The continuum limit of matrix models with the FRGE

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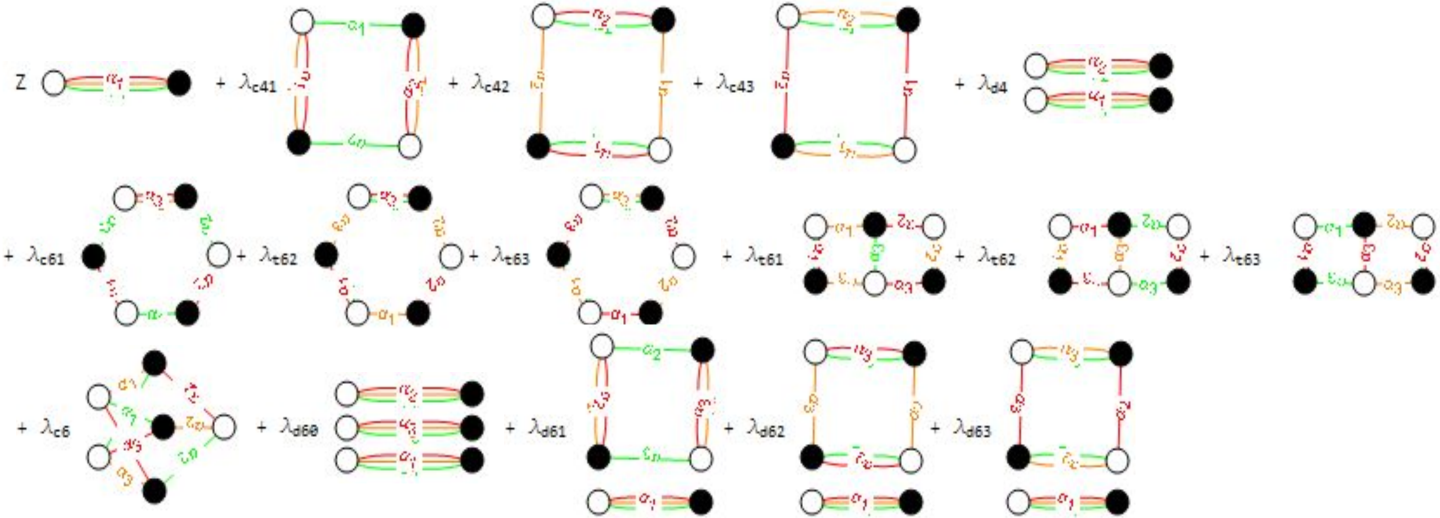


ESI workshop: *"Matrix Models for Noncommutative Geometry and String Theory"* Wien, July 09-13, 2018

In this talk:

No noncommutative geometry, only non-critical ST (if you want), instead motivated by direct approaches to Quantum Gravity

based on work with A. Eichhorn: Phys.Rev. D88 (2013) 084016,
Phys.Rev. D90 (2014) no.10, 104039
Ann. Inst. Henri Poncaré (2018) arXiv:1701.03029

and with J. Ben Geloun: arXiv:1606.04044

and A. Pereirra, D. Oriti and J. Ben Geloun: PRD (in print)

As well as work in progress with A. Eichhorn, A. Pereirra, J. Limma, V. Vitelli, A. Castro and J. Ben Geloun

On the road to Quantum Gravity

Each approach has built-in *features* and inherent *difficulties*
⇒
Combine approaches to use built-in feature of one to solve difficulty of another approach

Plan for this talk

1. **Preparation**: Continuum limit of lattice quantum gravity as large N limit

2. **Tool**: FRGE as a tool for large N limit

3. **Setup**: a) theory space
   b) regulator ⇒ how to take the large N limit
   c) symmetries ⇒ Ward Identities
   d) scaling dimension

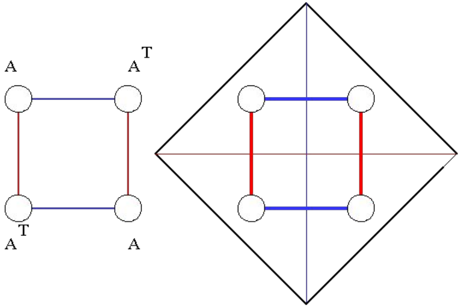
4. **Results**: a) Single trace approximation ⇒ Double scaling limit, multicritical points
   b) Tadpole approximation ⇒ All critical points and precision results
   c) Universality and Optimization

5. **Generalization**: tensor models ⇒ First results

Preparation: Continuum Limit of Lattice QG

Consider matrix model action

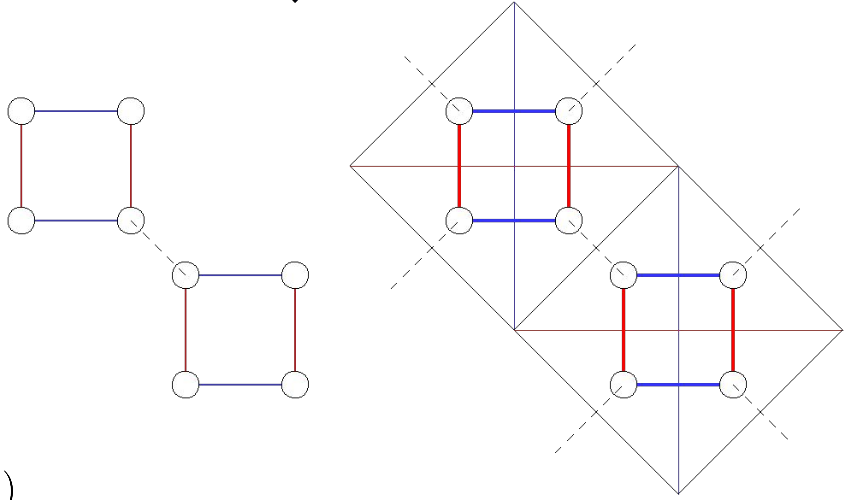
$$S[M] = \frac{1}{2}\mathrm{Tr}(A.A^T) + \frac{g}{4\,N}\mathrm{Tr}(A.A^T.A.A^T)$$



for real matrices A to generate the partition function

$$Z = \int [dM]_N \, \exp(-S[M]) = \sum_\gamma A(\gamma) = \sum_{\Delta(\gamma)} e^{-(-\ln(A(\gamma)))}$$

$\Rightarrow$ can be interpreted as a partition function for random square-tesselations with weights $-\ln(A(\gamma(\Delta)))$ expressible by the Regge action:



$$S_{\mathrm{Regge}}(\Delta) = k_d\, N_d(\Delta) - k_{d-2}\, N_{d-2}(\Delta)$$

where $k_{d-2} - \alpha k_d \propto 1/G$ and $k_d \propto \Lambda/G$

with $k_{d-2} = \ln(N)$ and $k_d = \ln(g) - \frac{d(d-1)}{4}\ln(N)$

Preparation: (contd.)

we want to take the continuum limit $a \to 0$ of the tesselation by squares at fixed volume $\langle V \rangle = a_o^d \langle N_d \rangle$

$\Rightarrow$ take matrix size N to infinity $\Rightarrow$ *G vanishes!*

$\Rightarrow$ For finite G we need a critical scaling of $g(N)$ with matrix size N in continuum limit.

$\Rightarrow$ to investigate the continuum limit of gravity on a random lattice we need to investigate the double scaling limit of the matrix model partition function

$$Z = \int [dM]_N \exp(-S[M]) = \sum_{\gamma} A(\gamma) = \sum_{\Delta(\gamma)} e^{-(-\ln(A(\gamma)))}$$

(see e.g. Brezin, Zinn-Justin: PLB 288 (1992) 54; C. Ayala: PLB 311 (1993) 55)

Tool: Functional Renormalization Group Equation (FRGE)

Partition function $e^{W_k[J]} = \int [d\phi]_\Lambda e^{-S_o[\phi]-\frac{1}{2}\phi - R_k.\phi + J.\phi} \Rightarrow$ field vacuum expectation value $\phi = \frac{\delta W_k}{\delta J}$

with an IR suppression term $\frac{1}{2}\phi.R_k.\phi$ (scale-dependent "mass" term of order *k* for IR d.o.f.)

effective average action $\Gamma_k[\phi] = (\phi.J_k[\phi] - W_k[J_k[\phi]]) - \frac{1}{2}\phi.R_k.\phi$

obeys a flow equation $\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k \left(\frac{\delta^2 \Gamma_k[\phi]}{\delta\phi\,\delta\phi} + R_k\right)^{-1}\right)$

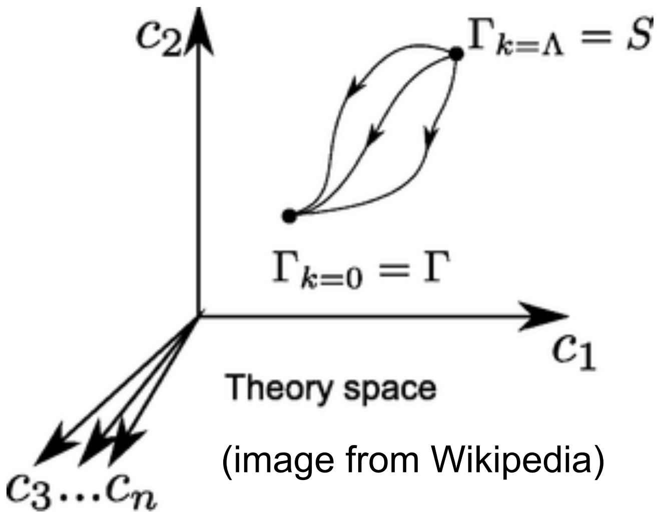
(see e.g. Wetterich *Phys. Lett. B*, **301**: 90)

Interpretation:

1. UV limit: saddle point around $\frac{1}{2}\phi.R_k.\phi$ gives $\Gamma_{k\to\Lambda\to\infty}[\phi] \to S_o[\phi]$
2. IR limit: suppression term drops out $\Gamma_{k\to 0}[\phi] \to \Gamma[\phi]$

$\Rightarrow$ interpolation between bare action and quantum effective action

$\Rightarrow$ tool for systematic investigation of bare actions (limits $k \to \Lambda \to \infty$)



(image from Wikipedia)

Tool: FRGE to find large N universality classes

The standard use of the FRGE is to investigate:

1. UV limit: saddle point around $\frac{1}{2}\phi.R_k.\phi$gives $\Gamma_{k\to\Lambda\to\infty}[\phi] \to S_o[\phi]$
2. IR limit: suppression term drops out $\Gamma_{k\to 0}[\phi] \to \Gamma[\phi]$

Hence, more generally it is an interpolation between bare action and quantum effective action,
OR simply a tool to convert a high-dimensional *integration problem* into a *flow problem*.

$\Rightarrow$ use this approach for matrix model partition function

$$e^{-W_N[J]} = \int [dM]_\Lambda e^{-S[M]-\frac{1}{2}M.R_N.M+J.M}$$

such that the effective average action $\Gamma_N[M] := \sup_J \{J_N[M].M - W_N[J_N[M]]\} - \frac{1}{2}M.R_N.M$

satisfies the FRGE: $N\partial_N \Gamma_N[M] = \frac{1}{2}\mathrm{Tr}\left(\frac{N\partial_N R_N}{\Gamma^{(2)}[M]+R_N}\right)$

Where it is now very important that $\Delta_N S[M] = \frac{1}{2}M.R_N.M$ is not an IR suppression term (i.e. has NO dimension)
BUT only an **abstract** suppression factor that we use to convert the integral into a flow

Setup I: Theory Space

Bare action: $S_{matrix} = \frac{1}{2}\mathrm{Tr}(M.M^T) + \frac{g}{N}\mathrm{Tr}(M.M^T.M.M^T)$

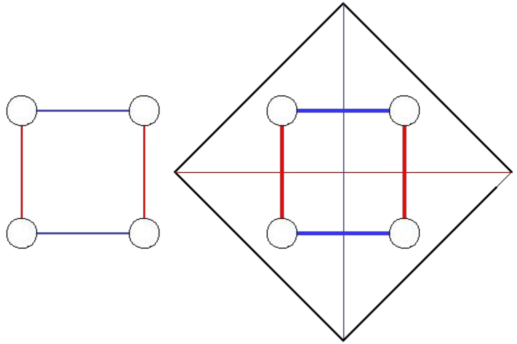
$\Rightarrow$ Fundamental field: real matrix M

Fundamental symmetry: Invariance under bi-orthogonal trf. $M \to O_1.M.O_2^T$

$\Rightarrow$ generates even effective operators (i.e. of form $\mathrm{Tr}(M^{2n_1})...\mathrm{Tr}(M^{2n_k})$)

$\Rightarrow$ theory space $\Gamma_k[M] = f_k\left(\mathrm{Tr}(M^2), \mathrm{Tr}(M^4), \mathrm{Tr}(M^6), ...\right)$ has no occurrence of scale

Graphic representation:



Setup II: Regulator

There is no natural scale, but we want to give a large suppression factor to matrix entries in upper-left corner (small index values)

$\Rightarrow$ invent a Laplacian, e.g. $\Delta M_{ab} := (a+b)\, M_{ab}$

$\Rightarrow$ we can construct a suppression term for the upper-left corner (with shape analogous to Litim's cut-off)

$$\Delta_N S[M] = M_{ab} R_N(a,b) M_{ab} \qquad \text{with} \quad R_N(a,b) = Z\left(\frac{2N}{a+b} - 1\right)\theta\left(1 - \frac{2N}{a+b}\right)$$

Notice: The introduction of the Laplacian breaks bi-orthogonal symmetry!

$\Rightarrow$ broken Ward-Identity

Setup III: Symmetries and Ward Identities

Assume an invariance of the bare action under a symmetry generated by $\mathcal{G}_\epsilon\, S[\phi] = \epsilon^B \frac{\delta S[\phi]}{\delta \phi^A} f_B^A[\phi]$

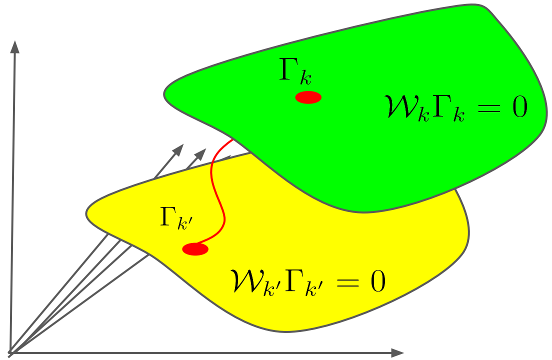
(for simplicity assume invariance of the measure $[d\phi]_\Lambda$ under this symmetry)

$\Rightarrow$ Legendre transform yields $\mathcal{W}_k = \mathcal{G}_\epsilon \Gamma_k - \frac{1}{2}\mathcal{G}_\epsilon \left(\langle \phi.R_k.\phi\rangle - \phi.R_k.\phi\right)$

Which ensures that the effective action satisfies the correct Ward identity $\lim_{k\to 0} \mathcal{W}_k \Gamma_k = \mathcal{W}\Gamma = 0$

Moreover: analogous to derivation of FRGE one finds

$$\partial_k \mathcal{W}_k \Gamma_k = -\frac{1}{2}\mathrm{Tr}\left((\Gamma_k^{(2)} + R_k)^{-1}.\partial_k R_k.(\Gamma_k^{(2)} + R_k).(\mathcal{W}_k\Gamma_k)^{(2)}\right)$$



$\Rightarrow$ if initial effective average action satisfies initial WTI then the effective action satisfies the normal WTI

$\Rightarrow$ symmetry improved flow by solving mWTI and using symmetric couplings as coordinates on $\mathcal{W}_k\Gamma_k = 0$

Setup IV: Scaling Dimension

There is no "canonical dimension" of operators in this theory space, but since we are interested in the large *N* limit,

we have to impose that the beta functions admit a *1/N* expansion, i.e. $\beta_{g_i} = b_i^1(g_1, \ldots) + 1/N\, b_i^2(g_1, \ldots) + O(1/N^2)$

this fixes the scaling of the operators, by generating upper and lower bounds that admit only one solution at the end.

E.g. tadpole of one $g_4\, Tr(M^4)^{(2)}$ flows into *Z*

and two-vertex diagram with two $g_4\, Tr(M^4)^{(2)}$ flows into g_4

$\Rightarrow$ once dimension of g_4 is fixed one can fix dimensions of all other operators.

$\Rightarrow$ for couplings defined as $\Gamma_N[M] = \sum g^i_{n_1 \ldots n_i} \mathrm{Tr}(M^{2n_1}) \ldots \mathrm{Tr}(M^{2n_i})$

one obtains the canonical dimension from *1/N* expandability as

$$\dim(g^i_{n_1 \ldots n_i}) = N^{i-1+\sum_{k=1}^{i} n_k}$$

More generally: One can express the scaling dimension as function of faces and number of matrices/tensors

Results I: Matrix Model - Single Trace Truncation

1. Single trace truncation: $\Gamma_N[M] = \frac{Z}{2}\mathrm{Tr}(M^2) + \sum_{n\geq 2} \frac{\bar{g}_{2n}}{2n}\mathrm{Tr}(M^{2n})$ with dimensionless couplings $\bar{g}_i = Z^{\frac{i}{2}} N^{\frac{i}{2}-1} g_i$

   $\Rightarrow$ beta functions: $\eta = g_4 \, [\dot{R}\, P^2]$

$$\beta(g_{2n}) = ((1+\eta)n - 1)\, g_{2n} + 2n \sum_{i;\vec{m}:\sum m_k = n} (-1)^{\sum_i m_i} [\dot{R} P^{1+\sum_i m_i}] \begin{pmatrix} \sum_i m_i \\ m_1 m_2 \ldots \end{pmatrix} \prod_i g_{2(i+1)}^{m_i}$$

   Finding fixed points with one relevant direction, but $\theta \approx 1.0, \ldots, 1.1$ instead of analytic $\theta = 0.8$ *(in all truncations)*
   (and all other crit. exponents near negative integers and aligned with $g_{2n} \, : \, n > 4$)

2. Multitrace truncation: only $\mathrm{Tr}(M^2)\mathrm{Tr}(M^{2n})$ flow into single-trace operators at large N

   $\Rightarrow$ include $g_{2,2}$ and $g_{2,4}$ in truncation, but critical exponents actually get *worse*:

$$\theta_1 = 1.21, \; \theta_2 = -0.69, \; \theta_3 = -1.01, \; \theta_4 = -1.88$$

   (inclusion of further multitrace operators does not improve result)

Results II: Matrix Model - Tadpole Approximation

O(N) symmetry is generated by $M \to O^T.M.O = \phi + \epsilon\,[M,A] + O(\epsilon^2)$

and leads to Ward-identity $\mathcal{W}_N \Gamma_N[M] = \mathcal{G}_\epsilon \Gamma_N[M] - \mathrm{tr}_{op}\left(\frac{[A,R_N]}{\Gamma_N^{[2]}[M]+R_N}\right) = 0$

Observation: Tadpole approximation of flowing WTI vanishes
(i.e. no index dependence of tadpoles of index-independent operators!)

1. Tadpole approximation of single trace truncation

   $\eta = 2 g_4\, x \qquad \beta(g_{2n}) = ((n-1)+n\eta)g_{2n} - 2\,n\,x\,g_{2(n+1)}$

   $\Rightarrow$ find $\theta_1 = 1$ (is *20%* off, but all further multicritical exponents with good accuracy)

2. Tadpole approximation with multirace operators

   including multitrace operators g_{2n}, $g_{2,2n}$, $g_{4,2n}$, $g_{2,2,2n}$ in truncation gives arbitrarily close values to $\theta_1 = \frac{4}{5}$

   and multicritical exponents also in *O(1%)* precision

Results III: Scheme dependence and Optimization

The tadpole approximation contains only traces of the form $\mathrm{Tr}\left(P.\dot{R}.P.F[M]\right)$
⇒ this reduces to one single constant under most projection rules, say *x*

⇒ FP values of "dimensionless" ratios such as $\frac{g_6}{g_4^2}$ and critical exponents can not depend on *x*

Optimization: in two-vertex approximation one has a ratio between *x* and $\mathrm{Tr}\left(P.\dot{R}.P.F[M].P.F[M]\right)$

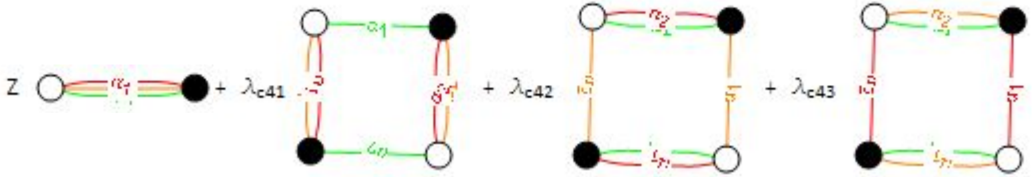
⇒ one can optimize this ratio s.t. e.g. $\frac{g_6}{g_4^2}$ has minimal scheme dependence

Optimization leads to an improvement of multi-critical exponents ⇒ deviations less than *1%*

However: two-vertex beta functions possess structure that contains no new effective operators at double scaling limit ⇒ still *20%* deviation from analytic values
⇒ go to three-vertex beta functions to optimize double scaling limit

Generalization to Tensor Models

Colored Tensor Models work analogous to matrix models:
$\Rightarrow$ example bare action for complex rank 3 model:



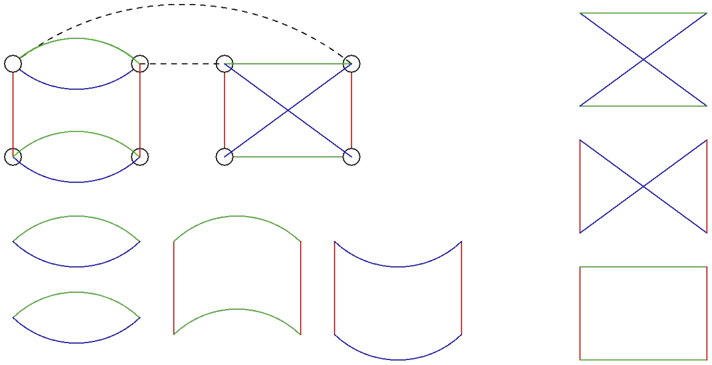
Feynman graphs have a dual geometric interpretation as triangulations of piecewise linear 3D manifolds

Regulator term: $\Delta T_{abc} = (a+b+c)\, T_{abc}$

Scaling dimension from 1/N expandibility:

$$s(\gamma) = 3 - \frac{1}{2}(3p(\gamma) - F(\gamma))$$

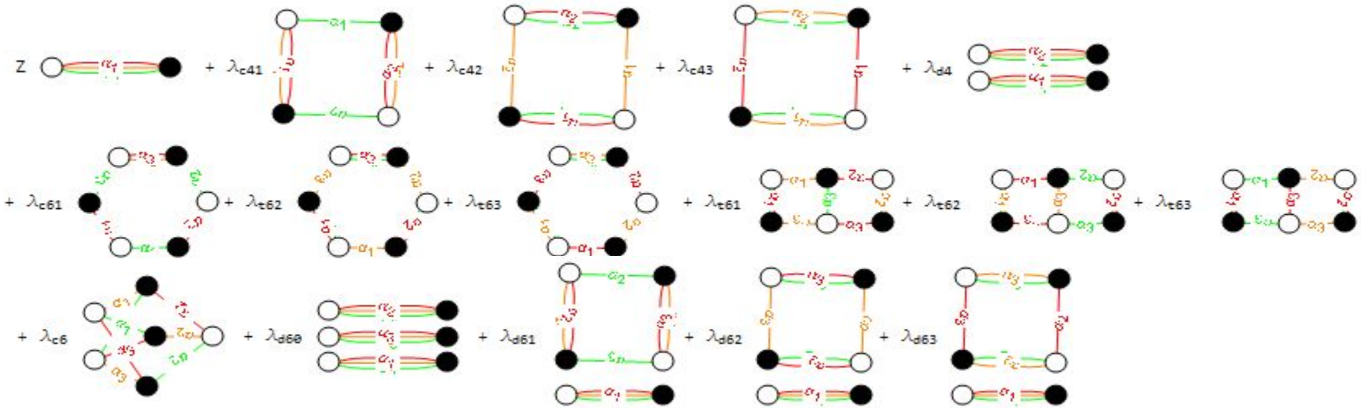
where p = #tensor pairs and F = #faces



(example: rank 3 real model)

Generalization to Tensor Models: First Results

Consider truncation:



1. Double scaling limit (known analytic result): $\theta = D - 2 = 1$

   There appears a fixed point with a single relevant direction and $\theta = 0.9, ..., 2.2$ in all truncations and approximation schemes and compatible with color symmetry.

   (FRGE candidate for double scaling limit)

2. Multiscaling limit: There appears another fixed point with two relevant direction that appears in various truncations and approximations

Summary

1. We use the duality of matrix/tensor models and lattice gravity to explore the lattice continuum as a large N limit
   ⇒ study universality classes using FRGE and replacing scale by matrix size N
   ⇒ model does not have scale, FRGE is now dimesnionless a tool to investsigate large N
   ⇒ 1/N expandibility of beta functions is a necessary requirement

2. Matrix model results:
   a. FRGE is a tool to find asymptotic safety in GW model (previous work with A. Sfondrini: IJMP A 26 (2011) 4009)
   b. FRGE finds double scaling limit and multicritical points in pure matrix models
   c. FRGE achieves numerical accuracy when Ward-ID is respected

3. Tensor model results:
   a. Setup can be applied to tensor models
   b. FRGE finds various scaling limits in real and complex tensor models and tensor field theories
   c. Ongoing work with A. Eichhorn, A. Perreira, J. Lumma, V. Vitelli, A. Castro and J. Ben Geloun

Thank you !