Signature change in matrix model solutions

Various classical solutions to $D(< 10)$-dimensional IKKT-like Lorentzian matrix models are examined in the commutative limit. The solutions are associated with smooth $d(< D)$-dimensional manifolds in this limit, and their associated induced metric (and Steinacker's effective metric) are computed. The signature of these metrics can vary when quadratic and cubic terms are included in the bosonic action. Regions with Lorenzian signature can serve as toy models for cosmological space-times, complete with cosmological singularities which are associated with the signature change. The singularities are resolved away from the commutative limit. Toy models of open and closed cosmological space-times are given for $d = 2\&4$. Finally, we speculate on the application of the fuzzy $d = 4$ hyperboloid solution to a noncommutative verison of the AdS/CFT correspondence principal.

Signature change believed to be a feature of quantum gravity

Sarkharov Sov.Phys.JETP 60 (1984) 214-218
Gibbons, Hartle Phys.Rev. D42 (1990) 2458-2468

discussed in different contexts: loop quantum gravity, causal dynamical triangulation, string theory, …

claim: also appears for classical matrix model solutions (with indefinite target metric)

disclaimers: won't work with 10d maximally SUSY IKKT, rather lower dimensional toy models,
just bosonic sector
won't consider fluctuations, stability issues

program: look at various solutions to classical equation of motion in continuum (commutative, semiclassical) limit - associated with emergence of smooth Poisson manifolds

compute induced and effective metrics

signature change common feature if additional terms included in matrix action

regions with Lorentzian signature serve as crude cosmological modles
- complete with cosmological singularities (at singularity change);
resolved away from commutative limit

Outline

2d solutions of 3d Lorentzian matrix models

(A)dS^2, Euclidean (A)dS^2 , S^1 X R^1 (boring)

deformed (A)dS^2, Euclidean (A)dS^2 , S^2

4d solutions of 8d matrix models with indefinite metric

$CP^{1,1}$, $CP^{0,2}$ (boring)

deformed $CP^{1,1}$, $CP^{0,2}$, CP^2

6d solution

$CP^{1,2}$ - projects down to H^4

possible application to cosmology (Steinacker's talk) and non-commutative AdS/CFT

3d `IKKT-inspired’ model – bosonic sector

infinite dimensional Hermitean matrices $X^\mu,\ \mu = 0, 1, 2$

Lorentzian background metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$

action $$S(X) = \frac{1}{g^2}\text{Tr}\Big(-\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{ia}{3}\epsilon_{\mu\nu\lambda}X^\mu[X^\nu, X^\lambda]\Big)$$

vary X^μ $$[[X_\mu, X_\nu], X^\nu] + ia\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] = 0$$

symmetries:

2+1 Lorentz invariance $X^\mu \to \Lambda^\mu{}_\nu X^\nu$

unitary gauge transformations $X^\mu \to U X^\mu U^\dagger$

translations $X^\mu \to X^\mu + \text{constant} \times \mathbb{1}$

solutions

- Non-commutative (A)dS^2/ Euclidean (A)dS^2

$$[X_\mu, X_\nu] = -ia\epsilon_{\mu\nu\lambda}X^\lambda \qquad X^\mu X_\mu \;\; \text{fixed}$$

P-M Ho & M Li hep-th/0004072, hep-th/0005268
D Jurman, H Steinacker arXiv:1309.1598
A. Chaney, L. Lu, A.S. arXiv:1511.06816

→ UIR's of su(1,1)

$X^\mu X_\mu$ > 0 Non-commutative (A)dS^2 (principal, supplemental series)
< 0 Non-commutative Euclidean (A)dS^2 (discrete series)

- Non-commutative cylinder

Chaichan, Demichev, Presnajder, Tureanu hep-th/0007156, Phys.Lett. B515 (2001) 426-430
Balachandran, Govindarajan, Martins, Teotonio-Sobrinho hep-th/0410067
A.S. arXiv:1404.2549

$$[X_0, X_\pm] = \pm 2a\, X_+ \qquad X_+X_- \;\; \text{fixed}$$

$$[X_+, X_-] = 0\,, \qquad X_\pm = X_1 \pm X_2$$

→ UIR's of 2d Euclidean algebra

Commutative (semi-classical) limit

$$X^\mu \to \text{commuting coordinates } x^\mu$$

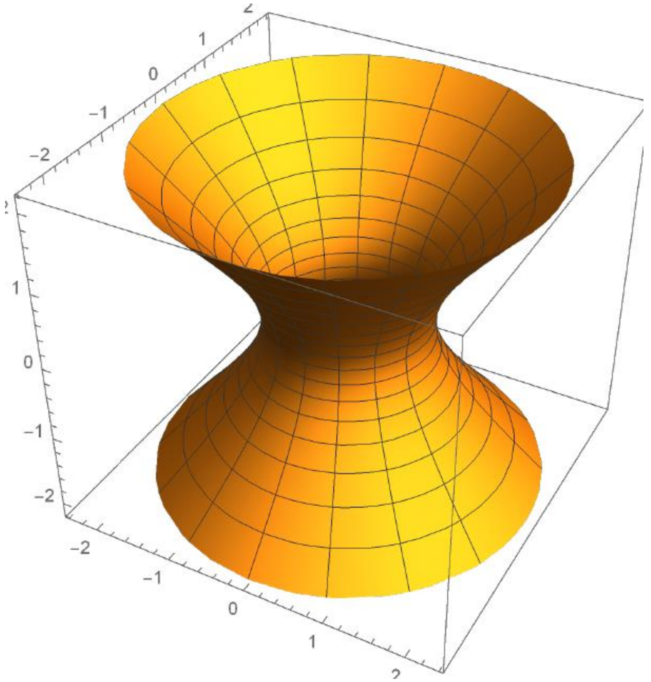
$$[\ ,\ ] \to i\hbar\{\ ,\ \} \qquad\qquad a \to -\hbar$$

Non-commutative (A)dS^2 → (A)dS^2

$$x^\mu x_\mu = r^2$$

$$\{x_\mu, x_\nu\} = \epsilon_{\mu\nu\lambda} x^\lambda$$

- preserves SO(2,1) isometry



parametrization

$$(\tau, \sigma),\ -\infty < \tau < \infty,\ 0 \le \sigma < 2\pi.$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = r \begin{pmatrix} \sinh\tau \\ \cosh\tau\cos\sigma \\ \cosh\tau\sin\sigma \end{pmatrix}$$

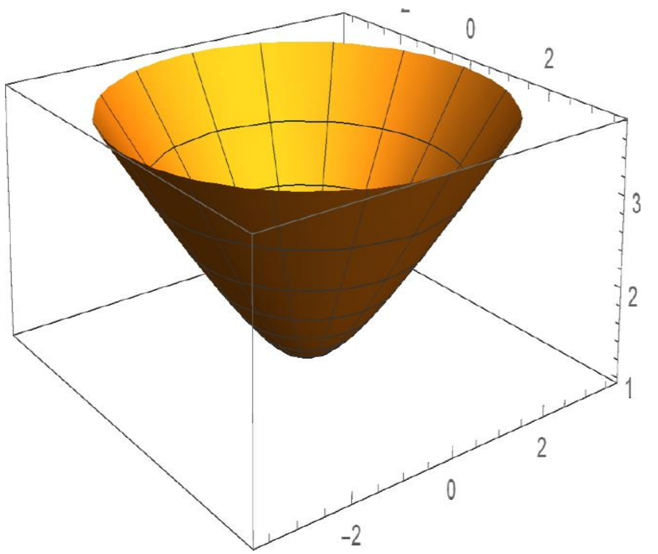
induced metric

$$ds^2 = r^2\left(-d\tau^2 + \cosh^2\tau\, d\sigma^2\right)$$

Non-commutative Euclidean (A)dS^2 → Euclidean (A)dS^2

$$x^\mu x_\mu = -r^2$$

$$\{x_\mu, x_\nu\} = \epsilon_{\mu\nu\lambda} x^\lambda$$



parametrization

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = r \begin{pmatrix} \cosh\tau \\ \sinh\tau\cos\sigma \\ \sinh\tau\sin\sigma \end{pmatrix}$$

induced metric

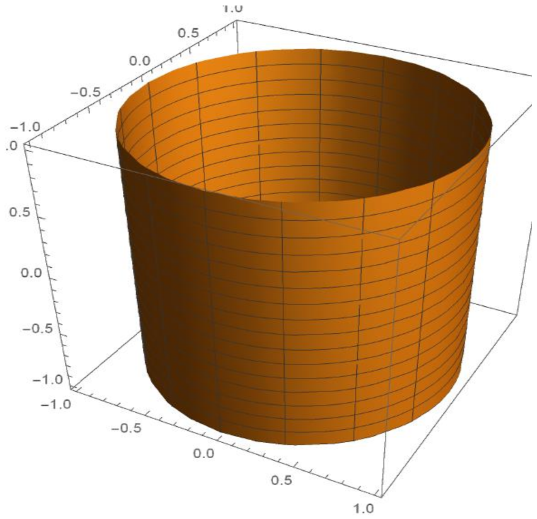
$$ds^2 = r^2\left(d\tau^2 + \sinh^2\tau\, d\sigma^2\right)$$

Non-commutative cylinder → cylinder

$$(x^1)^2 + (x^2)^2 = \rho^2$$

$$\{x^0, x^1\} = -\frac{1}{\rho} x^2 \qquad \{x^0, x^2\} = \frac{1}{\rho} x^1$$

$$\{x^1, x^2\} = 0$$



parametrization

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \tau \\ \rho\cos\sigma \\ \rho\sin\sigma \end{pmatrix}$$

induced metric

$$ds^2 = -d\tau^2 + \rho^2 d\sigma^2$$

previous examples don't exhibit signature change

- now add quadratic term:

can result from IR regularization S-W Kim, J Nishimura, Tsuchiya arXiv:1108.1540, arXiv:1110.4803

$$S(X) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{ia}{3}\epsilon_{\mu\nu\lambda}X^\mu[X^\nu, X^\lambda] + \frac{b}{2}X_\mu X^\mu\Big)$$

$$[[X_\mu, X_\nu], X^\nu] + ia\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] + bX_\mu = 0$$

commutative limit

$$X^\mu \to x^\mu \qquad [\ ,\ ] \to i\hbar\{\ ,\ \} \qquad a \to \hbar\alpha \qquad b \to \hbar^2\beta$$

$$-\{\{x_\mu, x_\nu\}, x^\nu\} - \alpha\epsilon_{\mu\nu\lambda}\{x^\nu, x^\lambda\} + \beta x_\mu = 0$$

New solutions

$Deformed\ AdS^2$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = r \begin{pmatrix} \sinh\tau \\ \rho\cosh\tau\cos\sigma \\ \rho\cosh\tau\sin\sigma \end{pmatrix} \qquad \{\tau,\sigma\} = \frac{1}{r\,\cosh\tau}$$

solution provided: $\alpha = -\frac{1}{2} \qquad \beta = \rho^2$

induced metric

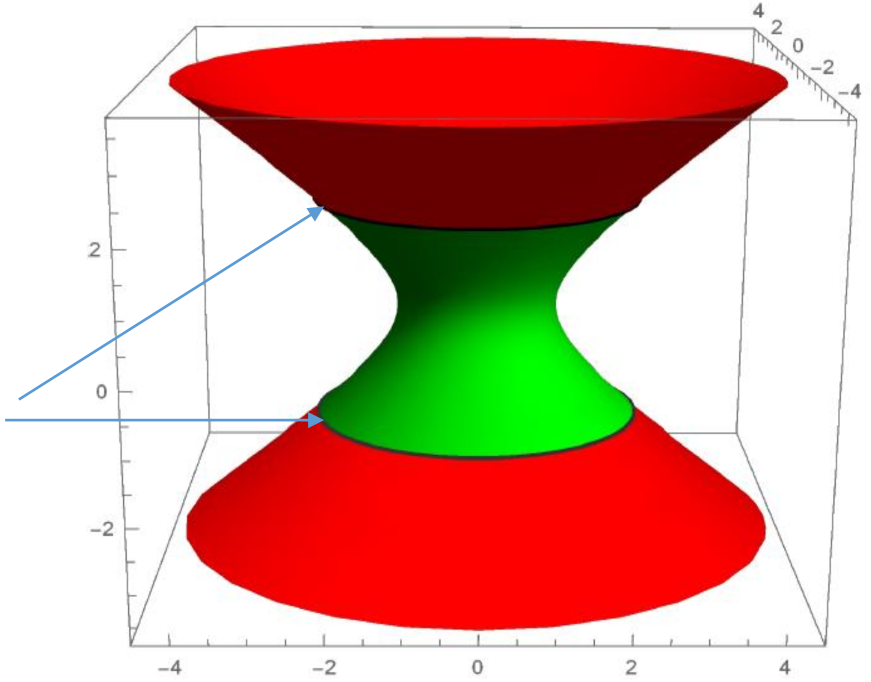
$$ds^2 = r^2\cosh^2\tau\left((-1+\rho^2\tanh^2\tau)\,d\tau^2 + \rho^2\,d\sigma^2\right)$$

Signature change when $\rho^2 > 1$

$$\tau = \tau_\pm = \pm\tanh^{-1}|\tfrac{1}{\rho}|$$

Euclidean $\tau > \tau_+ \text{ and } \tau < \tau_-$

Lorentzian $\tau_- < \tau < \tau_+$



Deformed Euclidean AdS^2

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = r \begin{pmatrix} \cosh\tau \\ \rho\sinh\tau\cos\sigma \\ \rho\sinh\tau\sin\sigma \end{pmatrix}$$

$$\{\tau,\sigma\} = \frac{1}{r\,\sinh\tau}$$

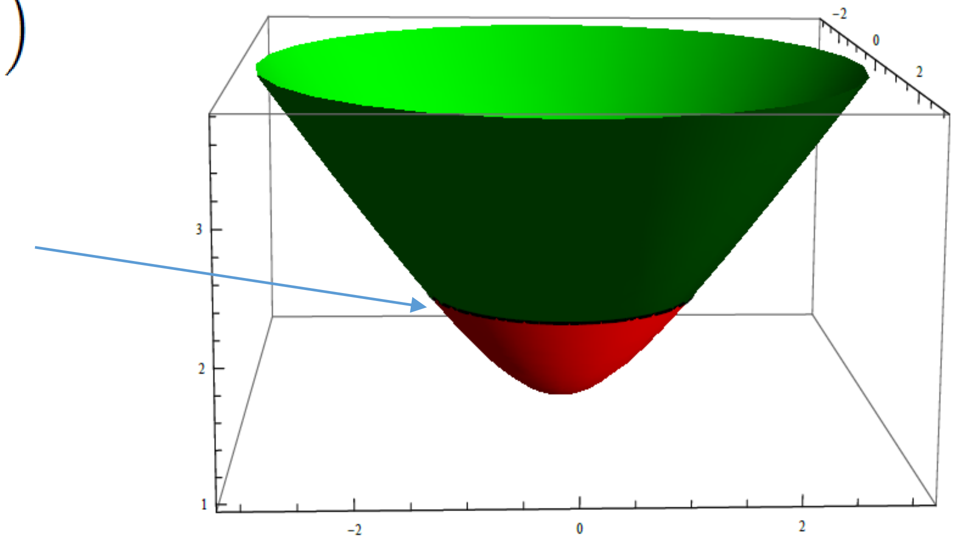
solution provided: $\alpha = -\frac{1}{2}$ $\beta = \rho^2$

induced metric

$$ds^2 = r^2\sinh^2\tau\left((\rho^2\coth^2\tau - 1)\,d\tau^2 + \rho^2\,d\sigma^2\right)$$

Signature change when $\rho^2 < 1$

$$\tau = \tau_+ = \tanh^{-1}|\rho|$$



Euclidean $\tau < \tau_+$

Lorentzian $\tau > \tau_+$

Fuzzy sphere solves Euclidean matrix model

- also ***Lorentzian*** matrix model!

A.Chaney, L. Liu, A.S. arXiv:1506.03505

commutative limit $$(x^0)^2 + (x^1)^2 + (x^2)^2 = r^2$$

$$\{x^0, x^1\} = x^2 \qquad \{x^1, x^2\} = x^0 \qquad \{x^2, x^0\} = x^1$$

solution provided: $\alpha = -\frac{1}{2} \qquad \beta = -1$

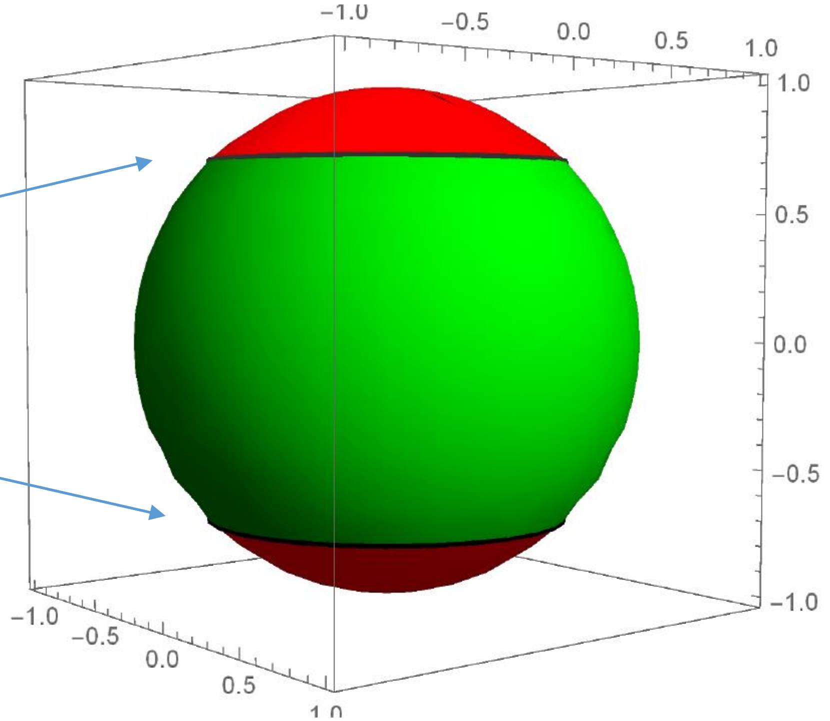
parametrization
$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = r \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix}$$

induced metric $ds^2 = r^2\left(\cos 2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2\right)$

signature change at $\theta = \frac{\pi}{2}, \frac{3\pi}{4}$

Euclidean $0 < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \pi$

Lorentzian $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$



Summary

examined matrix solutions which are UIRR's of su(1,1), su(2), E2

last 2 examples crudely describe 2d quantum cosmologies

signature changes can occur in commutative limit when quadratic term is added - cosmological singularities on 1-brane

resolved away from commutative limit

singularities occur at non-zero spatial scales (time not defined for smaller scales)

Next: generalizations to 4d space-times?

Try non-commutative version of complex projective spaces

commutator algebra closes

VP Nair and S Randjbar-Daemi hep-th/9802187
G Alexanian, AP Balachandran, Immirizi, Ydri hep-th/0103023
AP Balachandran, B Dolan, J-L Lee, X Martin, D O'Connor hep-th/0107099
T Azuma, S Bal, K Nagao, J Nishimura hep-th/0405277
D Karabali, VP Nair and S Randjbar-Daemi hep-th/0407007
H Grosse, H Steinacker hep-th/040789
AP Balachandran, S Kurkcuoglu, S Vaidya hep-th/0511114

3 candidates: $CP^2 = CP^{2,0}$, $CP^{1,1}$, $CP^{0,2}$

$CP^{p,q}$ = $SU(p+1,q)/U(p,q)$

non-commutative versions in K. Hasabe arXiv:1207.1968

complex coordinates $z_i, \; i = 1, ..., p+q+1$

constraint $H^{2q,2p+1}$ $$\sum_{i=1}^{p+1} z_i^* z_i - \sum_{i=p+2}^{p+q+1} z_i^* z_i = 1$$

identification $z_i \sim e^{i\beta} z_i$

$$CP^{1,1} = SU(2,1)/U(1,1)$$

$z_i,\ i=1,2,3$ $\qquad$ $z^i z_i^* = 1\ ,$ $\qquad$ metric Ξ =diag$(1,1,-1)$

$su(2,1)$ Gell-Mann matrices $\tilde\lambda_a,\ a=1-8$ $\qquad$ $\tilde\lambda_a \Xi = \Xi \tilde\lambda_a^\dagger$

$$\mathrm{tr}\,\tilde\lambda_a\tilde\lambda_b \;=\; [\tilde\lambda_a]^i{}_j[\tilde\lambda_b]^j{}_i \;=\; 2\eta_{ab}$$

$$[\tilde\lambda_a,\tilde\lambda_b] = 2i\tilde f_{abc}\tilde\lambda^c\ ,\qquad \eta = \mathrm{diag}(1,1,1,-1,-1,-1,-1,1)$$

$$\tilde f_{123} = 1 \qquad \tilde f_{845} = \tilde f_{867} = -\frac{\sqrt{3}}{2} \qquad \tilde f_{147} = \tilde f_{165} = \tilde f_{246} = \tilde f_{257} = \tilde f_{345} = \tilde f_{376} = -\frac{1}{2}$$

classical Schwinger construction $\qquad$ $x^a = z_i^*[\tilde\lambda^a]^i{}_j z^j$

semiclassical limit of noncommutative CP^{1,1}

- just add compatible Poisson structure

$$\{z^i, z_j^*\} = -i\delta_j^i \qquad \{z^i, z^j\} = \{z_i^*, z_j^*\} = 0$$

Then $z^i z_i^* - 1 \approx 0$ is a first class constraint generating phase equivalence

su(2,1) Poisson algebra for x^a $\qquad \{x_a, x_b\} = 2\tilde{f}_{abc}x^c$

examine 8d matrix

breaks SO(4,4) to SU(2,1)

$$S(X) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[X_a, X_b][X^a, X^b] + \frac{2ia}{3}\tilde{f}_{abc}X^aX^bX^c\Big)$$

$$[[X_a, X_b], X^b] + ia\tilde{f}_{abc}[X^b, X^c] = 0$$

semi-classical limit, $X^\mu \to x^\mu$, $a \to \hbar\alpha$ $\;[\,,\,] \to i\hbar\{\,,\,\}$

$$-\{\{x_a, x_b\}, x^b\} - \alpha\tilde{f}_{abc}\{x^b, x^c\} \;= 0$$

solved by su(2,1) Poisson algebra with $\alpha = 2$

induced metric from 8d

$$ds^2 \;= dx^a dx_a \;= 4\Big(dz_i^* dz^i - |z_i^* dz^i|^2\Big)$$

indefinite version of Fubini-Study metric

local coordinates (ζ_1, ζ_2) $\quad \zeta_1 = \frac{z^1}{z^3} \quad \zeta_2 = \frac{z^2}{z^3}\;, \quad z^3 \neq 0$

$$\frac{1}{4}ds^2 \;= \frac{|d\zeta_1|^2 + |d\zeta_2|^2}{|\zeta_1|^2 + |\zeta_2|^2 - 1} - \frac{|\zeta_1^* d\zeta_1 + \zeta_2^* d\zeta_2|^2}{(|\zeta_1|^2 + |\zeta_2|^2 - 1)^2}$$

or Euler-like angles θ, ϕ, ψ and $\tau \in R_+$ $\quad 0 \leq \theta < \pi,\; 0 \leq \phi < 2\pi,\; 0 \leq \psi < 4\pi$

$$\zeta_1 = e^{i(\psi+\phi)/2} \coth\tau\, \cos\frac{\theta}{2} \qquad \zeta_2 = e^{i(\psi-\phi)/2} \coth\tau\, \sin\frac{\theta}{2}\;,$$

induced metric $$ds^2 = g_{\tau\tau}\, d\tau^2 + g_{\theta\theta}\,(d\theta^2 + \sin^2\theta d\phi^2) + g_{\psi\psi}\,(d\psi + \cos\theta d\phi)^2$$

$$g_{\tau\tau} = -4 \qquad g_{\theta\theta} = \cosh^2\tau \qquad g_{\psi\psi} = -\cosh^2\tau\sinh^2\tau$$

2 space-like directions, 2 time-like directions

Repeat for $CP^{0,2} = SU(2,1)/U(2)$

$$z^i z_i^* = -1$$

semi-classical equations again solved by $\{x_a, x_b\} = 2\tilde{f}_{abc}x^c$ with $\alpha = 2$

induced metric $$ds^2 = dx^a dx_a = 4\Big(-dz_i^* dz^i - |z_i^* dz^i|^2\Big)$$

or using $$\zeta_1 = e^{i(\psi+\phi)/2}\tanh\tau\,\cos\frac{\theta}{2} \qquad \zeta_2 = e^{i(\psi-\phi)/2}\tanh\tau\,\sin\frac{\theta}{2}$$

$$g_{\tau\tau} = -4 \qquad g_{\theta\theta} = -\sinh^2\tau \qquad g_{\psi\psi} = -\cosh^2\tau\sinh^2\tau$$

Euclidean signature

Both solutions satisfy sourceless Einstein equations with $\Lambda = \frac{3}{2}$; ***No signature change***

Steinacker: *Relevant metric in the semi-classical limit may not be the induced metric* $g_{\mu\nu}$

arxiv:1003.4134 *Rather, it is the one that appears in the coupling to matter: `effective metric'* $\gamma_{\mu\nu}$

Non-commutative action for massless scalar field

$$-\frac{1}{k^2}\mathrm{Tr}[X^\mu,\Phi][X_\mu,\Phi]$$

semi-classical limit $X^\mu \to x^\mu$ $\Phi \to \phi$ $\mathrm{Tr} \to \int \frac{d^n\sigma}{\sqrt{|\det\Theta|}}$ $[\,,\,] \to i\hbar\{\,,\,\}$ $k \to \hbar\kappa$

$$\frac{1}{\kappa^2}\int \frac{d^n\sigma}{\sqrt{|\det\Theta|}}\{x^\mu,\phi\}\{x_\mu,\phi\} \Longleftrightarrow \frac{1}{\kappa^2}\int d^n\sigma\,\sqrt{|\det\gamma|}\,\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

symplectic two-form $\Omega = \frac{1}{2}[\Theta^{-1}]_{\mu\nu}\,d\sigma^\mu \wedge d\sigma^\nu$

$$\boxed{\sqrt{|\det\gamma|}\,\gamma^{\mu\nu} = \frac{1}{\sqrt{|\det\Theta|}}\,[\Theta^T g\Theta]^{\mu\nu}}$$

$$CP^{1,1} \qquad \Omega_{CP^{1,1}} = -\frac{1}{2} d\left(\cosh^2\tau\,(d\psi + \cos\theta\, d\phi)\right)$$

$$CP^{0,2} \qquad \Omega_{CP^{0,2}} = -\frac{1}{2} d\left(\sinh^2\tau\,(d\psi + \cos\theta\, d\phi)\right)$$

Kahler 2-forms

For both cases $g_{\mu\nu} = \gamma_{\mu\nu}$

Next add quadratic term:

$$S(X) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[X_a, X_b][X^a, X^b] + \frac{ia}{3}\tilde{f}_{abc}X^a[X^b, X^c] + 6\tilde{b}X_aX^a\Big)$$

$$[[X_a, X_b], X^b] + ia\tilde{f}_{abc}[X^b, X^c] + 12\tilde{b}X_a = 0$$

semi-classical limit $\tilde{b} \to \hbar^2\tilde{\beta}$.

$$-\{\{x_a, x_b\}, x^b\} - \alpha\tilde{f}_{abc}\{x^b, x^c\} + 12\tilde{\beta}x_a = 0$$

New solutions

Deformations of $CP^{1,1}$ and $CP^{0,2}$

modify ansatz

$$x_{1-3} = \mu\, z_i^*[\tilde{\lambda}_{1-3}]^i{}_j z^j$$

$$x_{4-7} = z_i^*[\tilde{\lambda}_{4-7}]^i{}_j z^j$$

$$x_8 = \nu\, z_i^*[\tilde{\lambda}_8]^i{}_j z^j\ ,$$

satisfies matrix equations for

$$\alpha = 2\mu\,\frac{\tilde{\beta}^2-\tilde{\beta}-1-\gamma}{2\tilde{\beta}+1} \qquad \mu = \sqrt{\frac{\tilde{\beta}^3-4\tilde{\beta}^2-6\tilde{\beta}+\tilde{\beta}\gamma-2}{2(\tilde{\beta}^2+4\tilde{\beta}+2)}} \qquad \nu = \frac{\alpha}{2(1+\tilde{\beta})}\ ,$$

where

$$\gamma = \sqrt{\tilde{\beta}^4-12\tilde{\beta}^3-22\tilde{\beta}^2-12\tilde{\beta}-2}$$

reality in 3 disconnected regions

$$i')\ \ -3.414 \lesssim \tilde{\beta} \lesssim -0.746 \qquad ii')\ \ -0.603 \lesssim \tilde{\beta} \lesssim -0.586 \qquad iii)\ \ 13.67 \lesssim \tilde{\beta}$$

induced metric

$$ds^2 = g_{\tau\tau}\, d\tau^2 + g_{\theta\theta}\,(d\theta^2 + \sin^2\theta d\phi^2) + g_{\psi\psi}\,(d\psi + \cos\theta d\phi)^2$$

deformed $CP^{1,1}$

$$g_{\tau\tau} = 4\Big((\mu^2+\nu^2-2)\cosh^2\tau\sinh^2\tau - 1\Big)$$

$$g_{\theta\theta} = \cosh^2\tau\,(\mu^2\cosh^2\tau - \sinh^2\tau)$$

$$g_{\psi\psi} = -\cosh^2\tau\sinh^2\tau$$

$g_{\tau\tau}$ changes sign when $(\mu^2+\nu^2-2)\sinh^2\tau\cosh^2\tau = 1$ (red curves)

$g_{\theta\theta}$ changes sign when $\tanh^2\tau = \mu^2$ (green curves)

$\mathrm{sign}(g_{\tau\tau}, g_{\theta\theta}) = (-,+)$

two space-like directions

two time-like directions

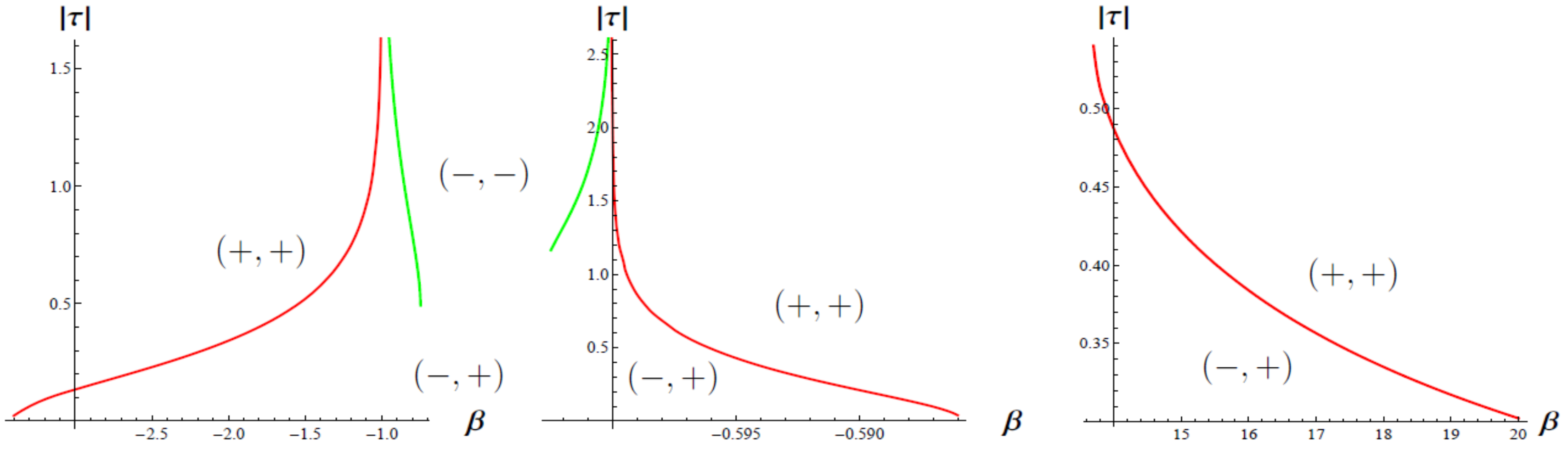
$\mathrm{sign}\ (g_{\tau\tau}, g_{\theta\theta}) = (+,+)$

Lorentzian signature

$d\psi + \cos\theta d\phi$ time-like

$\mathrm{sign}(g_{\tau\tau}, g_{\theta\theta}) = (-,-)$

Euclidean signature



(a) region i')   (b) region ii')   (c) region iii)

deformed $CP^{0,2}$

$$g_{\tau\tau} = 4\Big((\mu^2+\nu^2-2)\cosh^2\tau\sinh^2\tau - 1\Big)$$

$$g_{\theta\theta} = \sinh^2\tau\left(\mu^2\sinh^2\tau - \cosh^2\tau\right)$$

$$g_{\psi\psi} = -\cosh^2\tau\sinh^2\tau\ ,$$

$g_{\tau\tau}$ changes sign when $(\mu^2+\nu^2-2)\sinh^2\tau\cosh^2\tau = 1$ (red curves)

$g_{\theta\theta}$ changes sign when $\coth^2\tau = \mu^2$ (green curves)

$\mathrm{sign}(g_{\tau\tau}, g_{\theta\theta}) = (-,-)$

Euclidean signature

$\mathrm{sign}(g_{\tau\tau}, g_{\theta\theta}) = (+,+)$

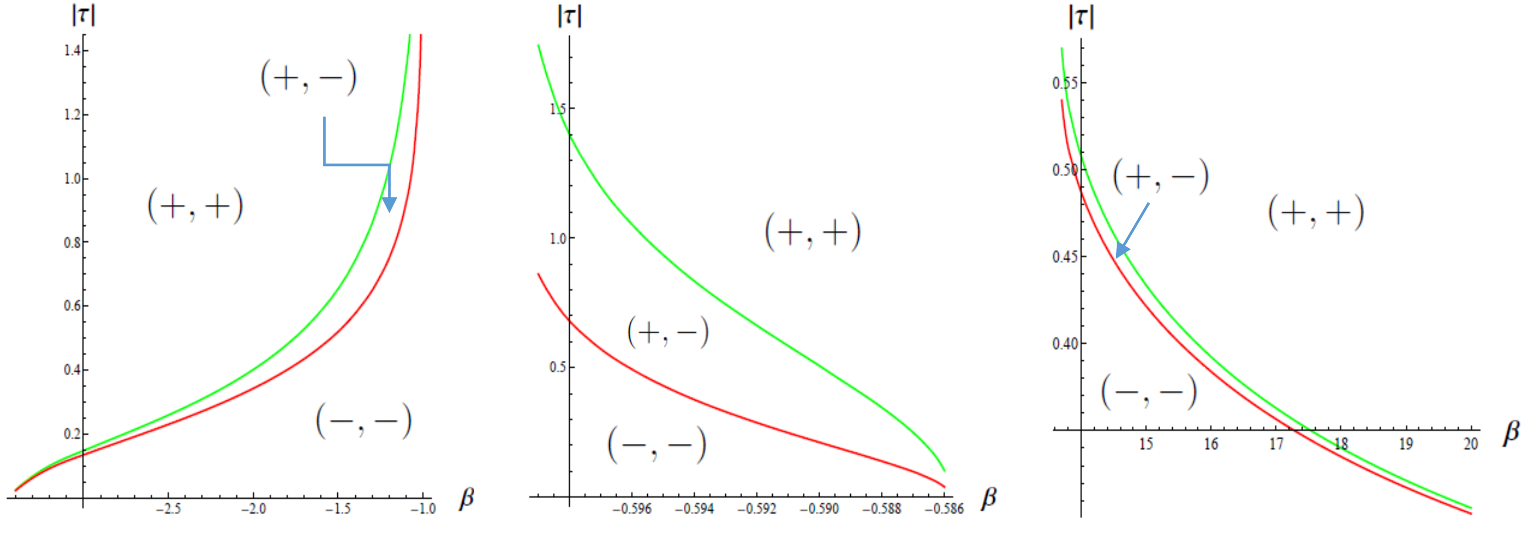
Lorentzian signature

$d\psi + \cos\theta d\phi$ time-like

$\mathrm{sign}(g_{\tau\tau}, g_{\theta\theta}) = (+,-)$

Lorentzian signature

$d\tau$ time-like



(a) region i') (b) region ii') (c) region iii)

effective metric

$$ds^2_{\text{eff}} = \gamma_{\tau\tau}\, dt^2 + \gamma_{\theta\theta}\,(d\theta^2 + \sin^2\theta\, d\phi^2) + \gamma_{\psi\psi}\,(d\psi + \cos\theta\, d\phi)^2$$

deformed $CP^{1,1}$

$$\Omega_{CP^{1,1}} = -\frac{1}{2} d\left(\cosh^2\tau\,(d\psi + \cos\theta\, d\phi)\right)$$

$$\frac{\gamma_{\tau\tau}}{\sqrt{|\det\gamma||\det\Theta|}} = -1$$

$$\frac{\gamma_{\theta\theta}}{\sqrt{|\det\gamma||\det\Theta|}} = \frac{1}{4(\mu^2 - \tanh^2\tau)}$$

$$\frac{\gamma_{\psi\psi}}{\sqrt{|\det\gamma||\det\Theta|}} = -\frac{1}{4(\operatorname{sech}^2\tau\, \operatorname{csch}^2\tau + 2 - \mu^2 - \nu^2)}$$

sign change in $\gamma_{\psi\psi}$ when $(\mu^2 + \nu^2 - 2)\sinh^2\tau\cosh^2\tau = 1$ (red curves)

sign change in $\gamma_{\theta\theta}$ when $\tanh^2\tau = \mu^2$ (green curves)

$\operatorname{sign}(\gamma_{\psi\psi}, \gamma_{\theta\theta}) = (-,+)$

two space-like directions
two time-like directions

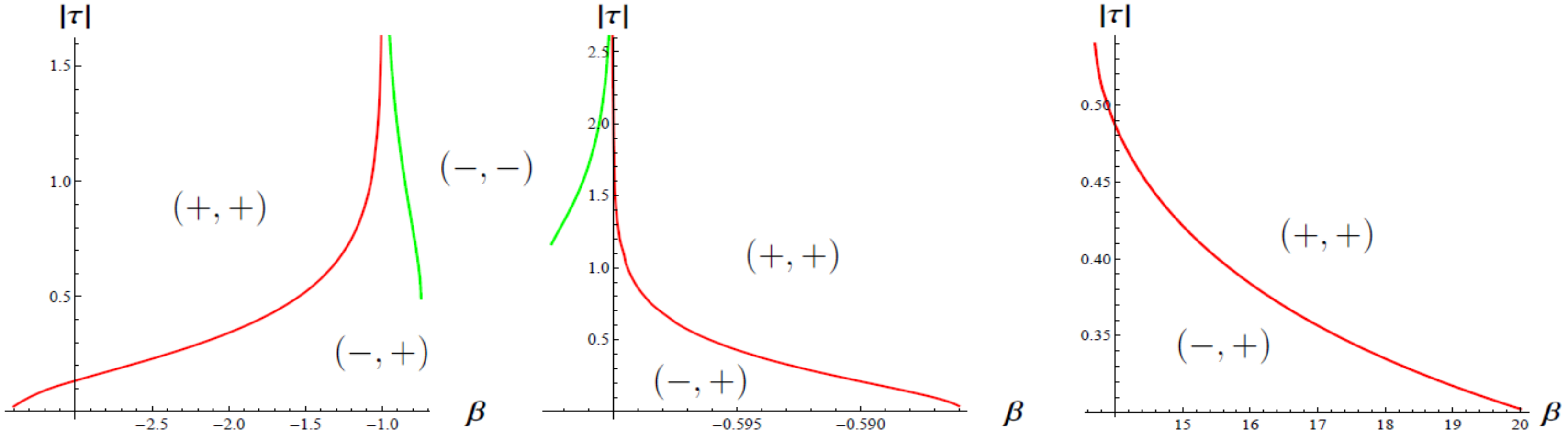
$\operatorname{sign}(\gamma_{\psi\psi}, \gamma_{\theta\theta}) = (-,-)$

Euclidean signature

$\operatorname{sign}(\gamma_{\psi\psi}, \gamma_{\theta\theta}) = (+,+)$

Lorentzian signature

$d\tau$ time-like



(a) region i') (b) region ii') (c) region iii)

deformed $CP^{0,2}$

$$\Omega_{CP^{0,2}} = -\frac{1}{2}\, d\left(\sinh^2\tau\,(d\psi+\cos\theta\, d\phi)\right)$$

$$\frac{\gamma_{\tau\tau}}{\sqrt{|\det\gamma||\det\Theta|}} = -1$$

$$\frac{\gamma_{\theta\theta}}{\sqrt{|\det\gamma||\det\Theta|}} = \frac{1}{4(\mu^2-\coth^2\tau)}$$

$$\frac{\gamma_{\psi\psi}}{\sqrt{|\det\gamma||\det\Theta|}} = \frac{1}{4(\mu^2+\nu^2-2-4\,\mathrm{csch}^2 2\tau)}$$

sign change in $\gamma_{\psi\psi}$ when $(\mu^2+\nu^2-2)\sinh^2\tau\cosh^2\tau = 1$ (red curves)

sign change in $\gamma_{\theta\theta}$ when $\coth^2\tau = \mu^2$ (green curves)

$\mathrm{sign}(\gamma_{\psi\psi},\gamma_{\theta\theta}) = (-,-)$

Euclidean signature

$\mathrm{sign}(\gamma_{\psi\psi},\gamma_{\theta\theta}) = (+,-)$

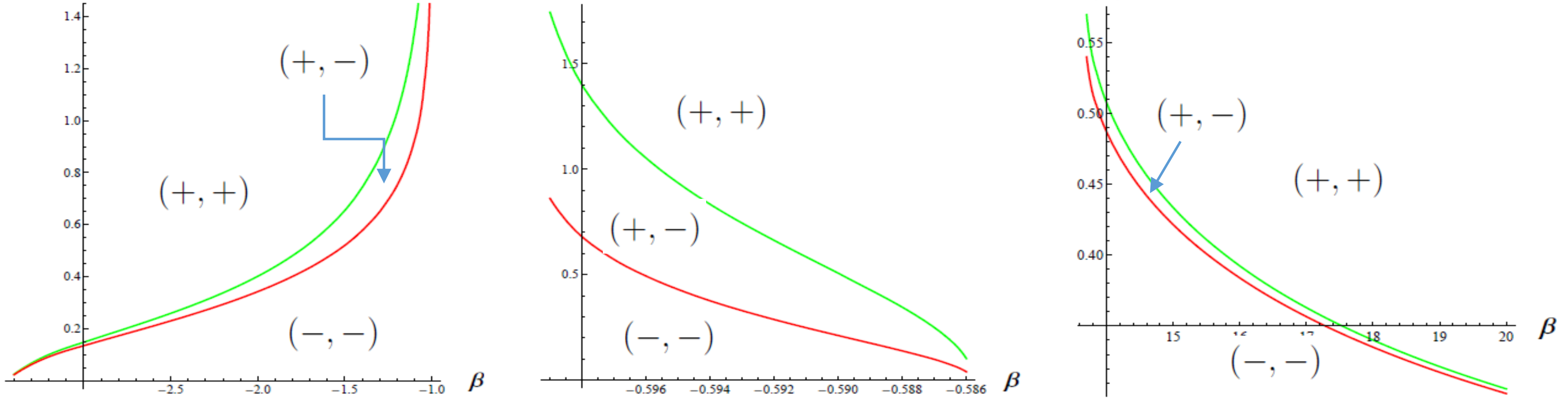
Lorentzian signature

$d\psi+\cos\theta\, d\phi$ time-like

$\mathrm{sign}(\gamma_{\psi\psi},\gamma_{\theta\theta}) = (+,+)$

Lorentzian signature

$d\tau$ time-like



(a) region i')

(b) region ii')

(c) region iii)

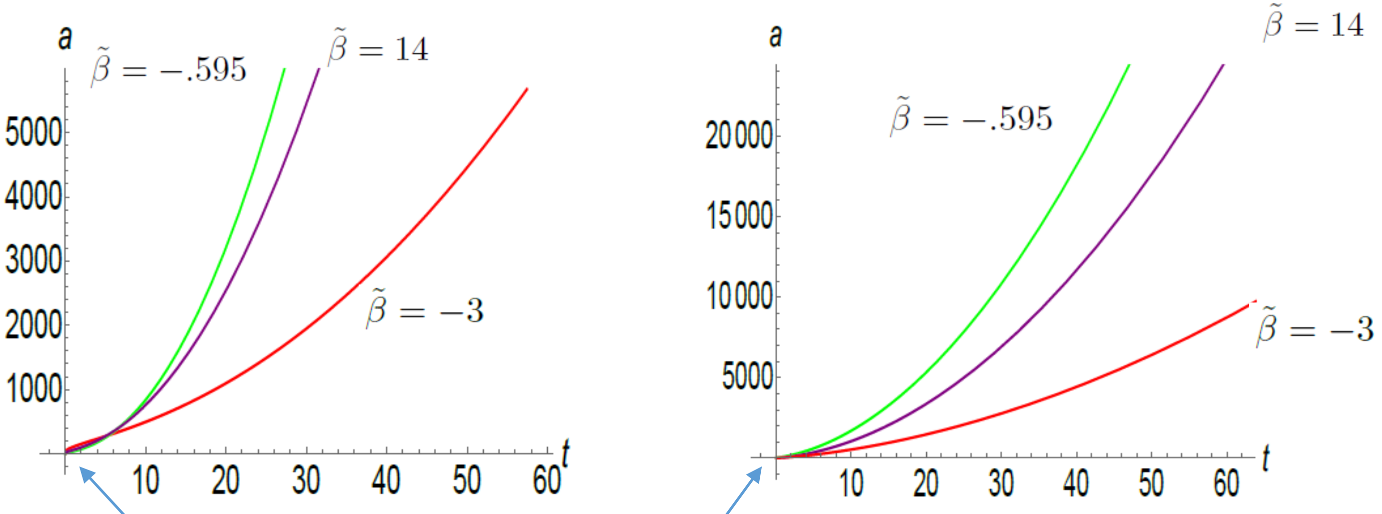
Lorentz phase (τ time-like) describes expanding space-times

$$\tau\text{–slice} = S^3 \qquad |\zeta_1|^2 + |\zeta_2|^2 = f(\tau)$$

introduce spatial scale $\quad a(|\tau|)^3 = \int_{S^3} \sqrt{\det \gamma^{(3)}}\, d\theta d\phi d\psi$

time in co-moving frame $\quad t(\tau) = \int_{\tau_0}^{\tau} \sqrt{-\gamma_{\tau\tau}(\tau')}\, d\tau'$

a vs t in regions of Lorentzian signature



(a) $CP^{1,1}$ (b) $CP^{0,2}$

Signature changes are cosmological singularities

– resolved away from commutative limit

$$CP^2 = SU(3)/U(2)$$

$$z_i^* z_i = 1 \qquad\qquad z_i \sim e^{i\beta} z_i$$

$x^\alpha = z_i^*[\lambda^\alpha]_{ij} z_j$ $\qquad$ λ^α = su(3) Gell-Mann matrices $\qquad$ $[\lambda^\alpha, \lambda^\beta] = 2i f^{\alpha\beta\gamma}\lambda^\gamma$

solves $\{\{x^\alpha, x^\beta\}, x_\beta\} + i\alpha f^{\alpha\beta\gamma}\{x_\beta, x_\gamma\} + \beta x^\alpha = 0$ with 8d Euclidean metric,

but also with 8d *Lorentzian* metric $\eta = \mathrm{diag}(1,1,1,1,1,1,1,-1)$

$\beta = -6\tilde{\alpha}^2$

A. Chaney & A.S.
arXiv:1511.06816

choose local coordinates $\quad \zeta_1 = e^{\frac{i}{2}(\psi+\phi)} \cos\frac{\theta}{2} \tan\tau \qquad \zeta_2 = e^{\frac{i}{2}(\psi-\phi)} \sin\frac{\theta}{2} \tan\tau\ , \qquad 0 \le \tau \le \frac{\pi}{2}$

induced metric from 8d Minkowski space

$$ds^2 = g_{\tau\tau}\, d\tau^2 + g_{\theta\theta}\,(d\theta^2 + \sin^2\theta d\phi^2) + g_{\psi\psi}\,(d\psi + \cos\theta d\phi)^2$$

$$g_{\tau\tau} = 4\Big(1 - \frac{3}{2}\sin^2 2\tau\Big) \qquad g_{\theta\theta} = 4\sin^2\tau \qquad g_{\psi\psi} = 4\sin^2\tau\cos^2\tau$$

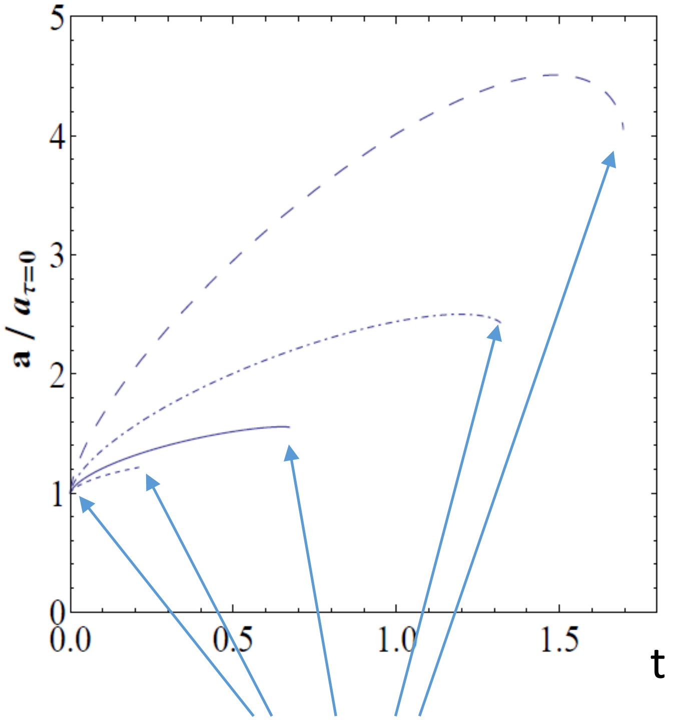
signature change at $\sin^2 2\tau = \frac{2}{3}$ $\qquad \tau_- \approx .478\,, \quad \tau_+ \approx 1.09$

Euclidean signature for $0 < \tau < \tau_-$ and $\tau > \tau_+$

Lorenzian signature for $\tau_- < \tau < \tau_+$

can be extended to deformed CP^2

a vs t
in regions of
Lorentzian signature



singularities occur at non-zero spatial scales
(time not defined for smaller scales)

Signature changes are cosmological singularities

– resolved away from commutative limit

conclusion: If additional terms included in bosonic action,

signature change not uncommon feature of matrix model solutions

open questions:

Stability; role of fermion sector; how to get more realistic cosmological models?

Previous 4d examples not fully homogeneous and isotropic

Maximally symmetric space: non-commutative (A)dS^4, ie, fuzzy H^4
(Steinacker&co) arXiv:1709.10480, arXiv:1710.11495, arXiv:1806.05907

Algebra respects isometries, but has to be extended for closure

indefinite version of fuzzy S^4

H Grosse, C Klimcik, P Presnajder hep-th/9602115
S.Ramgoolam hep-th/0105006
D. O'Connor, J Medina hep-th/0212170
Y. Kimura hep-th/0204256

Besides cosmology, there may be another application:

non-commutative version of holography

AdS/CFT correspondence

Proposal for strong/weak duality:

exact equivalence ?

quantum gravity in asymptotically AdS_{d+1} ⟺ CFT_d on boundary

Goal: make the bulk non-commutative to introduce possible quantum gravity effects

implications for dual theory on boundary?

Non-commutative AdS_2 /CFT_1 examined in A. Pinzul, A.S. arXiv:1707.04816

found Killing vectors on non-commutative space reduced to commutative ones near boundary

--> NC AdS_2 is asymptotically AdS_2 - correspondence principal should apply

constructed boundary correlators from on-shell bulk action

all results so far consistent with conformal symmetry

boundary 2-pt function agrees with commutative result (up to rescaling)

Idea: Repeat for non-commutative version of AdS_4 /CFT_3

Here: Review non-commutative AdS_4 (semi-classical limit)

Examine boundary limit (where CFT_3 , if exists, lives)

Claim: boundary is commutative

Euclidean AdS^4 (H^4)

embedding coordinates x^μ, $\mu = 1 - 5$

$x^\mu x_\mu = -\ell_0^2$ $\qquad\qquad$ $\eta = \text{diag}(+,+,+,+,-)$

$SO(4,1)$ isometry

Killing vectors $K^{\mu\nu}(= -K^{\nu\mu})$

$$[K^{\mu\nu}, K^{\rho\sigma}] = \eta^{\mu\rho}K^{\nu\sigma} - \eta^{\nu\rho}K^{\mu\sigma} - \eta^{\mu\sigma}K^{\nu\rho} + \eta^{\nu\sigma}K^{\mu\rho}$$

action on the embedding coordinates

$$K^{\mu\nu}x^\rho = \eta^{\mu\rho}x^\nu - \eta^{\nu\rho}x^\mu$$

Introduce Poisson structure

which preserve isometries (maps to conformal symmetries on the boundary?)

introduce 10 additional generators $x^{\mu\nu} = (-x^{\nu\mu})$ to close algebra

$so(4,2)$

$$\{x^\mu, x^\nu\} = -4x^{\mu\nu}$$

$$\{x^{\mu\nu}, x^\rho\} = \eta^{\mu\rho}x^\nu - \eta^{\nu\rho}x^\mu$$

$$\{x^{\mu\nu}, x^{\rho\sigma}\} = \eta^{\mu\rho}x^{\nu\sigma} - \eta^{\nu\rho}x^{\mu\sigma} - \eta^{\mu\sigma}x^{\nu\rho} + \eta^{\nu\sigma}x^{\mu\rho}$$

action of the Killing vectors

$$K^{\mu\nu} = \{x^{\mu\nu}, \cdot\} = -\tfrac{1}{4}\{\{x^\mu, x^\nu\}, \cdot\}$$

secondary and tirciary constraints

$$x_\mu x^{\mu\nu} = 0 \qquad\qquad -4x^{\mu\nu}x_\mu{}^\rho + x^\nu x^\rho + \ell_0^2\eta^{\nu\rho} = 0$$

realization using $CP^{1,2}$

S^2 bundle over Euclidean AdS^4

$$\mathtt{z}^a \mathtt{z}^*_a = 1\ , \qquad a = 1,2,3,4, \qquad \eta^{(c)} = \mathrm{diag}(++,-,-)$$

$$\mathtt{z}^a \sim e^{i\beta} \mathtt{z}^a$$

Introduce 4×4 $\gamma-$matrices for $SO(4,1)$ $\qquad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}\ ,$

$$\gamma^{1-3} = \begin{pmatrix} & \sigma_{1-3} \\ \sigma_{1-3} & \end{pmatrix}\ , \qquad \gamma^4 = \begin{pmatrix} & -i\mathbb{1}_2 \\ i\mathbb{1}_2 & \end{pmatrix}\ , \qquad \gamma^5 = \begin{pmatrix} i\mathbb{1}_2 & \\ & -i\mathbb{1}_2 \end{pmatrix}$$

projection to Euclidean AdS^4 $\qquad x^\mu = i\mathtt{z}^*_a \, [\gamma^\mu]^a{}_b \, \mathtt{z}^b$

$$x^\mu x_\mu = -(\mathtt{z}^a \mathtt{z}^*_a)^2 = -1$$

additional constrants satisfied using $\qquad x^{\mu\nu} = -\mathtt{z}^*_a \, [\Sigma^{\mu\nu}]^a{}_b \, \mathtt{z}^b \qquad \Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$

add Poisson structure $\{z^a, z^*_b\} = -i\delta^a_b$ $\{z^a, z^b\} = \{z^*_a, z^*_b\} = 0$

to recover $so(4,2)$ Poisson bracket algebra for x^μ, $x^{\mu\nu}$

local coordinates $\zeta^i = z^i/z^4$, $i = 1,2,3$ $z^4 \neq 0$

$\zeta^i\zeta^*_i = 1 + 1/|z^4|^2$, and so $\zeta^i\zeta^*_i \geq 1$ metric diag$(+,+,-)$

$$
\begin{aligned}
x^1 &= i\,\frac{\zeta^*_1 - \zeta^1 + \zeta^2\zeta^*_3 + \zeta^3\zeta^*_2}{\zeta^i\zeta^*_i - 1} \\
x^2 &= \frac{\zeta^*_1 + \zeta^1 + \zeta^2\zeta^*_3 - \zeta^3\zeta^*_2}{\zeta^i\zeta^*_i - 1} \\
x^3 &= i\,\frac{-\zeta^*_2 + \zeta^2 + \zeta^1\zeta^*_3 + \zeta^3\zeta^*_1}{\zeta^i\zeta^*_i - 1} \\
x^4 &= \frac{\zeta^*_2 + \zeta^2 - \zeta^1\zeta^*_3 + \zeta^3\zeta^*_1}{\zeta^i\zeta^*_i - 1} \\
x^5 &= -\frac{1 + |\zeta^1|^2 + |\zeta^2|^2 + |\zeta^3|^2}{\zeta^i\zeta^*_i - 1}
\end{aligned}
$$

lower hyperboloid H^4

$\partial\, CP^{1,2}$

boundary limit

$\zeta^i\zeta^*_i = |\zeta^1|^2 + |\zeta^2|^2 - |\zeta^3|^2 \to 1$

five-dimensional hyperboloid $H^{2,3}$

Poisson brackets on the six-dimensional phase space

$$\{\zeta^i, \zeta^*_j\} = i(\zeta^k\zeta^*_k - 1)(\zeta^i\zeta^*_j - \delta^i_j)$$

vanish in the boundary limit

Upon quantization, boundary of non-commutative $CP^{1,2}$ is commutative

Finally, project: $\partial\, CP^{1,2} \to \partial\,\text{Euclidean}\, AdS^4$

Future project:

quantize using a star product

asymptotically AdS^4 ?

Non-commutative Killing vectors $K^{\mu\nu}_\star = -\frac{1}{4}[[x^\mu, x^\nu]_\star, \cdot\,]_\star$

boundary limit $\quad K^{\mu\nu}_\star \;\to\; K^{\mu\nu}$?

If true , can compute boundary correlators for the dual theory and check conformality

the end