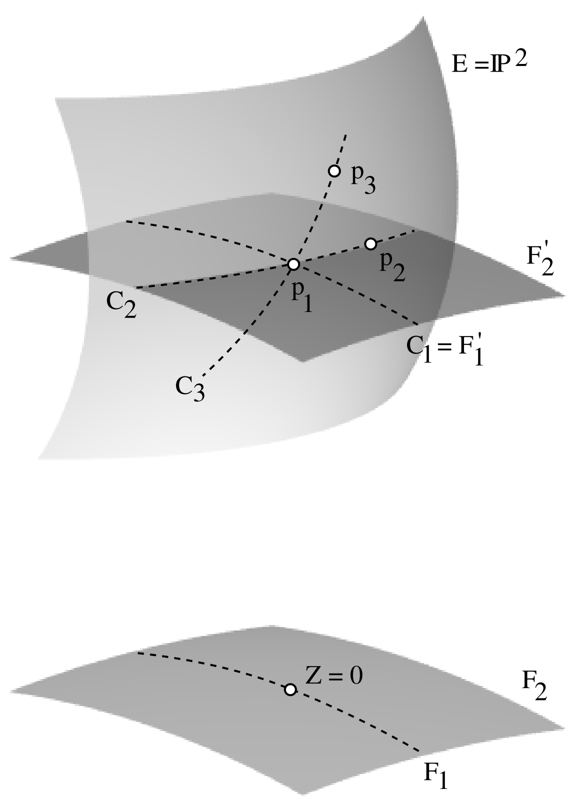

Figure 2: Flags in $\mathbb{A}^3$ and their transforms under blowup.

[As E and H'_i intersect transversally, G_i is regular of dimension i. Note that G_i only depends on $\mathrm{T}_0 H_i$ and that F'_i is regular of dimension i. It is clear that $G_i \subseteq G_{i+1}$ for $i \leq k-2$ and $i \geq k$. Moreover, $G_{k-1} = E \cap H'_{k-1} = E \cap F'_k \subseteq F'_k = G_k$ so that $G_0 \subseteq \ldots \subseteq G_n$ defines indeed a flag in W'.]

For arbitrary Z transversal to $\mathcal{F}$ choose V regular in W transversal to Z of complementary dimension and such that $V \cap F_i$ has complementary dimension to $Z \cap F_i$ in F_i for all i. Then $\tilde{F}_i = F_i \cap V$ defines a flag on V, abusing slightly the indexing. Choose an isomorphism $Z \times V \cong W$ which maps $(Z \cap F_i) \times (V \cap F_i)$ onto F_i.

[To do this, let m be minimal with $Z \cap F_m \subsetneq F_m$, so that $F_i \subseteq Z$ for $i < m$. Choose V_m in F_m transversal to $Z \cap F_m$ of complementary dimension, and an isomorphism $(Z \cap F_m) \times V_m \cong F_m$. If $Z \cap F_{m+1} = Z \cap F_m$, choose a regular curve U_{m+1} in F_{m+1} transversal to F_m and an isomorphism $F_m \times U_{m+1} \cong F_{m+1}$. Let V_{m+1} be the image of $V_m \times U_{m+1}$ in F_{m+1}.]

[If $Z \cap F_m \subsetneq Z \cap F_{m+1}$, set $V_{m+1} = V_m$. In both cases, V_{m+1} is transversal to $Z \cap F_{m+1}$ in F_{m+1} and of complementary dimension. We have an isomorphism $(Z \cap F_{m+1}) \times V_{m+1} \cong F_{m+1}$ extending $(Z \cap F_m) \times V_m \cong F_m$. By induction we construct $V_m, \ldots, V_n = V$ and an isomorphism $Z \times V \cong W$ which extends all $(Z \cap F_i) \times V_i \cong F_i$, and such that $V_i = V \cap F_i$.]

Let V' be the local blowup of V in 0 at the point corresponding to $\mathbf{Q}'$ so that $Z \times V' \cong W'$. Let $\tilde{\mathcal{G}}$ be the flag in V' induced from $\tilde{\mathcal{F}}$. Define $\mathcal{G}$ in W' by the images of $(Z \cap F_i) \times \tilde{G}_i$. This gives a flag in W' which does not depend on the choice of V. It is checked that the construction is canonical.

[Let U be a second choice, replacing V. By construction, there is an automorphism $\psi : W \to W$ subordinate to $\mathcal{F}$ and preserving Z and L which maps V onto U. Hence $V_i = V \cap F_i$ is mapped onto $U \cap F_i$, which is the corresponding flag on U. The induced automorphism maps $\mathcal{G}$ on the induced flag of $Z \times U'$, whence the assertion.]

Let $x_1, \ldots, x_n$ be subordinate to $\mathcal{F}$ in $\mathbf{R}$. For point blowups, the line L is defined in $\mathrm{T}_0 W$ by the ideal $<x_1 + t_1 x_k, \ldots, x_{k-1} + t_{k-1} x_k, x_{k+1}, \ldots, x_n>$ for certain $t_i \in \mathbf{K}$, since $x_{k+1}, \ldots, x_n$ belong to the ideal defining L whereas x_k does not, by definition of k. Set

$$
\begin{aligned}
y_i &= x_i / x_k - t_i \qquad && \text{for } i < k, \\
y_i &= x_i && \text{for } i \geq k.
\end{aligned}
$$

The y_i are coordinates in $\mathbf{R}'$ which are subordinate to $\mathcal{G}$. This is a less intrinsic way to define $\mathcal{G}$.

For arbitrary centers and coordinates $x_1, \ldots, x_n$ in $\mathbf{R}$ which are subordinate to $\mathcal{F}$ and so that Z is defined by $<x_i, i \in J>$ with $J \subset \{1, \ldots, n\}$, one may take for V the regular subscheme of W defined by the ideal $<x_i, i \notin J>$. The induced coordinates y_i on V' for $i \in J$ are subordinate to $\tilde{\mathcal{G}}$. Setting $y_i = x_i$ for $i \notin J$, they are complemented to coordinates on W' which are subordinate to $\mathcal{G}$. For arbitrary subordinate coordinates in W, the induced subordinate coordinates in W' are obtained by applying the appropriate automorphisms of $\mathbf{R}$ and $\mathbf{R}'$.

Subordinate coordinates are sufficiently flexible to render local blowups monomial (Thm. 5(a)), and sufficiently specific to define significant invariants (Thm. 5(b)).

Gauss-Bruhat decomposition

We introduce subgroups of $\mathbf{G} = \mathrm{Aut}\,\mathbf{R}$ as follows. For any regular system of parameters x in $\mathbf{R}$ identify $g \in \mathbf{G}$ with the vector g^* in $\mathbf{M} \cdot \mathbf{K}[[x]]^n$ of components $g_i = gx_i$, the composition corresponding to substitution of series. Let ε be a monomial order on $\mathbb{N}^n$, i.e., a well-ordering compatible with addition. It induces an ordering $\leq$ of the monomials of $\mathbf{K}[[x]]$ by comparison of their exponents.

Given x and ε, the subgroups $\mathbf{B}_\varepsilon$ and $\mathbf{U}_\varepsilon$ of $\mathbf{G}$ are formed by those automorphisms g for which $g_i - x_i$ consists of monomials $\geq x_i$, respectively $< x_i$, for all i. It is clear that $\mathbf{B}_\varepsilon$ and $\mathbf{U}_\varepsilon$ are closed under composition, and that the inverse of a linear automorphism in $\mathbf{B}_\varepsilon$ or $\mathbf{U}_\varepsilon$ belongs again to $\mathbf{B}_\varepsilon$, respectively $\mathbf{U}_\varepsilon$. For automorphisms g tangent to the identity, define recursively $h^0 = \mathrm{Id}_{\mathbf{R}}$ and $h^{k+1} = \mathrm{Id}_{\mathbf{R}} - (g - \mathrm{Id}_{\mathbf{R}})h^k$. By induction, h^k belongs to $\mathbf{B}_\varepsilon$, respectively $\mathbf{U}_\varepsilon$, if g does. As h^k coincides with g^{-1} up to degree k, also g^{-1} does. Hence $\mathbf{B}_\varepsilon$ and $\mathbf{U}_\varepsilon$ are subgroups of $\mathbf{G}$.

We say that ε is graded if it preserves the partial order on $\mathbb{N}^n$ given by $|\alpha| = \alpha_1 + \ldots + \alpha_n$. In this case, $\mathbf{U}_\varepsilon$ is a finite dimensional algebraic group and consists of linear automorphisms. All automorphisms tangent to the identity then belong to $\mathbf{B}_\varepsilon$. In general, the intersection B_ε of $\mathbf{B}_\varepsilon$ with $G = \mathrm{Gl}_n(\mathbf{K})$ is a Borel subgroup of G, and $U_\varepsilon = \mathbf{U}_\varepsilon \cap G$ is the unipotent radical of the opposed Borel subgroup. For the inverse lexicographic order lix, $\mathbf{U}_{lix}$ consists of Jonquière automorphisms, and $\mathbf{B}_{lix}$ of those g for which $g_i \in <x_i, \ldots, x_n>$. Given a flag $\mathcal{F}$ in $\mathbf{R}$ and subordinate coordinates x, $\mathbf{B}_{lix}$ consists of those automorphisms g for which gx is again subordinate.

Let S_n denote the permutation group. The following decomposition of the automorphism group $\mathbf{G}$ generalizes the classical Gauss-Bruhat decomposition of reductive groups.

Theorem 2. *For any monomial order* ε *and any coordinates* x *in* $\mathbf{R}$ *one has*

$$\mathbf{G} = \mathbf{B}_\varepsilon \mathbf{U}_\varepsilon S_n = \mathbf{U}_\varepsilon \mathbf{B}_\varepsilon S_n.$$

If ε *is graded, then also* $\mathbf{G} = \mathbf{B}_\varepsilon S_n \mathbf{U}_\varepsilon$. *In general,* $\mathbf{B}_\varepsilon S_n \mathbf{U}_\varepsilon \subsetneq \mathbf{G}$.

The assertion can be interpreted as the formal analog of the theorem of Jung and van der Kulk on the description of automorphisms of the polynomial ring $\mathbf{K}[x,y]$ in two variables as compositions of linear and Jonquière automorphisms [24, 25]. The corresponding statement in dimension ≥ 3 has been recently disproved [29, 32].

Proof. Applying a permutation to $\mathbb{N}^n$ conjugates $\mathbf{B}_\varepsilon$ and $\mathbf{U}_\varepsilon$ and does not affect the decomposition. It allows to assume $x_1 < \ldots < x_n$.

(a) We prove that for any $g \in \mathbf{G}$ there exist $b \in \mathbf{B}_\varepsilon$ and $s \in S_n$ such that $sgb \in \mathbf{U}_\varepsilon$. Inversion then implies the first equality. Let k be the maximal index $\leq n+1$ for which there exist $b \in \mathbf{B}_\varepsilon$ and $s \in S_n$ such that $(sgb)_j - x_j$ consists of monomials $< x_j$ for all $j < k$ and such that $d = (sgb)^{-1}$ has invertible partial derivative $\partial_k d_k$. If $k = n+1$ we are done. Assume $k \leq n$. Define c by $c_i = x_i$ for $i \neq k$ and c_k equal the series obtained from d_k by deleting all monomials $< x_k$. This is an automorphism in $\mathbf{B}_\varepsilon$. Set $h = sgb$. Then

$$(hc)_k = (c^* h^*)_k = (d^* h^*)_k + ((c^* - d^*) h^*)_k = x_k + (c_k - d_k) h^*.$$

The series $c_k - d_k$ consists of monomials $x^\alpha < x_k$ and therefore does not depend on $x_k, \ldots, x_n$. Hence

$$(c_k - d_k) h^* = (c_k - d_k)(h_1, \ldots, h_{k-1}, 0, \ldots, 0).$$

Let $x^{\beta_j} \leq x_j$ be a monomial of h_j for $j < k$. If $x^\alpha < x_k$, then

$$(x^{\beta_1}, \ldots, x^{\beta_{k-1}}, 0, \ldots, 0)^\alpha \leq (x_1, \ldots, x_{k-1}, 0, \ldots, 0)^\alpha = x^\alpha < x_k.$$

As $(hc)_j = h_j$ for $j < k$ and setting $\tilde{b} = bc \in \mathbf{B}_\varepsilon$ we see that $(sg\tilde{b})_j - x_j$ consists of monomials $< x_j$ for all $j \leq k$. This implies that the matrix defining the linear part of $(sg\tilde{b})$ has block form $\binom{A\,B}{0\,C}$ where A is unipotent upper triangular of size $k \times k$ and where C is invertible. The linear part of $(sg\tilde{b})^{-1}$ has the same form. Hence there exists a permutation t of the last $n-k$ coordinates such that $\tilde{d} = (\tilde{s}g\tilde{b})^{-1}$ has

$\partial_{k+1}\tilde{d}_{k+1}$ invertible, where $\tilde{s} = ts$. Also $(\tilde{s}g\tilde{b})_j - x_j$ consists of monomials $< x_j$ for all $j \leq k$, hence k was not maximal.

(b) For any $g \in \mathbf{G}$ there exist $b \in \mathbf{B}_\varepsilon$ and $s \in S_n$ such that $gsb \in \mathbf{U}_\varepsilon$. Let k be maximal such that there are $b \in \mathbf{B}_\varepsilon$ with $b_j = x_j$ for $j \geq k$ and $s \in S_n$ such that $(gsb)_j - x_j$ consists of monomials $< x_j$ for all $j < k$ and such that $\partial_k d_k$ is invertible for $d = (gsb)^{-1}$. Assume $k \leq n$. As before we find $c \in \mathbf{B}_\varepsilon$ such that $(gsbc)_j - x_j$ consists of monomials $< x_j$ for $j \leq k$. Choose a permutation t of the last $n-k$ coordinates with $\tilde{d} = (gsbct)^{-1}$ having $\partial_{k+1}\tilde{d}_{k+1}$ invertible. Then bc belongs to $\mathbf{B}_\varepsilon$ and satisfies $(bc)_j = x_j$ for all $j > k$. Hence $\tilde{b} = t^{-1}bct$ does. Set $\tilde{s} = st$. It follows that $(g\tilde{s}\tilde{b})_j - x_j$ consists of monomials $< x_j$ for all $j \leq k$, contradicting the maximality of k.

(c) If ε is graded, write $g \in \mathbf{G}$ as $g = hl$ with h tangent to the identity and l linear. Then $h \in \mathbf{B}_\varepsilon$ and $l = bsu$ by the classical Bruhat decomposition, hence $g \in \mathbf{B}_\varepsilon S_n \mathbf{U}_\varepsilon$. For the strict inclusion, take for ε the lexicographic order in $\mathbb{N}^2$ with $x < y$. Then $g = (y + x^2, x) \notin \mathbf{B}_\varepsilon S_n \mathbf{U}_\varepsilon$.

Maximal initial ideals

For x a regular system of parameters in $\mathbf{R}$, identify elements f of $\mathbf{R}$ with their series expansion $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$. Given a monomial order ε on $\mathbb{N}^n$, we denote by $\mathrm{in}_{x\varepsilon} f$ its initial monomial with exponent $\exp_{x\varepsilon} f$, and by $\mathrm{in}_{x\varepsilon}\mathbf{I}$ the initial ideal of an ideal $\mathbf{I}$ of $\mathbf{R}$. Elements of $\mathbf{I}$ are called a (minimal, ordered) standard basis of $\mathbf{I}$ if their initial monomials generate $\mathrm{in}_{x\varepsilon}\mathbf{I}$ (and are of minimal length, respectively, are minimal and numbered with increasing initial monomials). An element $f \in \mathbf{R}$ is called reduced with respect to $\mathbf{I}$ if no monomial of f belongs to $\mathrm{in}_{x\varepsilon}\mathbf{I}$.

Let $\mathcal{E}_n$ be the set of monomial ideals in $\mathbf{K}[[x]]$. Given ε, every $\mathbf{J} \in \mathcal{E}_n$ will be identified with its ordered sequence of vertices $A = (\alpha_1, \ldots, \alpha_p, \infty, \ldots)$, where $x^{\alpha_1}, \ldots, x^{\alpha_p}$ is the ordered minimal monomial generator system of $\mathbf{J}$. The lexicographic order on the sequences of ordered vertices defines a well-order on $\mathcal{E}_n$ which we denote again by ε or $\leq$.

[Assume there were an infinite strictly decreasing sequence $\mathbf{J}_i$ in $\mathcal{E}_n$ with ordered vertices A_i. As the sequence of first vertices is decreasing in $\mathbb{N}^n$ and ε is a well-ordering there exists an infinite subset M_1 of indices for which the first vertices are equal. As all $\mathbf{J}_i$ are different, only finitely many second vertices can be infinite. Hence there exists an infinite subset $M_2 \subseteq M_1$ for which also the second vertices are equal. By induction one finds for any k decreasing subsets M_k and k elements in $\mathbb{N}^n$ which form the first k vertices of $\mathbf{J}_m$ for all m in M_k. Then the monomial ideals generated by the first k elements form a strictly increasing sequence of ideals with respect to inclusion, contradiction.]

For given x and ε, the set $\mathcal{IN}_\varepsilon(\mathbf{I}) = \{\mathrm{in}_{x\varepsilon} g\mathbf{I},\ g \in \mathbf{G}\}$ equals the set of all initial ideals of $\mathbf{I}$ with respect to different coordinate choices, cf. Lemma 3 below. It does not depend on x. Its minimum $\min_\varepsilon \mathbf{I}$ exists in $\mathcal{E}_n$ and is an invariant of $\mathbf{I}$ in $\mathbf{R}$ which is defined for any ε. For infinite ground fields it equals the generic initial ideal. The existence of maximal initial ideals is more delicate.

Theorem 3. *For every monomial order ε on $\mathbb{N}^n$ the maximal initial ideal* $\max_\varepsilon \mathbf{I} = \max \mathcal{IN}_\varepsilon(\mathbf{I})$ *exists and does not depend on x.*

It is clear that $\max_\varepsilon \mathbf{I}$ is coordinate independent. By the semicontinuity of initial ideals [7] it is realized on a closed stratum of the stratification of $\mathbf{G}$ given by the constancy of the initial ideal (e.g. with respect to the coefficientwise topology on $\mathbf{G}$). For graded monomial orders, $\mathcal{IN}_\varepsilon(\mathbf{I})$ is finite and the existence of $\max_\varepsilon \mathbf{I}$ obvious.

Lemma 1. *Let h^k be a sequence in $\mathbf{K}[[x]]$ and let x^α be the supremum of the initial monomials of the h^k ($=0$ if unbounded). Then, after replacing h^k by a subsequence, $h^k \in \mathbf{M}^{(\alpha)} + \mathbf{M}^k$ for all k, where $\mathbf{M}^{(\alpha)} = \langle x^\gamma,\ \gamma \geq \alpha\rangle$.*

Proof. Let Γ_k be the set of γ in the support of h_k with $\gamma < \alpha$. Set $d_k = \min\{|\gamma|,\ \gamma \in \Gamma_k\}$. If the d_k's were bounded, we could find - replacing h^k by a subsequence - an element γ belonging to all Γ_k, and then $\mathrm{in}_{x\varepsilon} h^k \leq x^\gamma < x^\alpha$ for all k. Therefore, for any k, there exists an m_k such that $d_{m_k} \geq k$. Replacing h^k by h^{m_k} gives $h^k \in \mathbf{M}^{(\alpha)} + \mathbf{M}^k$.

Proof of the theorem. Let o be maximal with $\mathbf{I} \subseteq \mathbf{M}^o$ and let $\gamma \in \mathbb{N}^n$ be the largest n-tuple with $|\gamma| = o$. Then $\mathcal{IN}_\varepsilon(\mathbf{I})$ is bounded above by $\langle x^\gamma \rangle$. Let $\mathbf{J} = (x^{\alpha_1}, \ldots, x^{\alpha_q})$ be its supremum, $\alpha_1 < \ldots < \alpha_q$. The supremum exists because $\mathcal{E}_n$ is well-ordered. Let m be maximal such that $\mathbf{J}$ and $\mathrm{in}_{x\varepsilon} g\mathbf{I}$ coincide in the first $m-1$ vertices for some $g \in \mathbf{G}$. Fix such a g. If $m \le q$, then $\text{in}_{x\varepsilon} g\mathbf{I} < \mathbf{J}$. Choose $g^k \in \mathbf{G}$ with $\mathbf{J} = \sup_k \text{in}_{x\varepsilon} g^k\mathbf{I}$ and such that $\text{in}_{x\varepsilon} g\mathbf{I} \le \text{in}_{x\varepsilon} g^k\mathbf{I} < \mathbf{J}$ for all k. All $\text{in}_{x\varepsilon} g^k\mathbf{I}$ coincide with $\mathbf{J}$ in the first $m-1$ vertices. Let $h_1^k, \ldots, h_{q_k}^k$ be an ordered standard basis of $g^k\mathbf{I}$ so that $\text{in}_{x\varepsilon} h_i^k = x^{\alpha_i}$ for $i < m$ and $\text{in}_{x\varepsilon} h_m^k < x^{\alpha_m}$, with $\sup_k \text{in}_{x\varepsilon} h_m^k = x^{\alpha_m}$. By the lemma we may assume $h_m^k \in \mathbf{M}^{(\alpha_m)} + \mathbf{M}^k$. Artin's Approximation Theorem [3] gives a $g \in \mathbf{G}$ such that the first m vertices of $\text{in}_{x\varepsilon} g\mathbf{I}$ are $\alpha_1, \ldots, \alpha_m$. It follows that m was not maximal.

[To apply the Approximation Theorem, it is necessary to transcribe the conditions on h_i^k and g^k into the existence of solutions of functional equations. These are as follows. (1) There exist $a_{ij}^k \in \mathbf{K}[[x]]$ such that $h_i^k = \sum_j a_{ij}^k \cdot g^k f_j$ for all $i \le m$, where the f_j are a fixed generator system of $\mathbf{I}$. (2) $h_i^k \in \mathbf{M}^{(\alpha_i)}$ for all $i < m$.]

[(3) $h_i^k - x^{\alpha_i} \in \mathbf{M}^{(\alpha_i)+}$ for all $i < m$, where $\mathbf{M}^{(\alpha)+} = <x^\gamma,\ \gamma > \alpha>$. (4) $h_m^k \in \mathbf{M}^{(\alpha_m)} + \mathbf{M}^k$. (5) $h_m^k - x^{\alpha_m} \in \mathbf{M}^{(\alpha_m)+} + \mathbf{M}^k$. Moreover, the exact solutions a_{ij} and g have to be such that $h_i = \sum_j a_{ij} \cdot g f_j$ form a standard basis of $g\mathbf{I}$ modulo $\mathbf{M}^{(\alpha_m)}$, i.e., so that $g\mathbf{I}$ has no elements with initial monomial outside $<x^{\alpha_1}, \ldots, x^{\alpha_m}> + \mathbf{M}^{(\alpha_m)}$. For this, one has to use $<h_1^k, \ldots, h_m^k> \equiv g^k\mathbf{I} \bmod \mathbf{M}^{(\alpha_m)} + \mathbf{M}^k$ and the Becker-Buchberger Criterion [4] with a similar argument as is given below.]

Hence there is a $g \in \mathbf{G}$ such that $\mathbf{J}$ and $\text{in}_{x\varepsilon} g\mathbf{I}$ coincide in the first q vertices. If $\text{in}_{x\varepsilon} g\mathbf{I}$ has precisely q vertices we are done. Else, $\text{in}_{x\varepsilon} g\mathbf{I} < \mathbf{J}$ and there exist $g^k \in \mathbf{G}$ such that $\mathbf{J} = \sup_k \text{in}_{x\varepsilon} g^k\mathbf{I}$ and $\text{in}_{x\varepsilon} g\mathbf{I} \le \text{in}_{x\varepsilon} g^k\mathbf{I} \le \mathbf{J}$. Then $\mathbf{J}$ and $\text{in}_{x\varepsilon} g^k\mathbf{I}$ coincide in the first q vertices. Let $h_1^k, \ldots, h_{q_k}^k$ be an ordered standard basis of $g^k\mathbf{I}$ with $h_{q+1}^k, \ldots, h_{q_k}^k$ reduced with respect to $\mathbf{J}$. It exists because $\mathbf{J} = \text{in}_{x\varepsilon} g\mathbf{I}$. The set $\Gamma_k = \bigcup_{i=q+1}^{q_k} \text{supp } h_i^k$ is contained in $\mathbb{N}^n \setminus (\bigcup_{i=1}^q \alpha_i + \mathbb{N}^n)$ by reducedness. If $d_k = \min \{|\gamma|,\ \gamma \in \Gamma_k\}$ were bounded, there would exist, after replacing g^k by a subsequence, an element γ belonging to all Γ_k. Then $\gamma \notin \bigcup_{i=1}^q \alpha_i + \mathbb{N}^n$ and reducedness would imply $\text{in}_{x\varepsilon} g^k\mathbf{I} \le \mathbf{J} + <x^\gamma> < \mathbf{J}$. Hence the d_k's are unbounded. Passing to a subsequence, we may assume $h_{q+1}^k, \ldots, h_{q_k}^k \in \mathbf{M}^k$ for all k.

This shows that the first q elements $h_1^k, \ldots, h_q^k$ form a standard basis of $g^k\mathbf{I}$ modulo $\mathbf{M}^k$, i.e., that their initial monomials generate $\text{in}_{x\varepsilon} g^k\mathbf{I}$ modulo $\mathbf{M}^k$. By the Becker-Buchberger Criterion [4, Thm. 4.1] we have for $i \le q$ and $k \in \mathbb{N}$:

$$<h_1^k, \ldots, h_q^k> \equiv g^k\mathbf{I} \mod \mathbf{M}^k,$$

$$\text{in}_{x\varepsilon} h_i^k = x^{\alpha_i},$$

for any relation r between $x^{\alpha_1}, \dots, x^{\alpha_q}$ there is an $s^k \in \mathbf{K}[[x]]^q$ with

$r \cdot h^k \equiv s^k \cdot h^k \bmod \mathbf{M}^k$ and $s_i^k \cdot \mathrm{in}_{x\varepsilon} h_i^k \in \mathbf{M}^{(\mathrm{in}_{x\varepsilon} r h^k)}$.

The Approximation Theorem gives $g \in \mathbf{G}$ and $h_1, \dots, h_q \in \mathbf{K}[[x]]$ such that for $i \leq q$:

$<h_1, \dots, h_q> = g\mathbf{I}$,

$\mathrm{in}_{x\varepsilon} h_i = x^{\alpha_i}$,

for any r as before there exists an $s \in \mathbf{K}[[x]]^q$ such that

$r \cdot h = s \cdot h$ and $s_i \cdot \mathrm{in}_{x\varepsilon} h_i \in \mathbf{M}^{(\mathrm{in}_{x\varepsilon} r h)}$.

This proves that $h_1, \dots, h_q$ form a standard basis of $g\mathbf{I}$ and hence $\mathrm{in}_{x\varepsilon} g\mathbf{I} = \mathbf{J}$. The theorem is proven.

Invariants

Most of the resolution invariants appearing in the literature are special cases or variants of the preceding constructions. Assume given, for any regular system of parameters x in $\mathbf{R}$, a map q_x from the set of ideals of $\mathbf{R}$ to a well-ordered set. To make $q_x(\mathbf{I})$ to an invariant of $\mathbf{I}$ in $\mathbf{R}$ it suffices to specify some distinguished coordinates x in $\mathbf{R}$. The many different ways such coordinates and invariants were selected historically all belong to two classes.

Flag invariants. The map $x \to q_x(\mathbf{I})$ is called a flag invariant of $\mathbf{I}$ in $\mathbf{R}$ with respect to $\mathcal{F}$ if it takes the same value for all subordinate coordinates. Thus, for x subordinate, $q_{\mathcal{F}}(\mathbf{I}) = q_x(\mathbf{I})$ depends on $\mathcal{F}$ but not on x. As flags persist under local blowup, also flag invariants do.

Initial ideals. Choosing minimizing and maximizing coordinates makes them to invariants. The first are abundant, the second provide more information. Initial ideals with respect to graded monomial orders and elimination orders are particularly significant. In geometric situations, it is natural to take minima and maxima by extending the ground field to its algebraic closure.

In both cases, the key to control the change of the invariants effectively under blowup is to realize them in coordinates for which the blowup is monomial. This allows to work with coordinate independent data in fixed coordinate systems, and to compute explicitly the invariants of the blown up singularities, cf. Theorems 4, 5 and 6. We give below a list of the most important resolution invariants.

(1) If ε is graded, the Hilbert-Samuel functions of $\mathbf{R}/\mathbf{I}$ and $\mathbf{R}/\mathrm{in}_{x\varepsilon}\mathbf{I}$ coincide. Therefore, $\min_\varepsilon \mathbf{I}$ and $\max_\varepsilon \mathbf{I}$ refine the Hilbert-Samuel function. The Hilbert-Samuel function does not determine the initial ideal [13, rem. 5, p. 549]. For two ideals $\mathbf{I}$ and $\mathbf{I}'$ of $\mathbf{R}$, $\mathrm{in}_{x\varepsilon}\mathbf{I}' < \mathrm{in}_{x\varepsilon}\mathbf{I}$ does not imply the lexicographic inequality between the Hilbert-Samuel functions, nor conversely.

(2) For ε graded and $\min_\varepsilon \mathbf{I} = <x^{\alpha_1}, \dots, x^{\alpha_p}>$ ordered, $\nu^\# = (|\alpha_1|, \dots, |\alpha_p|, \infty, \dots)$ refines Hironaka's ν^*. In contrast to ν^*, $\min_\varepsilon \mathbf{I}$ and $\nu^\#$ are upper semicontinuous with respect to localization at regular primes [16, 20, ex. 12].

(3) Let rev be the reverse graded order given by $\alpha < \beta$ if $|\alpha|^i = |\beta|^i$ for $i = k+1, \dots, n$ and $|\alpha|^k < |\beta|^k$ for some $k \leq n$, where $|\alpha|^i$ denotes the sum of the first i components of α. Then $\max_{rev} \mathbf{I}$ exists trivially and refines $n - \tau$, where τ is the minimal number of variables defining the tangent cone of the singularity [1, 16, 37]. Maximizing coordinates allow to specify those affine charts of the blown up scheme where the Hilbert-Samuel function or $\max_{rev} \mathbf{I}$ has not dropped.

(4) Let lix be the inverse lexicographic order on $\mathbb{N}^n$ with $x_1 < \ldots < x_n$. Denote by gix the corresponding graded lexicographic order, $\alpha <_{gix} \beta$ if $|\alpha| \leq |\beta|$, and where, in case of equality, $\alpha <_{lix} \beta$. Let π_i be the i-th projection on $\mathbb{N}^n$ and σ_i the transposition $(1, i)$. For $f \in \mathbf{R}$ the number $|\mathrm{exp}_{gix} f| - \sum \pi_i(\mathrm{exp}_{lix} \sigma_i f)$ equals the number $d(\Delta) - d(\tilde{\Delta})$ considered by Abhyankar, Hironaka and Spivakovsky [33]. It depends on the chosen coordinates and measures the distance of f from being a monomial up to a unit. One can make it to an invariant by taking the minimum over all coordinate choices. This minimum is not achieved in generic coordinates.

(5) Let $\mathbf{I}$ and $\mathbf{I}'$ be two ideals in $\mathbf{R}$ such that $\mathbf{R}/\mathbf{I}'$ and $\mathbf{R}/\mathbf{I}$ have embedding dimension $n' < n$. For any graded ε, one has $\mathrm{in}_{x\varepsilon}\mathbf{I}' < \mathrm{in}_{x\varepsilon}\mathbf{I}$ since $\mathrm{in}_{x\varepsilon}\mathbf{I}'$ contains a linear monomial, whereas $\mathrm{in}_{x\varepsilon}\mathbf{I}$ does not.

(6) Theorem 3 is valid for more general orders ε on $\mathbb{N}^n$. For instance, given a subset A of $\mathbb{N}^n$ and a well-order ε on it which is compatible with addition, maximal initial ideals exist, provided that all occurring initial ideals have their generators in A. This applies to construct maximal slopes of Newton polyhedra. For $o \in \mathbb{N}$, let $A = \{\alpha \in \mathbb{N}^n,\ \alpha^1 < o\}$, and let δ be some monomial order on $\mathbb{N}^{n-1} = 0 \times \mathbb{N}^{n-1}$. Define ε on A by $\alpha <_\varepsilon \beta$ if $\pi\alpha <_\delta \pi\beta$ in $\mathbb{N}^{n-1}$, where $\pi : A \to \mathbb{N}^{n-1}$ denotes the

projection with center $(o, 0, \ldots, 0)$. For $f = x_1^o + \sum_{\alpha \in A} c_\alpha x^\alpha$ in Weierstrass form, $\max_\varepsilon f$ refines the maximal slope of the Newton polyhedron of f with respect to δ [2, 6, rem. 1.11, 31].

(7) Invariants as maximal slopes are usually defined for restricted classes of coordinates, e.g., such that other invariants as $n - \tau$ are realized in these coordinates, or such that the defining equation has Weierstrass or Tschirnhaus form, or such that the coordinates define components of the exceptional divisor. Theorem 3 extends to these situations. For the first case, let ε be a graded monomial order, and let κ be arbitrary. Then the relative maxima max $\{\mathrm{in}_{x\kappa} g\mathbf{I},\ g \in \mathbf{G},\ \mathrm{in}_{x\varepsilon} g\mathbf{I} = \min_\varepsilon \mathbf{I}\}$ and max $\{\mathrm{in}_{x\kappa} g\mathbf{I},\ g \in \mathbf{G},\ \mathrm{in}_{x\varepsilon} g\mathbf{I} = \max_\varepsilon \mathbf{I}\}$ exist and do not depend on x.

The proof is similar, noting that a perturbation of an automorphism g in high degrees does not affect $\mathrm{in}_{x\varepsilon} g\mathbf{I}$ since ε is graded. The extension in the other cases is immediate. This provides a characteristic-free interpretation of weighted orders in the sense of Moh and Encinas-Villamayor as maximal initial ideals [26, 27, 11, 12, 20, 21, 22]. The hypersurface of maximal contact is replaced by coordinates which maximize the weighted order.

(8) Let $f \in \mathbf{K}[[x_1, x_2, x_3]]$ be in Weierstrass form with respect to x_1. Let o be its order at 0. Choose coordinates x_2, x_3 such that the projection of the Newton polyhedron of f from $(o, 0, 0)$ to $\mathbb{N}^2$ is minimal set-theoretically. Let $\mathcal{F}$ be the flag in $\mathbf{K}[[x_2, x_3]]$ of ideals $\mathbf{M}_1 = <x_3> \subseteq \mathbf{M}_0 = <x_2, x_3>$. Then for all monomial orders ε on $\mathbb{N}^2$ such that $x_2 < x_3$, the monomial $\mathrm{in}_{x\varepsilon} f$ is a flag invariant of f. Taking in particular the lexicographic order, this contains Hironaka's invariant β [17, 21]. His δ appears as a maximal slope as in (6). The coordinate changes Hironaka considers are those from $\mathbf{U}_\varepsilon$ for ε the lexicographic order on $\mathbb{N}^3$.

(9) For ε graded and $\mathbf{K}$ infinite, depth $\mathbf{R}/\mathbf{I}$ equals the number of variables appearing in the minimal monomial generator system of $\min_\varepsilon \mathbf{I}$ [13, p. 567, 14, p. 162, 30, II. Cor. 10].

[This is false for finite fields. Let $\mathbf{K} = \mathbf{F}_2$ and set $f = x^4y^4 + x^2y^6$. Then $\min_{gix} f = x^6y^2$ and not x^8 as would occur for infinite fields.]

(10) Let $f \in \mathbf{R}$ be a hypersurface with Newton polyhedron $N_x f \subseteq \mathbb{R}^n_+$ with respect to coordinates x. There exist various attempts to extract coordinate free information from $N_x f$ by choosing coordinates for which $N_x f$ is most remote from the origin. This can be systematized as follows. For ε a graded monomial order and $k \leq n$, denote by

κ the k-th elimination order with respect to ε, $\alpha <_\kappa \beta$ if $|\alpha|^k \leq |\beta|^k$ where, in case of equality, $\alpha <_\varepsilon \beta$. We have $x_{k+1} <_\kappa \ldots <_\kappa x_n <_\kappa x^\alpha <_\kappa x_1 <_\kappa \ldots <_\kappa x_k <_\kappa x^\beta$ for $\alpha, \beta \in \mathbb{N}^n$ with $\alpha^{k+1} + \ldots + \alpha^n \geq 2$ and $\beta^1 + \ldots + \beta^k \geq 2$. Then $\max_\kappa f$ describes the maximum over all coordinate choices of the minimal distance of $N_x f$ to the $(n-k)$-dimensional coordinate planes in $\mathbb{R}^n_+$.

(11) Let $\mathcal{F}$ be a flag in $\mathbf{R}$ and let x be subordinate to $\mathcal{F}$. For $f \in \mathbf{R}$, denote by $E^1_x f$ the vertices of the Newton polyhedron $N_x f$ of f which are selected by weights $w_1 < \ldots < w_n$, i.e., whose scalar product with $w = (w_1, \ldots, w_n)$ is minimal. Let $T^1_x f$ be the set of facets of $E^1_x f$, i.e., $(n-1)$-dimensional compact faces of the positive convex hull of $E^1_x f$. It will be shown in Theorem 5 that $E^1_x f$ and $T^1_x f$ only depend on $\mathcal{F}$ but not on x, are hence flag invariants of f with respect to $\mathcal{F}$. For $T \in T^1_x f$, denote by $v(T)$ the $(n-1)$-dimensional volume vector given as the exterior product of the edges of T, chosen so that $v(T)$ points inward $N_x f$. The components $v_i(T)$ of $v(T)$ are given as the maximal minors of the matrix whose rows are the edges of T, taken with alternating sign. In particular, these are homogeneous polynomials of degree $n-1$ in the components of the vertices of $N_x f$ of f. The vector $v(T)$ is a flag invariant of f with respect to $\mathcal{F}$ whose components satisfy $v_1 \geq \ldots \geq v_n \geq 0$. The proof goes by induction on n and projection to $\mathbb{N}^{n-1}$ or using the fact that T is the unique compact facet of its positive convex hull. One might take any affine independent n points T in $E^1_x f$ whose volume vector $v(T)$ lies in $\mathbb{N}^n$. Also the n-dimensional volume of the n-simplex spanned by T and the origin is an interesting flag invariant of f. We shall see in Theorem 5 that $v(T)$ always decreases under monomial blowup.

Automorphisms

Let Ω be the set of all monomial orders on $\mathbb{N}^n$ and let Λ be those for which $e_1 < \ldots < e_n$ for the natural basis of $\mathbb{N}^n$. Fix coordinates x in $\mathbf{R}$. Let S_n act from the left on $\mathbb{N}^n$, Ω and $\mathbf{R}$ by $(s\alpha)^i = \alpha^{s^{-1}i}$, $\alpha <_{s\varepsilon} \beta$ if $s^{-1}\alpha <_\varepsilon s^{-1}\beta$ and $s \cdot \sum c_\alpha x^\alpha = \sum c_{s^{-1}\alpha} x^\alpha$. Then $\Omega = S_n \Lambda$, $\operatorname{supp} tf = t \cdot \operatorname{supp} f$ and $\exp_{x,t\varepsilon} f = t \cdot \exp_{x\varepsilon} t^{-1} f$ for $t \in S_n$.

Let $\varepsilon \in \Omega$ and let lix denote the inverse lexicographic order. The following two lemmas are easy.

Lemma 2. (a) $\mathbf{B}_\varepsilon = \{g \in \mathbf{G},\ \mathrm{in}_{x\varepsilon} g_i = x_i$ *for all* $i\}$.
(b) $\mathbf{B}_{lix} \subseteq \mathbf{B}_\varepsilon$ *and* $\mathbf{U}_\varepsilon \subseteq \mathbf{U}_{lix}$ *if* $\varepsilon \in \Lambda$.

Proof. If $g \in \mathbf{B}_\varepsilon$ then trivially $\mathrm{in}_{x\varepsilon} g_i \geq x_i$. The induced automorphism $\bar{g}$ on $\mathbf{M}/\mathbf{M}^2$ has triangular form with non-zero entries on the diagonal, which implies that $\mathrm{in}_{x\varepsilon} g_i = x_i$. The inverse inclusion and (b) are immediate.

Automorphisms from the Borel factor do not affect initial ideals. This will be the clue to study the behaviour of initial ideals under blowup.

Lemma 3. (a) $\exp_{x\varepsilon} gf = \exp_{g^{-1}x\varepsilon} f$ *and* $\mathrm{in}_{x\varepsilon} gf = g \cdot \mathrm{in}_{g^{-1}x\varepsilon} f$ *for* $g \in \mathbf{G}$.
(b) $\mathrm{in}_{x\varepsilon} bf = \mathrm{in}_{x\varepsilon} f$ *for* $b \in \mathbf{B}_\varepsilon$.

Proof. Write $\exp_{x\varepsilon} gf = \exp_{x\varepsilon} g \sum c_\alpha x^\alpha = \exp_{x\varepsilon} \sum c_\alpha (g_1, \ldots, g_n)^\alpha = \exp_{x\varepsilon} \sum d_\alpha x^\alpha$ for certain d_α. Let $y = g^{-1}x$ and $h = g^{-1}$ with series $h_i = hy_i$ in y. Then $\exp_{y\varepsilon} f = \exp_{y\varepsilon} \sum c_\alpha x^\alpha = \exp_{y\varepsilon} \sum d_\alpha (h_1, \ldots, h_n)^\alpha = \exp_{x\varepsilon} f$. Write $\mathrm{in}_{x\varepsilon} gf = x^\alpha$ with $\alpha = \exp_{x\varepsilon} gf$, then $\mathrm{in}_{y\varepsilon} f = (g^{-1}x)^\alpha = g^{-1}(x^\alpha) = g^{-1} \cdot \mathrm{in}_{x\varepsilon} gf$. (b) is easy.

For $f \in \mathbf{R}$ and $t \in S_n$, denote by $E_x f$ the set of vertices of the Newton polyhedron of f and by $E_x^t f$ the sector of those vertices which are selected by weights $w_1, \ldots, w_n$ with $w_{t1} < \ldots < w_{tn}$. Let $T_x^1 f$ and $v(T)$ be defined as in (11) of the preceding section. Let $\mathbf{I}$ be an ideal of $\mathbf{R}$. We can now specify the coordinate changes which are necessary to realize the minimal and maximal initial ideals.

Theorem 4. (a) $E_x^t f = \{\exp_{x\varepsilon} f,\ \varepsilon \in t\Lambda\} = \{t \cdot \exp_{x\varepsilon} t^{-1} f,\ \varepsilon \in \Lambda\}$.
(b) *For coordinates y in $\mathbf{R}$ there exist $u \in \mathbf{U}_{lix}$ and $s \in S_n$ such that for all $t \in S_n$*

$$t^{-1} \cdot E_y^t f = (ts)^{-1} \cdot E_{ux}^{ts} f.$$

(c) *For x subordinate to a flag $\mathcal{F}$ in $\mathbf{R}$, $E_x^1 f$, $T_x^1 f$ and $\sum_{T \in T_x^1 f} v(T)$ are flag invariants of f with respect to $\mathcal{F}$.*
(d) *For any $\varepsilon \in \Omega$, there exist g and h in $\mathbf{U}_\varepsilon S_n$ such that*

$$\mathrm{min}_\varepsilon \mathbf{I} = \mathrm{in}_{x\varepsilon} g\mathbf{I} \quad \textit{and} \quad \mathrm{max}_\varepsilon \mathbf{I} = \mathrm{in}_{x\varepsilon} h\mathbf{I}.$$

If ε is graded, g and h can be chosen in $S_n U_\varepsilon$.

Proof. On finite subsets of $\mathbb{N}^n$, any monomial order is given as pullback of the natural order on $\mathbb{R}_+$ by some linear form $w : \mathbb{N}^n \to \mathbb{R}_+$. This implies (a). For (b), choose $g \in \mathbf{G}$ with $y = gx$ and write $g = sub \in S_n \mathbf{U}_{lix} \mathbf{B}_{lix}$ according to Theorem 2. Then, by Lemma 2(b) and Lemma 3(b),

$$\begin{aligned} E_y^t f &= \{\exp_{y,\varepsilon} f,\ \varepsilon \in t\Lambda\} \\ &= \{\exp_{y,t\varepsilon} f,\ \varepsilon \in \Lambda\} \end{aligned}$$

$$
\begin{aligned}
&= \{t \cdot \exp_{y,\varepsilon} t^{-1} f,\ \varepsilon \in \Lambda\} \\
&= \{t \cdot \exp_{x,\varepsilon} b^{-1} u^{-1} s^{-1} t^{-1} f,\ \varepsilon \in \Lambda\} \\
&= \{t \cdot \exp_{x,\varepsilon} u^{-1} s^{-1} t^{-1} f,\ \varepsilon \in \Lambda\} \\
&= \{t \cdot \exp_{ux,\varepsilon} s^{-1} t^{-1} f,\ \varepsilon \in \Lambda\} \\
&= \{ts^{-1} t^{-1} \cdot \exp_{ux,ts\varepsilon} f,\ \varepsilon \in \Lambda\} \\
&= ts^{-1} t^{-1} \cdot \{\exp_{ux,\varepsilon} f,\ \varepsilon \in ts\Lambda\} \\
&= ts^{-1} t^{-1} \cdot E^{ts}_{ux} f.
\end{aligned}
$$

Assertion (c) is then clear since, for g subordinate, i.e. in $\mathbf{B}_{lix}$, the above u and s are the identity and (d) follows from Theorem 2 and Lemma 3.

[In (d), permutations cannot be omitted. For the maximum, take $n = 3$ and the second elimination order κ with respect to gix. Let $f(x, y, z) = x^4 + yz^7 + y^5 z^2$ and let s be the permutation which commutes y with z. Then $\mathrm{in}_{gix} f = \mathrm{in}_{gix} sf = x^4$, whereas $\mathrm{in}_\kappa f = yz^7 <_\kappa y^2 z^5 = \mathrm{in}_{x\kappa} sf = \max_\kappa f$. Therefore a permutation is necessary to realize the gix-relative maximum $\max_\kappa f$. For the minimum, let $\mathbf{K}$ be a field of characteristic 2, $n = 3$ and set $\mathbf{I} = <x^8, z^8, x^2z^6, x^4y^4 + x^4z^4>$. The permutation $s(x, y, z) = (y, z, x)$ yields $s\mathbf{I} = <y^8, x^8, y^2x^6, y^4z^4 + y^4x^4>$ and therefore $\mathrm{in}_{x,gix} s\mathbf{I} = <x^8, y^8, x^6y^2, x^4y^4> = \min_{gix} \mathbf{I}$.]

[However, $\mathrm{in}_{x,gix} u\mathbf{I} > \min_{gix} \mathbf{I}$ for $u \in \mathbf{U}_{gix}$. Let $u(x, y, z) = (x, y + ax, z + bx + cy)$ be in U_{gix} different from the identity. The presence of x^8 in $\mathbf{I}$ reduces the calculation of $u\mathbf{I}$ to the case $u(x, y, z) = (x, y, z + y)$. Then $u\mathbf{I} = <x^8, z^8 + y^8, x^2z^6 + x^2y^6, x^4z^4>$ and $\mathrm{in}_{x,gix} u\mathbf{I} = <x^8, y^8, x^2y^6, x^4z^4> > \min_{gix} \mathbf{I}$.]

[Applying the permutation $t(x, y, z) = (y, x, z)$ gives $t\mathbf{I} = <y^8, z^8, y^2z^6, x^4y^4 + y^4z^4>$ and $\mathrm{in}_{x\varepsilon} t\mathbf{I} = <y^8, z^8, y^2z^6, x^4y^4> > \min_\varepsilon \mathbf{I}$. Similarly, we get $tu\mathbf{I} = <y^8, z^8 + x^8, y^2z^6 + x^6y^2, y^4z^4>$ with $\mathrm{in}_{x\varepsilon} tu\mathbf{I} = <x^8, y^8, x^6y^2, y^4z^4> > \min_\varepsilon \mathbf{I}$. This shows additionally that to realize the minimum it is not sufficient to allow only permutations in the variables appearing in the minimal monomial generator system of $\min_\varepsilon \mathbf{I}$.]

We next describe the effect of automorphisms from $\mathbf{U}_\varepsilon S_n$ on blowups. Let $\varphi : \mathbf{R} \to \mathbf{R}'$ be a residually rational local blowup of regular center $\mathbf{P}$ in $\mathbf{R}$ at $\mathbf{Q}'$ in $\mathbf{S}$. Let $\mathbf{Q}$ be an ideal in $\mathbf{R}$ whose class in $\mathbf{P}/\mathbf{MP}$ determines $\mathbf{Q}'$. There exist a regular system of parameters $x_1, \ldots, x_n$ in $\mathbf{R}$, a subset J of $\{1, \ldots, n\}$ and some $j \in J$ such that $\mathbf{P} = <x_i,\ i \in J>$ and $\mathbf{Q} = <x_i,\ i \in J \setminus j>$ [16, III.2, 22, appendix]. The blowup φ is then called monomial of type $j \in J$ with respect to $x_1, \ldots, x_n$. The regular system of parameters of $\mathbf{R}'$ given by $y_i = x_i/x_j$ for $i \in J \setminus j$ and $y_i = x_i$ for $i \notin J \setminus j$ is called the induced coordinate system in $\mathbf{R}'$. Automorphism groups of $\mathbf{R}'$ are marked by a prime. For automorphisms g of $\mathbf{R}$ and h of $\mathbf{R}'$ consider the diagram $(*)$

$$\begin{array}{ccc} \mathbf{R}' & \xrightarrow{h} & \mathbf{R}' \\ \uparrow \varphi & & \uparrow \psi \\ \mathbf{R} & \xrightarrow{g} & \mathbf{R} \end{array}$$

where ψ is some blowup of $\mathbf{R}$. Assume that $J = \{1, \ldots, j\}$, let gix be the graded inverse lexicographic order and let κ be the $(j-1)$-st elimination order with respect to gix.

Lemma 4. (a) *Any $g = su \in S_n U_{gix}$ induces an automorphism h of $\mathbf{R}'$ for which $(*)$ commutes with $\psi = s\varphi s^{-1}$, and φ is monomial of type $sj \in sJ$ with respect to gx.*
(b) *Any automorphism $h = sv \in S_n \mathbf{U}'_\kappa$ of $\mathbf{R}'$ is induced from an automorphism $g = su \in S_n \mathbf{U}_\kappa$ for which $(*)$ commutes with $\psi = s\varphi s^{-1}$, and φ is monomial of type $sj \in sJ$ with respect to gx.*

For φ monomial with respect to x and for arbitrary ε and $u \in U_\varepsilon$ there exists a $b \in B_\varepsilon$ such that bu preserves the monomiality of φ. This is useful to realize minima of initial ideals with respect to graded monomial orders.

Proof. (a) is clear because u fixes $\mathbf{P}$ and $\mathbf{Q} + \mathbf{P}^2$. For (b), the definition of κ implies that $v_i = y_i + r_i(y) + s_i(y)$ where s_i depends on y_l with $l \notin J \setminus j$, and where for $i \in J \setminus j$, r_i is linear depending on y_l with $l \in J \setminus \{i, j\}$, whereas $r_i = 0$ if $i \notin J \setminus j$.

[This holds because κ satisfies $x_j < \ldots < x_n < x^\alpha < x_1 < \ldots < x_{j-1} < x^\beta$ for any $\alpha, \beta \in \mathbb{N}^n$ with $\alpha^j + \ldots + \alpha^n > 0$ and $\beta^1 + \ldots + \beta^{j-1} > 0$.]

Define u in $\mathbf{G}$ by $u_i = x_i + r_i(x) + x_j s_i(x)$ if $i \in J \setminus j$ and $u_i = x_i + s_i(x)$ if $i \notin J \setminus j$. Then $v \in \mathbf{U}'_\kappa$ implies $u \in \mathbf{U}_\kappa$. Since φ is monomial of type $j \in J = \{1, \ldots, j\}$ with respect to x we get $y_j v_i = u_i$ for $i \in J \setminus j$ and $v_i = x_i$ for $i \notin J \setminus j$. This proves (b).

Assertions (a) and (b) hold for arbitrary $J \subseteq \{1, \ldots, n\}$ and $j \in J$ provided the monomial orders satisfy $x_i < x_j < x_l$ for all $i \in J \setminus j$ and $l \notin J$, respectively $x_j < x_i$ for all $i \neq j$ and $x_i < x_l x_m$ for all $i, l, m \in J$. Assertion (b) fails for $h = vs \in \mathbf{U}'_\kappa S_n$. Take $n = 2$, $J = \{1, 2\}$, $j = 2$, $s \in S_2$ the transposition and $v = (x_1 + x_2^2, x_2)$. Then $h = sv = (x_2 + x_1^2, x_1)$ maps the exceptional divisor $x_2 = 0$ onto $x_2 + x_1^2 = 0$ with singular image in W.

Blowup

Let $\varphi : \mathbf{R} \to \mathbf{R}'$ be a residually rational local blowup with regular center $\mathbf{P}$. If φ is monomial of type $j \in J$ with respect to coordinates x in $\mathbf{R}$ with induced coordinates y in $\mathbf{R}'$, the strict transform $\mathbf{I}'$ of $\mathbf{I}$ is generated by all $f' = \sum c_\gamma y^{\gamma^*}$ where $f = \sum c_\gamma x^\gamma \in \mathbf{I}$ and $* : \operatorname{supp} f \to \operatorname{supp} f'$ is given by

$$\gamma^{*i} = \gamma^i \qquad \text{for } i \neq j,$$

$$\gamma^{*i} = \gamma^i + |\gamma|^{J\setminus j} - \operatorname{ord}_{\mathbf{P}} f, \qquad |\gamma|^{J\setminus j} = \textstyle\sum_{i \in J\setminus j} \gamma^i.$$

Consider now the Newton polyhedron of some f. If it has some compact facet T (cf. (11) of the section on invariants), let $v(T)$ be the associated normal vector. We show that v decreases under any monomial blowup with respect to the componentwise order (i.e., independently of the dimension of the center). This reduces Hironaka's polyhedral game automatically to the case of Newton polyhedra without compact facets. As the blowup can be monomialized without affecting the flag this suggests to reduce the general resolution problem of hypersurfaces to the case where the Newton polyhedron has no compact facet of the described type.

Theorem 5. (a) *For any flag $\mathcal{F}$ in $\mathbf{R}$ transversal to $\mathbf{P}$ there exist subordinate coordinates x in $\mathbf{R}$ such that φ is monomial of type $j \in J$ with respect to x for some j and J.*

(b) *Let φ be monomial of type $j \in J$ with respect to x and let $\mathbf{I} = <f>$. For any facet T of the Newton polyhedron $N_x f$ of f the induced simplex T^* satisfies*

$$v_i(T^*) = v_i(T) \qquad \text{for } i \notin J \setminus j,$$

$$v_i(T^*) = v_i(T) - v_j(T) \qquad \text{for } i \in J \setminus j.$$

T^* is a facet of the Newton polyhedron $N_y f'$ of f' if and only if $v_i(T^*) \geq 0$ for all i.

(c) *Let $\mathcal{F}'$ be the flag induced from $\mathcal{F}$ in $\mathbf{R}'$ and let $T \in T^1_{\mathcal{F}} f$. Then T^* belongs to $T^1_{\mathcal{F}'} f'$ if and only if $v_i(T^*) \geq v_{i+1}(T^*) \geq 0$ for all i.*

Remark. In general, T^* is not a facet of $N_y f'$. If $\mathbf{P} = \mathbf{M}$ and $T \in T^1_{\mathcal{F}} f$, then $T^* \in T^1_{\mathcal{F}'} f'$ can at most happen in one chart, namely for $j = 1$. This suggests to consider the number of compact facets of the Newton polyhedron as an induction invariant. If this number remains constant, no facet disappears and one may consider the $(n-1)$-dimensional volume $\sum_{T\in T^1_{\mathcal{F}} f} v(T)$ of the compact part of $N_x f$ as a second invariant. However, it is not clear how to make both coordinate independent.

Proof. Start with any subordinate coordinates x in $\mathbf{R}$. Then the center Z is defined in W by certain series $a_1, \dots, a_k$ in $\mathbf{K}[[x]]$. Let m be the smallest index for which x_m appears linearly in one of the a_i's, say a_1. Write $a_1 = x_m + b$. There exists (compare with the proof of the Gauss-Bruhat decomposition) a subordinate coordinate change in $\mathbf{R}$ such that, in the new coordinates, $b \in \mathbf{K}[[x_1, \dots, x_{m-1}]]$. By minimality of m, $b \in \mathbf{M}^2$. As $Z \cap F_{m-1}$ is regular of ideal $(b, a_2, \dots, a_k, x_m, \dots, x_n)$ the choice of m implies $b = 0$ and $a_1 = x_m$. Replacing W by W_1 defined by $x_m = 0$ and Z by $Z \cap W_1$ induction applies. Hence there exist subordinate coordinates $x_1, \dots, x_n$ in $\mathbf{R}$ and a set of indices J such that Z is given by $<x_i,\ i \in J>$.

A similar argument shows that there exists $j \in J$ such that $\mathbf{Q} = <x_i,\ i \in J \setminus j>$.

[It can be assumed that $\mathbf{Q}$ is generated by linear forms $a_1, \dots, a_{k-1}$ in x_i with $i \in J$. Let $m \in J$ be minimal such that x_m appears in one of the a_i's, say a_1. Write $a_1 = x_m + b$ with $b \in <x_i,\ i \in J \setminus m>$. A subordinate coordinate change reduces to $a_1 = x_m$ and induction applies.]

For (b) it suffices to observe that $v(T)$ is dual to T and transforms according to the inverse of the dual to $*$. The formula follows. Note that (b) also holds for the total transform of f. To see (c), recall that subordinate coordinate changes lie in B_{lix} and hence, by Lemma 2(b), in $\mathbf{B}_\varepsilon$. Lemma 3(b) implies the assertion.

The extension of (b) to lower dimensional faces of $N_x f$ cannot be expected without specification of the center of blowup [34, 20].

The regular center $\mathbf{P}$ is permissible with respect to an ideal $\mathbf{I}$ of $\mathbf{R}$ if $\mathrm{ord}_{\mathbf{M}} f = \mathrm{ord}_{\mathbf{P}} f$ for the elements f of a standard basis of $\mathbf{I}$ with respect to a graded monomial order ε [16, Lemma 9, chap. II, Thm. 6, p. 251, Thm. 2, p. 195, Cor 1., Prop. 1, p. 184]. If φ is monomial of type $j \in J$ with respect to coordinates x, this is equivalent to $|\gamma|^J \geq |\alpha|$ for all f and all $\gamma \in \mathrm{supp}\ f$.

Lemma 5. *Let $\varepsilon \in \Lambda$ and let κ be the k-th elimination order with respect to ε for $k \leq n$ fixed. Let j_0 be the minimal height among the $\mathbf{I}$-permissible ideals. Assume $\mathbf{K}$ infinite.*

(a) $j_0 \geq n - \mathrm{depth}\, \mathbf{R}/\mathbf{I}$.

(b) *If* $k \geq n - \operatorname{depth} \mathbf{R}/\mathbf{I}$ *and if there exist coordinates* x *such that* $\operatorname{in}_{x\kappa}\mathbf{I} = \operatorname{in}_{x\varepsilon}\mathbf{I} = \min_\varepsilon \mathbf{I}$, $\mathbf{P} = <x_1, \ldots, x_k>$ *is permissible for* $\mathbf{I}$.
(c) *For* $f \in \mathbf{R}$, $k \geq j_0$ *and coordinates* x *with* $\operatorname{in}_{x\kappa} f = \max_\kappa f$, $\mathbf{P} = <x_1, \ldots, x_k>$ *is permissible for* f.

Proof. The number of variables appearing in the minimal monomial generator system $x^{\alpha_1}, \ldots, x^{\alpha_p}$ of $\min_\varepsilon \mathbf{I}$ equals $n - \operatorname{depth} \mathbf{R}/\mathbf{I}$ [13, 14, 30], whence (a). For (b), let $f_1, \ldots, f_p$ be an ordered standard basis of $\mathbf{I}$ with respect to ε and x. From $\operatorname{in}_{x\kappa}\mathbf{I} = \operatorname{in}_{x\varepsilon}\mathbf{I}$ follows $\gamma \geq_\kappa \alpha_i$ for any $\gamma \in \operatorname{supp} f_i$. As $\varepsilon \in \Lambda$ and $k \geq n - \operatorname{depth} \mathbf{R}/\mathbf{I}$ we have $\alpha_i \in \mathbb{N}^k$, say $|\alpha_i|^k = |\alpha_i|$. This gives $|\gamma|^k \geq |\alpha_i| = \operatorname{ord}_{\mathbf{M}} f_i$ for all γ. Hence $<x_1, \ldots, x_k>$ is permissible. (c) is obvious.

The techniques developed so far allow to control explicitly the behaviour of initial ideals under blowup. This information is collected in the next two theorems.

Theorem 6. *Let* $\varphi : \mathbf{R} \to \mathbf{R}'$ *be a residually rational local blowup with center* $\mathbf{P}$ *permissible with respect to* $\mathbf{I}$. *For any graded monomial order* ε *on* $\mathbb{N}^n$ *one has*

$$\min_\varepsilon \mathbf{I}' \leq_\varepsilon \min_\varepsilon \mathbf{I},$$

and

$$\max_\varepsilon \mathbf{I}' \leq_\varepsilon \max_\varepsilon \mathbf{I}.$$

The first inequality is strict if $\mathbf{K}$ *is infinite and* $\operatorname{height} \mathbf{P} = n - \operatorname{depth} \mathbf{R}/\mathbf{I}$.

This refines slightly the infinitesimal semicontinuity of the Hilbert-Samuel function of $\mathbf{R}/\mathbf{I}$, though it does not imply it. The second inequality contains as a particular case the infinitesimal semicontinuity of $n - \tau$. In contrast to [16], we do not need the persistence of standard bases under blowup. The assertions of the theorem extend to residually algebraic blowups if the mimima and maxima are taken over the algebraic closure of $\mathbf{K}$. Theorem 8 on localization then implies that the pair $(\min_\varepsilon \mathbf{I}, \max_\varepsilon \mathbf{I})$ does not increase lexicographically under any permissible blowup, cf. [5, Thm (0E), p. 42, 6].

Proof. Permuting the components of $\mathbb{N}^n$ allows to assume $\varepsilon \in \Lambda$. Choose coordinates x in $\mathbf{R}$ for which φ is monomial of type $j \in J = \{1, \ldots, j\}$ with $j = \operatorname{height} \mathbf{P}$. Let $g \in \mathbf{G}$ be an automorphism of $\mathbf{R}$ which realizes $\min_\varepsilon \mathbf{I}$. By Theorem 4, g may be chosen in $S_n U_\varepsilon$. By Lemma 2, $U_\varepsilon \subseteq U_{gix}$. Lemma 4 implies that φ is monomial with respect to gx. Replace x by gx and let y be the induced coordinates in $\mathbf{R}'$. Since $\operatorname{in}_{x\varepsilon}\mathbf{I} = \min_\varepsilon \mathbf{I}$ and $\operatorname{in}_{y\varepsilon}\mathbf{I}' \geq \min_\varepsilon \mathbf{I}'$ it suffices to show that $\operatorname{in}_{y\varepsilon}\mathbf{I}' \leq \operatorname{in}_{x\varepsilon}\mathbf{I}$.

Let f_i be an ordered standard basis of $\mathbf{I}$ with respect to ε, with strict transforms f_i' and initial monomials x^{α_i} and $y^{\alpha_i'}$ respectively. We may assume $f_i - x^{\alpha_i}$ reduced with respect to $\mathbf{I}$. Then $\alpha_i' \leq \alpha_i^* \leq \alpha_i$ for all i. (i) If equality holds for all i, $\operatorname{in}_{x\varepsilon}\mathbf{I}$ is contained in $\operatorname{in}_{y\varepsilon}\mathbf{I'}$ (upon writing x for y) and Lemma 6 below gives $\operatorname{in}_{y\varepsilon}\mathbf{I'} \leq \operatorname{in}_{x\varepsilon}\mathbf{I}$. (ii) If not, let m be the first index with $\alpha_m' < \alpha_m$. From $\alpha_i' = \alpha_i$ for $i < m$ follows $\alpha_i^* = \alpha_i$ and $|\alpha_i|^{J\setminus j} = |\alpha_i|$. This implies $\alpha_i \in \mathbb{N}^{J\setminus j}$ for $i < m$, i.e., components with index outside $J \setminus j$ are zero. Let $\gamma \in \operatorname{supp}\ f_m$ be the exponent with $\gamma^* = \alpha_m'$. Since f_m is reduced with respect to $f_1, \ldots, f_{m-1}$ it does not belong to $\bigcup_{i<m} \alpha_i + \mathbb{N}^n$. As γ^* differs from γ only in the j-th component and $\alpha_i \in \mathbb{N}^{J\setminus j}$, also $\gamma^* = \alpha_m' \notin \bigcup_{i<m} \alpha_i + \mathbb{N}^n$. Then $\operatorname{in}_{y\varepsilon}\mathbf{I'} < \operatorname{in}_{x\varepsilon}\mathbf{I}$ follows from

Lemma 6. *Let $\mathbf{J}$ and $\mathbf{J'}$ be monomial ideals. Denote by $(\alpha_1, \ldots)$ the ordered vertices of $\mathbf{J}$ with respect to ε. Assume that there is an $x^{\alpha'} \in \mathbf{J'}$ such that for some m:*

$$\alpha' < \alpha_m,$$

$$\alpha' \notin \textstyle\bigcup_{i<m} \alpha_i + \mathbb{N}^n,$$

$$x^{\alpha_i} \in \mathbf{J'} \quad \text{for } i < m.$$

Then $\mathbf{J'} < \mathbf{J}$. This holds in particular if $\mathbf{J} \subsetneq \mathbf{J'}$.

Proof of the lemma. Let $(\beta_1, \ldots)$ be the ordered vertices of $\mathbf{J'}$, and let k be the first index for which $\alpha_k \neq \beta_k$. Then $\alpha_k \notin \bigcup_{i<m} \alpha_i + \mathbb{N}^n = \bigcup_{i<m} \beta_i + \mathbb{N}^n$. If $k < m$, $x^{\alpha_k} \in \mathbf{J'}$ implies that $\alpha_k \in \bigcup_{i \geq m} \beta_i + \mathbb{N}^n$. Therefore $\beta_m = \min \bigcup_{i\geq m} \beta_i + \mathbb{N}^n \leq \alpha_k$. This, in turn, gives $\beta_k < \alpha_k$ and $\mathbf{J'} < \mathbf{J}$. Observe that $\alpha_k < \infty$ and $\beta_k = \infty$ is impossible. If $k \geq m$, then $\alpha' \in \bigcup_{i\geq m} \beta_i + \mathbb{N}^n$ and $\beta_m \leq \alpha' < \alpha_m$. Therefore $k = m$ and $\mathbf{J'} < \mathbf{J}$. If $\mathbf{J} \subsetneq \mathbf{J'}$, take $x^{\alpha'} \in \mathbf{J'} \setminus \mathbf{J}$ arbitrary and $m \in \mathbb{N}$ with $\alpha_m = \infty$.

We continue with the proof of the theorem. Assume that $\mathbf{P}$ has height $j = n - \operatorname{depth} \mathbf{R}/\mathbf{I}$. Choose coordinates x with $\operatorname{in}_{x\varepsilon}\mathbf{I} = \min_\varepsilon \mathbf{I}$. Then the monomials $\operatorname{in}_{x\varepsilon} f_i$ depend on j variables, namely $x_1, \ldots, x_j$, because $\varepsilon \in \Lambda$. Moreover, $\alpha_i \notin \mathbb{N}^{j-1}$ for some i, say $|\alpha_i|^{j-1} < |\alpha_i|^j = |\alpha_i|$. Hence $\mathbf{P} = <x_1, \ldots, x_j>$. Then $\alpha_i' \leq \alpha_i^* < \alpha_i$ and $\min_\varepsilon \mathbf{I'} < \min_\varepsilon \mathbf{I}$ follows (same proof as in (ii) above).

For the maximum, choose coordinates x in $\mathbf{R}$ such that φ is monomial of type $1 \in J = \{1, \ldots, j\}$. Let y be the induced coordinates in $\mathbf{R'}$. Let $h \in \mathbf{G'}$ be an automorphism of $\mathbf{R'}$ which realizes $\max_\varepsilon \mathbf{I'}$. As before, we may take $h = sv \in S_n U_\varepsilon' \subseteq S_n U_{gix}'$. By Lemma 4, h is induced by an automorphism $g = su$ of $\mathbf{R}$. Replacing x by

gx and y by hy allows to assume $\mathrm{in}_{y\varepsilon}\mathbf{I}' = \max_\varepsilon \mathbf{I}'$ and φ monomial of type $s1 \in sJ$. Then $\max_\varepsilon \mathbf{I}' = \mathrm{in}_{y\varepsilon}\mathbf{I}' \leq \mathrm{in}_{x\varepsilon}\mathbf{I} \leq \max_\varepsilon \mathbf{I}$ follows.

[Observe that the minimum had to be realized in $\mathbf{R}$, the maximum in $\mathbf{R}'$, cf. with [20, sec. 5]. Sometimes it can be shown that if the maximum is realized in $\mathbf{R}$ it is automatically realized in $\mathbf{R}'$ [17, Rem. (5), p. 126].]

If the Hilbert-Samuel function decreases, also $\min_\varepsilon \mathbf{I}$ decreases. This follows from the theorem and the fact that $\min_\varepsilon \mathbf{I}$ determines the Hilbert-Samuel function. The converse is not true. Take $\varepsilon = gix$ in $\mathbb{N}^3$ with $x < y < z$, $J = \{1, 2, 3\}$ and $j = 2$. Then $f = z^2 + xy^2$ has strict transform $f' = z^2 + xy$, but $\min_\varepsilon f' = xy < \min_\varepsilon f = z^2$. The assumption that the blowup is residually rational is mandatory for the maximum [20, ex. 7].

Theorem 7. *Let $\varphi : \mathbf{R} \to \mathbf{R}'$ be a residually rational local blowup with permissible center $\mathbf{P}$ with respect to $\mathbf{I}$. Assume that* $\operatorname{height} \mathbf{P} = j_0$ *and $\mathbf{K}$ infinite. Let ε be a graded monomial order and let κ be the $(j_0 - 1)$-st elimination order with respect to ε. Assume given coordinates x in $\mathbf{R}$ for which φ is monomial of type $j_0 \in J = \{1, \ldots, j_0\}$ and $\mathrm{in}_{x\varepsilon}\mathbf{I} = \min_\varepsilon \mathbf{I}$. Let y be the induced coordinates in $\mathbf{R}'$. If $\min_\varepsilon \mathbf{I}' = \min_\varepsilon \mathbf{I}$, then*

$$\mathrm{in}_{y\kappa}\mathbf{I}' <_\kappa \max_\kappa \mathbf{I}.$$

Here, $\max_\kappa \mathbf{I}$ can also be the relative maximum with respect to $\min_\varepsilon \mathbf{I}$. It cannot be expected to have in general $\max_\kappa \mathbf{I}' <_\kappa \max_\kappa \mathbf{I}$ without further specification of the center [33].

Lemma 7. *Let ε and κ be any monomial orders on $\mathbb{N}^n$.*
(a) $\mathrm{in}_{x\varepsilon}\mathbf{I} \leq_\varepsilon \mathrm{in}_{x\kappa}\mathbf{I} \leq_\kappa \mathrm{in}_{x\varepsilon}\mathbf{I}$.
(b) *If $\mathrm{in}_{x\varepsilon}\mathbf{I} = \mathrm{in}_{x\kappa}\mathbf{I}$ there exists a common reduced ordered standard basis f_i of $\mathbf{I}$ such that $\mathrm{in}_{x\varepsilon}f_i = \mathrm{in}_{x\kappa}f_i$ for all i.*
(c) *If $\mathrm{in}_{x\varepsilon}\mathbf{I} \neq \mathrm{in}_{x\kappa}\mathbf{I}$ let r be the first index where the ordered vertices of $\mathrm{in}_{x\varepsilon}\mathbf{I}$ and $\mathrm{in}_{x\kappa}\mathbf{I}$ differ. There exist $f_1, \ldots, f_r$ in $\mathbf{I}$ such that the initial monomials of $f_1, \ldots, f_r$ are the first r vertices of $\mathrm{in}_{x\varepsilon}\mathbf{I}$ and $\mathrm{in}_{x\kappa}\mathbf{I}$ respectively and such that f_i is reduced with respect to $f_1, \ldots, f_{i-1}$ for all $i \leq r$.*

Proof of the lemma. Let g_i and h_j denote reduced ordered standard bases of $\mathbf{I}$ with respect to ε and κ ($1 \leq i \leq p$, $1 \leq j \leq q$) with initial monomials $\mathrm{in}_{x\varepsilon}g_i = x^{\alpha_i}$ and $\mathrm{in}_{x\kappa}h_j = x^{\beta_j}$. If $\mathrm{in}_{x\varepsilon}\mathbf{I} = \mathrm{in}_{x\kappa}\mathbf{I}$ then $p = q$ and $\alpha_i = \beta_i$ for all i. By reducedness, no monomial of $g_i - h_i$ belongs to $\mathrm{in}_{x\varepsilon}\mathbf{I}$. Hence $g_i = h_i$. If $\mathrm{in}_{x\varepsilon}\mathbf{I} \neq \mathrm{in}_{x\kappa}\mathbf{I}$ the index r is less or equal $\min\{p, q\}$ for else there would exist a strict inclusion between $\mathrm{in}_{x\varepsilon}\mathbf{I}$ and $\mathrm{in}_{x\kappa}\mathbf{I}$, which is impossible. Fix some $k \leq r$ and set $\mathrm{in}_{x\kappa}g_k = x^\gamma$, $\mathrm{in}_{x\varepsilon}h_k = x^\delta$. Reducedness implies

$$\gamma,\ \delta \notin \bigcup_{l<k}\ \alpha_l + \mathbb{N}^n = \bigcup_{l<k}\ \beta_l + \mathbb{N}^n.$$

From $x^\gamma \in \mathrm{in}_{x\kappa}\mathbf{I}$ and $x^\delta \in \mathrm{in}_{x\varepsilon}h_k$ follows $\gamma \in \bigcup_{l\geq k}\ \beta_l + \mathbb{N}^n$ and $\delta \in \bigcup_{l\geq k}\ \alpha_l + \mathbb{N}^n$. As the α_l and β_l are ordered, $\gamma \geq_\kappa \beta_k$ and $\delta \geq_\varepsilon \alpha_k$. This implies for all $k \leq r$:

$$(*) \qquad \mathrm{in}_{x\varepsilon}g_k = x^{\alpha_k} \leq_\varepsilon \mathrm{in}_{x\varepsilon}h_k \quad \text{and} \quad \mathrm{in}_{x\kappa}h_k = x^{\beta_k} \leq_\kappa \mathrm{in}_{x\kappa}g_k.$$

For $k < r$ we get $x^{\alpha_k} \leq_\varepsilon \mathrm{in}_{x\varepsilon}h_k \leq_\varepsilon \mathrm{in}_{x\kappa}h_k = x^{\beta_k}$ and $x^{\beta_k} \leq_\kappa \mathrm{in}_{x\kappa}g_k \leq_\kappa \mathrm{in}_{x\varepsilon}g_k = x^{\alpha_k}$ hence equality everywhere. Set $f_k = g_k$ for all $k < r$. For $k = r$ three cases occur: if $\mathrm{in}_{x\varepsilon}g_r = \mathrm{in}_{x\varepsilon}h_r$ then $\mathrm{in}_{x\varepsilon}h_r \leq_\varepsilon \mathrm{in}_{x\kappa}h_r \leq_\kappa \mathrm{in}_{x\varepsilon}h_r$ and $\alpha_r \neq \beta_r$ implies strict inequalities. Then set $f_r = h_r$. If $\mathrm{in}_{x\kappa}g_r = \mathrm{in}_{x\kappa}h_r$ the argument is symmetric and set $f_r = g_r$. If $\mathrm{in}_{x\varepsilon}g_r \neq \mathrm{in}_{x\varepsilon}h_r$ and $\mathrm{in}_{x\kappa}g_r \neq \mathrm{in}_{x\kappa}h_r$ the inequalities from $(*)$ imply $\mathrm{in}_{x\varepsilon}g_r <_\varepsilon \mathrm{in}_{x\varepsilon}h_r$ and $\mathrm{in}_{x\kappa}h_r <_\kappa \mathrm{in}_{x\kappa}g_r$. Therefore $f_r = g_r + h_r$ satisfies $\mathrm{in}_{x\varepsilon}f_r = \mathrm{in}_{x\varepsilon}g_r$ and $\mathrm{in}_{x\kappa}f_r = \mathrm{in}_{x\kappa}h_r$. As $\alpha_r \neq \beta_r$ the inequalities $\mathrm{in}_{x\varepsilon}f_r \leq_\varepsilon \mathrm{in}_{x\kappa}f_r \leq_\kappa \mathrm{in}_{x\varepsilon}f_r$ are strict. Observe finally that in all three cases the resulting f_r is reduced with respect to $f_1, \ldots, f_{r-1}$.

Proof of the theorem. It suffices to show $\mathrm{in}_{y\kappa}\mathbf{I}' < \mathrm{in}_{x\kappa}\mathbf{I}$. We may assume $\varepsilon \in \Lambda$. From $\min_\varepsilon \mathbf{I}' = \min_\varepsilon \mathbf{I}$ follows by Theorem 6 and Lemma 5(a) that $j_0 > n - \operatorname{depth}\mathbf{R}/\mathbf{I}$. By Lemma 5, $\mathrm{in}_{x\kappa}\mathbf{I} <_\kappa \mathrm{in}_{x\varepsilon}\mathbf{I}$. By Lemma 7(a), $\mathrm{in}_{x\kappa}\mathbf{I} \leq_\kappa \mathrm{in}_{x\varepsilon}\mathbf{I}$. If equality would hold, Lemma 5(b) and $j_0 - 1 \geq n - \operatorname{depth}\mathbf{R}/\mathbf{I}$ would imply $<x_1, \ldots, x_{j_0-1}>$ permissible. Let $(\alpha_1, \ldots)$ and $(\beta_1, \ldots)$ be the ordered vertices of $\mathrm{in}_{x\varepsilon}\mathbf{I}$ and $\mathrm{in}_{x\kappa}\mathbf{I}$, and let r be the first index where $\alpha_r \neq \beta_r$. Choose $f_1, \ldots, f_r$ as in the lemma and set $\mathrm{in}_{y\kappa}f_i' = y^{\beta_i'}$. Then $\beta_r <_\kappa \alpha_r <_\varepsilon \beta_r$ and $|\beta_r|^{j_0-1} < |\alpha_r|^{j_0-1}$. As $|\alpha_r|^{j_0-1} = |\alpha_r|$ because of $\varepsilon \in \Lambda$ and $j_0 - 1 \geq n - \operatorname{depth}\mathbf{R}/\mathbf{I}$ we get $\beta_r' \leq_\kappa \beta_r^* < \beta_r$. For $i < r$, $|\beta_i|^{j_0-1} \leq |\beta_i| = |\alpha_i|$ implies $\beta_i' \leq_\kappa \beta_i^* \leq_\kappa \beta_i$. Let $m \leq r$ be minimal with $\beta_m' <_\kappa \beta_m$. The argument (ii) of the proof of Theorem 6 gives $\mathrm{in}_{y\kappa}\mathbf{I}' <_\kappa \mathrm{in}_{x\kappa}\mathbf{I}$.

Coordinates as in the theorem always exist if $\varepsilon \in \Lambda$, see the proof of Theorem 6. If $\max_\kappa \mathbf{I}'$ could be realized in $\mathbf{R}'$ by automorphisms h which are induced from $\mathbf{R}$ and which keep φ monomial, we would have $\max_\kappa \mathbf{I}' < \max_\kappa \mathbf{I}$.

Localization

For a prime ideal $\mathbf{P}$ of $\mathbf{R}$ denote by $\mathbf{R}'$ the completed localization of $\mathbf{R}$ at $\mathbf{P}$. For coordinates $x_1, \ldots, x_n$ in $\mathbf{R}$ with $\mathbf{P} = <x_1, \ldots, x_r>$, identify $\mathbf{R}$ and $\mathbf{R}'$ with $\mathbf{K}[[x_1, \ldots, x_n]]$ and $\mathbf{K}'[[x_1, \ldots, x_r]]$ respectively, where $\mathbf{K}' = \mathbf{K}((x_{r+1}, \ldots, x_n))$. We write x and y for $x_1, \ldots, x_n$ and $x_1, \ldots, x_r$. Let $\mathbf{J}$ and $\mathbf{J}'$ be monomial ideals of $\mathbf{R}$ and $\mathbf{R}'$ with respect to x and y. Given a monomial order ε on $\mathbb{N}^n$, set $\mathbf{J}' \leq_\varepsilon \mathbf{J}$ if $\mathbf{J}'' \cdot \mathbf{R}^* \leq_\varepsilon \mathbf{J} \cdot \mathbf{R}^*$ as monomial ideals in $\mathbf{R}^* = \mathbf{K}'[[z_1, \ldots, z_n]]$, where $\mathbf{R}$ and $\mathbf{R}'$ are embedded canonically into $\mathbf{R}^*$, and where $\mathbf{J}''$ denotes the ideal generated by $\mathbf{J}'$ and $z_{r+1}, \ldots, z_n$. Identifying $\mathbb{N}^r$ with $\mathbb{N}^r \times 0^{n-r} \subseteq \mathbb{N}^n$, ε induces a monomial order on $\mathbb{N}^r$, also denoted by ε. Let $\min_\varepsilon \mathbf{I}$ denote the minimal initial ideal of $\mathbf{I}$ taken with respect to the algebraic closure of $\mathbf{K}$. We will see that it is upper semicontinuous with respect to localization.

Theorem 8. *Let $\mathbf{I}$ be an ideal in $\mathbf{R}$ and let $\mathbf{I}' = \mathbf{I} \cdot \mathbf{R}'$ with $\mathbf{P}$ prime in $\mathbf{R}$. For any graded monomial order ε in $\mathbb{N}^n$ one has*

$$\min_\varepsilon \mathbf{I}' \leq_\varepsilon \min_\varepsilon \mathbf{I}.$$

The inequality is strict, if $\mathbf{R}/\mathbf{I}$ has embedding dimension n and $\mathbf{P} \neq \mathbf{M}$.

Similar statements have been proven for the multiplicity [28, 28.3 and 40.1, 16, Lemma 7, p. 217, and Thm. 1, p. 218] and the Hilbert-Samuel function [5, Thm. 2, p. 77].

For the maximal initial ideals, the respective inequalities need not hold. Let $\mathbf{R} = \mathbf{K}[[x, y, z]]$, $\mathbf{P} = <x, y>$, $f = x^2 + xz^3 + y^3$ and $\varepsilon = lex$ with $z < y < x$. We then have $\max_\varepsilon f = \mathrm{in}_\varepsilon f = y^3$ and $\max_e f' \geq \mathrm{in}_\varepsilon h f' = x > y^3 = \max_\varepsilon f$ where $h \in \mathrm{Aut}\, \mathbf{R}'$ is given by $h^* = (x + z^{-3}x^2 + z^{-3}y^3, y)$.

In the graded case, take $\mathbf{R} = \mathbf{K}[[x, y, z]]$ with char $\mathbf{K} = 2$, $\mathbf{P} = <x, y>$ and $\varepsilon = gix$ with $x < y < z$. For $f = x^4y^2 + x^2y^4 + x^3y^3z + x^2y^4z^2 + xy^5z^3 + y^6z^2$ we have $\max_\varepsilon f' \geq \mathrm{in}_\varepsilon h f' = x^3y^3 > x^4y^2 = \max_\varepsilon f$ where $h \in \mathrm{Aut}\, \mathbf{R}'$ is given by $h^* = (x, y + z^{-1}x)$.

Proof. We may assume $\mathbf{P} \neq \mathbf{M}$. A standard argument of Nagata-Zariski reduces to the case $\mathbf{P}$ regular [16, Thm. 1, p. 218]. Permute the components of $\mathbb{N}^n$ so that $\varepsilon \in \Lambda$. Choose by Theorem 4 an automorphism $h = su \in S_n U_\varepsilon$ such that $\mathrm{in}_{x\varepsilon} h\mathbf{I} = \min_\varepsilon \mathbf{I}$.

Then u fixes $\mathbf{P}$. Replacing x by $h^{-1}x$ gives $\mathbf{P} = <x_{s1}, \ldots, x_{sr}>$ and $\text{in}_{x\varepsilon}\mathbf{I} = \min_\varepsilon \mathbf{I}$ by Lemma 3(a). We have to show that $\text{in}_{y\varepsilon}\mathbf{I}' \leq_\varepsilon \text{in}_{x\varepsilon}\mathbf{I}$.

For $\mathbf{I} = <f>$, let f' be the image of f in $\mathbf{R}'$ and set $\text{in}_{x\varepsilon} f = x^\alpha$ and $\text{in}_{y\varepsilon} f' = y^{\alpha'}$. Let $o : \mathbb{N}^n \to \mathbb{N}^r$ be the projection on the components of index $s1, \ldots, sr$. The coefficient of y^{α^o} in the expansion of f' is $\sum_\gamma c_\gamma x^{\gamma - \alpha^o}$, where $f = \sum_\gamma c_\gamma x^\gamma$ and the sum ranges over all $\gamma \in \mathbb{N}^n$ with $\gamma^o = \alpha^o$. It is non-zero because $c_\alpha \neq 0$. Then $\alpha' \leq_\varepsilon \alpha^o \leq_\varepsilon \alpha$ implies $\text{in}_{y\varepsilon} f' \leq_\varepsilon \text{in}_{x\varepsilon} f$. If $\mathbf{R}/<f>$ has embedding dimension n, f has order ≥ 2 and strict inequality follows by definition of the order between monomial ideals.

For $\mathbf{I}$ arbitrary, let $f_1, \ldots, f_p$ be the reduced ordered standard basis of $\mathbf{I}$ with respect to ε and x, i.e., f_i reduced with respect to $f_1, \ldots, f_{i-1}$ or all i. If $\text{in}_{x\varepsilon} f_i = \text{in}_{y\varepsilon} f'_i$ for all i, then $\text{in}_{x\varepsilon}\mathbf{I} \subseteq \text{in}_{y\varepsilon}\mathbf{I}'$ and the assertion follows from Lemma 6. If not, let m be the first index where the initial monomials differ. Set $\text{in}_{x\varepsilon} f_i = x^{\alpha_i}$ and $\text{in}_{y\varepsilon} f'_i = y^{\alpha'_i}$. Then $\alpha'_i = \alpha_i = \alpha_i^o$ for $i < m$ and $\alpha'_m <_\varepsilon \alpha_m$ by the hypersurface case. Write $\alpha'_m = \gamma^o$ for some $\gamma \in \text{supp } f_m$. Reducedness implies $\gamma \notin \bigcup_{i<m} \alpha_i + \mathbb{N}^n = \bigcup_{i<m} \alpha_i^o + \mathbb{N}^n$. As $\alpha_i^o \in \mathbb{N}^r$, also $\gamma^o \notin \bigcup_{i<m} \alpha_i^o + \mathbb{N}^n$, and we are done by Lemma 6. The strict inequality follows as for hypersurfaces.

References

[1] S. Abhyankar. Resolution of singularities of embedded algebraic surfaces. Acad. Press 1966.

[2] S. Abhyankar. Desingularization of plane curves. Proc. Symp. Pure Appl. Math. **40**, Amer. Math. Soc. 1983.

[3] M. Artin. On the solutions of analytic equations. Invent. Math. **5** (1968), 277-291.

[4] T. Becker. Stability and Buchberger criterion for standard bases in power series rings. J. Pure and Appl. Algebra **66** (1990), 219-227.

[5] B.-M. Bennett. On the characteristic function of a local ring. Ann. Math. **91** (1970), 25-87.

[6] E. Bierstone, P. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. **128** (1997), 207-302.

[7] E. Bierstone, P. Milman. Relations among analytic functions I. Ann. Inst. Fourier **37** (1987), 187-239.

[8] V. Cossart. Polyèdre caractéristique d'une singularité. Thèse d'Etat, Orsay 1987.

[9] D. Eisenbud. Commutative Algebra. Springer 1995.

[10] S. Encinas, H. Hauser. Strong resolution of singularities in characteristic zero. Comment. Math. Helv. **77** (2002), 421-445.

[11] S. Encinas, O. Villamayor. Good points and constructive resolution of singularities. Acta Math. **181** (1998), 109-158.

[12] S. Encinas, O. Villamayor. A course on constructive desingularization and equivariance. In: Resolution of Singularities. Ed.: H. Hauser et al. Progress in Math. vol. 181, 147-230. Birkhäuser 2000.

[13] A. Galligo, A.: A propos du Théorème de Préparation de Weierstrass. Springer Lecture Notes in Math. 409 (1973), 543-579.

[14] M. Giusti, M.: Some effectivity problems in polynomial ideal theory. Springer Lecture Notes in Computer Science 174 (1984), 159-171.

[15] H. Grauert, H.: Über die Deformation isolierter Singularitäten analytischer Mengen. Invent. Math. **15** (1972), 171-198.

[16] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. of Math. **79** (1964), 109-326.

[17] H. Hironaka. Desingularization of excellent surfaces. Notes by B. Bennett. Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Springer Lecture Notes in Math. 1101 (1984), 99-132.

[18] H. Hironaka. Idealistic exponents of singularity. In: Algebraic Geometry, the Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977, 52-125.

[19] H. Hironaka. Stratification and flatness. Real and complex singularities. Proc. Ninth Nordic Summer School. Sympos. Math., Oslo, 1976, 199-265. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[20] H. Hauser. Seventeen obstacles for resolution of singularities. In: Singularities. The Brieskorn Anniversary volume. Eds.: G.-M. Greuel et al. Progress in Math. vol. 162, 289-313. Birkhäuser 1997.

[21] H. Hauser. Excellent surfaces and their taut resolution. In: Resolution of Singularities. A research text book in tribute to O. Zariski. Eds.: H. Hauser

et al. Progress in Math. vol. 181, 341-374. Birkhäuser 2000.

[22] H. Hauser. The Hironaka Theorem on resolution of singularities. (Or: A proof that we always wanted to understand.) Bull. Amer. Math. Soc. **40** (2003), 323-403.

[23] H. Hauser. Mysterious resolution in positive characteristic. Preprint, Univ. Innsbruck 2000.

[24] H.W.E. Jung. Über ganze birationale Transformationen der Ebene. J. Reine Angew. Math. **184** (1942). 161-174.

[25] W. van der Kulk. On polynomial rings in two variables. Nieuw. Arch. Wisk. **1** (1953), 33-41.

[26] T.-T. Moh. On a Newton polygon approach to the uniformization of singularities of characteristic p. Proc. Conference on Singularities, La Rábida. Eds.: A. Campillo, L. Narváez. Progres in Math. 134, 49-93. Birkhäuser 1996.

[27] T.-T. Moh. On a stability theorem for local uniformization in characteristic p. Publ. RIMS Kyoto Univ. **23** (1987), 965-973.

[28] M. Nagata. Local rings. Robert Krieger Publishing Company 1975.

[29] M. Nagata. On automorphism group of $k[x,y]$. Lectures in Math. **5** (1972), Kinokuniya Book-Store Tokyo 1972.

[30] K. Pardue. Nonstandard Borel-fixed ideals. Thesis, Brandeis Univ. 1994.

[31] G. Regensburger. Die Auflösung von "ebenen" Kurvensingularitäten. Thesis, Univ. Innsbruck 1999.

[32] I. Shestakov, U. Umirbaev. The tame and the wild automorphisms of polynomial rings in three variables. J. Amer. Math. Soc. To appear.

[33] M. Spivakovsky. A solution to Hironaka's Polyhedra Game. In: Arithmetic and Geometry. Papers dedicated to I.R. Shafarevich on the occasion of his sixtieth birthday, vol. II (eds.: M. Artin, J. Tate). Birkhäuser 1983, 419-432.

[34] M. Spivakovsky. A counterexample to the theorem of Beppo Levi in three dimensions. Invent. Math. **96** (1989), 181-183.

[35] O. Villamayor. Constructiveness of Hironaka's resolution. Ann. Sci. Éc. Norm. Sup. Paris **22** (1989), 1-32.

[36] O. Villamayor. Patching local uniformizations, Ann. Scient. Éc. Norm. Sup. Paris **25** (1992), 629-677.

[37] O. Zariski. Reduction of singularities of algebraic three dimensional varieties. Ann. Math. **45** (1944), 472-542.

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