Covariant quantum spaces and higher-spin theory in the IKKT matrix model

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cost Action MP 1405
Quantum Structure of Spacetime

Motivation

requirements for fundamental theory (spacetime & matter):

- simple, constructive
- finite dof (per "volume"), pre-geometric (?)
- gauge theory

string theory: remarkable features

issues:

- 10D $\rightarrow$ compactification (why? 4D? landscape?)
- definition ?

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Matrix Models as fundamental theories of space-time & matter ?

- string theory (IKKT model (this talk!), BFSS model)
- NC gauge theory
- dim. red. of YM

matrix model describe dynamical NC spaces

prototype: fuzzy sphere

consider action

$$S = Tr\Big([Y_a, Y_b][Y^a, Y^b] - 2i\epsilon_{abc}Y^aY^bY^c - Y_aY^a\Big), \qquad Y^a \in Mat_N(\mathbb{C})$$

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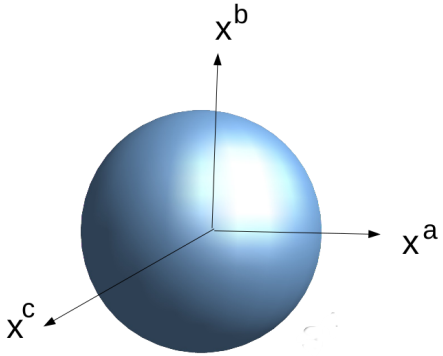
minimum:

$$[Y_a, Y_b] = i\epsilon_{abc} Y^c$$

i.e.: $Y^a = J^a_{(N)}$ .... generator of $\mathfrak{su}(2)$ acting on $\mathbb{C}^N$

describes quantization of S^2 with symplectic form $\omega_N = N\omega_1$

$$\sum_a Y_a Y^a = R_N^2$$



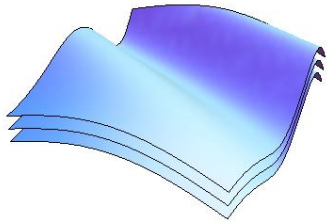
matrix models as NC gauge theories:

$$S = Tr\Big([Y^a, Y^b][Y_a, Y_b] \quad + \ldots\Big), \qquad Y^a \in End(\mathcal{H}),\ a, b = 1, \ldots, D$$

gauge invariance $Y^a \to UY^aU^{-1}$

choose NC "background" $Y^a = X^a$

$\phi \in End(\mathcal{H})$ ... function on $\mathcal{M}$



fluctuations $\to$ covariant coordinates

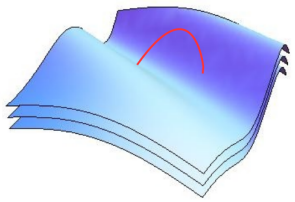
$$\boxed{Y^a = X^a + \mathcal{A}^a} \to UY^aU^{-1} = X^a + (U\mathcal{A}^aU^{-1} + \underbrace{U[X^a, U^{-1}]}_{\approx U\partial^a U^{-1}})$$

$$S = Tr\Big([Y^a, Y^b][Y_a Y_b]\Big) \sim \int_{\mathcal{M}} F_{\mu\nu}F^{\mu\nu}$$

fluctuations $\mathcal{A}^a$ of NC background in M.M. $\to$ gauge fields

<u>issue</u>:

UV/IR mixing: high-energy states string-like $\sim \ |x\rangle\langle y| \ \in End(\mathcal{H})$



quantum fluctuations (UV) $\rightarrow$ strong non-locality (IR)

except in $\mathcal{N} = 4$ SYM

$\equiv$ IKKT model, M.M. for string theory !

hence:

- M.M. describe dynamical NC spaces $\rightarrow$ gravity ?!
- fluctuations $\rightarrow$ gauge theory
- UV/IR $\rightarrow$ SUSY $\rightarrow$ unique model (= IKKT model)

goal:

- identify backgrounds (=suitable NC space solutions) s.t. fluctuations give good physics (& gravity!)
- (long-term)

  justify this with numerical simulations

cf. talk Nishimura

This talk: Lorentz / Euclidean invariance

- fully covariant 4D quantum space: fuzzy S^4
- leads to higher spin theory in M.M. (cf. Vasiliev)
- study resulting higher-spin theory (first steps)

HS, arXiv:1606.00769
M. Sperling & HS, arXiv:1704.02863
M. Sperling & HS, : arXiv:1707.00885

The IKKT model

IKKT or IIB model Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[Y,\Psi] = -Tr\left([Y^a,Y^b][Y^{a'},Y^{b'}]\eta_{aa'}\eta_{bb'} + \bar\Psi\gamma_a[Y^a,\Psi]\right)$$

$$Y^a = Y^{a\dagger} \in Mat(N,\mathbb{C}), \qquad a = 0,...,9, \qquad N \text{ large}$$

gauge symmetry $Y^a \to UY^aU^{-1}$, $SO(9,1)$, SUSY

proposed as non-perturbative definition of IIB string theory

origins:

- quantized Schild action for IIB superstring
- reduction of $10D$ SYM to point, N large
- $\mathcal{N}=4$ SYM on noncommutative $\mathbb{R}^4_\theta$

leads to "matrix geometry": ($\approx$ NC geometry)

- $S \sim Tr[X^a, X^b]^2 \;\Rightarrow$ configurations with small $[X^a, X^b] \neq 0$ should dominate

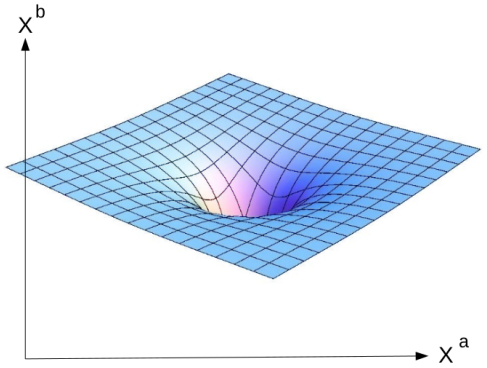
  i.e. "almost-commutative" configurations, geometry

- $\exists$ basis of quasi-coherent states $|x\rangle$,

  minimize $\sum_a \langle x|\Delta X_a^2|x\rangle$

$X^a \approx$ diag., spectrum $=: \mathcal{M} \subset \mathbb{R}^{10}$

$$\langle x|X^a|x'\rangle \approx \delta(x-x')x^a, \qquad x \in \mathcal{M}$$



embedding of branes in target space $\mathbb{R}^{10}$

$$X^a \sim x^a : \qquad \mathcal{M} \hookrightarrow \mathbb{R}^{10}$$

basic examples:

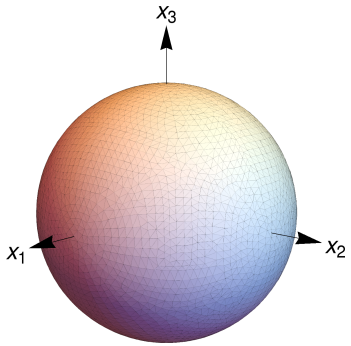
- The fuzzy 2-sphere S^2_N: Hoppe, Madore

  choose $X^a = J^a_{(N)}$ ... irrep of $SU(2)$ on $\mathcal{H} = \mathbb{C}^N$

$$[X^a, X^b] = i\varepsilon^{abc} X^c, \qquad X^a X_a = \frac{1}{4}(N^2 - 1) \ =: R^2_N \ .$$

quantized symplectic space (S^2, ω_N)

$$X^a \sim x^a : \qquad S^2 \hookrightarrow \mathbb{R}^3$$



fully covariant under $SO(3)$

functions on S^2_N: $\mathcal{A} = End(\mathcal{H}) = \bigoplus_{l=0}^{N-1} \underbrace{(2l+1)}_{\hat{Y}^l_m}$

- The Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$:

$$[X^a, X^b] = i\theta^{ab}\,\mathbf{1} \qquad \text{... Heisenberg algebra}$$

quantized symplectic space $(\mathbb{R}^4, \omega)$

admits translations, breaks rotations

functions: $\mathcal{A} = End(\mathcal{H}) \ni \phi = \int d^4k\, e^{ikX}\, \hat{\phi}(k)$

generic quantized symplectic space $\mathcal{M} \subset \mathbb{R}^D$:

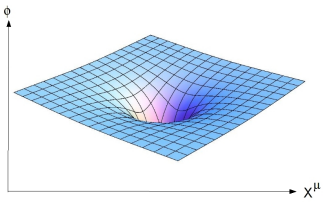
matrices X^a = quantized embedding maps

$$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$$

$[X^a, X^b] \sim i\theta^{ab}(x)$ ... Poisson tensor

fluctuations $X^a \to X^a + \mathcal{A}^a$ describe

- gauge theory on $\mathcal{M}$
  effective (open string) metric $G^{\mu\nu} \sim \theta^{\mu\mu'}\theta^{\nu\nu'} g_{\mu'\nu'}$
  dynamical geometry ($\to$ "emergent" gravity on $\mathcal{M}$)



review: H.S. arXiv:1003.4134 cf. Rivelles, H-S. Yang

- $\theta^{\mu\nu}$ breaks Lorentz/Euclidean invariance (in emerg. gravity)
  may enter at 1-loop (e.g. $R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$)

Blaschke, Steinacker arXiv:1003.4132

- "averaging" over $\theta^{\mu\nu}$ desirable

cf. Kawai-Kawana-Sakai arXiv:1610.09844

4D covariant quantum spaces

- $\exists$ fully $SO(5)$ covariant fuzzy four-sphere S^4_N

  Grosse-Klimcik-Presnajder; Castelino-Lee-Taylor; Ramgoolam; Kimura; Hasebe; Azuma-Bal-Nagao-Nisimura; Karabail-Nair; Zhang-Hu 2001 (QHE!) ...

  hidden "internal structure" $\rightarrow$ higher spin theory
- here:
  - work out lowest spin modes (spin 2!) on S^4_N in IKKT model
  - generalized S^4_Λ
- Lorentzian versions possible (future work)

basic message:

IKKT model $\rightsquigarrow$ Yang-Mills for higher spin algebra $\mathfrak{hs}$

$\exists$ action, suitable for quantization

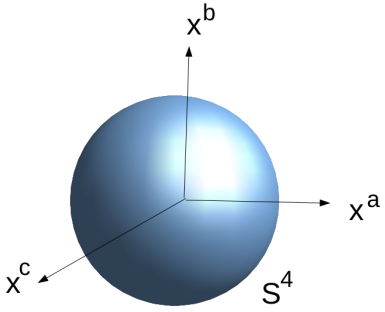
($\neq$ Vasiliev theory!)

"almost" contains gravity (...)

covariant fuzzy four-spheres

5 hermitian matrices $X_a,\ a = 1, ..., 5$ acting on $\mathcal{H}_N$

$$\sum_a X_a^2 = \mathcal{R}^2$$



covariance: $X_a \in End(\mathcal{H}_N)$ transform as vectors of $SO(5)$

$$\begin{aligned} [\mathcal{M}_{ab}, X_c] &= i(\delta_{ac} X_b - \delta_{bc} X_a), \\ [\mathcal{M}_{ab}, \mathcal{M}_{cd}] &= i(\delta_{ac}\mathcal{M}_{bd} - \delta_{ad}\mathcal{M}_{bc} - \delta_{bc}\mathcal{M}_{ad} + \delta_{bd}\mathcal{M}_{ac}) \ . \end{aligned}$$

$\mathcal{M}_{ab}$ ... $so(5)$ generators acting on $\mathcal{H}_N$

denote

$$[X^a, X^b] =: i\Theta^{ab}$$

<u>particular realization</u> $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ generators $\mathcal{M}_{ab}$:

$$X^a = r\mathcal{M}^{a6}, \qquad a = 1, ..., 5 \ , \qquad \Theta^{ab} = r^2 \mathcal{M}^{ab}$$

(cf. Snyder, Yang 1947)

- basic fuzzy 4-sphere $\mathcal{S}^4_N$:

Grosse-Klimcik-Presnajder 1996; Castelino-Lee-Taylor
Ramgoolam; Medina-o'Connor, Dolan, ...

choose $\mathcal{H}_N = (0,0,N)_{\mathfrak{so}(6)} \cong (\mathbb{C}^4)^{\otimes_S N}$

satisfies

$$X_a X_a = R^2 \mathbf{1}, \qquad R^2 \sim \tfrac{1}{4} N^2$$

$$\epsilon^{abcde} X_a X_b X_c X_d X_e = (N+2) R^2 \qquad \text{(volume quantiz.)}$$

- generalized fuzzy 4-spheres $\mathcal{S}^4_\Lambda$:

H.S, arXiv:1606.00769, M. Sperling & HS, arXiv:1704.02863

choose e.g. $\mathcal{H}_\Lambda = (n,0,N)_{\mathfrak{so}(6)}$

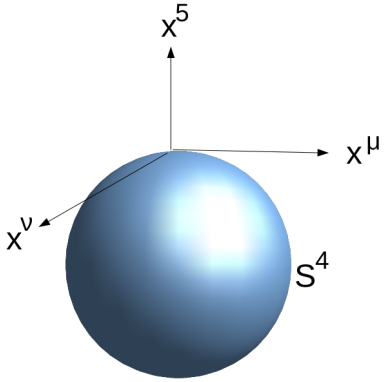
... thick sphere; $\mathcal{R}^2 := X_a X_a$ not sharp

bundle over S^4_N with fiber $\mathbb{C}P^2_n$ ($\rightarrow$ fuzzy extra dim's!)

local description: pick north pole $p \in S^4$

$\rightarrow$ tangential & radial generators

$$X^a = \begin{pmatrix} X^\mu \\ X^5 \end{pmatrix}, \qquad x^\mu \sim X^\mu,\ \mu = 1, ..., 4 \text{...tangential coords at } p$$



separate $SO(5)$ into $SO(4)$ & translations

$$\mathcal{M}^{ab} = \begin{pmatrix} \mathcal{M}^{\mu\nu} & \mathcal{P}^\mu \\ -\mathcal{P}^\mu & 0 \end{pmatrix}$$

rescale

$$P_\mu = \frac{1}{R} g_{\mu\nu} \mathcal{P}^\nu \qquad \text{(cf. Wigner contraction)}$$

algebra

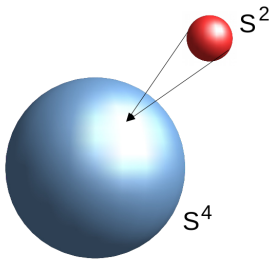
$$\begin{aligned} [P_\mu, X^\nu] &\simeq i\delta_\mu^\nu \frac{X^5}{R} \approx i\delta_\mu^\nu, \\ [P_\mu, P_\nu] &= \frac{i}{R^2} \mathcal{M}^{\mu\nu} \quad \rightarrow 0 \\ [X^\mu, X^\nu] &=: i\theta^{\mu\nu} = ir^2 \mathcal{M}^{\mu\nu} \quad \approx 0 \end{aligned}$$

cf. Snyder space !!

basic $\mathcal{S}^4_N$

semi-classical picture:

$$\begin{array}{ccc} \mathbb{C}P^3 & \ni & \psi \\ \downarrow & & \downarrow \\ S^4 & \ni & x^a = \overline{\psi}\Sigma^{a6}\psi \end{array}$$



Ho-Ramgoolam, Medina-O'Connor, Abe, ...

$x^a = \overline{\psi}\Sigma^{a6}\psi$ ... functions on $\mathbb{C}P^3$

$X^a = \overline{\Psi}\Sigma^{a6}\Psi, \qquad [\overline{\Psi}^\alpha, \Psi_\beta] = \delta^a_\beta$ ... functions on fuzzy $\mathbb{C}P^3_N$

fuzzy S^4_N is really fuzzy $\mathbb{C}P^3_N$ = twisted S^2 bundle over S^4!

Poisson tensor

$$\theta^{\mu\nu}(x,\xi) \sim -i[X^\mu, X^\nu]$$

rotates along fiber $\xi \in S^2$!

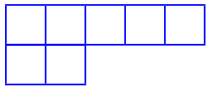
is averaged $[\theta^{\mu\nu}(x,\xi)]_0 = 0$ over fiber $\to$ local $SO(4)$ preserved,

4D "covariant" quantum space

fields and harmonics on S^4_N

"functions" on S^4_N:

$$\phi \in End(\mathcal{H}_N) \cong \bigoplus_{s \le n \le N} (n-s, 2s)_{\mathfrak{so}(5)} \quad = \bigoplus$$

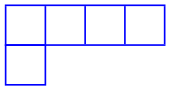


$(n, 0)$ modes = scalar functions on S^4:

$$\phi(X) = \phi_{a_1 \dots a_n} X^{a_1} \dots X^{a_n} \quad =$$

$(n, 2)$ modes = selfdual 2-forms on S^4

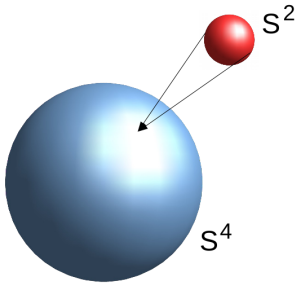
$$\phi_{bc}(X) \mathcal{M}^{bc} = \phi_{a_1 \dots a_n b;c} X^{a_1} \dots X^{a_n} \mathcal{M}^{bc} \quad =$$



etc.

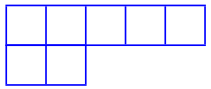
tower of higher spin modes, $s = 0, 1, 2, \dots, N$
from "twisted" would-be KK modes on S^2

(local $SO(4)$ acts non-trivially on S^2 fiber)



relation with spin s fields:

$$\phi \in End(\mathcal{H}_N) \cong \bigoplus_s \underbrace{\Big(\oplus_n \ (n-s, 2s) \Big)}_{\mathcal{C}^s} \quad = \bigoplus$$



$\exists$ isomorphism

$$\mathcal{C}^s \quad \cong \quad T^{*\otimes_S s} S^4$$

$$\phi^{(s)} = \phi^{(s)}_{b_1 \dots b_s; c_1 \dots c_s}(x)\, \theta^{b_1 c_1} \dots \theta^{b_s c_s} \quad \mapsto \quad \phi^{(s)}_{c_1 \dots c_s}(x) = \phi^{(s)}_{b_1 \dots b_s; c_1 \dots c_s} x^{b_1} \dots x^{b_s}$$

... "symbol" of $\phi \in \mathcal{C}^s$

M. Sperling & HS, arXiv:1707.00885

= symm., traceless, tangential, div.-free rank s tensor field on S^4

$$\begin{aligned} \phi_{c_1 \dots c_s}(x) x^{c_i} &= 0\,, \\ \phi_{c_1 \dots c_s}(x) g^{c_1 c_2} &= 0\,, \\ \partial^{c_i} \phi_{c_1 \dots c_s}(x) &= 0\,. \end{aligned}$$

semi-classical limit & higher spin

general functions on S^4_N:

$$\phi = \phi_{\underline{\alpha}}(x)\,\Xi^{\underline{\alpha}}, \qquad \Xi^{\underline{\alpha}} = \mathcal{M}\ldots\mathcal{M} \qquad \in \mathfrak{hs}$$

where

$$\Xi^{\underline{\alpha}} \in \mathfrak{hs} := \bigoplus_{s=0}^{\infty} (0,2s) \cong \oplus \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \end{array}$$

as vector space, (Poisson) Lie algebra

$$\{\mathcal{M}^{ab}, \mathcal{M}^{b_1c_1}\ldots\mathcal{M}^{b_sc_s}\} = g^{ab_1}\mathcal{M}^{bc_1}\ldots\mathcal{M}^{b_sc_s} \pm \ldots$$

relations

$$\mathcal{M}^{ab}\mathcal{M}^{ac} = \frac{R^2}{\theta}P_T^{ab}, \qquad \varepsilon_{abcde}\mathcal{M}^{ab}\mathcal{M}^{cd} = \frac{R}{\theta}x^e$$

functions on $S^4_N \cong \mathfrak{hs}$ - valued functions on S^4

cf. Vasiliev theory!

local representation & constraints near north pole $R(1,0,0,0,0)$:

- spin 1:

$$\begin{aligned}\phi^{(1)} &= \phi_{a_1\dots a_n b;c} x^{a_1}\dots x^{a_n}\theta^{bc} \qquad \in (n,2)\subset\mathcal{C}^1\\ &=: A_\mu(x)P^\mu + \omega_{\mu\nu}(x)\mathcal{M}^{\mu\nu}\end{aligned}$$

where

$$\omega_{\mu\nu} = -\frac{1}{2(n+2)}(\partial_\mu A_\nu - \partial_\nu A_\mu), \qquad \partial^\mu A_\mu = 0$$

- spin 2:

$$\begin{aligned}\phi^{(2)} &= \phi_{a_1\dots a_n bc;de} x^{a_1}\dots x^{a_n}\theta^{bd}\theta^{ce} \qquad \in (n,4)\subset\mathcal{C}^4\\ &=: h_{\mu\nu}(x)P^\mu P^\nu + \omega_{\mu:\alpha\beta}(x)P^\mu\mathcal{M}^{\alpha\beta} + \Omega_{\alpha\beta;\mu\nu}(x)\mathcal{M}^{\alpha\beta}\mathcal{M}^{\mu\nu}\end{aligned}$$

where

$$\begin{aligned}\partial^\mu h_{\mu\nu} &= 0\\ \omega_{\mu;\alpha\beta} &= -\tfrac{n+1}{(n+2)(n+3)}(\partial_\alpha h_{\mu\beta} - \partial_\beta h_{\mu\alpha}) \text{ ... lin. spin conn. of } h_{\mu\nu}\\ \Omega_{\alpha\beta;\mu\nu}(x) &= -\tfrac{1}{(n+2)(n+3)}\mathcal{R}_{\alpha\beta\mu\nu} \quad \text{... lin. curvature of } h_{\mu\nu}\end{aligned}$$

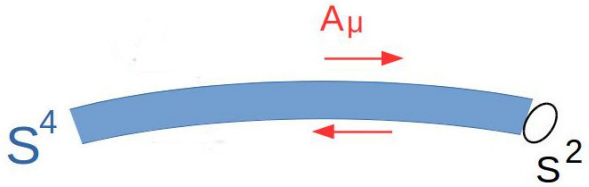
Fluctuation modes on S^4_Λ

organize tangential fluctuations at $p \in S^4$ as

$$\boxed{\mathcal{A}^\mu = \theta^{\mu\nu}\mathbf{A}_\nu}$$

where

$$\mathbf{A}_\nu(x) = A_\nu(x) + \underbrace{A_{\nu\rho}(x)P^\rho + A_{\nu\rho\sigma}(x)\mathcal{M}^{\rho\sigma}}_{A_{\nu ab}\mathcal{M}^{ab}\ \ldots SO(5)\ \text{connection}} + \ldots$$



rank 2 tensor field

$$A_{\nu\rho}(x) = \frac{1}{2}(h_{\nu\rho} + a_{\nu\rho}) \qquad h_{\nu\rho} = h_{\rho\nu} \quad \ldots \text{ metric fluctuation}$$

rank 3 tensor field

$$A_{\nu\rho\sigma}(x)\mathcal{M}^{\rho\sigma} \quad \ldots\ \mathfrak{so}(4) \text{ connection}$$

rank 1 field $\quad A_\nu(x) \quad \ldots \quad U(1)$ gauge field

gauge transformations:

$Y^a \to UY^aU^{-1} = U(X^a + \mathcal{A}^a)U^{-1}$ leads to $(U = e^{i\Lambda})$

$$\delta\mathcal{A}^a = i[\Lambda, X^a] + i[\Lambda, \mathcal{A}^a]$$

expand

$$\Lambda = \Lambda_0 + \frac{1}{2}\Lambda_{ab}\mathcal{M}^{ab} + \ldots$$

... $U(1) \times SO(5) \times \ldots$ - valued gauge trafos

diffeos from $\delta_v := i[v_\rho P^\rho, .]$

$$\begin{aligned} \delta h_{\mu\nu} &= (\partial_\mu v_\nu + \partial_\nu v_\mu) - v^\rho \partial_\rho h_{\mu\nu} + (\Lambda \cdot h)_{\mu\nu} \\ \delta A_{\mu\rho\sigma} &= \tfrac{1}{2}\partial_\mu \Lambda_{\sigma\rho}(x) - v^\rho \partial_\rho A_{\mu\rho\sigma} + (\Lambda \cdot A)_{\mu\rho\sigma} \end{aligned}$$

etc.

metric and vielbein

consider scalar field $\phi = \phi(X)$ (= transversal fluctuation $\mathcal{A}^a(X)$)

kinetic term

$$-[X^\alpha, \phi][X_\alpha, \phi] \sim e^\alpha \phi e_\alpha \phi = \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi,$$

vielbein

$$\begin{aligned} e^\alpha &:= \{X^\alpha, .\} = e^{\alpha\mu} \partial_\mu \\ e^{\alpha\mu} &= \theta^{\alpha\mu} \end{aligned}$$

Poisson structure $\rightarrow$ frame bundle!

metric

$$\gamma^{\mu\nu} = g_{\alpha\beta} e^{\alpha\mu} e^{\beta\nu} = \tfrac{1}{4} \Delta^4 g^{\mu\nu}$$

<u>averaging</u> over internal S^2:

$$[e^{\alpha\nu}]_0 = 0, \qquad [\bar{\gamma}^{\mu\nu}]_0 = \bar{\gamma}^{\mu\nu} = \frac{\Delta^4}{4} g^{\mu\nu}$$ ... SO(5) invariant !

perturbed vielbein: $Y^a = X^a + \mathcal{A}^a$

$$e^a \quad := \{Y^a, .\} \sim e^{a\mu}[\mathcal{A}]\partial_\mu \qquad \text{... vielbein}$$

$$e^{\alpha\mu}[\mathcal{A}] \quad \sim \theta^{\alpha\beta}(\delta^\mu_\beta + A_{\beta\rho}g^{\rho\mu}) + \tfrac{1}{r^2}\theta^{\alpha\nu}\theta^{\rho\sigma}\{A_{\nu\rho\sigma}, x^\mu\}$$

using $\{P^\rho, X^\mu\} \sim g^{\rho\mu}$ (!)

linearize & average over fiber $\rightarrow$

$$\gamma^{\mu\nu} \sim \; e^{\alpha\mu}[\mathcal{A}]e_\alpha^{\ \nu}[\mathcal{A}] \; = \; \bar{\gamma}^{\mu\nu} + [\delta\gamma^{\mu\nu}]_0$$

complication:

graviton is combination

$$h_{\mu\nu} := [\delta\gamma_{\mu\nu}]_0 = \frac{\Delta^4}{4}(A_{\mu\nu} + \partial^\rho A_{\mu\rho\nu} + \partial^\rho A_{\nu\rho\mu})$$

- basic S^4_N : $\partial^\rho A_{\mu\rho\nu} \approx nA_{\mu\nu}$ dominates $\rightarrow$ bad propagator
- generalized S^4_Λ : modes $A_{\mu\nu}, A_{\mu\rho\nu}$ independent, issue should be resolved (?)

action for spin 2 modes:

expand IKKT action to second oder in $\mathcal{A}^a$

$$S[Y] = S[X] + \frac{2}{g^2}\mathrm{Tr}\mathcal{A}_a \underbrace{\left((\Box + \frac{1}{2}\mu^2)\delta^a_b + 2[[X^a, X^b], .] - [X^a, [X^b, .]]\right)}_{\mathcal{D}^2} \mathcal{A}_b$$

for spin 2 modes $\mathcal{A}^\mu \sim \theta^{\mu\nu} A_{\nu\rho} P^\rho + \ldots$

$$\int \mathcal{A}\mathcal{D}^2\mathcal{A} \sim \int n^2 A^{\mu\nu} A_{\mu\nu} = \int h_{\mu\nu} h^{\mu\nu}$$

coupling to matter:

$$S[\mathrm{matter}] \sim \int_{\mathcal{M}} d^4x\, h^{\mu\nu} T_{\mu\nu}$$

$\rightarrow$ auxiliary field $h_{\mu\nu} \sim T_{\mu\nu}$!

"graviton" doesn't propagate, due to constraint $A_{\mu\nu}, A_{\mu\rho\nu}$

possible ways out:

1. there are 4 independent spin 2 modes on S^4_N !
2. 1-loop $\rightarrow$ induced gravity action $\sim \int h_{\mu\nu} \Box h^{\mu\nu}$
   can get (lin.) Einstein equations (fine-tuning ...)
3. better (?): generalized fuzzy sphere S^4_Λ
   - modes $A_{\mu\nu}, A_{\mu\rho\nu}$ independent
   - fuzzy extra dims

exact treatment of spin 2 modes on S^4_N:

Marcus Sperling & HS, arXiv:1707.00885

3 independent "graviton" modes

$$\begin{aligned} h_{\mu\nu}[A^B] &= -\, c\left(1+\tfrac{2}{n^2+7n+8}\right) T_{\mu\nu} \\ h_{\mu\nu}[A^C] &= -\, 3c\left(1+\tfrac{7}{3(2n+7)}-\tfrac{2}{3(n+8)}\right) T_{\mu\nu} \\ h_{\mu\nu}[A^D] &= -\, \tfrac{1}{3}c\left(1+\tfrac{2}{3(n-1)}-\tfrac{7}{3(2n+7)}\right) T_{\mu\nu} \end{aligned}$$

$$c = \frac{4}{5L_{NC}^4}\frac{g^2\ \mathrm{Vol}(\mathcal{S}^4)}{\dim(\mathcal{H})}$$

combined metric fluct:

$$h_{\mu\nu} = h_{\mu\nu}[A^B] + h_{\mu\nu}[A^C] + h_{\mu\nu}[A^D]$$

- leading contrib:

$$h^{(\mathrm{aux})}_{\mu\nu} = -G_N R^2 T_{\mu\nu} \quad \text{aux. field, too strong}$$

- subleading contrib:

$$\Box h^{(\mathrm{grav})}_{\mu\nu} = G_N T_{\mu\nu} \qquad \text{... lin. graviton!}$$

- strange contrib:

$$\begin{aligned} \sqrt{|\Box|}\, h^{(\mathrm{nonloc})}_{\mu\nu} &= -G_N R\, T_{\mu\nu} \\ h^{(\mathrm{nonloc})}_{\mu\nu} &\sim -\tfrac{1}{4\pi^2 r^3} + \ldots \qquad \text{... nonloc., strong} \end{aligned}$$

$$G_N = \frac{4}{5R^2 L_{NC}^4} \frac{g^2\ \mathrm{Vol}(\mathcal{S}^4)}{\dim(\mathcal{H})} =: L_{pl}^2$$

induced gravity scenario:

assume large induced gravity action $\int \Lambda^2 R \sim \int h_{\mu\nu} \Box h^{\mu\nu}$ (1-loop)

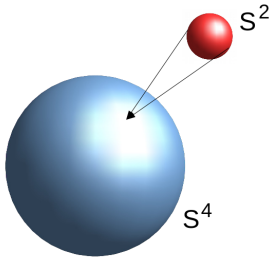
- auxiliary modes $\rightsquigarrow$ linearized gravitons, mass set by Λ^2
- remaining modes $\rightsquigarrow$ sub-leading, very long-range

(cf. conformal gravity)

generalized fuzzy sphere

basic sphere $\mathcal{S}^4_N$: $\Lambda = (0,0,N)_{\mathfrak{so}(6)}$, $H_\Lambda = N|\psi_0\rangle\langle\psi_0|$

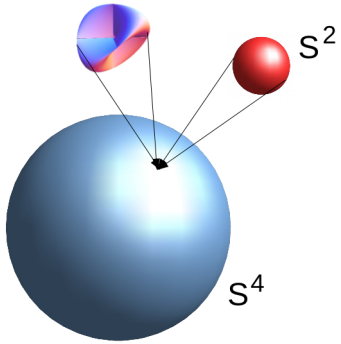
$$\begin{array}{ccc} \mathbb{C}P^3 & \ni & \psi \\ \downarrow & & \downarrow \\ S^4 & \ni & x^a = \overline{\psi}\gamma^a\psi \end{array}$$



$\mathbb{C}P^3$ is a S^2 bundle over S^4.

generalized $\mathcal{S}^4_\Lambda$: $\Lambda = (n,0,N)_{\mathfrak{so}(6)}$, $H_\Lambda = N|\psi_0\rangle\langle\psi_0| + n|\psi_1\rangle\langle\psi_1|$

$$\begin{array}{l} \mathcal{O}[\Lambda] \\ P\downarrow \\ \mathcal{O}[N\Lambda_1] \cong \mathbb{C}P^3 \xrightarrow{x^a} S^4 \,. \end{array}$$



$\mathcal{S}^4_\Lambda$ = $\mathbb{C}P^2$ bundle over $\mathbb{C}P^3 \cong \mathcal{S}^4 \times S^2$

generators of S^4_Λ:

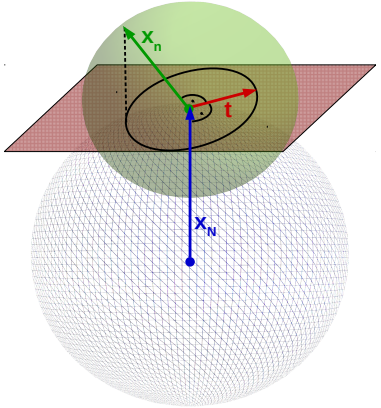
$$X^a = \pi_\Lambda(T^{a6}) \qquad \ldots\mathfrak{so}(6) \text{ generators, } \Lambda = (n, 0, N)$$
$$X_a X^a \approx \mathcal{R}^2$$

characteristic equation:

$$\left((i\mathcal{M}+2)^2 - \frac{(N-n)^2}{4}\right)\left(\left(i\mathcal{M}+\frac{3}{2}\right)^2 - \frac{(N+n+3)^2}{4}\right) = 0\,, \qquad \mathcal{M}^{ab} = \pi_\Lambda(T^{ab})$$

new $SO(5)$ vector generators:

$$T^a := (\mathcal{M}^2 - 2i\mathcal{M})^{a6} = T^{a\dagger}$$
$$Y^a := (\mathcal{M}^3 + \ldots)^{a6} = Y^{a\dagger}$$
$$T \cdot X = 0 = T \cdot Y$$



$T^a \approx$ "momentum" generators

commutation relations

$$
\begin{aligned}
[X^a, X^b] &= i\mathcal{M}^{ab}, \\
[X^a, Y^b] &= iY^{ab} = i(\mathcal{M}_N^{ab} - \mathcal{M}_n^{ab}) \ , \\
[X^a, T^b] &= -i\tilde{\mathcal{T}}^{ab} + i(\mathcal{R}^2 - c)g^{ab} \ , \\
[Y^a, Y^b] &= \tfrac{2i}{\det \tilde{A}} \Big(Y^{ab}(-2c + \mathcal{R}^2) - Y^a T^b + T^a Y^b + i \left(Y^a X^b - X^a Y^b \right) \Big) \ , \\
\left[T^a, T^b\right] &= \tfrac{i}{2} \det \tilde{A}\, Y^{ab} + i(2c - \mathcal{R}^2)\mathcal{M}^{ab} - i \left(X^a T^b - T^a X^b \right) + \tilde{\mathcal{T}}^{ab} - g^{ab} \\
\left[T^a, Y^b\right] &= i(Y^a X^b - X^a Y^b)
\end{aligned}
$$

embedding in Euclidean IKKT model (solutions!):

$SO(5)$ - invariant extra potential

$$S_{\text{eff}}[\mathcal{Y}] = \frac{1}{g^2}\text{Tr}\Big(-[\mathcal{Y}_A,\mathcal{Y}_B][\mathcal{Y}^A,\mathcal{Y}^B] + \mu_1^2\mathcal{R}_{(1)}^2 + \mu_2^2\mathcal{R}_{(2)}^2 + \lambda_1(\mathcal{R}_{(1)}^2)^2 + \lambda_2(\mathcal{R}_{(2)}^2)^2 + \mu_{12}\mathcal{R}_{(12)} + \dots \Big)$$

new $SO(5)$ covariant solution: e.g. ($\mathcal{R}_{(1)}^2 = \mathcal{Y}_{(1)}^a \cdot \mathcal{Y}_{(1)}^a$ etc.)

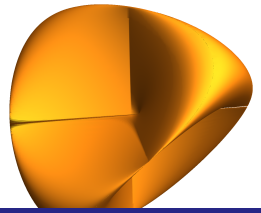
$$\mathcal{Y}^A = \begin{pmatrix} \mathcal{Y}_{(1)}^a \\ \mathcal{Y}_{(2)}^a \end{pmatrix} = \begin{pmatrix} Y^a \\ T^a \end{pmatrix}, \quad a = 1,\dots,5 ,$$

Y^a similar to X^a $\approx$ "thick" 4-sphere

S^4 with fuzzy extra dimensions

interesting for gravity & particle physics $\equiv$ "squashed $\mathbb{C}P^2 \times S^2$

(cf. HS arXiv:1504.05703 , HS & J. Zahn arXiv:1409.1440)



metric in x, y, t space worked out

towards gravity on generalized S^4_Λ

new modes, $A_{\mu\nu}$ independent from $A_{\mu\rho\sigma}$

neglect/ suppress mixing (incomplete) HS, arXiv:1606.00769
$\rightarrow$ lin. Einstein eq.

$$\mathcal{G}_{\mu\nu}[g+H] \approx -\frac{3}{R^2} g_{\mu\nu} + \frac{1}{2}\partial\cdot\partial\tilde{h}_{\mu\nu} \propto T_{\mu\nu}$$

drop background curvature $\sim \frac{1}{R^2}$ (& local effects)

more complete treatment needed

summary

- $\exists$ 4D covariant quantum spaces, e.g. fuzzy S^4_N
  $\rightarrow$ UV-regularized higher spin theory
  - UV cutoff $\leftrightarrow$ NC scale
    Poisson structure $\rightarrow$ frame bundle
  - closely related to Vasiliev theory, $\mathfrak{hs}$ algebra
- <u>in IKKT model</u>:
  - $\approx$ YM for $\mathfrak{hs}$, good UV behavior expected
  - (lin.) 4-D Einstein equations expected (?) for generalized $\mathcal{S}^4_\Lambda$
    (preliminary, needs more work ...)
  - induced gravity may play a role
  - IR modifications (massive?), additional modes
  - Minkowski version possible, needs more work
- if it works:

  quantum theory of 4D gravity, stringy, no compactification !

... there is no free lunch

but we have the ingredients!

<u>eom</u> for extra dimensions:

$$\begin{aligned}
\Box_{\mathcal{Y}} \mathcal{Y}^a_{(1)} &= -\tfrac{\mu_1^2}{2}\, \mathcal{Y}^a_{(1)} - \tfrac{\mu_{12}}{2}\, \mathcal{Y}^a_{(2)} - \lambda_1 \{\mathcal{R}_1^2, \mathcal{Y}^a_{(1)}\}_+ \\
\Box_{\mathcal{Y}} \mathcal{Y}^a_{(2)} &= -\tfrac{\mu_2^2}{2}\, \mathcal{Y}^a_{(2)} - \tfrac{\mu_{12}}{2}\, \mathcal{Y}^a_{(1)} - \lambda_2 \{\mathcal{R}_2^2, \mathcal{Y}^a_{(2)}\}_+
\end{aligned}$$

geometry: (quantized) coadjoint $SO(6)$ orbits

(co)adjoint orbits:

$$\mathcal{O}[\Lambda] = \{g \cdot H_\Lambda \cdot g^{-1};\ g \in SO(6)\} \quad \subset \mathfrak{so}(6) \ \cong \mathbb{R}^{15}$$

embedding functions:

$$\begin{aligned} m^{ab} :\ \mathcal{O}[\Lambda] \quad &\hookrightarrow \mathbb{R}^{15} \\ X \quad &\mapsto m^{ab} = tr(X\,\Sigma^{ab}) \end{aligned}$$

$$\Sigma^{ab} \in \mathfrak{so}(6)$$

quantized ("fuzzy") coadjoint orbits $\mathcal{S}^4_\Lambda$:

functions $m^{ab} \ \rightarrow$ generators $\mathcal{M}^{ab}$ on $\mathcal{H}_\Lambda$,

Λ ... (dominant) integral weight.

algebra of functions: $\mathcal{C}^\infty(\mathcal{O}) \ \rightarrow \ End(\mathcal{H}_\Lambda)$

semi-classical limit: $[.,.] \rightarrow i\{.,.\}$, same geometry as $\mathcal{O}[\Lambda]$