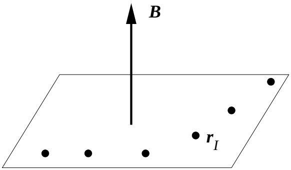
UV/IR DUALITY IN NONCOMMUTATIVE QUANTUM FIELD THEORY

Richard Szabo

Heriot–Watt University, Edinburgh
Maxwell Institute for Mathematical Sciences

Noncommutativity and Physics: Quantum Geometries and Gravity
Bayrischzell 2009

The Landau problem



$$\mathcal{L}_m = \frac{m}{2}\,\dot{\mathbf{x}}^2 - \frac{e}{c}\,\dot{\mathbf{x}}\cdot\mathbf{A}\ ; \qquad A_x = -\frac{B}{2}\,y\ ,\ A_y = \frac{B}{2}\,x$$

In strong field limit $e\,B \gg m$ (lowest Landau level projection):

$$\mathcal{L}_0 = -\frac{e\,B}{2c}\,(\dot{x}\,y - \dot{y}\,x)$$

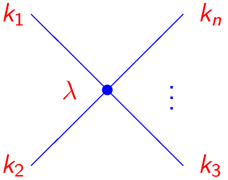
Canonical quantization gives **noncommutative space**:

$$[x,y] = \mathrm{i}\,\theta\ , \qquad \theta = \frac{\hbar\,c}{e\,B}$$

UV/IR mixing – The problem

- Interactions:

$$\widetilde{\phi}(k)\,\widetilde{\phi}(q) \;\longrightarrow\; \widetilde{\phi}(k)\,\widetilde{\phi}(q)\;\mathrm{e}^{\,\mathrm{i}\,k\times q}\;, \qquad k\times q \;=\; \tfrac{1}{2}\,k_\mu\,\theta^{\mu\nu}\,q_\nu$$



$$= \;\lambda\;\mathrm{e}^{\,\mathrm{i}\sum_{I<J} k_I\times k_J}$$

  with $k_1 + k_2 + \ldots + k_n \;=\; 0$; effective at energies E with $E\,\sqrt{\theta} \ll 1$
- **Non-planar graphs:** UV cutoff Λ $\Longrightarrow$ Effective IR cutoff $\Lambda_0 \;=\; \dfrac{1}{\theta\,\Lambda}$ (Minwalla, Van Raamsdonk & Seiberg '99)
- **The field theory cannot be renormalized!!!**

UV/IR mixing – The physics

- $\mathrm{e}^{\,\mathrm{i}\,k\cdot x}\star\phi(x)\star\mathrm{e}^{-\,\mathrm{i}\,k\cdot x} \;=\; \phi(x^\mu-\theta^{\mu\nu}\,k_\nu)$

  IR dynamics: "dipoles" with dipole moment $\Delta x^\mu \;=\; \theta^{\mu\nu}\,k_\nu$

  Like electron-hole bound state in strong magnetic field

  Dipoles interact by joining at their ends

  (Sheikh-Jabbari '99, Bigatti & Susskind '99)

$$W_k[\phi] \;=\; \mathrm{Tr}\,\exp\big(\,\mathrm{i}\,|k|\,\phi(x)\big)$$

- UV dynamics: Elementary quantum fields ϕ, pointlike momenta k_μ

- UV/IR "duality" (Rey '02)

UV/IR mixing – The cure

(Langmann & RS '02, Grosse & Wulkenhaar '04)

- Covariant version renders UV, IR regimes indistinguishable
- Make UV/IR "duality" symmetric:

$$k_\mu \longmapsto K_\mu = k_\mu + B_{\mu\nu}\, x^\nu \qquad \text{("Landau" momenta)}$$

$B_{\mu\nu}$ = "magnetic" background

"Noncommutative momentum space": $[K_\mu, K_\nu] = 2\,\mathrm{i}\, B_{\mu\nu}$

- **Grosse–Wulkenhaar model:** Real Euclidean scalar $\lambda\, \phi^{\star 4}_{2d}$-theory in background harmonic oscillator potential:

$$\partial_\mu^2 \longmapsto \partial_\mu^2 + \frac{\omega^2}{2}\, \widetilde{x}_\mu^{\,2}\,, \qquad \widetilde{x}_\mu = 2\theta^{-1}_{\mu\nu}\, x^\nu$$

QFT symmetric under Fourier transformation of fields: $k_\mu \leftrightarrow \widetilde{x}_\mu$

Renormalization

(Langmann, RS & Zarembo '04; Grosse & Wulkenhaar '05; Rivasseau *et al.* '05 . . .)

- Covariant model is renormalizable to all orders in λ!
- Described by matrix model (no spacetime!) with cutoff the matrix size N (degeneracy of Landau levels)
  $N \times N$ matrix model is related to an integrable KP-hierarchy
- At $\omega = 1$ (self-dual point), $\beta_{\lambda,\omega} = 0 \implies$ renormalized coupling flows to finite bare coupling (wavefunction renormalization compensates coupling constant renormalization, $\lambda\,\phi^4$ invariant)
- **No Landau ghost (renormalons)!** (without asymptotic freedom)
- Non-perturbative completion believed possible

Classical duality

- Charged scalar fields $\phi(x)$ on Euclidean $\mathbb{R}^{2d}$:

$$S[\phi] \ = \ \int \mathrm{d}^{2d}x \ \Big(\phi^\dagger \ \left(D_\mu^2 + \mu^2\right) \ \phi + g^2 \, \phi^\dagger \star \phi \star \phi^\dagger \star \phi\Big)$$

$$D_\mu \ = \ \frac{1}{\sqrt{2}} \left(-\,\mathrm{i}\,\partial_\mu + B_{\mu\nu}\, x^\nu\right)$$

- Invariant under duality transformation of order 2:

$$\begin{aligned} \phi(x) \ &\longrightarrow \ \widehat{\phi}(x) \ = \ \sqrt{|\det(B)|} \ \widetilde{\phi}(B\cdot x) \\ \theta \ &\longrightarrow \ \widehat{\theta} \ = \ -4B^{-1}\,\theta^{-1}\,B^{-1} \\ g \ &\longrightarrow \ \widehat{g} \ = \ 2^d \, \big|\det(B\,\theta)\big|^{-1/2}\, g \end{aligned}$$

- Self-dual point: $\theta \ = \ 2B^{-1}$

Quantum duality

- Generating functional of connected Green's functions:

$$\mathcal{G}(J) \ = \ -\log \frac{Z[J]}{Z[0]}$$

$$Z[J] \ = \ \int \mathcal{D}\phi \ \mathcal{D}\phi^\dagger \ \exp\Big(-S[\phi] - \int \mathrm{d}^{2d}x \ \big(\phi^\dagger \, J + \phi \, J^\dagger\big)\Big)$$

- **Formally** invariant under duality transformation of Schwartz functions $\phi \ \longmapsto \ \widehat{\phi}$ on $\mathbb{R}^{2d}$:

$$\mathcal{G}(J; B, g, \theta) \ = \ \mathcal{G}(\,\widehat{J}; B, \widehat{g}, \widehat{\theta}\,)$$

- Requires duality invariant regularization $\mathcal{G} \ \longrightarrow \ \mathcal{G}_\Lambda$
  — all Feynman diagrams converge

Quantum duality

- Expand fields in "matrix basis" $f_{n,m} \in L^2(\mathbb{R}^2),\ n,m = 0,1,\dots$ of Landau wavefunctions:

$$\phi(x) \ = \ \sum_{n,m} f_{n,m}(x)\ \phi_{n,m}$$

$$D_\mu^2 f_{n,m} \ = \ 2B\left(n+\tfrac{1}{2}\right) f_{n,m} \ =: \ E_n\, f_{n,m}\ , \qquad D_\mu^2\big|_{B\to -B} f_{n,m} \ = \ E_m\, f_{n,m}$$

- For suitable cut-off function F, replace free propagator:

$$C(n,m) \ = \ \left(E_n+\mu^2\right)^{-1} \ \longrightarrow \ C_\Lambda(n,m) \ = \ \left(E_n+\mu^2\right)^{-1} F\left(\Lambda^{-2}\left(E_n+E_m\right)\right)$$

- Feynman diagrams $= \displaystyle\sum_{n_1,m_1,\dots,n_K,m_K} \ \prod_{k=1}^K C_\Lambda(n_k,m_k) \ \times \ \text{(vertices)}$

  (finite sums)

Matrix model

- Mapping to a matrix model $(d = 2\ ,\ \theta = 2B^{-1})$:

$$f_{n,m} \star f_{n',m'} = \delta_{m,n'}\ f_{n,m'}\ , \qquad \int \mathrm{d}^2x\ f_{n,m} = \delta_{n,m}$$

$$S[\phi] = \mathrm{Tr}\left(\phi^\dagger\,\mathcal{B}\,\phi + \mu^2\,\phi^\dagger\,\phi + g^2\,\left(\phi^\dagger\,\phi\right)^2\right)$$

$$\phi = (\phi_{n,m})\ , \qquad \mathcal{B}_{n,m} = \theta^{-1}\left(n + \tfrac{1}{2}\right)$$

- QFT has $U(\infty)$ symmetry $\phi \longrightarrow U^\dagger\,\phi\,U$ and is $N \longrightarrow \infty$ limit of $N \times N$ complex matrix model in external field:

$$Z_N = \int\ \prod_{n,m=1}^{N} \mathrm{d}\phi_{n,m}\ \mathrm{d}\phi^\dagger_{n,m}\ \mathrm{e}^{\,-S[\phi]}$$

- Related to Kontsevich–Penner model

Analytic continuation to Minkowski signature

- Naively $x^0 \longrightarrow \pm\,\mathrm{i}\,t$, $B_{0i} \longrightarrow \pm\,\mathrm{i}\,E_i$, but this is "wrong"

- Perturbative dynamics of (non-covariant) NCFT **cannot** be obtained by Wick rotation (Bahns *et al.* '02, Liao & Sibold '02, Rim & Yee '03)

- Time-ordering and two-point function do not combine into Feynman propagators in non-planar graphs

- Renormalization properties (in S-matrix framework) very different
  — UV/IR mixing may be far less severe or even absent (Bahns '07)

Analytic continuation to Minkowski signature – Results

(Fischer & RS '09)

- There is a dense domain $\phi \in \Phi \subset L^2(\mathbb{R}^2)$ and "electric Landau wavefunctions" $f^{\pm}_{n,m} \in \Phi'$, $n,m = 0,1,\dots$ such that

$$\phi(x) = \frac{1}{2} \sum_{n,m} \left(f^{+}_{n,m}(x)\, \phi^{-}_{n,m} + f^{-}_{n,m}(x)\, \phi^{+}_{n,m} \right)$$

$$D^2_\mu f^{\pm}_{n,m} = \pm\,\mathrm{i}\, E_n\, f^{\pm}_{n,m}\ , \qquad D^2_\mu\big|_{B\to -B} f^{\pm}_{n,m} = \pm\,\mathrm{i}\, E_m\, f^{\pm}_{n,m}$$

$$f^{\pm}_{n,m}{}^{*} = f^{\mp}_{m,n}\ , \quad \langle f^{\pm}_{n,m} | f^{\mp}_{n',m'} \rangle = \delta_{m,n'}\, \delta_{n,m'}\ , \quad f^{\pm}_{n,m} \star f^{\pm}_{n',m'} = \delta_{m,n'}\, f^{\pm}_{n,m'}$$

- Unitarity and causality: Both matrix bases required to ensure:

  1. Stability (manifestly real action)
  2. CT-invariance ($\phi^{\mp}_{n,m} = C\, T \phi^{\pm}_{n,m}$)

Analytic continuation to Minkowski signature – Results

- Quantum duality: Regulated propagators in Minkowski space:

$$C^{\pm}(n,m) \;=\; \big\langle\, {\phi^{\pm}_{m,n}}^{*}\,\phi^{\mp}_{m,n}\big\rangle$$
$$\longrightarrow \quad C^{\pm}_{\Lambda}(n,m) \;=\; 2\,\mathrm{i}\,\big(\pm\,\mathrm{i}\,E_n+\mu^2\big)^{-1}\,F\,\big(\Lambda^{-2}\,|E_n+E_m|\big)$$

Represent incoming (resp. outgoing) particles (resp. antiparticles)

- Coupled complex two-matrix model: At self-dual point:

$$S \;=\; \frac{1}{2}\sum_{s=\pm}\,\mathrm{Tr}\,\Big(4s\,\phi_s^{\dagger}\,\mathrm{i}\,\mathcal{B}\,\phi_{-s}+\mu^2\,\phi_s^{\dagger}\,\phi_{-s}+g^2\,\big(\phi_s^{\dagger}\,\phi_{-s}\big)^2\Big)$$

$GL(\infty)\times GL(\infty)$ symmetry: $\phi_s \longmapsto \phi_s\,U_s\ , \qquad \phi_s^{\dagger} \longmapsto U_{-s}^{-1}\,\phi_s^{\dagger}$

CT-symmetry: $\big(\phi_s\,,\,\phi_s^{\dagger}\big) \longmapsto \big(\phi_{-s}\,,\,\phi_{-s}^{\dagger}\big) \quad , \quad \theta \longmapsto -\,\theta$

Inverted harmonic oscillator

(Chruscinski '04)

$$H = \frac{1}{2}\left(P^2 - \omega^2\, Q^2\right)$$

- Related to usual harmonic oscillator by complex scaling $\omega \longrightarrow \pm\, \mathrm{i}\,\omega$
- H selfadjoint on $L^2(\mathbb{R})$ with $\mathrm{Spec}(H) = \mathbb{R}$, but has generalized eigenfunctions with **imaginary** eigenvalues
- Occur as residues of original eigenfunctions analytically continued to complex energy plane
  — closing contour of integration in eigenfunction expansion gives analog of **discrete** expansion in Landau wavefunctions
- Analogous to (controversial) Bohm–Gadella theory of resonant states in quantum mechanics

Rigged Hilbert spaces

$$\Phi \ \subset \ \mathcal{H} \ \subset \ \Phi'$$

$\Phi \ = \$ dense subspace of Hilbert space $\mathcal{H}$ with dual Φ'

- Generalized eigenvectors: $\langle \phi | A F_\lambda \rangle \ := \ \langle A\phi | F_\lambda \rangle \ = \ \lambda \, \langle \phi | F_\lambda \rangle$

  where $\lambda \in \mathbb{C} \,, \ \phi \in \Phi \,, \ F_\lambda \in \Phi' \,, \ A \in \mathrm{End}(\mathcal{H})$

- Gel'fand–Maurin Theorem: For any $|\phi\rangle \in \Phi$, there exists $|F_\lambda\rangle \in \Phi'$ such that

$$|\phi\rangle \ = \ \int_{\mathrm{Spec}(A)} \mathrm{d}\mu(\lambda) \ |F_\lambda\rangle \, \langle F_\lambda | \phi \rangle$$

- **Example:** For inverted oscillator $\mathcal{S}(\mathbb{R}) \ \subset \ L^2(\mathbb{R}) \ \subset \ \mathcal{S}'(\mathbb{R})$

Resonance expansion

- By P-invariance, each $\mathcal{E} \in \mathrm{Spec}(H)$ has 2-fold degenerate eigenfunctions $\chi_\pm^{\mathcal{E}}$, $\eta_\pm^{\mathcal{E}}$ given by parabolic cylinder functions (only two linearly independent), so for any $\phi \in \mathcal{S}(\mathbb{R})$:

$$\phi(q) = \sum_{s=\pm} \int \mathrm{d}\mathcal{E}\ \chi_s^{\mathcal{E}}(q)\ \langle \chi_s^{\mathcal{E}} \,|\, \phi \rangle = \sum_{s=\pm} \int \mathrm{d}\mathcal{E}\ \eta_s^{\mathcal{E}}(q)\ \langle \eta_s^{\mathcal{E}} \,|\, \phi \rangle$$

- H also has generalized eigenfunctions $f_n^\pm$ with discrete eigenvalues $\pm\,\mathrm{i}\,\theta^{-1}\left(n+\frac{1}{2}\right)$, $n = 0, 1, \dots$, occuring as residues of $\chi_\pm^{\mathcal{E}}$ / $\eta_\pm^{\mathcal{E}}$ in upper / lower complex half-plane

- In suitable domain $\phi \in \Phi \subset \mathcal{S}(\mathbb{R})$:

$$\phi(q) = \frac{1}{2} \sum_{s=\pm} \sum_{n=0}^{\infty} f_n^s(q)\ \langle f_n^{-s} | \phi \rangle$$

Configuration space Φ

$$\mathcal{S}_\alpha^\alpha(\mathbb{R}) \ \subset \ L^2(\mathbb{R}) \ \subset \ \mathcal{S}_\alpha^\alpha(\mathbb{R})'$$

$\mathcal{S}_\alpha^\alpha(\mathbb{R}) \ = \$ Gel'fand–Shilov space with $\alpha \geq \frac{1}{2}$
$\mathcal{S}_\alpha^\alpha(\mathbb{R})' \ = \$ space of tempered ultra-distributions of Roumieu type

Gel'fand–Shilov spaces: Entire functions $\phi(q)$ on $\mathbb{C}$ restricted to $\mathbb{R}$, with $\left\| q^m \, \partial_q^n \phi \right\|_\infty \leq C \, M^{n+m} \, n^{\alpha \, n} \, m^{\alpha \, m}$

$\mathcal{S}_\alpha^\alpha(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \ = \ \mathcal{S}_\infty^\infty(\mathbb{R})$ closed under Fourier transformation, star-product (Soloviev '07, Chaichian *et al.* '08), basis given by harmonic oscillator wavefunctions (Lozanov-Crvenković & Perišić '07)

Theorem (Fischer–RS): For any $\phi \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$, one has $\lim_{\mathcal{E}\to\infty} \left\langle \eta_\pm^{\mathcal{E}} \middle| \phi \right\rangle \ = \ 0$ (resp. $\lim_{\mathcal{E}\to\infty} \left\langle \chi_\pm^{\mathcal{E}} \middle| \phi \right\rangle \ = \ 0$) over $\mathcal{E}$ in upper (resp. lower) complex half-plane

Open problems

- Physical meaning of "electric Landau wavefunctions" $f^{\pm}_{n,m}$ — relation to time-ordered perturbation theory?
- Resonance expansion **proves** electric-magnetic duality $B_{0i} \longrightarrow \pm\, \mathrm{i}\, E_i$ of QED effective action — simple explanation of electric-type noncommutativity like lowest Landau level projection?
- Analytic continuation of Grosse–Wulkenhaar model to Minkowski signature (inverted harmonic oscillator potential)
  — renormalization?
- Meaning of duality covariance, beyond Moyal spaces:
  1. UV/IR duality as metaplectic representations of Heisenberg group (Grosse–Wulkenhaar model on solvable symmetric spaces) (Bieliavsky, Gurau & Rivasseau '08)
  2. UV/IR mixing on κ-deformed space (Grosse & Wohlgenannt '06) — related to quantum group dualities?