Fuzzy geometry via noncommutative frames: fuzzy de Sitter space

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Outline

Physical motivation to study noncommutative spaces is the idea that spacetime at small scales, or ‘quantum spacetime’, has some kind of geometric structure which is different from that of a manifold: perhaps discrete. Quantum geometry might be effective or emergent, or fundamental.

A discrete structure physicists are very familiar with is that of an algebra, e.g. the algebra of operators in quantum mechanics or a Lie algebra. We shall assume that noncommutative space is described by an algebra of operators.

To describe classical fields on a noncommutative space and their equations of motion we need to define derivatives, i.e. smoothness. Ideally, one aims to introduce noncommutative differential geometry and further, identify geometry with gravity.

There are several different approaches to this: we will work within the noncommutative frame formalism of Madore (CUP 1995).

A paradigmatic example of a noncommutative space, for its various properties, is the fuzzy sphere.

To construct it one uses its symmetry: we generalize that construction. The motivation is to obtain 4-dimensional noncommutative spacetimes with spherical symmetry and thus find examples of realistic noncommutative configurations of the gravitational field (cosmology, black holes). As a first step, we define fuzzy spaces of maximal symmetry.

We will first give an oveview of basic elements of the noncommutative frame formalism, and then show how can it be used to describe four-dimensional noncommutative de Sitter space, its geometry and its metric structure, using the de Sitter group SO(1,4) and its unitary irreducible representations.

The talk is based on the work with J. Madore and D. Latas, arXiv: 1508.06058, 1709.05158 & in progress.

Noncommutative space is an algebra $\mathcal{A}$ generated by a set of hermitian operators, coordinates x^μ,

$$[x^\mu, x^\nu] = i\kbar J^{\mu\nu}(x).$$

We either have an abstract position algebra or its concrete representation.

The 'structure' of this space can be described by the spectra of its coordinates. There are however other ways to understand/define a noncommutative space: its symmetries, the set of coherent states, and very importantly, the commutative limit.

Diffeomorphisms on a noncommutative space are functions on the algebra. Obviously, changes of coordinates change their spectrum: in the following we will identify the terms 'fuzzy' and 'noncommutative', not presuming either discrete spectra or finite-dimensional representations.

Differential structure of $\mathcal{A}$ is given by the momentum algebra.
Momenta p_α define a set of vector fields e_α, the free falling frame

$$e_\alpha f = [p_\alpha, f],$$

as the commutator satisfies the Leibniz rule.

Often in noncommutative geometry one identifies the set of momenta with the set of coordinates. But if we want to include in description, as a particular case, the usual geometry, we do not do so.

On commutative manifold, the moving frame is $e_\alpha f = e_\alpha^\mu (\partial_\mu f)$, that is

$$p_\alpha = e_\alpha^\mu \, \partial_\mu, \quad e_\alpha^\mu = [p_\alpha, x^\mu] \, .$$

Momenta are outside the coordinate algebra $\mathcal{A}$ (e_α are outer derivatives).
The space of vector fields has dimension equal to dimension of spacetime.

In the noncommutative case the set of vector fields is not linear over $\mathcal{A}$.
The set of dual 1-forms θ^α, which defines differential d, has linear structure

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta, \quad df = (e_\alpha f)\theta^\alpha.$$

In order to introduce orthonormality of the local frame, $g^{\alpha\beta}{=}\eta^{\alpha\beta}{=}\mathrm{const}$, impose

$$[f, \theta^\alpha] = 0.$$

Metric is then extended by linearity, $g^{\mu\nu} = g(dx^\mu, dx^\nu) = e^\mu_\alpha\, e^\nu_\beta\, \eta^{\alpha\beta}$.

Condition $[f, \theta^\alpha] = 0$ is not invariant under linear transformations:
the formalism breaks the local Lorentz invariance and, as it is generally the case in noncommutative geometry, admits more differential calculi.

Conditions like $d^2 = 0$, and in particular, consistency of geometric and algebraic structures give a number of consistency constraints: one of them is the relation $2P^{\alpha\beta}{}_{\gamma\delta}p_\alpha p_\beta - F^\beta{}_{\gamma\delta}p_\beta - K_{\beta\gamma} = 0.$

Laplacian of a scalar function is defined naturally,

$$\Delta f = \eta^{\alpha\beta}[p_\alpha, [p_\beta, f]].$$

It is possible, and rather straightforward, to define exterior multiplication (with $\theta^\alpha\theta^\beta \neq \theta^\beta\theta^\alpha$ in general), and differential-geometric quantities like connection, covariant derivative and curvature, by formulae analogous to those given in Cartan's description of geometry.

Therefore one can describe scalar, spinor and gauge fields (as matrix models), as well as curved gravitational backgrounds (as ground states); and perhaps more.

In short: coordinates with their relations define position space $\mathcal{A}$ and its algebraic properties, momenta and their relations define differential and geometric properties.

A new element in the noncommutative setup is that (some or all) momenta can belong to the initial algebra, $p_\alpha \in \mathcal{A}$: therefore the relation (# coordinates)=(# momenta)=spacetime dimension is not automatic. That is, structures different from the symplectic structure of classical mechanics are possible (and perhaps meaningful?)

On matrix spaces, derivations are always inner.

By a specific choice of p_α we define the tangent space and thereby the spacetime dimension.

One can extend the construction of the fuzzy sphere to other homogeneous spaces: concretely, we will analyze 4-dimensional fuzzy de Sitter space.

In the commutative case, de Sitter space can be defined as imbedding

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda}$$

in the flat 5-dimensional space

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2.$$

It has maximal symmetry. We (and originally Gazeau, Mourad & Queva 2006, Jurman & Steinacker 2014) define fuzzy de Sitter space using the algebra of its symmetry group SO(1,4).

The algebra of the SO(1,4) group has 10 generators $M_{\alpha\beta}$ and also describes conformal symmetry in 3d. Commutation relations are

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}).$$

$\alpha, \beta, ... = 0, 1, 2, 3, 4$; we use signature $\eta_{\alpha\beta} = \mathrm{diag}(+----)$.

It has two Casimir operators

$$\mathcal{Q} = -\frac{1}{2}\,M_{\alpha\beta}M^{\alpha\beta}, \qquad \mathcal{W} = -W_\alpha W^\alpha.$$

W_α are quadratic in algebra generators

$$W_\alpha = \tfrac{1}{8}\,\epsilon_{\alpha\beta\gamma\delta\eta}M^{\beta\gamma}M^{\delta\eta}.$$

In the Inönü-Wigner contraction limit, W_α reduce to components of the Pauli-Lubanski vector of the Poincaré group.

The second Casimir relation $\mathcal{W}$=const is analogous to the imbedding of 4d commutative de Sitter space in 5 flat directions. We thus introduce coordinates as $x^\alpha = \ell\, W^\alpha$ and define fuzzy de Sitter space to be a unitary irreducible representation (UIR) of the de Sitter algebra.

The quartic Casimir gives the value of the cosmological constant, $\eta_{\alpha\beta}\, x^\alpha x^\beta = 3/\Lambda$: we take $\ell^2 = \bar{k}^2\Lambda/3$ as a quantization condition.

Coordinates x^α are quadratic in the group generators and do not close into a Lie or quadratic algebra under commutation,

$$[W^\alpha, W^\beta] = -\frac{i}{2}\, \epsilon^{\alpha\beta\gamma\delta\eta}\, W_\gamma M_{\delta\eta}\, .$$

However, one can show that in the UIR's they generate the whole algebra:

$$i\, \mathcal{W} M^{\rho\sigma} = [W^\rho, W^\sigma] + \frac{1}{2}\, \epsilon^{\alpha\mu\rho\sigma\tau} W_\tau [W_\alpha, W_\mu].$$

One can work out the flat noncommutative limit of fuzzy de Sitter space, by considering a small neighbourhood of the north pole

$$x^4 \approx \sqrt{\frac{3}{\Lambda}}, \qquad x^0,\, x^i \approx 0$$

for $\Lambda \to 0$, that is in the Inönü-Wigner contraction limit.

The result is the dual of the Snyder algebra,

$$[x^i, x^j] \sim ia^2 \epsilon^{ijk} M_{0k}, \quad [x^0, x^i] \sim ia^2 \epsilon^{ijk} M_{jk}, \quad a^2 = \bar{k}/(2\mu^2).$$

On the other hand, there are two choices of momenta that give geometries (i.e. metric, curvature) of the de Sitter space in the commutative limit.

We discuss here the one defined by four momenta:

$$ip_0 = \sqrt{\Lambda}\, M_{04}\,, \qquad ip_i = \sqrt{\Lambda}\,(M_{i4} + M_{0i}).$$

From the given definitions we find

$$dx^0 = x^4\theta^0 + x^j\theta^j, \quad dx^i = (x^0 + x^4)\theta^i, \quad dx^4 = x^0\theta^0 - x^j\theta^j$$

so the corresponding line element is

$$ds^2 = -(\theta^0)^2 + (\theta^i)^2 = -d\tau^2 + e^{2\tau}\, dx^i dx^i\,.$$

We can identify the cosmological time as $\tau/\ell = -\log(x^0 + x^4)/\ell\,.$

In the conformal group notation, $\pi_i = M_{i4} + M_{0i}$ are translations, M_{04} is dilatation and $M_{i4} - M_{0i}$ are special conformal transformations. Rotations are denoted by $L_i = 1/2\,\epsilon_{ijk} M_{jk}$.

Unitary irreducible representations of de Sitter group are known, found by Thomas 1941, Newton 1950, Dixmier 1961 by induction from (k, k') representations of the SO(4) subgroup.

They are denoted by two quantum numbers (s, ρ or ν, q) and fall into following categories:

- Principal continuous series: $\rho \geq 0$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

  $\mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} + \rho^2)$
- Complementary continuous series: $\nu \in R, \ \ |\nu| < \frac{3}{2}, \ \ s = 0, 1, 2 \dots$

  $\mathcal{Q} = -s(s+1) + \frac{9}{4} - \nu^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} - \nu^2)$
- Discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}, 2 \dots, \ \ q = s, s-1, \dots 0 \text{ or } \frac{1}{2}$

  $\mathcal{Q} = -s(s+1) - (q+1)(q-2), \quad \mathcal{W} = s(s+1)q(q-1)$

We would like next to determine the spectra of coordinates in some of these UIR's. However, matrix elements of $M_{\alpha\beta}$ do not help much to diagonalize W^α, as matrices are ifinite-dimensional and off-diagonal.

A better possibility would be to use Hilbert space representations. They exist for Class I UIR's (principal and complementary series) and are used to construct coherent states (Perelomov 1986); however, they all have an unwanted property $W^0 = 0$, which implies $W^\alpha = 0$.

There is also Moylan Hilbert space representation (1982) of all UIR's of the principal continuous series, which uses representation spaces of the UIR's of positive mass of the Poincaré group. We will solve the eigenvalue problems of coordinates for $(\rho, s = \frac{1}{2})$ UIR's.

Another, more general approach to the spectrum problem is to use group and representation theory and the (existing) branching rules for SO(1,4): we will also show these results.

The representation space for $(\rho, s = \frac{1}{2})$ principal continuous UIR is the Hilbert space of Dirac bispinors $\psi(\vec{p})$ which satisfy the Dirac equation. (The corresponding representation of the Poincaré group is the Bargmann-Wigner representation.) The scalar product is given by

$$(\psi, \psi) = \int \frac{d^3p}{2p_0}\, \psi^\dagger \gamma^0 \psi.$$

If we work with the Dirac representation of γ-matrices, we have

$$\psi(\vec{p}) = \begin{pmatrix} \varphi(\vec{p}) \\ -\dfrac{\vec{p}\cdot\vec{\sigma}}{p_0 + m}\,\varphi(\vec{p}) \end{pmatrix}, \qquad (\psi, \psi') = \int \frac{d^3p}{p_0}\,\frac{2m}{p_0 + m}\,\varphi^\dagger \varphi' .$$

This representation has very nice and subtle properties (which we learned gradually). For example, the eigenvalue problem

$$M\psi = \lambda\psi$$

for an operator of the form $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is equivalent to

$$\frac{p_0 + m}{2m}\left(A - \frac{p_k\sigma^k}{p_0 + m}\, A\, \frac{p_i\sigma^i}{p_0 + m} + [B, \frac{p_k\sigma^k}{p_0 + m}] \right) \varphi = \lambda\varphi$$

$(p_0 = \sqrt{-p_i p^i + m^2}\,)$ for the spinor $\varphi(\vec{p})$.

Because of the scalar product, hermiticity is often nontrivial i.e. not obvious.

The group SO(1,4) group generators in the $(\rho, s = \frac{1}{2})$ UIR are given by

$$M_{ij} = L_{ij} + S_{ij}, \qquad S_{ij} = \frac{i}{4}\,[\gamma_i, \gamma_j]$$

$$M_{0i} = L_{0i} + S_{0i}, \qquad S_{0i} = \frac{i}{4}\,[\gamma_0, \gamma_i]$$

$$M_{40} = -\frac{\rho}{m}\,p_0 + \frac{1}{2m}\,\{p^i, M_{0i}\}$$

$$M_{4k} = -\frac{\rho}{m}\,p_k - \frac{1}{2m}\,\{p^0, M_{0k}\} - \frac{1}{2m}\,\{p^i, M_{ik}\}$$

with

$$
\begin{aligned}
L_{ij} &= i\left(p_i\,\frac{\partial}{\partial p^j} - p_j\,\frac{\partial}{\partial p^i}\right)\\
L_{0i} &= ip_0\,\frac{\partial}{\partial p^i}\\
L_{40} &= -\frac{\rho}{m}\,p_0 + \frac{1}{2m}\,\{p^i, L_{0i}\}\\
L_{4k} &= -\frac{\rho}{m}\,p_k - \frac{1}{2m}\,\{p^0, L_{0k}\} - \frac{1}{2m}\,\{p^i, L_{ik}\}
\end{aligned}
$$

$L_{\alpha\beta}$ are the group generators in the $(\rho, s = 0)$ UIR.

One can easily check that $M_{\alpha\beta}$, $S_{\alpha\beta}$, $L_{\alpha\beta}$ are hermitian with respect to the Bargmann-Wigner scalar product.

We obtain

$$W^0 = -\frac{1}{2m}\begin{pmatrix} (\rho - \frac{i}{2})p_i\sigma^i + i\, p_0^2 \frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk} p_0 p_i \frac{\partial}{\partial p^j}\sigma_k + \frac{3i}{2}p_0 \\ \epsilon^{ijk} p_0 p_i \frac{\partial}{\partial p^j}\sigma_k + \frac{3i}{2}p_0 & (\rho - \frac{i}{2})p_i\sigma^i + ip_0^2 \frac{\partial}{\partial p^i}\sigma^i \end{pmatrix}$$

$$W^4 = -\frac{1}{2}\begin{pmatrix} ip_0 \frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk} p_i \frac{\partial}{\partial p^j}\sigma_k + \frac{3i}{2} \\ \epsilon^{ijk} p_i \frac{\partial}{\partial p^j}\sigma_k + \frac{3i}{2} & ip_0 \frac{\partial}{\partial p^i}\sigma^i \end{pmatrix}$$

To find the eigenvalues of W^0 we do not have to solve differential equations: from the matrix elements of Thomas-Newton we easily find that the spectrum of W^0 is discrete and that the eigenvalues are equal to $k(k+1) - k'(k'+1)$.

Because of symmetry, the spectra of W^4 and W^i are the same.

W^4 and $W^0 + W^4$ commute with the angular momentum L_i so we can choose their eigenfunctions in the form

$$\varphi(\vec{p}) = \frac{f(p)}{p}\,\varphi_{jm} + \frac{h(p)}{p}\,\chi_{jm}$$

where

$$\varphi_{jm} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\; Y_{j-1/2}^{m-1/2} \\ \sqrt{\frac{j-m}{2j}}\; Y_{j-1/2}^{m+1/2} \end{pmatrix}, \quad \chi_{jm} = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\; Y_{j+1/2}^{m-1/2} \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\; Y_{j+1/2}^{m+1/2} \end{pmatrix}$$

and Y_l^m are spherical harmonics in momentum space, $p = |\vec{p}|$, etc.

Introducing variable $x = p_0/m$ and replacing $f = (x^2-1)^{1/4}\,\tilde{F}$, $h = (x^2-1)^{1/4}\,\tilde{H}$ for W^4 we obtain a set of Legendre equations

$$(x^2-1)\,\tilde{F}'' + 2x\,\tilde{F}' - \frac{j^2}{x^2-1}\,\tilde{F} = 2i\lambda(2i\lambda-1)\tilde{F}\,,$$

$$(x^2-1)\,\tilde{H}'' + 2x\,\tilde{H}' - \frac{(j+1)^2}{x^2-1}\,\tilde{H} = 2i\lambda(2i\lambda-1)\tilde{H}\,.$$

Regular solutions to these equations exist for every real λ and they give

$$f_{\lambda j} = A\left(\frac{p}{m}\right)^{1/2} P^{-j}_{-2i\lambda}\left(\frac{p_0}{m}\right), \quad h_{\lambda j} = A\,(2i\lambda - j - 1)\left(\frac{p}{m}\right)^{1/2} P^{-j-1}_{-2i\lambda}\left(\frac{p_0}{m}\right)$$

The corresponding eigenfunctions of W^4 are orthogonal and normalized to δ-function. The spectrum of W^4 is continuous, the real line.

We do similarly for $W^0 + W^4$. Introducing variable $z = \sqrt{\frac{(p_0 - m)}{(p_0 + m)}}$ and

$$f = (\frac{p_0}{m} + 1)^{-i\rho - (2j+1)/4} (\frac{p_0}{m} - 1)^{(2j+1)/4}\, \tilde{F}$$
$$h = (\frac{p_0}{m} + 1)^{-i\rho + (2j+1)/4} (\frac{p_0}{m} - 1)^{-(2j+1)/4}\, \tilde{H}$$

and we obtain Bessel equations

$$z^2\, \tilde{F}'' + z\, \tilde{F}' + \left(4\lambda^2 z^2 - j^2\right) \tilde{F} = 0$$
$$z^2\, \tilde{H}'' + z\, \tilde{H}' + \left(4\lambda^2 z^2 - (-j-1)^2\right) \tilde{H} = 0\,.$$

The corresponding regular solutions are

$$\varphi_{\lambda jm} = \left(\frac{p_0}{m} + 1\right)^{-i\rho - 1} \sqrt{\frac{\pi}{4\lambda z}} \left(C\, J_j(2\lambda z)\, \varphi_{jm} + D\, J_{j+1}(2\lambda z)\, \chi_{jm}\right).$$

As solutions for λ and $-\lambda$ are proportional, $\lambda \in (0, \infty)$. The spectrum of cosmological time τ is continuous, the real line.

A more elegant way to obtain these spectra is given by group theory.

From definition $W_\alpha = 1/8\,\epsilon_{\alpha\beta\gamma\delta\eta} M^{\beta\gamma} M^{\delta\eta}$, we recognize that W^0 is one of two Casimir operators of the SO(4) subgroup of SO(1,4), while W^4 is a Casimir operator of the SO(1,3) subgroup of SO(1,4). Similarly, $W^0 + W^4$ is one of the Casimir operators of the E(3) subgroup of SO(1,4).

Therefore we can obtain the eigenvalues of W^0 (i.e. W^4, τ) when we reduce the representation of SO(4) (resp. SO(1,3), E(3)) given by our principal continuous UIR of SO(1,4), to the irreducible ones.

The rules of reduction are the branching rules. They are known for reduction from SO(1,4) to SO(1,3) representations, are straightforward for reduction from SO(1,4) to SO(4), and I could not find any for reduction from SO(1,4) to E(3).

Formula for SO(1,4) $\rightarrow$ SO(1,3) was obtained by Ström 1968:

$$\mathcal{H}^{s\pm} = (2\pi^4)^{-2} \int_{-\infty}^{\infty} \sum_{s_0=s,s-1,\dots} \mathcal{H}^{s\pm}(s_0,\nu)\,(s_0^2+\nu^2)\,d\nu$$

where s_0 and ν label the UIR's of the Lorentz group.

It means that the representation space $\mathcal{H}^{s\pm}$ of the (ρ, s) UIR of SO(1,4) is decomposed into a direct integral and a direct sum of the (ν, s_0) UIR's of SO(1,3): $\nu \in (-\infty, +\infty)$ is continuous and s_0, $|s_0| \leq s$, is discrete.

The eigenvalue of W^0 which corresponds to each of representations in the decomposition is $s_0\nu \in (-\infty, \infty)$.

Continuity of the spectrum of $W^0 + W^4$ also follows from the group theory. Namely, relations in the algebra give

$$e^{-i\alpha M_{04}}(W^0 + W^4)e^{i\alpha M_{04}} = e^{\alpha}(W^0 + W^4)$$

that is, dilatation maps between the eigenstates $|\lambda\rangle$ of $W^0 + W^4$,

$$e^{i\alpha M_{04}}|\lambda\rangle = |e^{\alpha}\lambda\rangle$$

for each real α: therefore the spectrum of $W^0 + W^4$ is continuous.

It remains to show that a nonzero λ exists, and and that the eigenvalues are positive: we showed it in a concrete representation.

To summarize:

- We discussed properties of fuzzy de Sitter space, both geometric (spectra of coordinates) and differential-geometric (metric, curvature), in the principal continuous series UIR's.
- It is perhaps possible to analyse other representations, e.g. discrete series.
- The construction can be directly generalized to homogeneous spaces in higher dimensions: in part, because properties of the principal continuous series of all SO(1,n) are similar.
- Perhaps we do not need to introduce (so many) new mathematical structures to describe spacetime at small scales?