Branes, Dominoes and Words

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Matrix Models for non-commutative geometry
and string theory, ESI 2018

"BPS operators in N=4 SO(N) super Yang-Mills theory: plethysms, dominoes and words" arXiv:1804.11090[hep-th],
C. Lewis-Brown, S. Ramgoolam (LBR2018)

Introduction

$\mathcal{N} = 4$ super Yang-Mills theory in 4D :

$$U(N) \leftrightarrow AdS_5 \times S^5$$

$$SO(N) \leftrightarrow AdS_5 \times S^5/Z_2$$

The Z_2 is an orientifold projection : spacetime Z_2 combined with worldsheet orientation reversal.

$$X^\mu(\tau, \sigma) \to -X^\mu(\tau, 2\pi - \sigma)$$

Local gauge invariant operators in CFT are related to quantum states in the CFT, and in the dual AdS.

Half-BPS operators are constructed from a complex matrix $Z = X_1 + iX_2$.

Traces and products of traces give a good basis when $n < N$.

$$
\begin{array}{rcl}
ADS & \leftrightarrow & CFT \\
\text{single particle} & \leftrightarrow & \text{single traces, e.g. } tr(Z^n) \\
\text{multi-particle states} & \leftrightarrow & \text{multi-traces e.g.} tr(Z^{n_1})tr(Z^{n_2})
\end{array}
$$

This relies on an orthogonality at large N

$$\langle trZ^n trZ^m tr(Z^\dagger)^{n+m}\rangle \sim \frac{1}{N}$$

Follows from Gaussian Matrix model or free field theory:

$$\langle Z^i_j (Z^\dagger)^k_l \rangle = \delta^k_i \delta^l_j$$

For n, m of order $\sqrt{N}$, this orthogonality breaks down. In particular for $n \sim N$ where we have quantum states in the AdS corresponding to giant graviton branes, trace structures no longer a good way to identify brane states. Initial steps : subdeterminant operators.

Balasubramanian, Berkooz, Naqvi, Strassler, 2001 - hep-th/0107119

Exactly orthogonal basis for 2-point functions was identified. The basis elements were labelled by Young diagrams R, with $l(R) \leq N$.

Corley, Jevicki, Ramgoolam, 2001 - hep-th/0111222 (CJR2001)

$$\langle \mathcal{O}_R(Z) \mathcal{O}_S(Z^\dagger) \rangle = \delta_{RS} f_R$$

f_R is a product of weights for boxes of Young diagram R.
$f_R = \prod_{i,j} (N - i + j)$

$f(R)$ vanishes for $l(R) > N$ (stringy exclusion principle). Young diagram basis encodes the finite N relations between matrix invariants in this simple cutoff.

The Young diagram basis was interpreted in terms of giant graviton branes and the interaction was calculated in terms of Littlewood-Richardson coefficients (CJR2001)

$$\langle \chi_R(Z)\chi_S(Z)\chi_T(Z^\dagger)\rangle = g(R,S,T)f_T$$

$g(R,S,T)$ is the number of times that V_T ($U(N)$ irrep for Young diagram T) appears in the tensor product of $V_R \otimes V_S$.

There is a combinatoric Littlewood-Richardson rule, for assembling T from R, S which calculates g. (sketched in David's talk)

Remarkable fact: For R, S having n, m boxes, T has $n+m$ boxes: g can be expressed in terms of characters of S_n, S_m, S_{n+m}

$$g(R_1, R_2, R_3) = \frac{1}{n!m!} \sum_{\sigma_1 \in S_n, \sigma_2 \in S_m} \chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2)\chi_{R_3}(\sigma_1 \circ \sigma_2)$$

Origin of above -Schur-Weyl duality : We know irreps of $U(N)$ can be obtained from the fundamental by taking the n-fold tensor product amd restricting to different symmetry types. Symmetry types $\rightarrow$ irreducible reps of S_n.
Precise connection between $U(N)$ and S_n irreps in $V_N^{\otimes n}$.

$$V_N^{\otimes n} = \bigoplus_{\substack{R \vdash n \\ l(R) \leq N}} V_R^{(U(N))} \otimes V_R^{(S_n)}$$

Implies e.g.

$N^n = \sum_R Dim_N R \;\; d_R^{(n)}$

And the symmetric group formula for $U(N)$ tensor product multiplicities.

All the finite N relations between traces, encoded in $l(R) \leq N$.

Some of the key connections between symm-group stuff and unitary group stuff in this talk ..

The orthogonal basis of half-BPS gauge invariant operators in $N = 4$ SYM theories with $SO(N)$ gauge groups was obtained recently.

Caputa, de Mello Koch, Diaz, arXiv:1303.7252 [hep-th] ; arXiv:1301.1560 [hep-th]

For SO, $Z^T = -Z$, so

$$trZ^i = tr(Z^T)^i = (-1)^i trZ^i$$

Implies i has to be even.

For any multi-trace,

$$\prod_i (trZ^i)^{k_i}$$

$n = \sum_i ik_i$ is even.

Orthogonal basis elements are labelled by Young diagrams t with $n/2$ boxes.

$$\langle \mathcal{O}_{t_1}(Z)\mathcal{O}_{t_2}(Z^\dagger)\rangle = \delta_{t_1,t_2} f_{t_1}(N)$$

The orientifold projection map on traces

$$\begin{aligned} \mathcal{P}(trZ^i) &= 0 \text{ if } i \text{ is odd} \\ \mathcal{P}(trZ^i) &= trZ^i \text{ if } i \text{ is even} \end{aligned}$$

What does it look like on the Young diagram (Giant Graviton Brane) basis ?

LBR2018 :

$$\mathcal{P}(\mathcal{O}_R(Z)) = \sum_t c_R^t \, \mathcal{O}_t(Z)$$

$$c_R^t = \sum_{p \vdash n} \frac{1}{Sym\ p} \chi_R(2p) \chi_t(p)$$

This is related to plethysm multiplicities of Young diagrams.

What are plethysm multiplicities ?

$$V_t \otimes V_t = \bigoplus_R g(t,t,R) V_R$$

The part of $V_t \otimes V_t$, restricted to be symmetric under exchange, is a representation of the diagonal $U(N)$.

$$Sym^2(V_t) = \bigoplus_R g^+(t,t,R) V_R$$

Likewise the anti-symmetric part:

$$\Lambda^2(V_t) = \bigoplus_R g^-(t,t,R) V_R$$

Using Schur-weyl duality, we show (LBR2018)

$$c_R^t = g^+(t,t,R) - g^-(t,t,R)$$

In general, the LR coefficients $t^{\otimes k} \to R$ can be refined according to symmetry types, i.e. Young diagrams of S_k.

$$g^{Y}(t^{\otimes k}; R)$$

Finding a combinatoric interpretation for general Y, t, R is a long-standing open problem. But for $k = 2$, there are known algorithms for

$$g^{Y=\square\square}(t^{\otimes 2}; R) = g^{+}(t, t, R)$$

$$g^{Y=\begin{smallmatrix}\square\\\square\end{smallmatrix}}(t^{\otimes 2}; R) = g^{-}(t, t, R)$$

$$c_R^t = g^{+}(t, t, R) - g^{-}(t, t, R)$$

in terms of domino tilings of Young diagrams.

Carre and Leclerc, 1995

OUTLINE

Part 1 These results are based on the use of symmetric groups to organise the enumeration and correlators of gauge invariant operators.

I will sketch some key ideas - hopefully of some general interest beyond ads/cft.

Philosophy Permutations give a dual way of doing matrix theory without matrices.

Part 2
Then describe how domino combinatorics leads to results in line with expectations from brane physics.

Part 3

The quarter-BPS problem is a 2-matrix problem. These permutation methods lead to nice large N generating functions. There are relations to free algebras generated by two letters, which can be understood by refining the counting of single traces according to periodicity.

A perspective which leads to common structures in the counting for both $U(N)$ and $SO(N)$ counting.

Part 1: One complex matrix

Consider holomorphic functions of a complex matrix Z. Transforming in the adjoint of a $U(N)$ gauge symmetry.

$$Z \to UZU^\dagger$$

Gauge invariant polynomial holomorphic functions of Z are traces. e.g. $\mathrm{tr}\, Z^3, (\mathrm{tr}\, Z^2)\mathrm{tr}\, Z, (\mathrm{tr}\, Z)^3$
Permutations $\to$ traces

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}}$$

$$\begin{aligned} \sigma = (1)(2)(3) &\longrightarrow (\mathrm{tr}\, Z)^3 \\ \sigma = (1,2)(3) &\longrightarrow (\mathrm{tr}\, Z^2)\mathrm{tr}\, Z \\ \sigma = (1,2,3) &\longrightarrow \mathrm{tr}\, Z^3 \end{aligned}$$

For $\sigma = (1,2,3)$, we have

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} = \mathrm{tr} Z^3$$

For $\sigma = (1,3,2)$

$$\begin{aligned} \mathcal{O}_\sigma(Z) &= Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2} \\ &= Z^{i_1}_{i_3} Z^{i_3}_{i_2} Z^{i_2}_{i_1} = \mathrm{tr} Z^3 \end{aligned}$$

This equivalence can be described in terms of a permutation group property

$$(1,2,3) = (2,3)(1,3,2)(2,3)$$

Conjugate permutations produce the same trace.

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z)$$

Equivalence easy to see from diagrams:

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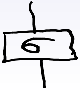
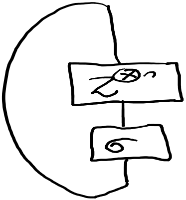
$z^{i_1}_{j_1} \cdots z^{i_n}_{j_n} \longrightarrow$ 

$\delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} \longrightarrow$ 

$\mathcal{O}_\sigma(Z) =$ 

$= \mathrm{tr}_{V^{\otimes n}} (\sigma Z^{\otimes n})$

The two-point function of arbitrary multi-trace operators

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z^\dagger)\rangle = \sum_{\gamma,\tau\in S_n} \delta(\sigma_1\gamma\sigma_2\gamma^{-1}\tau)N^{C_\tau}$$

This is a pairing on permutations.

$$\langle \sigma_1|\sigma_2\rangle_{QFT} = \sum_{\gamma,\tau\in S_n} \delta(\sigma_1\gamma\sigma_2\gamma^{-1}\tau)N^{C_\tau}$$

At large N, it is just the group theoretic pairing on conjugacy classes

$$\langle \sigma_1|\sigma_2\rangle = \sum_{\gamma\in S_n} \delta(\sigma_1\gamma\sigma_2\gamma^{-1})$$

Hence orthogonality of trace structures at large N - also known as large N factorization.

A basis of functions of $\sigma \in S_n$ is given by representation theory: For each Young diagram R with n boxes, irrep

$$D^R(\sigma) : V_R \to V_R$$

Choosing a basis in R, we have $D^R_{IJ}(\sigma)$ which are functions on S_n.
As R runs over irreps, I, J over basis elements, we have a basis for $\mathbb{C}(S_n)$.

$$n! = \sum_R d_R^2$$

The functions invariant under conjugation are

$$\chi_R(\sigma) = \sum_I D^R_{II}(\sigma) = tr_R(\sigma)$$

The corresponding operators (Fourier transform)

$$\mathcal{O}_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \; \mathcal{O}_\sigma(Z)$$

have the nice orthogonality properties under free field 2-point function.

For $SO(N)$ same strategy :

1. Use permutations to construct gauge invariants.
2. Describe the equivalences on the permutations.
3. Use rep theory to find the Fourier basis of functions on the equivalence classes.
4. Form linear combinations of permutation-labelled operators : they are orthogonal under free field 2-point functions, and encode finite N relations.

Diagrams → Equivalences

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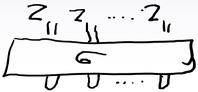
$$Z_{i_1 i_2} \cdots Z_{i_{2n-1} i_{2n}} \to Z_{11} \cdots Z_{11}$$

$$\mathcal{O}_\sigma(Z) =$$



$$\sigma \sim (-1)^{\gamma} \, \gamma \, \sigma \, \mu$$

$$\mu \in S_2^{\times n} \rtimes S_n = S_n[S_2]$$
$$\gamma \in S_n[S_2]$$

$H = S_n[S_2]$ is the wreath product of S_n with S_2. The symmetry of $(1,2)(3,4), \cdots, (2n-1, 2n)$. A subgroup of S_{2n}.

Functions invariant under this equivalence are

$$F_T(\sigma) = \sum_{IJ} \langle sign(H)|T, I\rangle D^T_{IJ}(\sigma)\langle R, J|\phi(H)\rangle$$

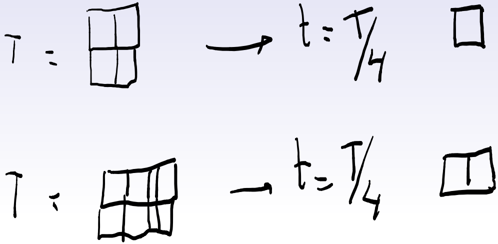
Project on the right to invariant of H. Project on the left to the sign rep of H.

T contains the invt of H, with multiplicity one, if it has even length rows. T contains the sign of H, with multiplicity one, if it has even length columns.

Mc Donald,“ Symmetric functions and Hall polynomials”

Hence T is made of multiples of 4 boxes.

Young diagram T of $2n$ boxes and Young diagram t of $n/2$ boxes.



T has $2n$ boxes

t has $\frac{n}{2}$ boxes.

Trace basis for gauge invariant operators with n copies of Z :

$$\mathcal{O}_p(Z) = (trZ^2)^{p_1}(trZ^4)^{p_2}\cdots(trZ^{n/2})^{p_{n/2}}$$

$$\sum_{i=1}^{n/2} 2ip_i = n$$

The numbers $\{p_1, p_2, \cdots, p_{n/2}\}$ are such that

$$1.p_1 + 2p_2 + \cdots = \sum_{i=1}^{\frac{n}{2}} ip_i = \frac{n}{2}$$

i.e. they define a partition of $n/2$.
Trace basis is labelled by partitions of $n/2$. For each $p \vdash n/2$ there is a $\sigma^{(p)} \in S_{2n}$ which produces the corresponding trace.

It is known

Ivanov V. N. Bispherical functions on the symmetric group associated with the hyperoctahedral subgroup. Journal of Mathematical Sciences, 96(5), 1999.

that the matrix elements of these σ_p are expressible in terms of characters in $t = T/4$.

$$\sum_{IJ} B_I^{T\to sign(H)} D_{IJ}^T(\sigma^{(p)}) B_J^{R\to\phi(H)} = \frac{2^{l(p)}}{2^n n!}\sqrt{\frac{(2n)!}{d_T}}\chi_t(p)$$

$\chi_t(p)$is character of a permutation in $S_{n/2}$ with cycles $[1^{p_1}, 2^{p_2}, \cdots]$

$$\mathcal{O}_t(Z) \propto \sum_{\substack{p \\ p_1+2p_2+\cdots=n/2}} \frac{1}{Sym(2p)} \chi_t(p) \quad (trZ^2)^{p_1}(trZ^4)^{p_2}\cdots$$

$Sym(2p) = \prod_{i=1}^{n/2}(2i)^{p_i}p_i!$

Orientifold projection

Apply projection to $\mathcal{O}_R(Z)$. Drop odd traces. Express the even traces in terms of $\mathcal{O}_t$.

Result

$$\mathcal{P}(\mathcal{O}_R) = \sum_t c_R^t \mathcal{O}_t$$

where c_R^t is expressed in terms of characters :

$$c_R^t = \sum_{p \vdash n/2} \frac{1}{Sym(p)} \chi_t(p) \chi_R(2p)$$

$\chi_R(2p)$ is character of a permutation in S_n with cycle structure $[2^{p_1}, 4^{p_2}, \cdots]$

To prove that this is $g^+(t,t,R) - g^-(t,t,R)$, use Schur-Weyl duality, which relates $U(N)$ rep theory to symmetric group rep theory.

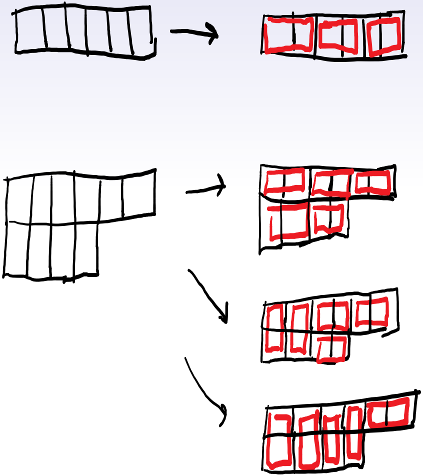
Carre and Leclerc (1995) showed that this difference of Plethysm coefficients has an interpretation in terms of domino combinatorics.

$$c_R^t = sign(R) Yam(R, t)$$

$sign(R) = (-1)^{R_1+R_3+\cdots}$. For fixed R all non-vanishing c_R^t have same sign (as directly verified by GAP computation of the character sum).

$Yam(R, t)$ is the number of Yamanouchi Domino tilings of Young diagram R with evaluation t.

Examples : Domino tilings of Young diagrams.



Yamanouchi Domino Tableaux are labelled domino tilings, obeying some rules:

- The numbers are weakly increasing along rows.
- Strictly increasing down columns.
- Reading the numbers, from right to left and going down, we have a word : $w_1 w_2 .... w_n$. Any truncation of this word $w_1 w_2 \cdots w_s$ should contain no more 2's than 1's, no more 3's than 2's etc.

The evaluation of the Yamanouchi domino tableau is the list $[t_1, t_2, ...]$ of the numbers of $1, 2, \cdots$ etc. which form row lengths of Young diagram t.

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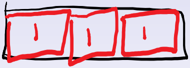
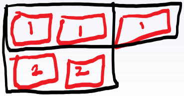


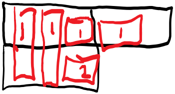
$\rightarrow$ $R = [6]$

$t = [3]$

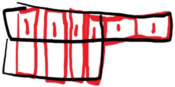


$\rightarrow$ $R = [6, 4]$

$t = [3, 2]$



$\rightarrow$ $t = [4, 1]$



$\rightarrow$ $t = [5]$

What does this have to do with branes ?

Giant gravitons are supersymmetric brane states

Mc Greevy, Susskind, Toumbas - 2000

which are S^3 embedded in S^5 of $AdS_5 \times S^5$, rotating on a disc in the S^5.

Dual giants large in AdS_5 were subsequently found.

It was proposed that sub-determinant operators are duals of giant gravitons expanding in S^5.

Balasubamanian, Berkooz, Naqvi, Strassler (2001)

$$\mathcal{O}_L^{subDet} = \epsilon_{i_1,\cdots i_L, i_{L+1},\cdots, i_N} Z_{1 i_1} Z_{2 i_2} \cdots Z_{L, i_L}$$

Review : Giant gravitons and Young diagrams

Young diagram basis :

Corley, Jevicki, Ramgoolam (2001)

Single long column Young diagrams(subdeterminants) - single giants large in S^5.

Multi-column Young diagrams - multiple giants in S^5.

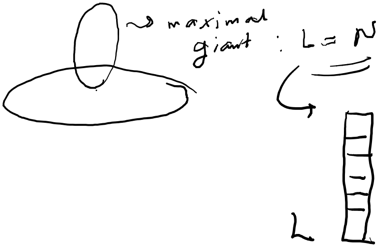
Single long row Young diagrams: Dual giants large in AdS_5. Multiple row Young diagrams : multiple dual giants.

Subsequent evidence using strings attached to giants, e.g.

Balasubramanian, Berenstein, Feng, Huang (2005), de Mello Koch, Dessein, Giataganas, Mathwin (2011) ; de Mello Koch, Ramgoolam (2012)

Subsequent evidence using correlators of brane/gravitons :
Bissi, Kristjansen, Young, Zoubos (2011) ; Hai Lin (2012)

$$\underbrace{X_1^2 + X_2^2}_{\text{disc}} = 1 - (\underbrace{X_3^2 + X_4^2 + X_5^2 + X_6^2}_{S^3})$$



Giant graviton interpretation of Domino combinatorics

When doing a Z_2 projection of a single giant graviton quantum state - particle moving on a disc - the Z_2 will project a single giant in the $U(N)$ theory to a single giant in the $SO(N)$ theory.
The Z_2 can also project two giants related as mirror images to a single giant.
In general we would expect, a combination of the two possibilities.
Domino combinatorics : Two-row giants can give project to one-row or two-row states.
In general

$$c_1(R)/2 \leq c_1(t) \leq c_1(R)$$

and

$$r_1(R)/2 \leq r_1(t) \leq r_1(R)$$

Another approach to brane understanding of the Domino rule. The $g(t, t, R)$ governs the interaction of branes in the $U(N)$ theory.

$$\langle \chi_t(Z)\chi_t(Z)\chi_R(Z^\dagger)\rangle = g(t, t, R) \quad f_R$$

$$g(t, t, R) = g^+(t, t, R) + g^-(t, t, R)$$

$$c_R^t = g^+(t, t, R) - g^-(t, t, R)$$

Question Can we explain, from the spacetime point of view, the change of sign as being an effect of the orientifold plane ?

The permutation approach to gauge invariants extends to multi-matrix systems. Quarter BPS operators at zero coupling are holomorphic gauge invariant functions of Z, Y.

D'Hoker, Heslop, Howe, Ryzhov (2003)

For gauge invariants in the $U(N)$ theory with n copies of Z and m copies of Y, the number of operators is

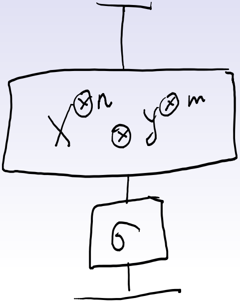
$$\sum_{\substack{R_1 \vdash m, R_2 \vdash n, R_3 \vdash (m+n) \\ l(R_3) \leq N}} (g(R_1, R_2, R_3))^2$$

Brown, Heslop, Ramgoolam (2007); de Mello Koch, Bhattacharrya, Collins (2008)

For $SO(N)$ theory:

$$\sum_{\substack{R_1 \vdash 2m, R_2 \vdash 2n, R_3 \vdash 2(m+n) \\ l(R_3) \leq N}} g(R_1, R_2, R_3)$$

R_1, R_2 have odd row lengths. R_3 has even row lengths.



$\sigma \in S_{n+m}$

$$\sigma \sim \gamma \sigma \gamma^{-1}$$

for $\gamma \in S_n \times S_m \subset S_{n+m}$

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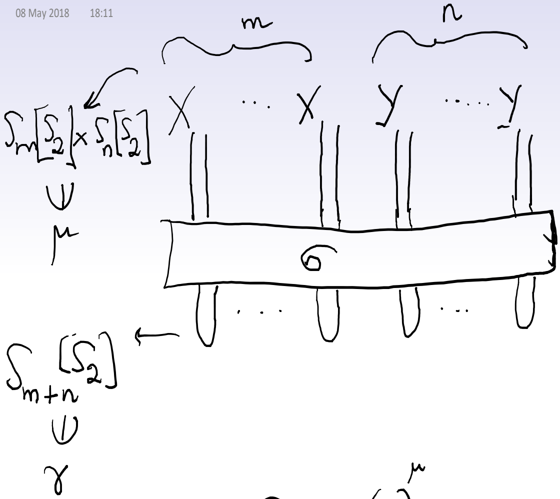


$$\sigma \sim (-1)^{\mu} \mu \sigma \gamma$$

Writing explicit generating functions for the number of operators at finite N is hard. But things simplify at large N ($n+m<N$).
For the $U(N)$ theory, the counting of single-traces is

$$Z_{st}(x,y) = -\sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log(1-x^d-y^d)$$

$\phi(d)$ is the Euler totient function (number of positive integers less than d and relatively prime to d).
The counting of multi-traces is

$$Z_{mt}(x,y) = PEXP(Z_{st}(x,y)) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i-y^i)}$$

Bianchi , F.A. Dolan, P.J. Heslop, H. Osborn (2006)
This is a very interesting structure :

$$Z_{mt}(x,y) = \prod_{i=1}^{\infty} Z_{root}(x^i,y^i)$$

$$Z_{root}(x,y) = \frac{1}{(1-x-y)}$$

Z_{root} itself has a counting interpretation:

$$Z_{root}(x,y) = \sum_{n,m} \frac{(n+m)!}{n!m!} x^n y^m$$

The coefficient of $x^n y^m$ counts words made from $\hat{x}, \hat{y}$, which span a free algebra.

$$(n,m) = (1,1) : \frac{2!}{1!1!} = 2 \to \hat{x}\hat{y}, \hat{y}\hat{x}$$
$$(n,m) = (2,1) : \frac{3!}{2!1!} = 3 \to \hat{x}^2\hat{y}, \hat{x}\hat{y}\hat{x}, \hat{y}\hat{x}^2$$

These free algebras generalize to partially commuting trace monoids, when we consider quiver generalizations of this 2-matrix counting.

“Quivers as Calculators,” Pasukonis, Ramgoolam (2013) ;

“Quivers, Words and Fundamentals,” Mattioli, Ramgoolam (2014)

The space of multi-trace operators is isomorphic as a graded vector space, to the counting in a product of free algebras generated by

$$\hat{x}_i, \hat{y}_i \quad i \in \{1, 2, \cdots\}$$

Multi-level letters. Letters at different levels commute.

But what is the isomorphism ? What do products of free algebras have to do with multi-traces ? What does $\frac{1}{(1-x-y)}$ have to do with traces ?

LBR2018: Counting Single traces of Z, Y can be refined according to periodicity of the traces. e.g.

$$\begin{array}{ll} tr(Z^2Y^2) & \text{periodicity1} \\ tr(ZYZY) & \text{periodiocity 2} \end{array}$$

The single traces of periodicity 1 are in 1-1 correspondence with Lyndon words. $f_L(x, y)$ is

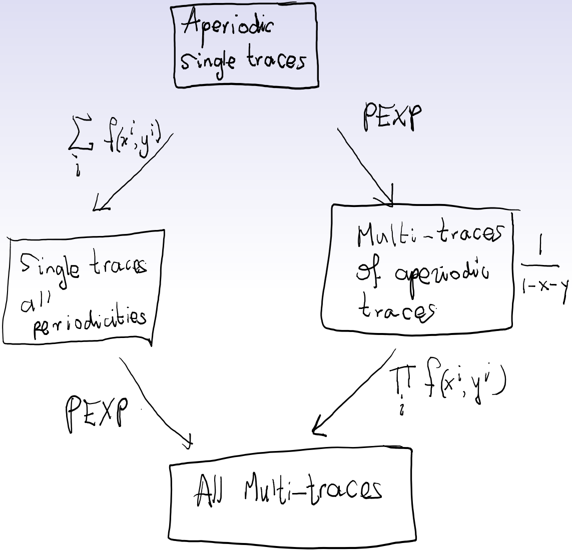
$$f_L(x, y) = -\sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log(1 - x^d - y^d)$$

$\frac{1}{(1-x-y)}$ is the plethystic exponential of $f_L(x, y)$. It counts the multi-traces of aperiodic traces.

$\mu(d)$ is the Mobius function. $\mu(1) = 1$. $\mu(d)$ is zero if d has repeated prime factors. $\mu(d) = (-1)^s$ is d has s distinct prime factors.

Refinement of traces by periodicity is useful for correlators.
There is an i! in the planar correlator of two single traces, coming from this periodicity.

$$\langle \mathrm{tr}(X^2Y)(X^2Y) \quad \mathrm{tr}Y^\dagger(X^\dagger)^2Y^\dagger(X^\dagger)^2\rangle$$



Aperiodic single traces
$\sum_i f(x^i, y^i)$
PEXP
Single traces all periodicities
Multi-traces of aperiodic traces
$\frac{1}{1-x-y}$
PEXP
$\prod_i f(x^i, y^i)$
All Multi-traces

Similar structure holds for $SO(N)$ theory. And generating functions (LBR-2018).
Single traces: (e.g. using cycle indices of D_{n+m} discrete group and Polya type argument)

$$H_{st}(x,y) = \frac{1}{2}\left[-\sum_{d=1}^{\infty}\frac{\phi(d)}{d}\log(1-x^d-y^d) + \frac{x^2+xy+y^2-x-y}{1-x^2-y^2}\right]$$

Multi-traces: (PEXP of above or directly from perm equivalences)

$$H_{mt}(x,y) = \prod_{i=1}^{\infty}\frac{1}{\sqrt{1-x^i-y^i}}\exp\left[\frac{x^{2i}+x^iy^i+y^{2i}-x^i-y^i}{2i(1-x^{2i}-y^{2i})}\right]$$

minimal periodicity single traces:

$$H_{mst}(x,y) = \frac{1}{2}\sum_{d=1}^{\infty}\mu(d)$$
$$\left[\frac{-1}{d}\log(1-x^d-y^d) + \frac{x^{2d}+x^dy^d+y^{2d}-x^d-y^d}{1-x^{2d}-y^{2d}}\right]$$

Vanishing aperiodic word.
$tr(X^2Y) = trY^T(X^T)^2 = -tr(X^2Y) = 0.$
Non-vanishing doubled version
$tr(X^2YX^2Y)$
The minimal periodicity traces include periodicity 1 and 2.
Multi-traces of minimal periodicity.

$$H_{mmt}(x,y)$$
$$= \frac{1}{\sqrt{1-x-y}}\prod_{i=1}^{\infty}\exp\left[\frac{1}{2i}\frac{x^{2i}+x^iy^i+y^{2i}-x^i-y^i}{1-x^{2i}-y^{2i}}\sum_{d|i}d\mu(d)\right]$$

SUMMARY AND QUESTIONS

Orientifold projection map on the Young diagram bases for half-BPS operators are related to :

- Characters of $S_{n/2}$ and S_n.
- Plethysm coefficients.
- Domino combinatorics

- Interpretation in terms of branes. Derivation from branes ?
- Derivation of change in sign from orientifold in space-time ?

$$c_R^t = g^+(t,t,R) - g^-(t,t,R)$$

There is a detailed Connection to branes and (LLM) geometries for half-BPS. Generalizing this to quarter BPS is an important open problem.

Need to solve the weak coupling problem (non-renormalization theorems not strong enough to make the free field results directly applicable at strong coupling).

For gauge-invariants of multi-matrix traces, we observed relations to free algebras, based on facts about word combinatorics(Chen-Fox-Lyndon theorem).

Aperiodic (or minimally periodic) traces are in some sense fundamental building blocks. Reflected in counting as well as planar correlators.

How does the word combinatorics enter correlators at sub-leading orders in $1/N$? in the action of 1-loop dilatation operator on traces ?

The connection between permutations and unitary groups, involving Schur-Weyl duality, underlie many of the results described here.

Permutations give a powerful dual picture for working with gauge invariants of matrices in ads/cft. Also beyond one-loop (Beisert, de Mello Koch, Berenstein, and others ...).

Perhaps this also holds for other questions in matrix theory (BFSS, BMN, IKKT ..) ?