Semi-Universal Unfoldings and Orbits of the Contact Group

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1 Introduction

Two germs of complex analytic varieties X_1, X_2 defined in $(\mathbb{C}^n, 0)$ by analytic map germs $h_1, h_2 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are analytically isomorphic if and only if

$$h_2 = (h_1 \circ \varphi) \cdot a$$

for some $\varphi \in \mathrm{Aut}(\mathbb{C}^n, 0)$ and some invertible matrix a with entries in the ring $\mathcal{O}_n$ of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. Thus, in singularity theory one is lead to study the action of the contact group $K = \mathrm{Aut}(\mathbb{C}^n, 0) \ltimes \mathrm{GL}_p(\mathcal{O}_n)$ on the space $\mathcal{O}_n^p$.

In case of a (finite dimensional) Lie group G acting on a manifold Y the following is well known and easily proven using the Inverse Mapping Theorem: Given any point $y \in Y$ and transversals $N \subseteq Y$ to the orbit yG, resp. $M \subseteq G$ to the stabilizer G_y, the restriction $N \times M \to Y$ of the action map $Y \times G \to Y$ is a local analytic isomorphism at the point $(y, 1)$. Thus any $y' \in Y$ near y has a unique "normal form" with respect to M in the slice N near y, and the element from M transforming y' into its normal form is unique locally.

Although the contact group K can be viewed, in a natural way, as an infinite dimensional Lie group, the property just described does not hold true for the action of K on $\mathcal{O}_n^p$, see section 6 for an explicit example. This is due to the failure of the Inverse Mapping Theorem for analytic maps between infinite dimensional spaces. In this paper we study the action of K on $\mathcal{O}_n^p$ near a map germ h of isolated singularity type. We shall introduce the notion of vicinity. A vicinity U of h is a set containing h which, in general, is too "thin" to be a neighborhood of h. But it is "fat" enough to have the following property: Any analytic "curve" $\gamma : T \to \mathcal{O}_n^p$, parametrized by a finite dimensional space T, with $\gamma(0) = h$ stays inside U for $t \in T$ close to 0. We shall then show that for suitable transversals $N \subseteq \mathcal{O}_n^p$ to the orbit hK, resp. $M \subseteq K$ to the stabilizer K_h, the restriction $N \times M \to \mathcal{O}_n^p$ of the action map $\mathcal{O}_n^p \times K \to \mathcal{O}_n^p$ is an isomorphism in a vicinity of the point $(h, 1)$. The proof relies on the Inverse Mapping Theorem in Banach spaces and the Grauert-Hironaka Division Theorem. As a Corollary we obtain Mather's Theorem on the existence of the semi-universal unfolding. Moreover we show that the orbit hK is isomorphic, in a vicinity of h, to a submanifold of $\mathcal{O}_n^p$.

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2 An Inverse Mapping Theorem in Banach scales

All topological vector spaces in this paper are assumed to be Hausdorff, locally convex, and sequentially complete. The reader is supposed to be familiar with the general theory of manifolds modelled on such spaces and analytic maps between them, see [BS], [He], [HM 3], [Mi].

Let E be a manifold and $a \in E$. A subset $U \subseteq E$ containing a is called a *vicinity* of a if for all analytic maps $\gamma : T \to E$, $T \subseteq \mathbb{C}^k$ an open neighborhood of 0, with $\gamma(0) = a$ the preimage $\gamma^{-1}(U)$ is a neighborhood of 0. A map $f : U \to F$ defined on a vicinity U of a with values in a manifold F is called *LF-analytic* at a if for all γ as above the composition $f \circ \gamma$ is analytic in a neighborhood of 0. This notion goes back to Luigi Fantappiè. If E and F are linear spaces and f is LF-analytic at a one can define its Gâteaux-differential $\mathrm{T}_a f : E \to F$ by

$$\mathrm{T}_a f(v) = \lim_{t\to 0} \frac{f(a+tv) - f(a)}{t}.$$

By an *LF-isomorphism* we mean a bijective map between vicinities of a and $f(a)$ such that f and its inverse are LF-analytic at a, resp. $f(a)$.

Clearly, a neighborhood is a vicinity. To describe a more general example consider a *Banach scale*, i. e. a topological vector space E as above together with pseudonorms $|\ |_s$, $s \in \mathbb{R}$ and $s > 0$, such that

1. if $s < s'$ then $|\ |_s \le |\ |_{s'}$,
2. $E_s := \{a \in E,\ |a|_s < \infty\}$ is a Banach space with norm $|\ |_s$,
3. $E = \bigcup_s E_s$ as a topological vector space with the final topology.

If $\gamma : T \to E$ is an analytic map as above then by [P 1, Proposition 2] there is an open neighborhood $T' \subseteq T$ of 0 such that γ maps T' into E_s for some s and that, in fact, $\gamma : T' \to E_s$ is a Banach space valued analytic map. Consequently, if $U_s \subseteq E_s$ is a neighborhood of a for small s then the union $U = \bigcup_s U_s$ is a vicinity of a, called a *scaled vicinity*. If $U_s \subseteq U_{s'}$ for $s' < s$ then their union U will be a neighborhood. In fact, the assumption implies that the set

$$(\bigcup_s U_s) \cap E_{s'} = U_{s'} \cup \bigcup_{s<s'} (U_s \cap E_{s'})$$

is a neighborhood of a in $E_{s'}$ as $E_{s'} \subseteq E_s$ is continuous for $s < s'$. A map between Banach scales E and F is called *scale-analytic* at a point a if a admits a scaled vicinity $U = \bigcup_s U_s$ such that f restricts to analytic maps $f_s : U_s \to F_s$. Then obviously f is LF-analytic at a and the Gâteaux-differentials $\mathrm{T}_a f_s$ are the restrictions of $\mathrm{T}_a f$.

Inverse Mapping Theorem. *Let $E = \bigcup_s E_s$, $F = \bigcup_s F_s$ be Banach scales, $f : W \to F$ a map in a vicinity $W \subseteq E$ of 0 with $f(0) = 0$. Assume that f is scale-analytic at 0 and that the Gâteaux-differentials $l_s = \mathrm{T}_0 f_s : E_s \to F_s$ of the restrictions f_s are bijective. Then there are vicinities $U \subseteq E$ and $V \subseteq F$ of 0 such that f restricts to an LF-isomorphism $f : U \to V$.*

Proof. The Gâteaux-differential $l = \mathrm{T}_0 f : E \to F$ of f is bijective. By the Closed Graph Theorem the l_s are isomorphisms of Banach spaces. Hence upon replacing f by $l^{-1} \circ f$ we may assume that $E = F$ and $\mathrm{T}_0 f = \mathrm{id}_E$. Define recursively scale-analytic maps g_i by

$$g_0 = 0, \quad g_{i+1} = \mathrm{id}_E - (f - \mathrm{id}_E) \circ g_i.$$

One can check that the power series expansions of g_i and g_{i+1} coincide up to order i. Thus one obtains a formal power series $\bar{g}$ which equals the expansion of each g_i up to order i. Now the usual proof of the Inverse Mapping Theorem in Banach spaces (see e. g. [U, 1.23]) gives an open neighborhood $V_s \subseteq E_s$ of 0 such that $\bar{g}$ converges against an analytic map $g_s : V_s \to E_s$ which is inverse to f_s restricted to a suitable open neighborhood $U_s \subseteq E_s$ of 0. As all g_s are induced by the same formal power series $\bar{g}$ they coincide on the intersections $V_s \cap V_{s'}$ and thus define a map g on $V = \bigcup_s V_s$ which is inverse to f restricted to $U = \bigcup_s U_s$. By construction, g is LF-analytic at 0.

Remarks. (a) We shall use the following notation. If h is an analytic map between Banach spaces with power series expansion $h = \sum_k h_k$ near 0 then

$$|h|_r = \sum_k |h_k| r^k$$

for $r > 0$, where $|h_k|$ is the usual norm of the k–homogeneous polynomial h_k.

(b) Given suitable norm estimates for the power series expansion of f one can calculate radii r'_s such that the statement of the Theorem holds true for the balls $\mathrm{B}_s(0, r'_s)$ in place of U_s. To be more precise, assume that $E = F$ and $\mathrm{T}_0 f = \mathrm{id}_E$. If

$$|f_s - \mathrm{id}_{E_s}|_{r_s} \leq c_s \cdot r_s$$

with small numbers $c_s, r_s > 0$ then one can take

$$r'_s = (\frac{1}{e} - \frac{c_s \cdot e}{1 - c_s \cdot e}) \cdot (\frac{1}{e} - c_s) \cdot r_s,$$

see [HM 3, proof of Theorem 6.2]. If c_s and r_s can be chosen independent of s one has $U_s \subseteq U_{s'}$ for $s' < s$. Then U is a neighborhood of 0 in E and $f : U \to f(U)$ is an analytic isomorphism.

3 The action of the contact group

The space $\mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\} = \mathbb{C}\{x\}$ of convergent power series in n variables becomes a topological vector space as follows: Fix a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive reals. For a power series $a = \sum_\alpha a_\alpha x^\alpha$ and $s \in \mathbb{R}, s > 0$ define its pseudonorm

$$|a|_s = \sum_\alpha |a_\alpha| s^{\lambda\alpha} \in \mathbb{R} \cup \{\infty\}.$$

This defines a Banach scale structure on $\mathcal{O}_n$. The underlying topological vector space is independent of the choice of λ.

Fix a natural number p and let

$$K = \mathrm{Aut}(\mathbb{C}^n, 0) \ltimes \mathrm{GL}_p(\mathcal{O}_n)$$

be the contact group. The semidirect product is with respect to the natural right action of $G = \mathrm{Aut}(\mathbb{C}^n, 0)$ on $\mathrm{GL}_p(\mathcal{O}_n)$. If m_n denotes the maximal ideal of $\mathcal{O}_n$ then K is an open subset of $\mathbf{k} = m_n \cdot \mathcal{O}_n^n \oplus \mathcal{O}_n^{p^2}$ and an infinite dimensional Lie group in the sense of [Mi, section 5]. It acts analytically on the space $\mathcal{O}_n^p$ of rows from the right by

$$h(\varphi, a) = (h \circ \varphi) \cdot a.$$

For fixed $h \in \mathcal{O}_n^p$ the stabilizer

$$K_h = \{(\varphi, a) \in K,\ (h \circ \varphi) \cdot a = h\}$$

is a submanifold, hence a Lie subgroup of K [HM 3, Theorem 4.1]. The tangent space $\mathrm{T}_1(K_h)$ equals

$$\mathbf{k}_h = \{(v, w) \in \mathbf{k},\ v \cdot \partial h + h \cdot w = 0\}$$

where ∂h is the Jacobian matrix of h. Of course, the tangent spaces $\mathbf{k}$ and $\mathbf{k}_h$ of the Lie groups K and K_h have natural Lie algebra structures. They will not be needed in the present paper.

Now consider the orbit hK. It is not clear that this is a submanifold of $\mathcal{O}_n^p$. Nevertheless, Mather called $m_n \cdot j(h) + (h) \cdot \mathcal{O}_n^p$ the tangent space of the orbit. Here $j(h)$ is the submodule of $\mathcal{O}_n^p$ generated by the partial derivatives of h and (h) is the ideal of $\mathcal{O}_n$ generated by the components of h. If one defines for any subset $A \subseteq \mathcal{O}_n^p$ containing h the tangent space $\mathrm{T}_h(A)$ as the set of tangent vectors of analytic curves $\gamma : (\mathbb{C}, 0) \to (\mathcal{O}_n^p, h)$ lying in A, then one can prove [HM 1] that

$$\mathrm{T}_h(hK) = m_n \cdot j(h) + (h) \cdot \mathcal{O}_n^p.$$

Observe that $\mathrm{T}_1(K_h)$ and $\mathrm{T}_h(hK)$ are $\mathcal{O}_n$-submodules of $\mathbf{k}$ and $\mathcal{O}_n^p$, respectively. Hence by [J, Theorem 3.1] they admit topological direct complements

$$\mathrm{T}_1(K_h) \oplus M = \mathbf{k} \quad \text{and} \quad \mathrm{T}_h(hK) \oplus N = \mathcal{O}_n^p.$$

We may view $M' = 1 + M$ and $N' = h + N$ as transversals to K_h and hK. Now the main result of this paper is as follows:

Theorem. *Assume that $h \in \mathcal{O}_n^p$ is of isolated singularity type, i. e. that $\mathrm{T}_h(hK)$ has finite codimension in $\mathcal{O}_n^p$. Let N be a topological direct complement of $\mathrm{T}_h(hK)$ in $\mathcal{O}_n^p$. Then there are a topological direct complement M of $\mathbf{k}_h = \mathrm{T}_1(K_h)$ in $\mathbf{k}$, vicinities $W \subseteq \mathcal{O}_n^p$ of h, $V \subseteq M' = 1 + M$ of the unit $1 \in K$ and a neighborhood $U \subseteq N' = h + N$ of h such that the action map*

$$\mathcal{O}_n^p \times K \to \mathcal{O}_n^p : (g, \varphi, a) \mapsto (g \circ \varphi) \cdot a$$

restricts to an LF-isomorphism $U \times V \to W$.

Remark. The vicinities cannot be replaced, in general, by ordinary neighborhoods, see example (c) in section 6.

Proof of the Theorem. Consider the analytic map

$$\omega : \mathcal{O}_n^p \oplus \mathbf{k} \to \mathcal{O}_n^p : (g, v, w) \mapsto ((h + g) \circ (x + v)) \cdot (1 + w).$$

The Gâteaux-differential $l = \mathrm{T}_0\omega : \mathcal{O}_n^p \oplus \mathbf{k} \to \mathcal{O}_n^p$ is given by

$$l(b, v, w) = b + v \cdot \partial h + h \cdot w.$$

The map $\beta : \mathbf{k} \to \mathcal{O}_n^p$ defined by $\beta(v, w) = v \cdot \partial h + h \cdot w$ has kernel $\mathbf{k}_h$ and image $\mathrm{T}_h(hK)$. Hence the restriction $l : N \oplus M \to \mathcal{O}_n^p$ is bijective for arbitrary topological direct complements N of $\mathrm{T}_h(hK)$ and M of $\mathbf{k}_h$. In order to apply the Inverse Mapping Theorem we have to choose a suitable M and introduce Banach scale structures on N, M and $F = \mathcal{O}_n^p$ such that the restriction $\omega : N \oplus M \to F$ becomes scale-analytic at 0 and the Gâteaux-differentials $l_s = \mathrm{T}_0\omega_s : N_s \oplus M_s \to F_s$ are bijective.

Fix a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of positive reals and another positive real λ_0 such that $\lambda_0, \lambda_1, \dots, \lambda_n$ are $\mathbb{Z}$-linearly independent. They induce Banach scale structures on $\mathcal{O}_n$ and $F = \mathcal{O}_n^p$:

$$|b|_s = \sum_\alpha |b_\alpha| s^{\lambda\alpha} \quad \text{for} \quad b = \sum_\alpha b_\alpha x^\alpha \in \mathcal{O}_n$$

$$|b|_s = \sum_i s^{\lambda_0(i-1)} |b_i|_s \quad \text{for} \quad b = (b_1, \dots, b_p) \in \mathcal{O}_n^p.$$

Moreover, $\mathbf{k} = m_n \cdot \mathcal{O}_n^n \oplus \mathcal{O}_n^{p^2}$ becomes a Banach scale by

$$|c|_s = \sum_i |c_i|_s \quad \text{for} \quad c \in \mathbf{k} \quad \text{with components} \quad c_i.$$

The finite dimensional space N is a Banach scale by $N_s = N$ for all s. Unfortunately, ω is *not* scale-analytic on $\mathcal{O}_n^p \oplus \mathbf{k}$. Since N is finite dimensional all $h+g$, $g \in N$, converge on a common polydisc. Using [HM 3, Proposition 7.2.c] one concludes that the restriction $\omega : N \oplus \mathbf{k} \to F$ is scale-analytic at 0. By a variant of the Grauert-Hironaka Division Theorem [HM 3, Theorem 5.2.b] the $\mathcal{O}_n$-linear map β admits a continuous linear section $\tau : I \to \mathbf{k}$, compatible with the scaling, where I denotes the image of β. Here it is essential how the Banach scale structure on $\mathcal{O}_n^p$ was defined and that $\lambda_0, \lambda_1, \dots, \lambda_n$ are $\mathbb{Z}$-linearly independent, cf. [HM 3, Remark a in section 5]. As $\mathrm{id}_{\mathbf{k}} - \tau \circ \beta$ is a continuous projector onto $\mathbf{k}_h = \ker \beta = \ker \tau \circ \beta$ the image of τ is a topological direct complement M of $\mathbf{k}_h$. Since τ is compatible the restrictions $\beta_s : M_s \to I_s$ are bijective. The finite dimensional space N is contained in F_s for small s, hence $F_s = N \oplus I_s$. Thus the Gâteaux-differentials l_s equal $\mathrm{id}_N \times \beta_s : N \oplus M_s \to N \oplus I_s$ and are bijective.

4 Semi-universal unfoldings

The Theorem of the preceding section yields an alternative proof of the existence of the semi-universal unfolding. For that purpose we need

Proposition. *Let $T \subseteq \mathbb{C}^k$ be an open neighborhood of 0 and let $\eta : T \to \mathcal{O}_n^p$ be an analytic map with $\eta(0) = h \in \mathcal{O}_n^p$. Then η is locally at 0 given by an unfolding H of h, i. e. there is an analytic map $H : T' \times U \to \mathbb{C}^p$, $T' \times U \subseteq T \times \mathbb{C}^n$ open, with*

$$\eta(t)(x) = H(t, x).$$

Conversely, by this equation any unfolding $H : T' \times U \to \mathbb{C}^p$ of h induces an analytic map $\eta : T' \to \mathcal{O}_n^p$ with $\eta(0) = h$.

Proof. [HM 3, Proposition 7.1]

Theorem (Mather). *Assume that $h \in \mathcal{O}_n^p$ is of isolated singularity type. Let $g_1, \dots, g_k$ be elements of $\mathcal{O}_n^p$ whose images form a $\mathbb{C}$-basis of $\mathcal{O}_n^p/(m_n \cdot j(h) + (h) \cdot \mathcal{O}_n^p)$. Then*

$$H(s, x) = h(x) + \sum_i s_i \cdot g_i(x)$$

is a semi-universal unfolding of h with respect to the contact group K.

Proof. The subspace $N \subseteq \mathcal{O}_n^p$ generated by $g_1, \ldots, g_k$ is a topological direct complement of the tangent space $\mathrm{T}_h(hK) = m_n \cdot j(h) + (h) \cdot \mathcal{O}_n^p$. By the Theorem of the preceding section we can find an LF-analytic map

$$\sigma : W \to N \times K$$

which is a right inverse of the map

$$N \times K \to \mathcal{O}_n^p : (g, \varphi, a) \mapsto ((h+g) \circ \varphi) \cdot a.$$

Now let $G : T \times U \to \mathbb{C}^p$ be an arbitrary unfolding of h with induced map $\gamma : T \to \mathcal{O}_n^p$, $\gamma(t)(x) = G(t,x)$. The composition $\sigma \circ \gamma$ is analytic in a neighborhood of 0. Its first component defines an unfolding

$$H'(t,x) = h(x) + \sum_i \alpha_i(t) \cdot g_i(x)$$

of h which, by construction, is K-equivalent to G and which is obtained from H by the base change α. It remains to check that the tangent map of the base change is unique. But this is easily seen taking derivatives with respect to t at $t = 0$ and using the assumption that $g_1, \ldots, g_k$ are linearly independent modulo $m_n \cdot j(h) + (h) \cdot \mathcal{O}_n^p$.

Remark. The semi-universality of $H(s,x)$ means: For any unfolding $G(t,x)$ of h there are a base change $s = \alpha(t)$ and an unfolding $(\varphi(t,x), a(t,x))$ of the unit $1 \in K$ such that

$$G(t,x) = H(\alpha(t), \varphi(t,x)) \cdot a(t,x).$$

Our proof gives a little more: There is a map σ defined in a vicinity of h such that for all $G(t,x)$ the required $\alpha(t)$ and $(\varphi(t,x), a(t,x))$ can be obtained from σ by inserting G.

5 Orbits of the contact group

As already mentioned in section 3, it is not clear that the K-orbits are submanifolds of the topological vector space $\mathcal{O}_n^p$, but one has

Theorem. *Assume that $h \in \mathcal{O}_n^p$ is of isolated singularity type. Then there are a closed linear subspace $F \subseteq \mathcal{O}_n^p$ admitting a topological direct complement and an LF-isomorphism $f : W \to f(W)$ between vicinities W of h and $f(W)$ of 0 mapping $hK \cap W$ onto $F \cap f(W)$.*

Proof. For suitable topological direct complements N of $\mathrm{T}_h(hK)$ in $\mathcal{O}_n^p$ and M of $\mathbf{k}_h$ in $\mathbf{k}$ the analytic map

$$\omega : N \oplus M \to \mathcal{O}_n^p : (g, v, w) \mapsto ((h+g) \circ (x+v)) \cdot (1+w)$$

restricts to an LF-isomorphism $U \times V \to W$ with vicinities $W \subseteq \mathcal{O}_n^p$ of h, $V \subseteq M$ of 0 and a neighborhood $U \subseteq N$ of 0. We claim that

$$\omega(0 \times V) = hK \cap W$$

if the vicinities are small enough. In fact, by [HM 1, Theorem 1] the orbit hK intersects the finite dimensional space $h + N$ in a (locally closed) analytic subset of $h + N$. Then $\mathrm{T}_h(hK) \cap N = 0$ implies that h is an isolated point of $(h+N) \cap hK$. Thus we may assume $(h+U) \cap hK = \{h\}$. Now take $h' \in hK \cap W$. Then there are $g \in U$ and $(v,w) \in V$ such that $h' = ((h+g)\circ(x+v))\cdot(1+w)$. In particular $h+g \in (h+U)\cap hK$, hence $g=0$ and $h' \in \omega(0\times V)$. Therefore $hK \cap W \subseteq \omega(0\times V)$, and the reversed inclusion is obvious. By the Closed Graph Theorem in Banach scales (see [G, chapter 4.1.5, Theorem 2]) the Gâteaux-differential

$$l = \mathrm{T}_0\omega : N \oplus M \to \mathcal{O}_n^p$$

is an isomorphism of topological vector spaces. Thus the composition $f = l \circ \omega^{-1}$ defined on W has the desired properties.

6 Remarks and examples

(a) There are analogous results if the contact group K is replaced by its subgroup $G = \mathrm{Aut}(\mathbb{C}^n, 0)$, i. e. for right equivalence instead of contact equivalence. The corresponding tangent spaces are $\mathbf{g} = m_n \cdot \mathcal{O}_n^n$, $\mathrm{T}_1(G_h) = \mathbf{g}_h = \{v \in \mathbf{g},\ v\cdot\partial h = 0\}$ and $\mathrm{T}_h(hG) = m_n \cdot j(h)$. However, the methods do not apply to the case of A-equivalence as an appropriate Inverse Mapping Theorem is lacking.

(b) The arguments of sections 3, 4 and 5 cannot be extended to the case of map germs h which are not of isolated singularity type. Then the complement N of the tangent space of the orbit is no longer finite dimensional and the map $\omega : N \oplus \mathbf{k} \to \mathcal{O}_n^p$ will not be scale-analytic. The above version of the Inverse Mapping Theorem uses the scaling in an essential way. There exist other Inverse Mapping Theorems for analytic maps between non-Banach spaces [C], [Ham], [L], [P 2] but it is not clear that one of those can be applied in the present situation. The existence of a semi-universal unfolding for non-isolated singularities has been proven in [Hau] by different methods.

(c) We show by an example that in the Theorem of section 3 the vicinities V and W can, in general, not be chosen to be neighborhoods. This signifies that the map $\omega : N \oplus M \to \mathcal{O}_n^p$ is not a local analytic isomorphism. Consider $h = x^2 \in \mathcal{O}_1$ with respect to right equivalence. Then $\mathbf{g}_h = 0$, $M = m_1$, $\mathrm{T}_h(hG) = m_1^2$ and $N = \mathbb{C} \oplus \mathbb{C}x$. We have to study the map

$$\omega : N \oplus M \to \mathcal{O}_1 : (t_0 + t_1 x, a) \mapsto (x^2 + t_0 + t_1 x)(x+a) = x^2 + 2ax + a^2 + t_0 + t_1 x + t_1 a.$$

Its Gâteaux-differential at $(t_0 + t_1 x, a)$ is given by

$$N \oplus M \to \mathcal{O}_1 : (s_0 + s_1 x, v) \mapsto 2vx + 2av + s_0 + s_1 x + s_1 a + t_1 v.$$

For $a = 0$ and arbitrary $t_1 \neq 0$ its kernel contains $(-x, x\cdot(t_1+2x)^{-1})$. Hence ω cannot be a local analytic isomorphism at 0.

(d) The normal form of h' in the vicinity of h is not unique if one does not restrict to the transversal submanifold $M' = 1 + M$. To see this, let $h = xy \in \mathcal{O}_2$. The stabilizer K_h is generated by the permutation of variables (y,x) and the elements $(ax, by, 1/ab) \in K \subseteq m_2 \cdot \mathcal{O}_2^2 \oplus \mathcal{O}_2$ with arbitrary units $a, b \in \mathcal{O}_2^*$. Since $\mathrm{T}_h(hK) = m_2^2$ we can take $N = \mathbb{C}\oplus\mathbb{C}x\oplus\mathbb{C}y$. The map $\beta : m_2\cdot\mathcal{O}_2^2 \oplus \mathcal{O}_2 \to \mathcal{O}_2$ is given by $\beta(v_1, v_2, w) = yv_1 + xv_2 + xyw$. A possible section is given by $\tau(g) = (v_1, v_2, 0)$ if $g \in m_2^2$ is written uniquely as $g = yv_1 + xv_2$ with $v_1 \in \mathbb{C}\{x,y\}$, $v_2 \in \mathbb{C}\{x\}$. Hence

$$M' = 1 + M = \{(x + v_1, y + v_2, 1),\ v_1 \in \mathbb{C}\{x,y\},\ v_2 \in \mathbb{C}\{x\} \text{ of order } \geq 1\}$$

is a submanifold transversal to K_h. Observe that $H(t,x,y) = xy + tx$ and $H'(t,x,y) = xy + (t + t^2)x$ are two unfoldings of h in N which are equivalent via the unfolding

$((1+t)x,(1+t)^{-1}y,1)$ in K_h. But of course there is no unfolding in M' which transforms H into H'.

References

[BS] J. Bochnak, J. Siciak: Analytic functions in topological vector spaces. Studia Math. **39** (1971), 77 - 112.

[C] J. F. Colombeau: Différentiation et bornologie. Thesis, Université de Bordeaux 1973.

[G] A. Grothendieck: Topological vector spaces. Gordon and Breach 1973.

[Ham] R. S. Hamilton: The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. **7** (1982), 65 - 222.

[Hau] H. Hauser: La construction de la déformation semi-universelle d'un germe de variété analytique complexe. Ann. Sci. Éc. Norm. Sup. (4) **18** (1985), 1 - 56.

[He] M. Hervé: Analyticity in infinite dimensional spaces. De Gruyter 1989.

[HM 1] H. Hauser, G. Müller: Analytic curves in power series rings. Compos. Math. **76** (1990), 197 - 201.

[HM 2] H. Hauser, G. Müller: Automorphism groups in local analytic geometry, infinite dimensional Rank Theorem and Lie groups. C. R. Acad. Sci. Paris, I. Ser. **313** (1991), 751 - 756.

[HM 3] H. Hauser, G. Müller: A Rank Theorem for analytic maps between power series spaces. Publ. Math. IHES **80** (1994), 95 - 115.

[J] M. Jurchescu: On the canonical topology of an analytic algebra and of an analytic module. Bull. Soc. Math. France **93** (1965), 129 - 153.

[L] J. Leslie: On the group of real analytic diffeomorphisms of a compact real analytic manifold. Trans. Amer. Math. Soc. **274** (1982), 651 - 669.

[Ma] J. Mather: Notes on right equivalence. Preprint 1969.

[Mi] J. Milnor: Remarks on infinite-dimensional Lie groups. In: Relativité, groupes et topologie II (eds. B. S. DeWitt, R. Stora), pp. 1007 - 1057. Elsevier 1984.

[P 1] D. Pisanelli: The proof of the Frobenius theorem in a Banach scale. In: Functional analysis, holomorphy, and approximation theory (ed. G. I. Zapata), pp. 379 - 389. Marcel Dekker 1983.

[P 2] D. Pisanelli: The proof of the inversion mapping theorem in a Banach scale. In: Complex analysis, functional analysis and approximation theory (ed. J. Mujica), pp. 281 - 285. North- Holland 1986.

[U] H. Upmeier: Symmetric Banach manifolds and Jordan C*-algebras. North-Holland 1985.

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