Noncommutative Symmetry Reduction: Backgrounds and Quantum Fields

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Noncommutativity and physics:
Quantum geometry and gravity

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Why Noncommutative Geometry?

Basics of Noncommutative Geometry

Physics in Flat NC Space

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NC Symmetry Reduction

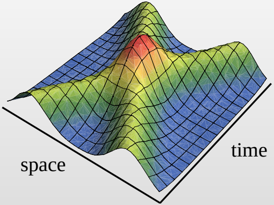
Dynamics of Symmetry Reduced Sectors

Field Fluctuations on NC Backgrounds

Why Noncommutative Geometry?

Einstein gravity

- ► based on smooth manifolds
  i. e. spacetime made out of points
- ↯ points not physical (black holes!)
- ∴ wrong category for short distance gravity



Quantum gravity

- ► motivation: “get rid of points”
- ► approaches: Strings, Loop Quantum Gravity, ...
- ► hardest problem: making contact with the “real world”

“Almost quantum” gravities (cf. EFT methods)

- ► intermediate step incorporating most important quantum effects
- ► ideas: infrared expansion, noncommutative geometry, ...
- ☺ NC Geometry without points and spacetime uncertainties built in

Basics of Noncommutative Geometry

- wanted: coordinate operators $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x})$

$\Rightarrow$ spacetime uncertainty relations $\Delta x^\mu \Delta x^\nu \neq 0$

- equivalently: use $\star$-products $f(x) \star g(x) \neq g(x) \star f(x)$
- examples:
  - Moyal-Weyl product:

$$f \star g = f e^{\frac{i\lambda}{2} \overleftarrow{\partial}_\mu \Theta^{\mu\nu} \overrightarrow{\partial}_\nu} g$$

  - Reshetikhin-Jambor-Sykora (RJS) product:

$$f \star g = f e^{\frac{i\lambda}{2} \overleftarrow{X}_\alpha \Theta^{\alpha\beta} \overrightarrow{X}_\beta} g \;, \quad [X_\alpha, X_\beta] = 0$$

- **NB:** RJS and Moyal-Weyl products are obtained from twists

$$\mathcal{F} = \exp\Big(-\frac{i\lambda}{2}\Theta^{\alpha\beta} X_\alpha \otimes X_\beta\Big) \in U\Xi \otimes U\Xi$$

Physics in Flat NC Space

Flat NC Space

- most straightforward construction, Moyal plane

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} = i\frac{C_{\mu\nu}}{{\Lambda_{NC}}^2}$$

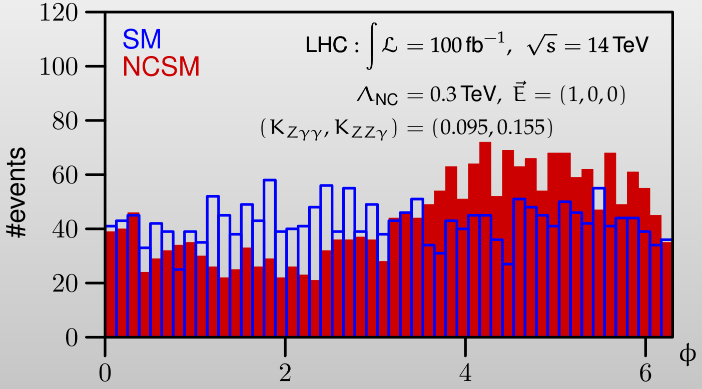
not excluded, as long as characteristic energy scale Λ_{NC} large and corresponding minimal area in the $e_\mu \wedge e_\nu$-plane

$$a_{NC} = l_{NC}^2 = 1/\Lambda_{NC}^2$$

small compared to the resolution of experiments.

- standard model of particle physics can be generalized to the Moyal plane using Seiberg-Witten maps [Wess et al.]
  - forbidden and rare decays [Munich/Zagreb group]
  - BBN [Zagreb group]
  - ☹ strong limits from isotropy
  - ☹ IR/UV mixing requires more work …

- Collider experiments [Würzburg group], e. g. azimuthal modulation at LHC



Using standard acceptance cuts and $85\,\text{GeV} < m_{\ell^+\ell^-} < 97\,\text{GeV}$, $200\,\text{GeV} < m_{\ell^+\ell^-\gamma} < 1\,\text{TeV}$, $0 < \cos\theta^*_\gamma < 0.9$,
$\cos\theta_Z > 0$ and $\cos\theta_\gamma > 0$ (favoring $\bar{q}q$ over $q\bar{q}$!)

Noncommutative Symmetries and Gravity

[Wess group, Madore, ...]

- classical spaces $\leftrightsquigarrow$ classical symmetries (Lie groups/algebras)
  - euclidean space $\leftrightsquigarrow$ euclidean group $SO(3) \ltimes \mathbb{R}^3$
  - Minkowski space $\leftrightsquigarrow$ Poincaré group $SO(3,1) \ltimes \mathbb{R}^4$
- NC spaces $\leftrightsquigarrow$ "quantum symmetries" (quantum groups/Hopf algebras)
  - q-euclidean space $\leftrightsquigarrow$ q-euclidean Hopf algebra
  - Moyal-plane $\leftrightsquigarrow$ θ-Poincaré Hopf algebra
- general feature:

  noncommutative spacetime $\leftrightsquigarrow$ noncocommutative Hopf algebra

  commutative spacetime $\leftrightsquigarrow$ Lie algebra $\rightarrow$ cocommutative HA
- Basic idea of (twisted) NC gravity:

  Einstein gravity $\leftrightsquigarrow$ diffeomorphism Lie algebra Ξ

  NC Einstein gravity $\leftrightsquigarrow$ deformed diffeomorphism Hopf algebra

$$(\Xi, [\ ,\ ]) \xrightarrow{\text{construct}} (U\Xi, \cdot, \Delta, S, \epsilon) \xrightarrow{\mathcal{F}} (U\Xi, \cdot, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \epsilon)$$

[Wess group]

✓ construction of cov. derivatives and curvature on NC manifolds

basic idea: deform everything using the twist ⇒ deformed covariant theory

⇒ NC Einstein equations:

$$\mathrm{Ric}_{ab} - \frac{1}{2} g_{ab} \star \mathfrak{R} = 8\pi G \, T_{ab}$$

- **NB:**
  - nonlocal and nonlinear equations of motion → i.g. complicated
  - ambiguities in defining Einstein equations ☹
  - ? relation to NC vielbein gravity [Chamseddine, Aschieri, Castellani]
- wanted: solutions of NC Einstein equations

NC Symmetry Reduction:

a first step towards solutions

Classical symmetry reduction:

- isometries $\hat{=}$ symmetry Lie algebra $\mathfrak{g}$
- represent $\mathfrak{g}$ in terms of vector fields Ξ
- demand $\mathcal{L}_{\mathfrak{g}}(\tau) = \{0\}$ for all symmetric tensor fields

NC symmetry reduction: [TO, AS: JHEP 0901:084,2009]

- isometries $\hat{=}$ symmetry Lie algebra $\mathfrak{g}$
- represent $\mathfrak{g}$ in terms of vector fields Ξ
- demand $\mathcal{L}_{\mathfrak{g}}(\tau) = \{0\}$ for all symmetric tensor fields

+ consistency condition: $\mathcal{L}_{\mathfrak{g}}(\tau \star \tau') = \{0\}$, if $\mathcal{L}_{\mathfrak{g}}(\tau) = \mathcal{L}_{\mathfrak{g}}(\tau') = \{0\}$!

- **NB:**
  - CC from nontrivial coproduct $\Delta_{\mathcal{F}}$ in Hopf algebra
  - restrictions among twist $\mathcal{F}$ and symmetry Lie algebra $\mathfrak{g}$
  - for RJS twists $[X_\alpha, \mathfrak{g}] \subseteq \mathfrak{g}$, $\forall_\alpha$ $\rightarrow$ classification! ☺

✓ FRW models, Schwarzschild black holes (& black branes, AdS, . . .)

Classification of deformed FRW models:

- $\mathfrak{g} = \mathrm{span}(p_i, L_i)$ with $p_i = \partial_i$ and $L_i = \epsilon_{ijk} x^j \partial_k$
- $[X_\alpha, \mathfrak{g}] \subseteq \mathfrak{g}$ gives $X_\alpha = X^0_\alpha(t)\partial_t + c^i_\alpha \partial_i + d^i_\alpha L_i + f_\alpha x^i \partial_i$
- taking $\alpha \in \{1, 2\}$ and demanding $[X_1, X_2] = 0$ we get

$\mathfrak{C}_{AB}$	$\mathbf{d}_1 = \mathbf{d}_2 = 0$	$\mathbf{d}_1 \neq 0$, $\mathbf{d}_2 = 0$
$f_1 = 0$,	$X_1 = X^0_1(t)\partial_t + c^i_1 \partial_i$	$X_1 = X^0_1(t)\partial_t + c^i_1 \partial_i + d^i_1 L_i$
$f_2 = 0$	$X_2 = X^0_2(t)\partial_t + c^i_2 \partial_i$	$X_2 = X^0_2(t)\partial_t + \kappa\, d^i_1 \partial_i$
$f_1 \neq 0$,	$X_1 = c^i_1 \partial_i + f_1 x^i \partial_i$	$X_1 = c^i_1 \partial_i + d^i_1 L_i + f_1 x^i \partial_i$
$f_2 = 0$	$X_2 = X^0_2(t)\partial_t$	$X_2 = X^0_2(t)\partial_t$
$f_1 = 0$,	$X_1 = X^0_1(t)\partial_t$	$X_1 = X^0_1(t)\partial_t + \frac{1}{f_2} d^j_1 c^k_2 \epsilon_{jki} \partial_i + d^i_1 L_i$
$f_2 \neq 0$	$X_2 = c^i_2 \partial_i + f_2 x^i \partial_i$	$X_2 = X^0_2(t)\partial_t + c^i_2 \partial_i + f_2 x^i \partial_i$

☺ there is an isotropic twist!

1. favorite model: $\left[\hat{t}, \hat{x}^i\right] = i\lambda \widehat{X(t)x^i}$
   - ▷ isotropic but nonhomogeneous model (interesting for CMB)
   - ▷ $X(t)$ can be used to tune away NC effects for large t
   - ▷ NC can drive gravity (see below!)
   - ☹ lies in the model class we understand less
2. next-to-favorite model: $\left[\widehat{\exp i\phi}, \hat{t}\right] = \lambda \widehat{\exp i\phi}$
   - ▷ discrete time spectrum $\sigma(\hat{t}) = \lambda(\mathbb{Z} + \delta)$
   - → singularity avoidance in cosmology!?!
   - ☺ we understand background dynamics (see below!)
   - ▷ nonisotropic model: maybe problems with CMB
3. less favored models: e. g. $\left[\hat{x}^i, \hat{x}^j\right] = i\lambda^{ij}\hat{1}$
   - ☹ NC scale growing with time
   - ☺ backgrounds and (Q)FT (see below!)
   - → nice playground for mathematical aspects (e. g. interacting fields)

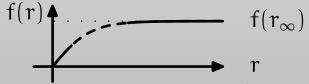
Classification of deformed black hole models:

- $\mathfrak{g} = \mathrm{span}(p^0, L_i)$ with $p^0 = \partial_t$ and $L_i = \epsilon_{ijk} x^j \partial_k$
- $[X_\alpha, \mathfrak{g}] \subseteq \mathfrak{g}$ gives $X_\alpha = (c^0_\alpha(r) + N^0_\alpha t)\partial_t + d^i_\alpha L_i + f_\alpha(r) x^i \partial_i$
- taking $\alpha \in \{1, 2\}$ and demanding $[X_1, X_2] = 0$ we get

$\mathfrak{B}_{AB}$	$f_2(r) = 0$	$f_2(r) \neq 0$
$N^0_1 = 0$, $N^0_2 = 0$	$X_1 = c^0_1(r)\partial_t + \kappa_1 d^i L_i$ $X_2 = c^0_2(r)\partial_t + \kappa_2 d^i L_i$	$X_1 = c^0_1 \partial_t + \kappa_1 d^i L_i$ $X_2 = c^0_2(r)\partial_t + \kappa_2 d^i L_i + f_2(r) x^i \partial_i$
$N^0_1 \neq 0$, $N^0_2 = 0$	$X_1 = (c^0_1(r) + N^0_1 t)\partial_t$ $X_2 = \kappa_2 d^i L_i$	$X_1 = (c^0_1(r) + N^0_1 t)\partial_t + \kappa_1 d^i L_i$ $X_2 = -\frac{1}{N^0_1} f_2(r) r c^{0\prime}_1(r)\partial_t + \kappa_2 d^i L_i + f_2(r) x^i \partial_i$
$N^0_1 = 0$, $N^0_2 \neq 0$	$X_1 = \kappa_1 d^i L_i$ $X_2 = (c^0_2(r) + N^0_2 t)\partial_t$	$X_1 = c^0_1(r)\partial_t + \kappa_1 d^i L_i$ (+ ODE for c^0_1) $X_2 = (c^0_2(r) + N^0_2 t)\partial_t + \kappa_2 d^i L_i + f_2(r) x^i \partial_i$

☺ there is a twist invariant under all BH symmetries!

1. isotropic model: $[\hat{t}, \hat{r}] = i\lambda \widehat{f(r)}$
   - ▷ isotropic and time translation invariant
   - ▷ $f(r)$ can be used to tune away NC effects for large r
2. discrete time model: $\left[\widehat{\exp i\phi}, \hat{t}\right] = \lambda \widehat{\exp i\phi}$
   - ▷ nonisotropic, but time translation invariant
   - ▷ quantization of time in orders of λ
   - ▷ can define (Q)FT on this background
3. discrete radius model: $[\exp i\phi \stackrel{\star}{,} r] = -2 \sinh\left(\frac{\lambda}{2} f(r) \partial_r\right) r \; \cdot \exp i\phi$
   - ▷ nonisotropic, but time translation invariant
   - ▷ $f(r) = r \;\rightarrow \sigma(\hat{r}) \sim \exp\big(\lambda(\mathbb{Z} + \delta)\big)$, not nice ☹
   - ▷ $f(r)$ generically



   $\rightarrow \sigma(\hat{r}) \approx \lambda(\mathbb{Z} + \delta)$ for large r , nicer ☺

- ► BH models solve NC Einstein equations using undeformed metric!
- ► cf. Schupp-Solodukhin NC black hole

Dynamics of Symmetry Reduced Sectors:

general properties and explicit solutions

Proposition (TO, AS: to appear)

Let $\mathcal{F} = \exp\left(-\frac{i\lambda}{2}\Theta^{\alpha\beta} X_\alpha \otimes X_\beta\right)$ be a $\mathfrak{g}$-compatible RJS twist. Then the symmetry reduced Riemannian geometry is undeformed if one $X_\alpha \in \mathfrak{g}$, for all pairs of vector fields connected by $\Theta^{\alpha\beta}$.

- most FRW, black hole (and black brane) models are solvable ☺
  - **NB:** this **does not** mean our models are trivial!
- ∃ examples with potential correction to backgrounds
- e. g. $\left[\hat{t}, \hat{x}^i\right] = i\lambda\hat{x}^i \;\Rightarrow$ NC Friedmann equations:

$$3\frac{\dot{A}(t-i\lambda)}{A(t-i\lambda)}\frac{\dot{A}(t+i\lambda)}{A(t+i\lambda)} + \frac{3}{2}\frac{\dot{N}(t)}{N(t)}\left(\frac{\dot{A}(t-i\lambda)}{A(t-i\lambda)} - \frac{\dot{A}(t+i\lambda)}{A(t+i\lambda)}\right) + \frac{3}{2}\left(\frac{\ddot{A}(t+i\lambda)}{A(t+i\lambda)} - \frac{\ddot{A}(t-i\lambda)}{A(t-i\lambda)}\right) = \rho(t)$$

$$-\frac{A(t)\dot{A}(t)\dot{A}(t-2i\lambda)}{A(t-2i\lambda)N(t-i\lambda)^2} + \frac{A(t)\dot{A}(t)\dot{N}(t-i\lambda)}{2N(t-i\lambda)^3} + \frac{3A(t)^2\dot{A}(t-2i\lambda)\dot{N}(t-i\lambda)}{2A(t-2i\lambda)N(t-i\lambda)^3} - \frac{A(t)\ddot{A}(t)}{2N(t-i\lambda)^2} - \frac{3A(t)^2\ddot{A}(t-2i\lambda)}{2A(t-2i\lambda)N(t-i\lambda)^2} = p(t)$$

- i. g. extremely complicated ☹,
  . . . but de Sitter space + cosmological constant solves it ☺

Field Fluctuations on NC Backgrounds:

a first step towards physics

Actions for Killing RJS deformed fields:

- fixed Riemannian manifold $(\mathcal{M}, g)$ with isometries $\mathfrak{g}$
- Definition: **Killing twist** $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g} \subseteq U\Xi \otimes U\Xi$
- nice feature: we have Hodge $*$ and thus actions!
- examples: (here $(\omega, \omega')_\star := \int \omega \wedge_\star *\omega'$ is SP on forms)
  - $S^\star_\Phi = -\frac{1}{2}(d\Phi, d\Phi)_\star - \frac{m^2}{2}(\Phi, \Phi)_\star - \sum_{k=3}^{N} \lambda_k (1, \Phi^{\star k})_\star$
  - $S^\star_{\mathrm{YM}} = \kappa \mathrm{Tr}(F, F)_\star \,, \quad F = dA - A \wedge_\star A$
- **NB:**
  - holds for **curved ST**; not restricted to Minkowski!
  - graded cyclicity $\to$ free actions are undeformed ☺
  - field equations for Φ:

$$d^\dagger d\Phi + m^2\Phi + \sum_{k=3}^{N} k\lambda_k \Phi^{\star(k-1)} = 0$$

Deformed covariant phasespace for free scalar field:

- **method:** deformed Poisson geometry [Aschieri, Lizzi, Vitale]

$$\{F,G\}_\star = \{\bar{f}^\alpha F, \bar{f}_\alpha G\} = -\{\bar{R}^\alpha G, \bar{R}_\alpha F\}_\star$$
$$\{F,\{G,H\}_\star\}_\star = \{\{F,G\}_\star, H\}_\star + \{\bar{R}^\alpha G, \{\bar{R}_\alpha F, H\}_\star\}_\star$$
$$\{F, G\star H\}_\star = \{F,G\}_\star \star H + \bar{R}^\alpha G \star \{\bar{R}_\alpha F, H\}_\star$$

- **ingredients for space-time deformations:**
  - cov. phasespace **Sol** $\hat{=}$ solutions of $d^\dagger d\Phi + m^2\Phi = 0$
  - Peierls bracket: (with $\Delta = \Delta^{av} - \Delta^{ret}$ as fundamental solution)

$$\{F,G\} = \int \mathrm{vol}_x \mathrm{vol}_y \frac{\delta F}{\delta\Phi(x)} \Delta(x,y) \frac{\delta G}{\delta \Phi(y)}$$

  - lift twist to **Sol**: $\sharp : \Xi \to \mathrm{Vec}(\mathbf{Sol})$, $v^\sharp = -\int \mathrm{vol}\, \mathcal{L}_v(\Phi)\frac{\delta}{\delta\Phi}$

$\Rightarrow$ deformed algebra $(\mathcal{A}, \star, \{\ ,\ \}_\star)$ generated by $\Phi(h) = \int \mathrm{vol}\, \Phi h$

  - $\Phi(h)\star\Phi(k) = \Phi(\bar{f}^\alpha h)\cdot\Phi(\bar{f}_\alpha k)$
  - $\{\Phi(h),\Phi(k)\}_\star = \{\Phi(h),\Phi(k)\} = \Delta(h,k)$

Free QFT on Killing RJS backgrounds:

- $\{\Phi(h), \Phi(k)\}_\star = \Delta(h,k) \overset{\text{quant}}{\longrightarrow} \left[\hat{\Phi}(h), \hat{\Phi}(k)\right]_\star = i\Delta(h,k)\hat{1}$
- Fock space construction:
  - usual one-particle HS $\mathcal{H} = \textbf{Sol}^{\mathbb{C}}_{\text{pos.}}$ with $(\psi_1, \psi_2)_{\mathcal{H}} = -i\Omega(\bar{\psi}_1, \psi_2)$
  - use isomorphism $\Delta : C_0^\infty(\mathcal{M}, \mathbb{C})/\text{Ker}(\Delta) \to \textbf{Sol}^{\mathbb{C}}$ and define

$$\mathcal{K} := \Delta^{-1}(\mathcal{H}) \,, \quad \text{with} \quad ([h],[k])_{\mathcal{K}} := i\Delta(\bar{h}, k)$$

$\Rightarrow$ deformed one-particle scalar product

$$([h],[k])^\star_{\mathcal{K}} = i\Delta(\bar{f}^\alpha \bar{h}, \bar{f}_\alpha k) = i\Delta(\bar{h}, k)$$

  - multi-particle states live in usual Fock space

$$|h_1, h_2, \dots, h_n\rangle_\star := \hat{a}^\dagger(h_1) \star \hat{a}^\dagger(h_2) \star \cdots \star \hat{a}^\dagger(h_n) \star |0\rangle$$

- generalization of existing results [Aschieri, Fiore, Wess, Zahn, ...] to **curved spacetimes** with Killing RJS twists

What about non-Killing twists (as required for cosmology)**?**

- actions so far only for $X_\alpha \in \mathfrak{g}$, $\forall_\alpha$ ☹ (b/c Hodge $*$ still missing)
- …but wave equations can be constructed, e. g.

$$\Box^\star \Phi + F[\Phi] = 0 \,, \quad \text{where} \quad \Box^\star = g^{ab} \star \left(\nabla^\star_{e_b} \nabla^\star_{e_a} - \Gamma_{ba}{}^{c} \star \nabla^\star_{e_c} \right)$$

- example: free scalar field on de Sitter space

$$\ddot{\Phi}(x) + 3H\dot{\Phi}(x) - e^{-2Ht} \Delta \tilde{\mathbf{\Phi}}(\mathbf{x}) + M^2 \Phi(x) = 0 \,, \quad \text{where}$$

1. $\tilde{\mathbf{\Phi}}(\mathbf{x}) = \exp\left(i\lambda(\partial_t - Hr\partial_r)\right)\Phi(x)$ for $\left[\hat{t}, \hat{x}^i\right] = i\lambda\hat{x}^i$
2. $\tilde{\mathbf{\Phi}}(\mathbf{x}) = \exp\left(i\lambda H\partial_\phi\right)\Phi(x)$ for $\left[\hat{t}, \widehat{\exp i\phi}\right] = \lambda\widehat{\exp i\phi}$

- **NB:**
  - ☺ i. g. linear but nonlocal equations
  - ☹ deformed Poisson geometry and quantization still open problem

Conclusions and outlook

- NCG is interesting step between classical and quantum gravity
- we found approach to NC symmetry reduction

$\longrightarrow$ cosmological, black hole (and black brane) solutions

- distinct NC effects depending on model, e. g.
  - discrete time spectra in cosmology
  - discrete radius spectra for black holes
- $\exists$ "realistic" models worth for cosmological studies
- free QFT on curved Killing RJS backgrounds

… still many open questions and undone calculations remain:

- symmetry reduction and solutions in NC vielbein gravity
- cosmological powerspectra and CMB predictions
- (Q)FT on curved non-Killing RJS backgrounds
- interacting QFT on curved Killing deformed backgrounds
- …

(gathering stamina for the long jog ahead . . .)