Fundamental Structures of M-Theory
(Vienna, July 2018)

The talk takes its title from [1] (see also [2-6]) where it was pointed out that the longitudinal coordinate in the orthonormal light-cone treatment of relativistic extended objects, {rather than being content with knowing the dynamics of the transverse degrees of freedom,} is actually a very interesting object – classically necessary to reconstruct the world-volume, constituting ~~these~~ a generalization of the Virasoro algebra, as well as a dynamical symmetry in which the sought internal Mass (-operator) appears as a structure constant in the commutation-relations of the longitudinal Lorentz-generators (just as the Hamiltonian of the Hydrogen-atom appears in the commutator of the

components of the Bernoulli-Laplace-Lenz-Runge vector, allowing the purely algebraic determination of its spectrum).
The following notes, written in February 2017 when visiting IHES (a full account, joint work with J. Arnlind and M. Kontsevich, is hoped to be finished by the end of this year), concern 'Quantized Minimal Surfaces':

In [ACH] the classical Weierstrass representation for minimal surfaces, utilizing the existence of isothermal parameters, was generalized to a non-commutative one, yielding triples of Weyl-algebra elements constituting non-commutative minimal surfaces (NMS).
In [AH] on the other hand a quantization of the Catenoid was constructed as formal power-series in $\hbar$, satisfying

$$\sum_{i=1}^{d} [X_i,[X_i,X_j]] = 0 \qquad j=1,2,3\,(\leq d) \qquad (1)$$

[ACH]: LMP 106 (2016) [AH]: PLB 723 (2013)

Let us now outline a general procedure* for constructing solutions of (1):

Start with an arbitrary minimal surface $\vec{x}(u,v)$ given in isothermal parametrization, $g_{ab}=\partial_a\vec{x}\cdot\partial_b\vec{x}=\sqrt{g}\,\delta_{ab}$; reparametrize as

$$\tilde{\vec{x}}(\tilde{u},\tilde{v}):=\vec{x}(u(\tilde{u},\tilde{v}),v(\tilde{u},\tilde{v})), \quad J:=\left|\frac{\partial\tilde{u}\tilde{v}}{\partial uv}\right| \stackrel{!}{=} \sqrt{g} \qquad (2);$$

~~let~~

$$[\hat{\tilde{u}},\hat{\tilde{v}}]=i\hbar\mathbb{1}, \quad \hat{X}_i := \tilde{x}_i(\hat{\tilde{u}},\hat{\tilde{v}}) \qquad (3)$$

will then satisfy (1), to lowest order in $\hbar$, due to

$$\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\,g^{ab}\partial_b\vec{x}\right)\equiv\{x_i,\{x_i,\vec{x}\}\} \qquad (4)$$

when

$$\{f,h\}:=\frac{1}{\sqrt{g}}\epsilon^{ab}\partial_a f\,\partial_b h \qquad (5);$$

note that (2) implies

$$\sqrt{\tilde{g}}=1, \qquad \{\tilde{u},\tilde{v}\}=1 \qquad (6),$$

i.e. formulates a transformation from isothermal parametrization to one where the determinant of the first fundamental form is constant (=1).

* presented at 'Quantum Spacetime 17' (Porto, February 3)

For Enneper-type surfaces the data entering the Weierstrass representation

$$\vec{x} = \mathrm{Re}\int \vec{\varphi}(z)\,dz \qquad (7)$$

are

$$\vec{\varphi} = \left((1-z^2)z^{N-2},\ i(1+z^2)z^{N-2},\ 2z^{N-1}\right) \qquad (8),$$

$N \in \mathbb{N}$, $N \geq 2$

the simplest case ($N=2$) giving

$$\vec{x}(u,v) = \begin{pmatrix} u - \frac{1}{3}u^3 + uv^2 \\ \frac{1}{3}v^3 - v - vu^2 \\ u^2 - v^2 \end{pmatrix} \qquad (9),$$

$$(g_{ab}) = (1+u^2+v^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while more generally, $S := |z|^2 = u^2+v^2$,

$$\sqrt{g} = \frac{1}{2}\vec{\varphi}\vec{\varphi}^* = S^{N-2}(1+S)^2 \qquad (10).$$

Trying to solve (2) with the Ansatz

$$\tilde{u} = u f(u^2+v^2) \left(= u\sqrt{\tfrac{\tilde{S}}{S}}\right)$$
$$\tilde{v} = v h(u^2+v^2) \left(= v\sqrt{\tfrac{\tilde{S}}{S}}\right) \qquad (11)$$

~~with $f \in \mathbb{R}$ ($\in \mathbb{R}H$) gives~~ gives

$$\tilde{S}' = (SH(S))' = S^{N-2}(1+S)^2 = S^N + 2S^{N-1} + S^{N-2} \qquad (12)$$

when $f = h$ $(= \sqrt{H})$.

Let us now restrict to $n=2$, giving

$$\tilde{S} = \frac{1}{3}S^3 + S^2 + S + C = \frac{1}{3}(1+S)^3 + C - \frac{1}{3} \tag{13}$$

which can be easily inverted to give

$$S = (3\tilde{S} + 1 - 3C)^{1/3} - 1 \tag{14}$$

hence yielding expressions for the inverse transformation

$$U = \tilde{U}\sqrt{\frac{S}{\tilde{S}}} = \tilde{U}\sqrt{\tilde{H}(\tilde{S})}, \quad V = \tilde{V}\sqrt{\frac{S}{\tilde{S}}} = \tilde{V}\sqrt{\tilde{H}(\tilde{S})} \tag{15}$$

$$(\tilde{S}\tilde{H})' = (S+1)^{-2} = (3\tilde{S}+1-3C)^{-2/3}$$

which is needed to obtain the X_i.

Defining $W = x_1 + i x_2$ and $z = u + iv$, resp.

$$W = X_1 + iX_2, \quad \Lambda = U + iV$$

one finds for this (Enneper) case

$$\{W, \bar{W}\} = \frac{2}{i}\left(1 - \frac{2}{(1+z\bar{z})}\right), \quad \{x_3, W\} = \frac{2}{i}\frac{z}{(1+z\bar{z})} \tag{16}$$

satisfying

$$\frac{1}{2}\{\{W,\bar{W}\},W\} = \{\{x_3,W\},x_3\}$$
$$\{\{x_3,W\},\bar{W}\} + \{\{x_3,\bar{W}\},W\} = 0 \tag{17}$$

while (1) resp.

$$\frac{1}{2}[W,[W,W^{\dagger}]] = [X_3,[X_3,W]]$$
$$[[X_3,W],W^{\dagger}] + [[X_3,W^{\dagger}],W] = 0 \tag{18}$$

Should be (at least in leading order) ~~be~~ solved by

$$W = \Lambda^{\dagger} - \frac{1}{3}\Lambda^3 , \qquad \chi_3 = \frac{1}{2}\left(\Lambda^2 + \Lambda^{\dagger 2}\right) \qquad (19)$$

with

$$\Lambda = f(\hat{N})\, T\, f(\hat{N}) \qquad (20)$$

where $T := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ and N is the number operator $\hat{N}|n\rangle = n|n\rangle$ (21);

Due to

$$z = \frac{\tilde{z}}{\sqrt{\tilde{s}}}\sqrt{\tilde{s}H(\tilde{s})} =: f(\tilde{s})\,\frac{\tilde{z}}{\sqrt{\tilde{s}}}\,f(\tilde{s}) \qquad (22)$$

~~with~~

$$f(\tilde{s}) = \left(\left(3\tilde{s}+1-3c\right)^{1/3} - 1\right)^{1/4} \qquad (23)$$

and for the non-commutative case it is convenient to fix the integration constant C to be

$$C \text{ [redacted] } -\hbar \qquad (24)$$

as ~~then~~ when writing

$$\Lambda_{n-1,n} = \lambda_n = f_{n-1}f_n , \qquad (\Lambda^{\dagger})_{n+1,n} = \bar{\lambda}_{n+1} = \bar{f}_n \bar{f}_{n+1} \qquad (25)$$

~~that~~ the 'boundary-condition' ("one-sidedness" of Λ) is automatically taken care of by explicitly having

$$f_{-1} = 0 , \quad \lambda_0 = 0 \qquad (26)$$

Before inserting (20) into (18) let us first look at the (slightly simpler to discuss) non-commutative version of

$$\frac{i}{2}\{z,\bar{z}\} = \frac{1}{(1+z\bar{z})^2}, \tag{27}$$

$$[\Lambda,\Lambda^+] = \frac{2\hbar}{\left(1+\frac{1}{2}(\Lambda\Lambda^+ + \Lambda^+\Lambda)\right)^2}$$

Inserting the Ansatz (25), one obtains the following recursion relations for $\delta_n := |\lambda_n|^2$:

$$(\delta_{n+1}-\delta_n)(2+\delta_{n+1}+\delta_n)^2 = 8\hbar \tag{28}$$,

which can easily be seen to have a unique solution $\delta_0 = 0 < \delta_1 < \delta_2 < \ldots$

On the other hand one can also check by Taylor-expansion that the equations

$$F_n(F_{n+1}-F_{n-1})\left(1+\frac{1}{2}F_n(F_{n+1}+F_{n-1})\right)^2 = 2\hbar \tag{29}$$

that one gets for $F_n := |f_n|^2$ are indeed satisfied (to lowest order in $\hbar$) by (cp (23)(24))

$$F_n^{(0)} = \left[(6(n+1)\hbar+1)^{1/3}-1\right] \tag{30}$$

which, as $n \to \infty$, grow like $(6nt_1)^{1/6}$ (consistent with (27)). For the higher Enneper-cases one would similarly get

$$[u,v] = \frac{i t_1}{(u^2+v^2)^{N-2}(1+u^2+v^2)^2} \qquad (31),$$

with $|\lambda_n|$ resp. F_n growing like $\left(2(N+1)nt_1\right)^{\frac{1}{2(N+1)}}$.

Before discussing (18), note that for all N

$$W = \frac{1}{N-1}(\Lambda^{\dagger})^{N-1} - \frac{1}{N+1}\Lambda^{N+1}, \quad X_3 = \frac{1}{N}\left(\Lambda^N + \Lambda^{\dagger N}\right) \qquad (32),$$

are such that for Λ of the form (20), $[W, W^{\dagger}]$ will be diagonal, and $[X_3, W]$ non-zero only on the first upper off-diagonal. Inserting (32) into (18) gives, even for the simplest case ($N=2$) relatively lengthy equations:

$$\frac{1}{2}\left[\Lambda^{\dagger} - \frac{1}{3}\Lambda^3, \frac{1}{9}[\Lambda^3, \Lambda^{\dagger 3}] - [\Lambda, \Lambda^{\dagger}]\right]$$

$$\overset{!}{=} \frac{1}{4}\left[\Lambda^2 + \Lambda^{\dagger 2}, [\Lambda^2, \Lambda^{\dagger}]\right] + \frac{1}{12}\left[\Lambda^2 + \Lambda^{\dagger 2}, [\Lambda^3, \Lambda^{\dagger 2}]\right]$$

$$\left[\left[\Lambda^2 + \Lambda^{\dagger 2}, \Lambda^{\dagger} - \frac{1}{3}\Lambda^3\right], \Lambda - \frac{1}{3}\Lambda^{\dagger 3}\right] + h.c. = 0$$

Simpler examples are

1) the Catenoid, $\vec{X}(u,v) = \begin{pmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{pmatrix}$ (33)

$$(g_{ab}) = \cosh^2 v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{g}\, \mathbb{1}$$

Due to the $U(1)$-symmetry, (2) can be solved by only reparametrizing v, i.e. $\tilde{u} = u$,

$$\tilde{v} = \frac{v}{2} + \frac{1}{4} \sinh(2v) + \text{const}$$

$$\frac{d\tilde{v}}{dv} = \cosh^2 v \qquad (34)$$

So that, up to ordering questions (cp. (8)), ~~and~~ representing $\tilde{u}$ by $i\hbar \frac{\partial}{\partial x}$, $\tilde{v}$ by x,

$$X_1 + iX_2 = W = w(x)\, e^{-\hbar \partial_x}, \quad X_3 = v(x) \qquad (35),$$

with w and v being deformations of $\cosh v_0(x)$ and $v_0(x)$ (satisfying $v_0' = \frac{1}{\cosh^2 v_0}$).

Inserting (35) into (18) one obtains finite-difference-equations for $v(x)$ and $w(x)$.

and, even simpler

Example 2: (Helicoid): $\vec{X}(u,v) = \begin{pmatrix} \sinh v \cos u \\ \sinh v \sin u \\ u \end{pmatrix}$

$$(g_{ab}) = (\partial_a \vec{X} \cdot \partial_b \vec{X}) = \cosh^2 v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{g}\,\mathbb{1}$$

$$\begin{pmatrix} \sinh v \cos u \\ \sinh v \sin u \\ u \end{pmatrix} = \begin{pmatrix} f(\tilde{v}) \cos \tilde{u} \\ f(\tilde{v}) \sin \tilde{u} \\ \tilde{u} \end{pmatrix}$$

reparametrize: $\vec{X}(u,v) = \underset{\sim}{X}(\tilde{u},\tilde{v})$

where $\left| \frac{\partial(\tilde{u}\tilde{v})}{\partial(uv)} \right| = \sqrt{g} = \cosh^2 v$;

e.g. $\tilde{u} = u$, $\tilde{v} = \frac{1}{2}\left(v + \frac{\sinh 2v}{2}\right)$ $\quad \frac{d\tilde{v}}{dv} = \cosh^2 v$

Solving $[[X_i, X_j], X_j] \overset{*}{=} 0$ via

$$X_i = \underset{\sim}{x}_i(\hat{U}, \hat{V})$$

$$[\hat{U}, \hat{V}] = i\hbar \mathbb{1}$$

$$\hat{U} = i\hbar \frac{\partial}{\partial x}, \quad \hat{V} = x$$

$$X_1 + iX_2 = F(\hat{V}=x)\, e^{i\hat{U}} = F(x)\, e^{-\hbar \frac{\partial}{\partial x}} =: W$$

$$X_3 = i\hbar \frac{\partial}{\partial x} = Z$$

$$* \Rightarrow 2F^2(x) - F^2(x+\hbar) - F^2(x-\hbar) = +\hbar^2 \frac{F''}{F}(x) \cdot 2$$

(as/from $-[[W,Z],Z] = \hbar^2 F'' e^{-\hbar\partial_x}$

$$\overset{!}{=}$$

$$\frac{1}{2}[[W,W^\dagger],W] = \frac{1}{2}\left(2WW^\dagger W - W^\dagger W^2 - W^2 W^\dagger\right)$$

$$= \frac{1}{2}\left(2F^2 - F_+^2 - F_-^2\right) F e^{-\hbar\partial_x}$$

One can easily check that

$$[[X_3, W], W^+] + [[X_3, W^+], W] = 0$$

is automatically satisfied in this (Helicoid) case, and that for

$$F(x) = \sinh v(x) =: S(x), \quad C(x) := \cosh v(x)$$

$$F(2F^2 - F_+^2 - F_-^2)$$
$$= -2\hbar^2 S\left[v'^2(C^2+S^2) + v'' SC + O(\hbar)\right]$$

while

$$2\hbar^2 F'' = 2\hbar^2\left[v'' C + v'^2 S\right],$$

and that the 2 expressions coincide to leading order when using

$$v(x) = v_0(x) + O(\hbar)$$

$$v_0' = \frac{1}{\cosh^2 v_0} = \frac{1}{C_0^2}, \quad v_0'' = \frac{-2S_0}{C_0^5}$$

3 further remarks concern

1) the fact that the $L_{\alpha=0,1,2}$ inherent in the symmetric Lax-pair formulation of non-periodic Toda-dynamics

$$\left(\dot{L}=[L, L_+-L_-],\ L=L_0+\underbrace{L_++L_-}_{=:L_1},\ L_2=\frac{1}{i}(L_+-L_-)\right)$$

satisfy $\ddot{L}_\alpha=[[L_\alpha, L_\beta], L_\gamma]\,\eta^{\beta\gamma}$ ← diag(1,−1,−1)

and that

2) in certain dynamical systems involving general matrices not only the eigenvalues (of hermitean $N\times N$ matrices X) are relevant, but also 'generalized eigenvalues', obtained by setting to zero determinants of submatrices of $(X-\lambda\mathbb{1})$.

3) in [J. Arnlind, J. Hoppe: Discrete Minimal Surface Algebras Sigma 6, 042, 2010]

the equations $[[X_i, X_j], X_j]=\mu_i X_i$

are discussed (and solutions constructed)

As a simple example of 2) one may take

$$X = \begin{pmatrix} \rho_1 & a_1 & c \\ a_1 & \rho_2 & a_2 \\ c & a_2 & \rho_3 \end{pmatrix}, \quad \dot X = \left[X, \begin{pmatrix} 0 & a_1 & c \\ -a_1 & 0 & a_2 \\ -c & -a_2 & 0 \end{pmatrix}\right] \qquad (36)$$

$$\dot\rho_1 = -2(a_1^2 + c^2), \quad \dot\rho_2 = 2(a_1^2 - a_2^2), \quad \dot\rho_3 = 2(c^2 + a_2^2)$$

$$\dot a_1 = a_1(\rho_1 - \rho_2) - 2a_2 c, \quad \dot a_2 = a_2(\rho_2 - \rho_3) + 2a_1 c, \quad \dot c = c(\rho_1 - \rho_3)$$

(note that this is <u>not</u> a/the periodic Toda system).

In this case not only the eigenvalues of X are conserved ((36) trivially implies $X(t) = R(t)\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} R^T(t)$) but also the 'generalized eigenvalue'

$$\lambda_4 := \rho_2 - \frac{a_1 a_2}{c},$$

obtained by putting to zero the determinant of the lower left 2×2 matrix of

$$X - \lambda\mathbb{1} = \begin{pmatrix} \rho_1 - \lambda & a_1 & c \\ a_1 & \rho_2 - \lambda & a_2 \\ c & a_2 & \rho_3 - \lambda \end{pmatrix}$$

It is known [7] that (36) describes an integrable Hamiltonian system on a coadjoint orbit of the group $G_\leq$ of (lower) triangular (unimodular) matrices

(, $\mathcal{L} = \mathcal{L}_{\leq} + \mathcal{A}$, $\mathcal{L}^* = \mathcal{S} + \mathcal{L}_{<}$, $X \in \mathcal{S}$)

the $C \neq 0$ orbits being characterized by λ_4 (and $\lambda_1 + \lambda_2 + \lambda_3 = \rho_1 + \rho_2 + \rho_3$).

$(\rho_3, \rho_1\rho_2 - a_1^2)$ and $(\rho_1, \rho_2\rho_3 - a_2^2)$ constitute closed subsystems and the functions

$$\lambda_i^2 - (\rho_1 + \rho_2)\lambda_i + (\rho_1\rho_2 - a_1^2) \text{ resp. } \lambda_i^2 - (\rho_2 + \rho_3)\lambda_i + (\rho_2\rho_3 - a_2^2),$$

which are the determinants of the upper left resp. lower right 2×2 part of $(X - \lambda_i \mathbb{1})_{i=1,2,3}$ have simple time-dependence, while the easiest way to explicitly solve (36) is to notice that the first and last column of $R^T = (\vec{u}, \vec{v}, \vec{w})$ satisfy

$$\dot{u}_i = (\rho_1 - \lambda_i) u_i \quad \left(\rho_1 = \sum_1^3 \lambda_k u_k^2\right)$$

$$\dot{w}_i = (\lambda_i - \rho_3) w_i \quad \left(\rho_3 = \sum \lambda_k w_k^2\right),$$

hence (using $\vec{u}^2(t) = 1 = \vec{w}^2(t)$)

$$u_i(t) = \frac{e^{-\lambda_i t} u_i(0)}{\sqrt{\sum e^{-2\lambda_k t} u_k^2(0)}}, \quad w_i(t) = \frac{e^{+\lambda_i t} w_i(0)}{\sqrt{\sum e^{+2\lambda_k t} w_k^2(0)}}$$

$\left(\vec{w} \cdot \vec{u} \overset{!}{=} 0 \, , \quad \vec{v} := \vec{w} \times \vec{u}\right)$

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[5] J. de Woul, J. Hoppe, D. Lundholm, M. Sundin A dynamical symmetry for supermembranes, JHEP 03 (2011) 134

[6] J. Hoppe, Relativistic Membranes, J. of Phys. A 46, 2013

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