Entanglement entropy on the fuzzy sphere

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- Entanglement Entropy is a measure of the quantum entanglement between two subsystems of a larger system
- It is widely used in information theory
- Geometric entanglement entropy is a universaly-defined observable in QFT
- In the last number of years, entanglement entropy has been used together with holography to understand the relationship between quantum structure of gauge theories and the corresponding geometry
- Spacetime = quantum entanglement + holography
- Would be great to use entanglement entropy to understand how spacetime arises in matrix models
- As a toy model, we will try to explore entanglement entropy in noncommutative spaces, focusing mostly on the fuzzy sphere

Outline

1. Entanglement entropy (EE)
   - Definition
   - Geometric EE in local QFT
   - Geometric EE in NC QFT
2. Geometric EE on the fuzzy sphere
   - Approach
   - Detour: mutual information
   - Results
3. Comparison to holographic results
4. The string model
   - String intepretation and EE
   - Quenches and the string model

Entanglement entropy: example

Consider a pair of spin 1/2 particles:

A: Hilbert space $\mathcal{H}_A$ with basis $|\uparrow\rangle_A$ and $|\downarrow\rangle_A$

B: Hilbert space $\mathcal{H}_B$ with basis $|\uparrow\rangle_B$ and $|\downarrow\rangle_B$

The total Hilbert space is $\mathcal{H}_A \otimes \mathcal{H}_B$, with basis

$$|\uparrow\rangle_A|\uparrow\rangle_B, \; |\uparrow\rangle_A|\downarrow\rangle_B, \; |\downarrow\rangle_A|\uparrow\rangle_B, \; |\downarrow\rangle_A|\downarrow\rangle_B$$

Consider two different states in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\uparrow\uparrow\rangle = |\uparrow\rangle_A|\uparrow\rangle_B, \quad \text{and} \quad \text{EPR} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B + \; |\downarrow\rangle_A|\uparrow\rangle_B\right)$$

If we have access only to spin **A**, we must describe it with a density matrix $\rho_A(\Psi) = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = \; {}_B\langle\uparrow|\Psi\rangle\langle\Psi|\uparrow\rangle_B + \; {}_B\langle\downarrow|\Psi\rangle\langle\Psi|\downarrow\rangle_B$

$$\rho_A(\uparrow\uparrow) = |\uparrow\rangle\langle\uparrow| \quad \text{and} \quad \rho_A(\text{EPR}) = \frac{1}{2}|\downarrow\rangle\langle\downarrow| + \frac{1}{2}|\uparrow\rangle\langle\uparrow|$$

EE: example continued

$\rho_A(\uparrow\uparrow)$ is a pure state, and can be described simply by $|\uparrow\rangle_A$.
Eigenvalues 1 and 0.

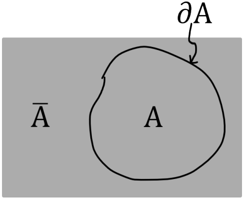
$\rho_A(\text{EPR})$ is not a pure state, and is not equivalent to any $\phi \in \mathcal{H}_A$.
No ϕ such that $\rho_A(\text{EPR}) = |\phi\rangle\langle\phi|$
Eigenvalues not 1 and 0.

These represt two extremes: complete entanglement (EPR) and no entanglement (pure state).
We measure the amount of entanglement by computing the entanglement entropy

$$S_A = -\text{Tr}\ \rho_A \ln \rho_A$$

$$S_A(\uparrow\uparrow) = 0 \quad \text{and} \quad S_A(\text{EPR}) = \ln 2$$

Geometric EE in a local QFT



Consider a QFT on some space with d spacial dimensions, and a region A with boundary ∂A in this space.

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\mathcal{H}_A$: Hilbert space of D of F inside A

$\mathcal{H}_{\bar{A}}$: Hilbert space of D of F outside A

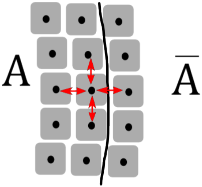
The vacuum EE is given by

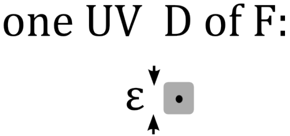
$$S = -\mathrm{Tr}_A \left(\rho_A \ln \rho_A\right) , \quad \text{with} \quad \rho_0 = \mathrm{Tr}_{\bar{A}} \, |0\rangle\langle 0|$$

Mark Srednicki 1993: such EE is proportional to the area of the boundary ∂A rather than the volume of the region A.

Area law: Leading UV divergent part of S is proportional to $\frac{|\partial A|}{\epsilon^{d-1}}$

Why is $S \sim \frac{|\partial A|}{\epsilon^{d-1}}$?





The Hamiltonian couples local degrees of freedom mostly with their nearest neighbours

Entanglement is 'monogamous': if any D of F is fully entangled with some subset of a system, it cannot also be entangled with any other subset.

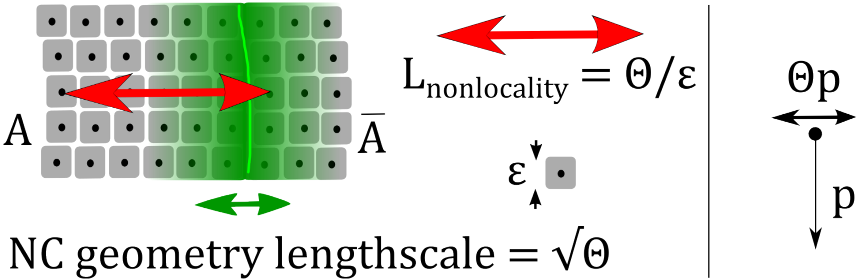
Quantum correlations between inside and outside are established via the boundary only

EE is proportional to the number of degrees of freedom that are close to the boundary of a region, which is $\frac{|\partial A|}{\epsilon^{d-1}}$

Geometric EE in NC QFT

There are four length scales to consider:

- UV cutoff ϵ
- noncommutative geometry length scale $\sqrt{\Theta}$, where $[\hat{x}, \hat{y}] = i\Theta$
- the effective nonlocality scale, Θ/ϵ (UV-IR connection)
- the size of the region (must be larger than ϵ and $\sqrt{\Theta}$)



We would expect that EE would undergo a transition from extensive (propotional to $|A|$) to area law (proportional to $|\partial A|$) when the size of the region is on the order of $L_{\text{nonlocality}}$.

Why is this interesting?

- Direct probe of the UV-IR connection
- Behaviour of entanglement entropy is linked to the ability of a quantum system to scramble information quickly: nonlocal theories might emulate the scrambling behaviour of stretched black hole horizons.
- Since EE is best understood in local theories, it is interesting to see which of its universal properties break down in the non-local setting

Fuzzy S^2

Scaled N-dimensional irrep of SU(2) (spin $J = (N-1)/2$)

$$X_i = R\frac{L_i}{J} \qquad [L_i, L_j] = i\epsilon_{ijk}L_k \qquad \sum_i L_i^2 = \frac{N^2-1}{4}$$

A scalar field ϕ is represented by a Hermitian $N \times N$ matrix

$$H = \frac{4\pi}{N}\frac{1}{2}\mathrm{Tr}\left((\dot{\phi})^2 - [L_i, \phi]^2 + \mu\phi^2\right)$$

In this model, we have

- $\epsilon \sim R/N$
- $\Theta = R^2/J \qquad (N\Theta \sim R^2)$
- $L_{\text{nonlocality}} = \Theta/\epsilon \sim R$

It will be difficult to observe the transition from extensive to area law behaviour, as even the largest possible region (half of the sphere) has diameter on the order of $L_{\text{nonlocality}}$.

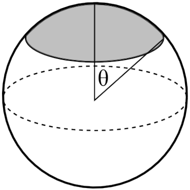
Fuzzy S^2: defining spherical cap regions

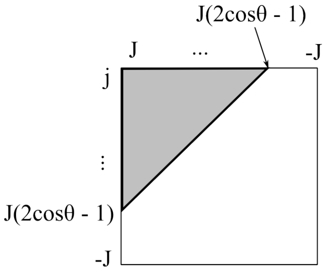
The D of F are N^2 coupled harmonic oscillators, whose positions correspond to the entries of ϕ.

We would like to compute EE for a spherical cap region.

Consider an operator $Z : \phi \to \frac{1}{2}(L_3\phi + \phi L_3)$

We associate inside of the spherical cap with the space spanned by eigenvectors of Z with eigenvalues greater than $J(2\cos\theta - 1)$





Fuzzy S^2: UV cutoff

To be able to control $L_{\text{nonlocality}}$ and have $L_{\text{nonlocality}} < R$, we have to control ϵ at fixed N.

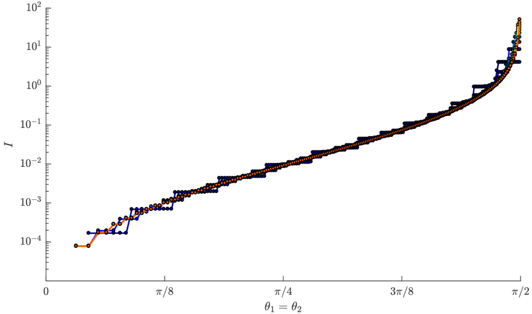
The Laplacian operator $\Delta : \phi \to [L_i, [L_i, \phi]]$ has eigenvalues $k(k+1)$ for k from 0 to $N(N-1)$.
Consider the subspace spanned by eigenvectors of Δ with eigvalues from 0 to $n(n-1)$ for $n < N$. Let P_n be a projection operator onto this subspace.
The corresponding UV cutoff is $\epsilon_n = R/n$, $L^n_{\text{nonlocality}} \sim \frac{n}{N} R$

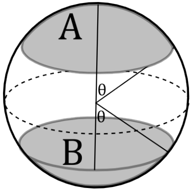
$PZ \neq ZP$: lowering the UV cutoff makes the boundary of the cap more 'fuzzy'
Consider then, the spectrum of PZP: the eigenvalues are still spread from $-J$ to J, so we will associate inside the cap with the space spanned by eigenvectors of PZP with eigenvalues greater than $J(2\cos\theta - 1)$.

Detour: mutual information I

To convince you that this is definition leads to sensible results, start with a quantity that does not appear to be affected by the UV-IR coupling: mutual information.

UV-finite quantity defined for two disjoint regions A and B:
$I(A, B) = S(A) + S(B) - S(A \cup B)$. Subtle cancelations!



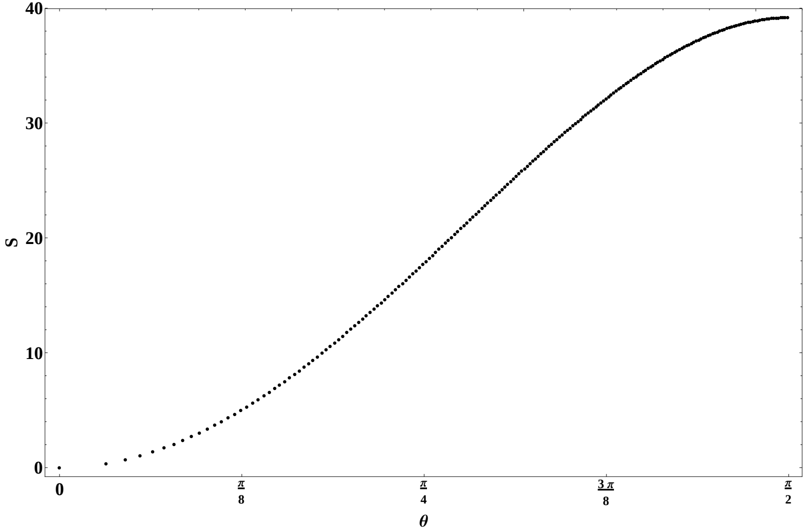


N=200
n=20, 40,. . .,200

Agrees with calculation on a discretized commutative sphere.

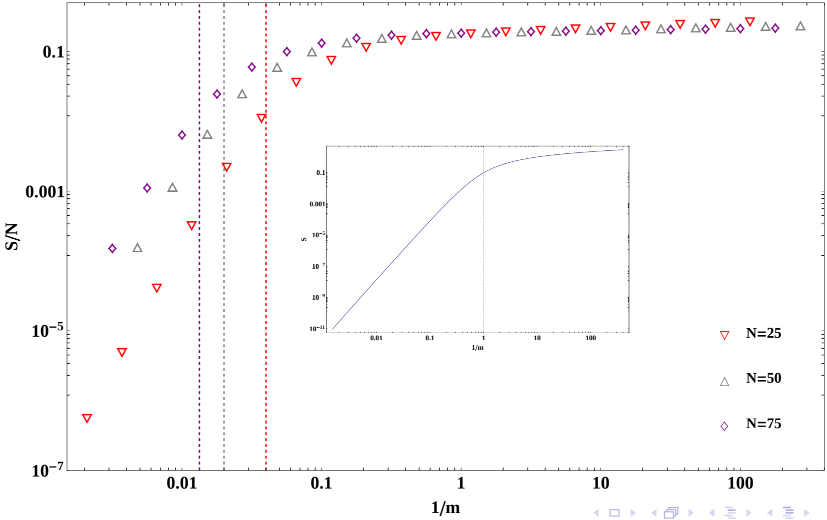
EE as a function of θ

$N = 100$, $n = N$



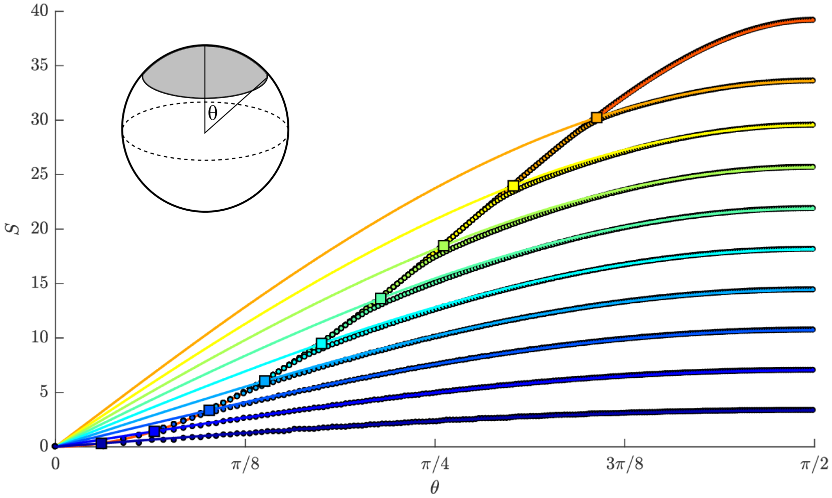
EE as a function of mass *m*

Inset shows EE between two HO as a function of their mass at fixed coupling



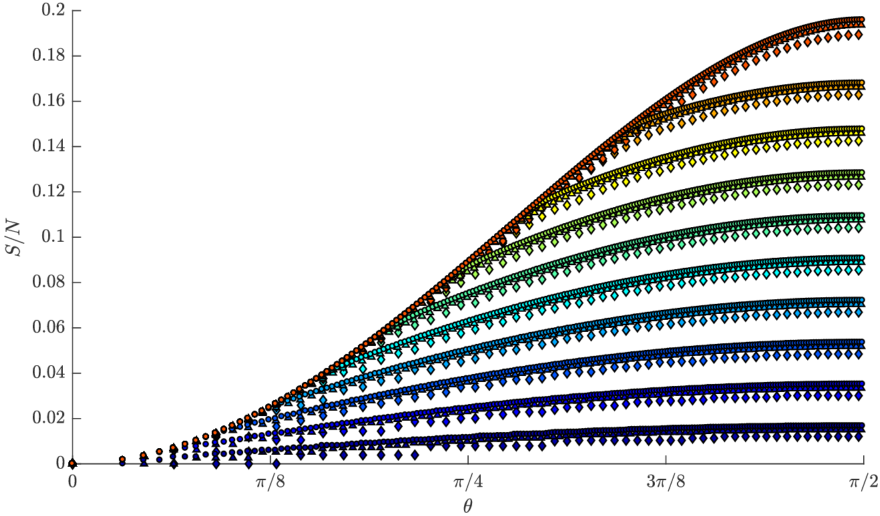
EE as a function of θ

$N = 200$, $n = 20, 40, \ldots, 200$ (blue to red)



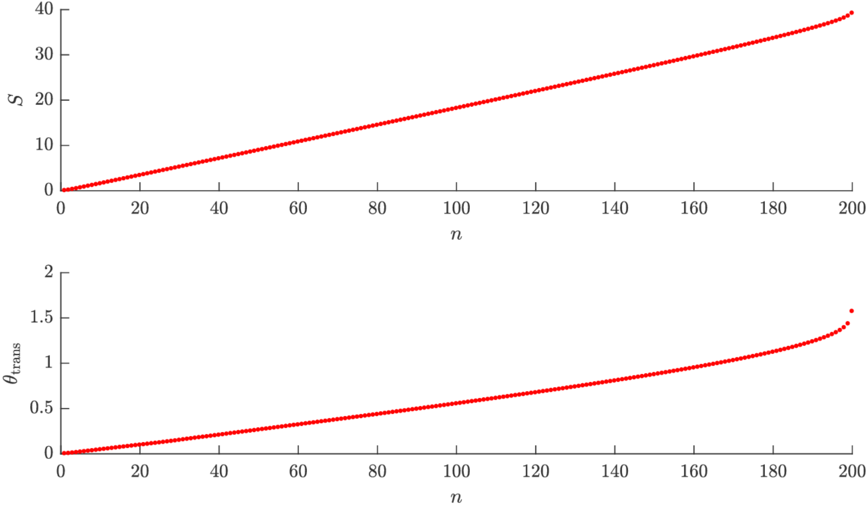
Scaled EE S/N as a function of θ

$N = 50, 100, 200$, $n = 20, 40, \ldots, 200$ (blue to red)



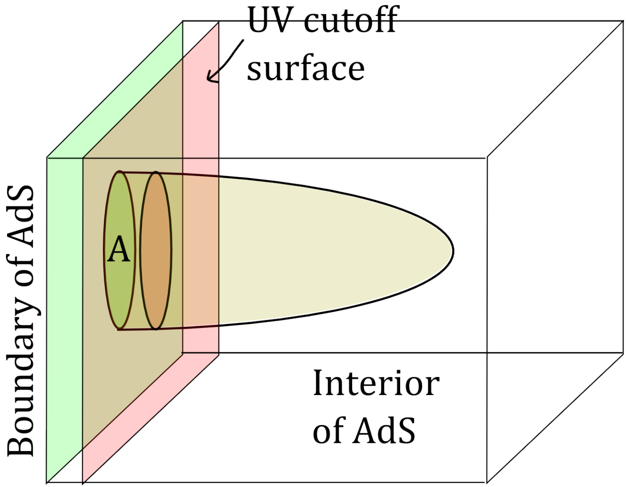
Transition angle as a function of n

Recall that we predicted that $L^n_{\text{noncommutativity}} \sim \frac{n}{N} R$.



Ryu-Takayanagi conjecture

EE of the region A is proportional to the area of the minimal surface in AdS supported by ∂A on the boundary: $S = \frac{A_{\text{minimal}}}{4G_N}$



Holographic dual to NC $\mathcal{N} = 4$ SYM in 3+1 dim

Bulk data:

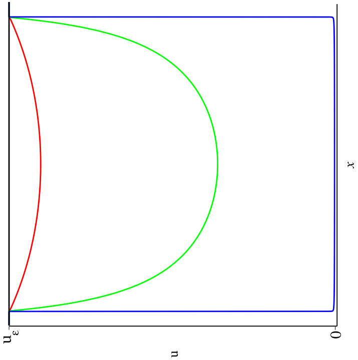
$$\begin{aligned}
\frac{ds^2}{R^2} &= u^2\left(-dt^2 + f(u)\left(dx^2+dy^2\right)+dz^2\right) + \frac{du^2}{u^2} + d\Omega_5^2\ , \\
e^{2\phi} &= g_s^2\ f(u)\ , \\
B_{xy} &= -\frac{1-f(u)}{\Theta} = -\frac{R^2}{\alpha'}\ a_\Theta^2 u^4 f(u)\ , \\
f(u) &= \frac{1}{1+(a_\Theta u)^4}\ ,
\end{aligned} \tag{1}$$

- B_{xy} is the only nonzero component of the NS-NS form background.
- x, y, z have units of length, while u has units of length inverse (energy).
- $a_\Theta = (\lambda)^{1/4}\sqrt{\Theta}$ is $\sqrt{\Theta}$ scaled by a power of the 't Hooft coupling λ. It can be thought of as the length scale of noncommutativity at strong coupling.

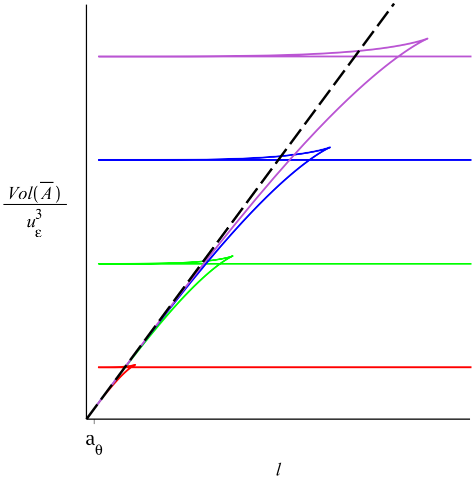
Extremal surfaces

In 3 spacial directions, with x-y being the noncommutative plane, consider a boundary region $A = \{(x, y, z) : 0 < x < l\}$. Distance in open string metric.

For every strip width, there are up to three extremal surfaces:



Holographic EE



Narrow strip:

$$|A_{\text{minimal}}| \sim \frac{W^2 l}{\epsilon^3}$$

Wide strip:

$$|A_{\text{minimal}}| \sim \frac{W^2 a_\Theta^2}{\epsilon^4}$$

Transition at

$$l = a_\Theta^2/\epsilon = \lambda^{1/2}\Theta/\epsilon$$

Clear transition from extensive to area law behaviour.
It would be very interesting to see whether the scale of noncommutativity is modified in a similar way on a fuzzy sphere when interactions are introduced (cf Okuno-Suzuki-Tsuchiya).

NCG as a theory of strings

Earlier, we mentioned that a particle moving on the NC plane is delocalized in the direction orthogonal to its motion.
Let the particle have momentum k in the x-direction, with $[\hat{x}, \hat{y}] = i\Theta$. Its wavefunction is

$$\hat{\phi}_k = \exp(ik\hat{x}) = \exp(-i(\Theta k)\hat{p}_y)$$

with

$$\langle x_2, y_2|\hat{\phi}_k|x_1, y_1\rangle = \exp(k(x_1 + x_2))\exp\left(\frac{(y_1 - y_2 - \Theta k)^2}{4\Theta}\right)$$

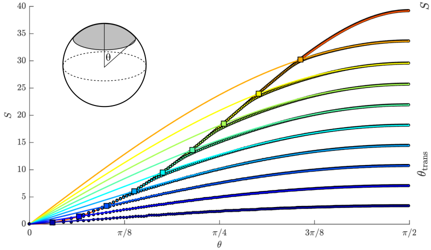
Using a bi-local presentation [Iso, Kawai, Kitazawa]:

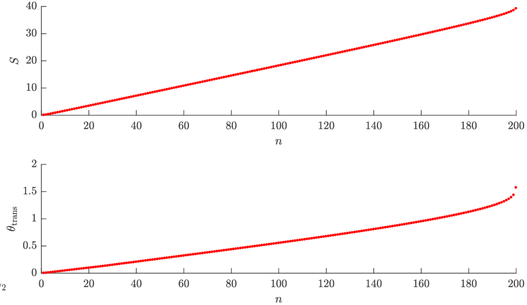
$$\hat{\phi} = \int_{x_1,y_1,x_2,y_2} \phi((x_1, y_1), (x_2, y_2))|x_1, y_1\rangle\langle x_2, y_2|$$

we see that $\hat{\phi}_k$ corresponds to a bi-local object with length Θk: a string of sorts.

Back to fuzzy sphere

Recall:





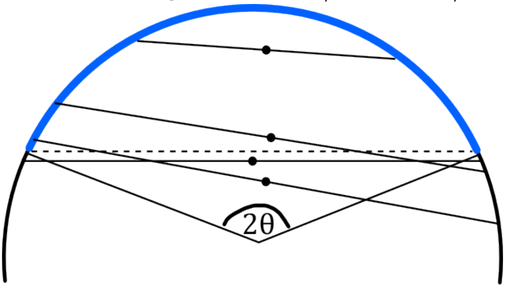
At a fixed region size θ, raising the UV cutoff increases EE only up to a point. Once n is larger than $R\theta$, increasing UV cutoff no longer increases S.

In the string intepretation, longer strings do not contribute to the EE.

String picture

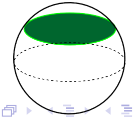
The length of the string is related to its momentum, so for an eigenvector of the Laplacian $\Delta : \phi \to [L_i, [L_i, \phi]]$ with eigenvalue $k(k+1)$, the string would have length $L = \Theta R/k = 2Rk/N$.

When a string is longer than $2R \sin \theta$, it no longer 'fits' in a polar cap region:



Our use of the middle point of the string to determine inside-vs-outside is motivated by EE calculations in SFT.

Intriguingly, when the UV cutoff is removed, EE is almost exactly proportional to the minimal area supported by the cap's boundary:



The string picture, which seems to use the embedding of the sphere into three-dimensional space rather than the intrinsic two-dimensional object might seem strange. It turns out to be crucial to model another phenomenon: evolution of EE after a quench.

EE after a quench

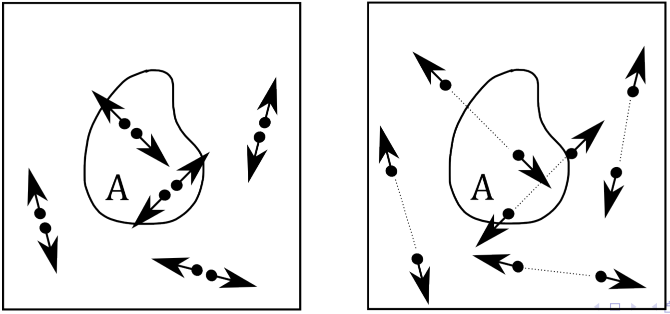
Example: mass quench. The mass parameter μ is changed at $t = 0$.

- Before the quench, $t < 0$, the system is in a vacuum state.
- After the quench, $t > 0$, the system is in an excited (squeezed) state.

In ordinary QFT, this can be modelled using a ballistic model:

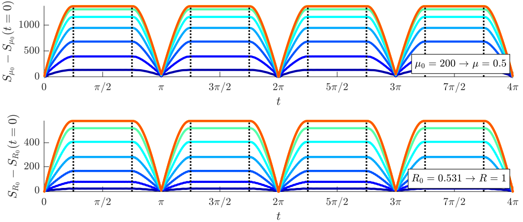
At t=0 entangled particle pairs are created everywhere:

These separate at the speed of light:

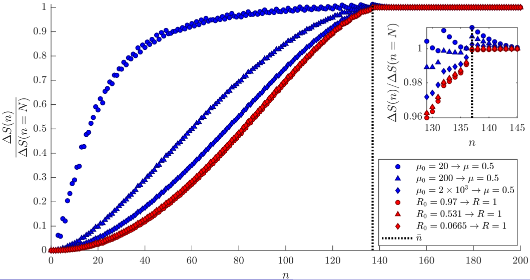


EE after a quench on a fuzzy sphere

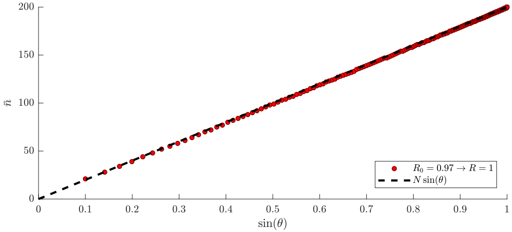
Excess entropy as a function of time. $N = 200$, $\theta = \pi/4$, $n = 20, 40, \ldots, 200$ (blue to orange)



Shows the same behaviour as vacuum entanglement entropy:



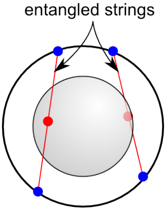
$n_{\max}$ vs $\sin(\theta)$

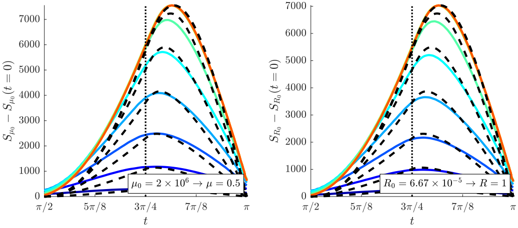


This plot strongly supports the relationship

$$L_{\max} = 2Rn_{\max}/N = 2R\sin\theta$$

A ballistic model applied to string middle points reproduces even more complicated behaviour, such as that after a local quench.





We are still trying to understand the details and the limitations of this model.

Perhaps by thinking of the BFSS matrix model as a String Field Theory, we could identify geometric EE observables. This would open a door to studying the emergent spacetime through the lens of EE.