T-Duality, fluxes and noncommutativity in closed string theory

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Mainly: arXiv:1802.07003 with Larisa Jonke, Fech Scen Khoo, Richard Szabo

Also: arXiv:1505.05457 with Larisa Jonke, Olaf Lechtenfeld

Matrix Models for Noncommutative Geometry and String Theory @ ESI Vienna

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“Lessons” for the Geometry of Spacetime and Quantum Gravity

- Geometry is Generalized (Noncommutativity, String, Matrices)
- Geometry is (maybe) Emergent (String, AdS/CFT, Matrices, ...)
- Geometry is (maybe) Doubled (Quantum Mechanics/Born Reciprocity, String/T-duality)

Doubling for Closed Strings

Circle compactifications $\rightsquigarrow$ Momentum and Winding modes with mass $\propto 1/R$ and R

Large radius limit $\rightsquigarrow$ Only momentum modes probe spacetime, and EFT is supergravity

$\rightsquigarrow$ measure lengths with position operators x

At QG scales, $R \sim \sqrt{\alpha'}$ $\rightsquigarrow$ both momentum and winding modes become important

e.g. in the Brandenberger-Vafa early universe scenario

$\rightsquigarrow$ position operators x and dual (to windings) $\tilde{x}$

Supergravity is certainly not enough here $\rightsquigarrow$ need (some kind of) Double Field Theory

e.g. proposals by Siegel ’93; Hull, Zwiebach ’09; Freidel, Leigh, Minic ’15; &c.

Symmetries

On one hand, we have diffeomorphisms and gauge transformations, as in field theory

But for closed strings, also T-duality, exchanging momenta $\leftrightarrow$ windings and $R \leftrightarrow 1/R$

N.B., T-duality is an asymmetric reflection: $X(\sigma, \tau) = X_L + X_R \xrightarrow{T} \tilde{X}(\sigma, \tau) = X_L - X_R$

When multiple (d) circle compactification, the T-duality symmetry group is $O(d, d; \mathbb{Z})$

$\rightsquigarrow$ The Double Field Theory should enjoy an $O(d, d; \mathbb{R})$ symmetry

$\rightsquigarrow$ The underlying geometric structure should contain/unify these symmetries

For open strings on D-branes

- Turn on B or F $\rightsquigarrow$ noncommutativity Douglas, Hull ’97; Chu, Ho ’98; Seiberg, Witten ’99

$$[X_1(\tau), X_2(\tau)] = i\theta_{12}\ , \quad \theta_{12} = -\frac{2\pi i\alpha'(B-F)}{1+(B-F)^2}\ .$$

- T-dual frame $\rightsquigarrow$ commutativity & D-branes at angles.

Lesson: New geometries arise in presence of non-trivial flux backgrounds.

Flux, Duality and Closed Strings

Left and right movers may experience different geometries (asymmetric strings).

T-duality reveals closed string backgrounds which are "non-geometric" (T-folds & co.)
e.g. Hull '04; Shelton, Taylor, Wecht '05; &c.

$$H_{ijk} \quad \to \quad f_{ij}^{k} \quad \to \quad Q_i^{jk} \quad \to \quad R^{ijk}$$

Generic closed string geometries argued to be noncommutative and nonassociative.
Lüst '10; Blumenhagen, Plauschinn '10; Mylonas, Schupp, Szabo '12

$$Q\text{-case} \quad [X^i, X^j] \sim Q^{ij}{}_k w^k$$

$$R\text{-case} \quad [X^i, X^j] \sim R^{ijk} p_k \qquad [X^i, X^j, X^k] \sim R^{ijk} \ .$$

Similar to particle in a non-constant magnetic field in QM. Jackiw '85; Bakas, Lüst '13

Enter Algebroids

- Courant Algebroids: unify Poisson and pre-symplectic structures
  Courant '90; Liu, Weinstein, Xu '95
  - Canonical example: $TM \oplus T^*M$, with a natural $O(d,d)$ metric, and fluxes as twists
- Generalized Complex Geometry: unify symplectic and complex structures
  Hitchin '02; Gualtieri '04
  - g and B on equal footing, Diffs and Gauge trafos as automorphisms of Courant bracket
  - Main additional player: a generalized metric:

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & B_{ik}\, g^{kj} \\ -g^{ik}\, B_{kj} & g^{ij} \end{pmatrix} .$$

Courant Algebroid vs. Doubling of coordinates

- Captures the symmetries, but not the doubling of coordinates
- But if the target is doubled, the symmetry would be $O(2d, 2d)$, i.e. too large

Double Field Theory

Siegel '93; Hull, Zwiebach '09

A proposal for a field theory invariant under $O(d,d)$; T-duality becomes manifest.

It uses doubled coordinates $(x^I) = (x^i, \tilde{x}_i)$, and all fields depend on both.

The $O(d,d)$ structure is associated to a (constant) $O(d,d)$-invariant metric

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \ , \quad h^t \eta h = \eta \ , \quad h \in O(d,d) \ ,$$

used to raise and lower $I = 1, \ldots, 2d$ indices.

Derivatives are also doubled accordingly: $(\partial_I) = (\partial_i, \tilde{\partial}^i)$.

The fields are the generalized metric $\mathcal{H}$ and invariant dilaton d ($e^{-2d} = \sqrt{-g}e^{-2\phi}$), with

Hohm, Hull, Zwiebach '10

$$S = \int \mathrm{d}x \mathrm{d}\tilde{x} e^{-2d} \left(\tfrac{1}{8} \mathcal{H}^{IJ} \partial_I \mathcal{H}^{KL} \partial_J \mathcal{H}_{KL} - \tfrac{1}{2} \mathcal{H}^{IJ} \partial_I \mathcal{H}^{KL} \partial_L \mathcal{H}_{KJ} - 2 \partial_I d \partial_J \mathcal{H}^{IJ} + 4 \mathcal{H}^{IJ} \partial_I d \partial_J d \right) \ .$$

DFT symmetries and constraints

Gauge transformations are included with a parameter $\epsilon^I = (\epsilon^i, \tilde{\epsilon}_i)$:

$$\begin{aligned} \delta_\epsilon \mathcal{H}^{IJ} &= \epsilon^K \partial_K H^{IJ} + (\partial^I \epsilon_K - \partial_K \epsilon^I)\mathcal{H}^{KJ} + (\partial^J \epsilon_K - \partial_K \epsilon^J)\mathcal{H}^{IK} := L_\epsilon \mathcal{H}^{IJ} \,, \\ \delta_\epsilon d &= -\tfrac{1}{2}\partial_K \epsilon^K + \epsilon^K \partial_K d \,, \end{aligned}$$

and $L_\epsilon \cdot$ is called the generalised Lie derivative. But S is not automatically invariant.

The theory is constrained.

- Weak constraint: $\Delta \cdot := \partial^I \partial_I \cdot = 0$; stems from the level matching condition.
- Strong constraint: $\partial^I \otimes \partial_I (\ldots) = 0$ on products on fields.

Strong constraint eliminates half coordinates $\rightsquigarrow$ DFT $\overset{\text{s.c.}}{\to}$ SUGRA

Alternatively, generalized vielbein $\mathcal{E}$ formulation $\mathcal{H}_{IJ} = \mathcal{E}^A{}_I \mathcal{E}^B{}_J S_{AB}$.

Siegel '93; Hohm, Kwak '10; Aldazabal et al. '11; Geissbuhler '11

- Allows to mildly dispense with the s.c. in generalized Scherk-Schwarz reductions

Questions to address

- What is the geometric structure of DFT and its relation to Courant algebroids?
- What is the Sigma-Model that captures the flux content of DFT?
  cf. Heller, Ikeda, Watamura '16
- What is the origin/role of DFT constraints and how does noncommutativity appear?

We want to answer these questions in the context of Membrane Sigma-Models

Membranes for Strings: Why?

- Already the familiar NSNS flux (field strength of B) lives in 3D (open membrane)
- Courant Algebroids correspond naturally to 3D Topological Field Theories
- Deformation quantization viewpoint acknowledging private communication with Peter Schupp
  - ("Closed") Fields $\rightsquigarrow$ Open Strings (Poisson Sigma-Model)
  - Closed Strings $\rightsquigarrow$ Open Membranes (Courant Sigma-Model)
  - Closed Membranes $\overset{?}{\rightsquigarrow}$ Open Tribranes (LAuth Sigma-Model)

Plan for the rest of the talk

1. Sigma-Models and Courant Algebroids
2. Doubled Membrane Sigma-Model
3. Universal description of geometric and non-geometric fluxes — NC/NA structure
4. (Almost) Algebroid Structures beyond Courant
5. Epilogue

Warm Up: (Twisted) Poisson Sigma-Model

Topological action for fields $X = (X^i) : \Sigma_2 \to M$ and $A \in \Omega^1(\Sigma_2; X^* T^* M)$

Schaller, Strobl '94; Ikeda '94

$$S_{\rm PSM}[X, A] = \int_{\Sigma_2} \Big(A_i \wedge {\rm d}X^i + \tfrac{1}{2}\Pi^{ij}(X) A_i \wedge A_j \Big)$$

Invariant under the gauge symmetry:

$$\begin{aligned} \delta X^i &= \Pi^{ji}\epsilon_j \ , \\ \delta A_i &= {\rm d}\epsilon_i + \partial_i \Pi^{jk} A_j \epsilon_k \ , \end{aligned}$$

provided that

$$\Pi^{l[i}\partial_l \Pi^{jk]} = 0 \quad \to \quad \Pi \text{ is a Poisson 2-vector}$$

Comments

- May be twisted by a 3-form H (Wess-Zumino term) $\rightsquigarrow$ twisted Poisson structure
  Klimcik, Strobl '01
  $$\Pi^{l[i}\partial_l \Pi^{jk]} = H_{lmn}\Pi^{li}\Pi^{mj}\Pi^{nk} \ .$$
- 2D case of AKSZ scheme of topological field theories (for $H = 0$ at least)
  Alexandrov, Kontsevich, Schwarz, Zaboronsky '95
- Deformation Quantization of Poisson manifolds $\sim$ Perturbation theory of PSM
  Kontsevich '97; Cattaneo, Felder '99

Courant Sigma-Model

Hofman, Park '02; Ikeda '02

Maps $X = (X^i) : \Sigma_3 \to M$, 1-forms $A \in \Omega^1(\Sigma_3, X^* E)$, and 2-form $F \in \Omega^2(\Sigma_3, X^* T^* M)$

$$S[X, A, F] \;=\; \int \Big(F_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \rho^i{}_I(X) A^I \wedge F_i + \tfrac{1}{6} T_{IJK}(X) A^I \wedge A^J \wedge A^K \Big) \;.$$

E is some vector bundle (here $TM \oplus T^* M$), η is the (constant) $O(d, d)$-invariant metric

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \;.$$

3D case of AKSZ scheme of topological field theories Roytenberg '06

Gauge Symmetries of the Courant Sigma-Model

The Courant Sigma-Model is invariant under the following gauge transformations Ikeda '02

$$
\begin{aligned}
\delta X^i &= \rho^i_J \epsilon^J \ , \\
\delta A^I &= \mathrm{d}\epsilon^I + \eta^{IN} T_{NJK} A^J \epsilon^K + \eta^{IJ} \rho^i{}_J t_i \ , \\
\delta F_m &= -\epsilon^J \partial_m \rho^i{}_J F_i + \tfrac{1}{2} \epsilon^J \partial_m T_{ILJ} A^I \wedge A^L + \mathrm{d}t_m + \partial_m \rho^j{}_J A^J t_j \ ,
\end{aligned}
$$

where ϵ and t are gauge parameters, provided that

$$
\begin{aligned}
&\eta^{KL} \rho^i{}_K \rho^j{}_L = 0 \\
&2\rho^l{}_{[I} \partial_{\underline{l}} \rho^k{}_{J]} - \rho^k{}_J \eta^{JL} T_{LIJ} = 0 \\
&4\rho^m{}_{[L} \partial_{\underline{m}} T_{IJK]} - 3\eta^{MN} T_{M[IJ} T_{KL]N} = 0 \ .
\end{aligned}
$$

These three conditions have an interesting relation to both physics and mathematics

- Coincide with the fluxes and Bianchi identities in sugra flux compactifications
- Coincide with the local form of the axioms of a Courant Algebroid

Courant Algebroid Axioms

Liu, Weinstein, Xu '95

$(E \xrightarrow{\pi} M, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho : E \to TM)$, such that for $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(M)$:

1. Modified Jacobi identity ($\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\rho(A)f$.)

$$[[A,B],C] + \text{c.p.} = \mathcal{D}\mathcal{N}(A,B,C)\ , \quad \text{where} \quad \mathcal{N}(A,B,C) = \tfrac{1}{3}\langle [A,B],C\rangle + \text{c.p.}\ ,$$

2. Modified Leibniz rule

$$[A, fB] = f[A,B] + (\rho(A)f)B - \langle A,B\rangle \mathcal{D}f\ ,$$

3. Compatibility condition

$$\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle [C,B] + \mathcal{D}\langle C,B\rangle, A\rangle\ ,$$

The structures also satisfy the following properties (they follow... Uchino '02):

4. Homomorphism

$$\rho[A,B] = [\rho(A), \rho(B)]\ ,$$

5. "Absence of strong constraint"

$$\rho \circ \mathcal{D} = 0 \quad \Leftrightarrow \quad \langle \mathcal{D}f, \mathcal{D}g\rangle = 0\ .$$

Naive Doubling

In order to incorporate the dual coordinates, we replace M with a doubled space $\mathcal{M}$.

A "large" CA $\mathbb{E}$ over $\mathcal{M}$ leads to a MSM with action ($I = 1, \dots, 2d$ and $\hat{I} = 1, \dots, 4d$)

$$S[\mathbb{X}, \mathbb{A}, \mathbb{F}] = \int \left(\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \tfrac{1}{2} \hat{\eta}_{\hat{I}\hat{J}} \mathbb{A}^{\hat{I}} \wedge \mathrm{d}\mathbb{A}^{\hat{J}} - \rho^I{}_{\hat{I}}(\mathbb{X}) \mathbb{A}^{\hat{I}} \wedge \mathbb{F}_I + \tfrac{1}{6} T_{\hat{I}\hat{J}\hat{K}}(\mathbb{X}) \mathbb{A}^{\hat{I}} \wedge \mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{K}} \right) .$$

In order to have some metric structure too, we add a general symmetric term on $\partial\Sigma_3$

$$S_{\text{sym}}[\mathbb{X}, \mathbb{A}] = \int_{\partial\Sigma_3} \tfrac{1}{2} g_{\hat{I}\hat{J}}(\mathbb{X}) \mathbb{A}^{\hat{I}} \wedge * \mathbb{A}^{\hat{J}} := \int_{\partial\Sigma_3} ||\mathbb{A}||_g .$$

- Previously we had $O(d, d)$ (η) but d-dimensional target
- Now we have $2d$-dimensional target but $O(2d, 2d)$ ($\hat{\eta}$)
- A DFT structure should be "in between", schematically:

$$\text{Large CA over } \mathcal{M} \xrightarrow{\mathrm{p}_+} \text{DFT Structure} \xrightarrow{\text{strong}} \text{Canonical CA over } M$$

Splitting and Projecting

A section $\mathbb{A} \in \mathbb{E}$ is

$$\mathbb{A} := \mathbb{A}_V + \mathbb{A}_F = \mathbb{A}^I \partial_I + \widetilde{\mathbb{A}}_I \mathrm{d}\mathbb{X}^I = \mathbb{A}_+^I e_I^+ + \mathbb{A}_-^I e_I^- \ ,$$

where we introduce the following combinations:

$$\mathbb{A}_\pm^I = \tfrac{1}{2}(\mathbb{A}^I \pm \eta^{IJ}\widetilde{\mathbb{A}}_J) \ , \quad e_I^\pm = \partial_I \pm \eta_{IJ}\mathrm{d}\mathbb{X}^J \ ,$$

This gives a decomposition of the generalized tangent bundle as

$$E = \mathbb{T}\mathcal{M} = L_+ \oplus L_- \ .$$

Then we consider a projection to the subbundle L_+ with O(d,d) vectors

$$\mathrm{p}_+ : E \longrightarrow L_+ \ , \quad (\mathbb{A}_V, \mathbb{A}_F) \longmapsto \mathbb{A}_+ := A = A_i(\mathrm{d}X^i + \tilde{\partial}^i) + A^i(\mathrm{d}\widetilde{X}_i + \partial_i) \ .$$

Projection of the symmetric bilinear of $\mathbb{E}$, leads to the O(d,d) invariant DFT metric:

$$\langle \mathbb{A}, \mathbb{B} \rangle_{\mathbb{E}} = \tfrac{1}{2}\eta_{\hat{I}\hat{J}}\mathbb{A}^{\hat{I}}\mathbb{B}^{\hat{J}} = \eta_{IJ}(\mathbb{A}_+^I\mathbb{B}_+^J - \mathbb{A}_-^I\mathbb{B}_-^J) \quad \mapsto \quad \eta_{IJ}A^I B^J = \langle A, B \rangle_{L_+} \ .$$

Projected Bracket

Using the projection, a closed bracket on L_+ is defined as

$$[\![A, B]\!]_{L_+} = \mathrm{p}_+\big([\mathrm{p}_+(\mathbb{A}), \mathrm{p}_+(\mathbb{B})]_E\big)$$

(N.B.: L_+ is not an involutive subbundle, thus neither a Dirac structure of $\mathbb{E}$.)

This agrees with the local formula for the so-called C-bracket, used in DFT
Siegel '93; Hull, Zwiebach '10

$$[\![A, B]\!]^J_{L_+} = A^K\,\partial_K B^J - \tfrac{1}{2}\,A^K\,\partial^J B_K - \{A \leftrightarrow B\}\ .$$

Thus, the map p_+ sends large CA structures to corresponding DFT structures.

Double Field Theory Sigma-Model

Applying this strategy to the Courant Sigma-Model, we obtain the action

agrees with the proposal of A.Ch., Jonke, Lechtenfeld '15

$$S[\mathbb{X}, A, F] = \int \Big(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ} A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J A^J \wedge F_I + \tfrac{1}{3} \widehat{T}_{IJK} A^I \wedge A^J \wedge A^K \Big) \ ,$$

where $\rho_+ : L_+ \to T\mathcal{M}$ is a map to the tangent bundle of $\mathcal{M}$.

The symmetric term undergoes a rather trivial projection:

$$S_{\text{sym}}[\mathbb{X}, A] = \int_{\partial \Sigma_3} \tfrac{1}{2} g_{IJ}(\mathbb{X}) A^I \wedge * A^J \ .$$

- Does it describe all types of fluxes in a unified way?
- What is the underlying mathematical structure that replaces the CA?
- What is the relation to the target space DFT and its constraint structure?

Examples: The 3-Torus Flux Chain

Consider a doubled torus as target of the DFT MSM and DFT structural data as

$$(\rho_+)^I{}_J = \begin{pmatrix} \rho^i{}_j & \rho^{ij} \\ \rho_{ij} & \rho_i{}^j \end{pmatrix} \qquad A^I = (q^i, p_i) \qquad T_{IJK} = \begin{pmatrix} H_{ijk} & f_{ij}{}^k \\ Q_i{}^{jk} & R^{ijk} \end{pmatrix} \qquad g_{IJ} = \begin{pmatrix} g_{ij} & g_i{}^j \\ g^i{}_j & g^{ij} \end{pmatrix} \ .$$

The goal is to describe the T-duality chain relating geometric and non-geometric fluxes

Shelton, Taylor, Wecht '05

$$H_{ijk} \xleftrightarrow{\mathrm{T}_k} f_{ij}{}^k \xleftrightarrow{\mathrm{T}_j} Q_i{}^{jk} \xleftrightarrow{\mathrm{T}_i} R^{ijk}$$

Also, to clarify the proposal for NC/NA deformations in non-geometric flux backgrounds

NSNS Flux & Geometric Flux

Choose

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & 0 \\ 0 & 0 \end{pmatrix} , \qquad T_{IJK} = \begin{pmatrix} H_{ijk} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad g_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} .$$

Then, taking the F-equations of motion, the membrane action reduces to

$$S_H[X] := \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k ,$$

which is the standard closed string model with NSNS flux as Wess-Zumino term.

Choose ($f_{ij}{}^k = -2\, E^\mu{}_{[i}\, E^\nu{}_{j]}\, \partial_\mu E^k{}_\nu$ structure constants of the 3D Heisenberg algebra)

$$(\rho_+)^M{}_J = \begin{pmatrix} E^\mu{}_j & 0 \\ 0 & 0 \end{pmatrix} , \qquad T_{IJK} = \begin{pmatrix} 0 & 2\, f_{ij}{}^k \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad g_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} .$$

The resulting action now becomes simply (using Maurer-Cartan $\mathrm{d}E^i = -\frac{1}{2}\, f_{jk}{}^i\, E^j \wedge E^k$)

$$S_f[X] := \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{ij}\, E^i \wedge *E^j ,$$

which is the action with T-dual target the Heisenberg nilmanifold.

The T-fold and Noncommutativity

To describe the globally non-geometric Q-flux frame we choose

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij}(X) \\ 0 & -\delta_i{}^j \end{pmatrix} \qquad T_{IJK} = \begin{pmatrix} 0 & 0 \\ Q_i^{jk} & 0 \end{pmatrix} \qquad g_{IJ} = \begin{pmatrix} 0 & \delta_3{}^j \\ 0 & g^{ij} \end{pmatrix} ,$$

with $g^{ij} = \mathrm{diag}(1,1,0)$ and $\beta^{ij}(X) = -Q_k{}^{ij}\, X^k$ with components $Q_3{}^{12} = Q = -Q_3{}^{21}$.

The same procedure leads, for $m = 1, 2$, to

$$\int_{\partial\Sigma_3} \big(\mathrm{d}\widetilde{X}_m \wedge \mathrm{d}X^m + Q\, X^3\, \mathrm{d}\widetilde{X}_1 \wedge \mathrm{d}\widetilde{X}_2 + \tfrac{1}{2}\, \mathrm{d}X^3 \wedge *\mathrm{d}X^3 + \tfrac{1}{2}\, \mathrm{d}\widetilde{X}_m \wedge *\mathrm{d}\widetilde{X}_m\big) \ .$$

This is shown to be equivalent to the T-fold action, obtained via Buscher rules

$$S_Q[X] = \int_{\partial\Sigma_3} \big(\tfrac{1}{2}\, \mathrm{d}X^3 \wedge *\mathrm{d}X^3 + \tfrac{1}{2(1+(Q\, X^3)^2)}\, \mathrm{d}X^m \wedge *\mathrm{d}X^m - \tfrac{Q\, X^3}{1+(Q\, X^3)^2}\, \mathrm{d}X^1 \wedge \mathrm{d}X^2\big)$$

The T-fold and Noncommutativity

From a different viewpoint, taking $\Sigma_3 = \Sigma_2 \times S^1$ and wrapping the membrane:

$$X^3(\sigma) = w^3\, \sigma^3 ,$$

a dimensional reduction of the topological action yields

$$S_{Q,w}[X, \widetilde{X}\,] := \int_{\Sigma_2} \left(\tfrac{1}{2}\, \mathrm{d}\widetilde{X}_m \wedge *\mathrm{d}\widetilde{X}_m + \mathrm{d}\widetilde{X}_m \wedge \mathrm{d}X^m + \tfrac{1}{2}\, Q_3{}^{mn}\, w^3\, \mathrm{d}\widetilde{X}_m \wedge \mathrm{d}\widetilde{X}_n \right) .$$

The topological sector contains $\theta = \frac{1}{2}\, \theta^{mn}\, \partial_m \wedge \partial_n + \partial_m \wedge \tilde{\partial}^m$, with Poisson brackets

$$\{X^m, X^n\}_\theta = \theta^{mn} = Q_3{}^{mn}\, w^3 \ , \qquad \{X^m, \widetilde{X}_n\}_\theta = \delta^m{}_n \qquad \text{and} \qquad \{\widetilde{X}_m, \widetilde{X}_n\}_\theta = 0 \ .$$

$\rightsquigarrow$ *Q*-flux leads to a closed string noncommutative geometry provided by a Wilson line exactly as in Lüst '10

$$\theta^{ij} = \oint_{C_k} Q_k{}^{ij}\, \mathrm{d}X^k \ .$$

R flux and nonassociativity

A frame with no conventional target space description in terms of standard coordinates

Realized in the membrane sigma-model upon choosing (with $\beta^{ij}(\widetilde{X}) = R^{ijk}\,\widetilde{X}_k$)

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij}(\widetilde{X}) \\ 0 & -\delta_i{}^j \end{pmatrix} \qquad T_{IJK} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk} \end{pmatrix} \qquad \text{and} \qquad g_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} .$$

This leads to the action, first proposed in Mylonas, Schupp, Szabo '12

$$S_R[X,\widetilde{X}] = \int_{\partial\Sigma_3} \big(\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\,R^{ijk}\,\widetilde{X}_k\,\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{X}_j + \tfrac{1}{2}\,g^{ij}\,\mathrm{d}\widetilde{X}_i \wedge *\mathrm{d}\widetilde{X}_j\big) .$$

$\rightsquigarrow$ 2-vector $\Theta^{IJ} = \begin{pmatrix} R^{ijk}\,\widetilde{X}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix}$ on the doubled space, with twisted Poisson bracket

$$\{X^i, X^j\}_\Theta = R^{ijk}\,\widetilde{X}_k\,, \qquad \{X^i, \widetilde{X}_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{\widetilde{X}_i, \widetilde{X}_j\}_\Theta = 0\,,$$

and the non-vanishing Jacobiator, a sign of nonassociativity in X-space,

$$\{X^i, X^j, X^k\}_\Theta := \tfrac{1}{3}\,\{\{X^i, X^j\}_\Theta, X^k\}_\Theta + \text{cyclic} = -R^{ijk}\,.$$

Towards the DFT Algebroid

In general, taking a parametrization of the ρ_+ components to be

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ B_{ij} & \delta_i{}^j + \beta^{jk} B_{ki} \end{pmatrix} ,$$

the relevant local expressions that replace the ones of the undoubled case are

$$\begin{aligned} &\eta^{IJ}\rho^K{}_I\rho^L{}_J = \eta^{KL} \\ &2\rho^L{}_{[I}\partial_{\underline{L}}\rho^K{}_{J]} - \eta^{LM}\rho^K{}_L\hat{T}_{MIJ} = \rho_{L[I}\partial^K\rho^L{}_{J]} \\ &4\rho^M{}_{[L}\partial_{\underline{M}}\hat{T}_{IJK]} + 3\eta^{MN}\hat{T}_{M[IJ}\hat{T}_{KL]N} = \mathcal{Z}_{IJKL} . \end{aligned}$$

- Expressions for fluxes and Bianchis of DFT when the strong constraint holds
  Geissbuhler, Marques, Nunez, Penas '13
- Conditions for gauge invariance of our MSM when the strong constraint holds
- They can be used to reverse-engineer a more general structure than CAs

A Word on the Generalized Metric

In general one obtains on-shell a string sigma-model with doubled target
as e.g. in Hull, Reid-Edwards '09

$$S_{\mathcal{H},\mathcal{F}}[\mathbb{X}] := \int_{\partial\Sigma_3} \tfrac{1}{2}\,\mathcal{H}_{IJ}\,\mathrm{d}\mathbb{X}^I \wedge *\mathrm{d}\mathbb{X}^J + \int_{\Sigma_3} \tfrac{1}{3}\,\mathcal{F}_{IJK}\,\mathrm{d}\mathbb{X}^I \wedge \mathrm{d}\mathbb{X}^J \wedge \mathrm{d}\mathbb{X}^K \ ,$$

where

$$\mathcal{H}_{IJ} := (\rho_+)_I{}^K\, g_{KL}\, (\rho_+)_J{}^L \quad \text{and} \quad \mathcal{F}_{IJK} := (\rho_+)_I{}^L\, (\rho_+)_J{}^M\, (\rho_+)_K{}^N\, \hat{T}_{LMN} \ .$$

$\mathcal{H}$ is then exactly the generalized metric, in various parametrizations, e.g.

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & -B_{ik}\, g^{kj} \\ g^{ik}\, B_{kj} & g^{ij} \end{pmatrix} , \qquad \text{for} \quad \beta = 0$$

$$\widetilde{\mathcal{H}}_{IJ} = \begin{pmatrix} g_{ij} & g_{ik}\, \beta^{kj} \\ -\beta^{ik}\, g_{kj} & g^{ij} - \beta^{ik}\, g_{kl}\, \beta^{lj} \end{pmatrix} , \qquad \text{for} \quad B = 0$$

The DFT Algebroid and other Relaxed Structures

A quadruple $(L_+, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle_{L_+}, \rho_+)$ satisfying $(\langle A, \mathcal{D}_+ f\rangle_{L_+} = \frac{1}{2}\rho_+(A)f)$

2. $[\![A, fB]\!] = f[\![A, B]\!] + (\rho_+(A)f)\, B - \langle A, B\rangle_{L_+} \mathcal{D}_+ f$,
3. $\langle [\![C, A]\!] + \mathcal{D}_+\langle C, A\rangle_{L_+}, B\rangle_{L_+} + \langle [\![C, B]\!] + \mathcal{D}_+\langle C, B\rangle_{L_+}, A\rangle_{L_+} = \rho_+(C)\langle A, B\rangle_{L_+}$,
5. $\langle \mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} = \frac{1}{4}\,\langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+}$.

Notably, the modified Jacobi, homomorphism and kernel properties are obstructed

In general, by relaxing properties one obtains a host of intermediate structures

cf. Vaisman '04; Hansen, Strobl '09; Bruce, Grabowski '16

$$\text{Pre-DFT algebroid} \xleftarrow{\not{5}} \text{Ante-Courant algebroid} \xleftarrow{\not{4}} \text{Pre-Courant algebroid} \xleftarrow{\not{1}} \text{Courant algebroid}$$

The DFT Algebroid is an example of pre-DFT Algebroid, for which

$$\text{Large Courant algebroid} \xrightarrow{\mathsf{p}_+} \text{DFT algebroid} \xrightarrow{5} \text{Courant algebroid}$$

Imposing that the RHS of property 5 is zero is exactly the strong constraint of DFT.

Comments on the Strong Constraint & Beyond DFT

- It's too strong. Essentially it merely reduces DFT to supergravity.
- It's not there in the original DFT and it has no obvious stringy origin.
- It's violated in certain (nonassociative) *R* flux models.
- It can be relaxed in generalised SS reductions (by a milder closure constraint.)

Options to formulate a sigma-model that goes beyond the standard DFT

- Depart from the constant η metric and consider a dynamical one $\eta(\mathbb{X})$
  cf. Freidel, Leigh, Minic '15; also Hansen, Strobl '09
- Make use of the additional symplectic structure related to the term $\omega = \mathrm{d}X \wedge \mathrm{d}\widetilde{X}$
  Vaisman '12; Freidel, Rudolph, Svoboda '17

Dynamical η

- The twist of the C-bracket is modified:

$$[\![A,B]\!]_\eta := \mathrm{p}_+ \left([\mathrm{p}_+(\mathbb{A}), \mathrm{p}_+(\mathbb{B})]_E \right) = [\![A,B]\!] + S(A,B) \ ,$$

where in local coordinate form

$$S(A,B) = S^L{}_{IJ} A^I B^J e_L^+ = \eta^{LK} \rho^M{}_{[I} \partial_{\underline{M}} \eta_{J]K} A^I B^J e_L^+ \ .$$

- However, the MSM is not modified, since $\langle [\![A,A]\!]_\eta, A \rangle_{L_+} = \langle [\![A,A]\!], A \rangle_{L_+}$.
- The gauge transformation of A^I is modified to

$$\delta_\epsilon A^I = \mathrm{d}\epsilon^I + \left(\eta^{IJ} \hat{T}_{JKL}(\mathbb{X}) + S^I{}_{KL}(\mathbb{X}) \right) A^K \epsilon^L$$

- We found that the strong constraint can be avoided provided that

$$\rho^K{}_{[I} \partial_K \eta_{L]J} = \rho_{JK} \rho_{N[I} \partial^K \rho^N{}_{L]} \ .$$

- We plan to understand this globally and find examples that solve this equation.

Epilogue

Take-Home Messages

- The geometric structure of DFT is between two Courant Algebroids
- A DFT Algebroid as a relaxed-CA structure; interpretation of strong constraint
- Membrane Sigma-Model compatible with flux formulation of DFT
- In principle, more general; with noncommutative/nonassociative deformations

Some Open Questions and Things-To-Do

- What is the theory without the strong constraint? Role of $\eta(\mathbb{X})$ and $\omega(\mathbb{X})$? Perhaps a relation to "Metastring Theory" or "Born Geometry"? Implications for stringy early-universe cosmology?
- One dimension higher? Closed Membranes, Exceptional Field Theory?
- Any relation to Matrix Models? Perhaps dynamical phase space A.Ch. '14

Back-up slide

Alternative definition of a Courant Algebroid

Ševera '98

Definition in terms of a bilinear, non-skew operation (Dorfman derivative)

$$[A, B] = A \circ B - B \circ A \, ,$$

notably satisfying instead of 1, the Jacobi identity (in Loday-Leibniz form):

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C) \, .$$

Axioms 2 and 3 do not contain $\mathcal{D}$-terms any longer,

$$\begin{aligned} A \circ fB &= f(A \circ B) + (\rho(A)f)B \, , \\ \rho(C)\langle A, B \rangle &= \langle C \circ A, B \rangle + \langle C \circ B, A \rangle \, . \end{aligned}$$

The two definitions are equivalent, as proven by Roytenberg '99