The Schwarzschild space-time

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Outline

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1 Introduction and Notation

A 'fuzzy space' defined by commutation relations

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x^\sigma)$$

Classical gravitational field : a moving frame $\tilde{e}^\mu_\alpha$

Lack of commutativity : a commutator $J^{\mu\nu}$

Fourier map : $x^\mu \mapsto p_\alpha$

Derivations : $e_\alpha f = [p_\alpha, f]$

Blurring : $\tilde{e}^\mu_\alpha \mapsto e^\mu_\alpha = [p_\alpha, x^\mu]$

(Dual) frame : θ^α $(\theta^\alpha(e_\beta) = \delta^\alpha_\beta)$

Necessarily : $\theta^\alpha f = f\theta^\alpha$

Quadratic relation

$$2P^{\alpha\beta}{}_{\gamma\delta}p_\alpha p_\beta - F^\beta{}_{\gamma\delta}p_\beta - K_{\gamma\delta} = 0$$

First order

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{\alpha\beta}_{\gamma\delta} + i\epsilon Q^{\alpha\beta}{}_{\gamma\delta}$$

Leibniz rule

$$i\hbar dJ^{\mu\nu} = [x^\mu, dx^\nu] - [x^\nu, dx^\mu]$$

or

$$i\hbar e_\alpha J^{\mu\nu} = [x^\mu, e^\nu_\alpha] - [x^\nu, e^\mu_\alpha]$$

2 The classical space-time

Manifold : $V = S^2 \times \mathbb{R}^2 \qquad S^2 \subset \mathbb{R}^3$

Coordinates : geodesic normal $x^i = x_a$

Frame : mirror the symmetry of the manifold

Static : $\theta^0 = \theta^0_0 dt \qquad \theta^0_0 = F(r,t)$

Radial component : θ^4

Angular components : lift to S^3

Consider the basis

$$\eta^a = H\epsilon^a{}_{bc}\xi^b dx^c + G\xi^a\xi_b dx^b$$

with : $\xi^a = r^{-1}x^a, \qquad r_S = Hr$

We find that

$$x_a\eta^a = G\xi_b dx^b$$

Either radial or part of SO_3

We set accordingly as frame

$$\theta^a = H\epsilon^a{}_{bc}\xi^b dx^c \qquad \theta^4 = Gdr \qquad \theta^0 = Fdt$$

The flat-space limit :

$$F = 1 \qquad G = 1 \qquad H = 1$$

Not the usual frame : projector $\pi^a_b = \delta^a_b - \xi^a\xi_b$ has 2 'square roots':

$$\epsilon^a{}_{dc}\xi^d\epsilon^c{}_{eb}\xi^e = \pi_{ab}$$

The corresponding metric is given by the line element

$$ds^2 = g_{\alpha\beta}\theta^\alpha\theta^\beta = H^2\pi_{ab}dx^a dx^b + (G\xi_a dx^a)^2 - F^2 dt^2$$

with

$$\pi_{ab}dx^a dx^b = r^2 d\Omega.$$

3 The algebra

Topological product : $S^2 \times \mathbb{R}^2$

Classical algebra : $\mathcal{C}^\infty(S^2) \otimes \mathcal{C}^\infty(\mathbb{R}^2)$

Blurring :

$$\mathcal{C}^\infty(S^2) \mapsto \mathcal{A}_1 \qquad \mathcal{C}^\infty(\mathbb{R}^2) \mapsto \mathcal{A}_2$$

Quantization condition : $\dfrac{4\pi r_S^2}{2\pi \hbar\!\!\!/ } = n$

Basic relation :

$$[\xi^a, \xi^b] = \frac{i\hbar\!\!\!/}{r_S^2} \epsilon^{abc} \xi_c \qquad \xi^a = \frac{\hbar\!\!\!/}{r_S^2} J^a \quad \xi^2 = 1$$

To be found: $[\xi^a, t]$, $[\xi^a, r]$, $[t, r]$

Important change : $\mathcal{A}_1$ depends on r

To incorporate the r dependence:

$$\mathcal{A}_1 = \bigoplus_1^\infty M_i$$

$\mathcal{A}_1$ must be an $\mathcal{A}_2$-bimodule

Let : $J = \{J_n\}$

Then

$$(rJ)_n = (Jr)_n = r_n J_n$$

$$(\Lambda J)_n = J_{n+1}$$

$$(J\Lambda)_n =$$

The Λ is unitary with respect to $(J_i, J_j) = c_2 \delta_{ij}$

It follows that $\Lambda^{-1} r \Lambda = r + \dfrac{\bar{k}}{4 r_S}$

Set $\Lambda = e^{2\pi i \mu t}$

This is the $\mathbb{R}^1_q$ algebra

Complete by incorporating the $\mathbb{R}^1_q$ calculus.

4 The momenta

We set : $i\bar{k}p_a = \alpha x_a$

Product : $\alpha = \dfrac{\mu r_S^2}{r}$

Asymptotically flat : $\alpha = \dfrac{r_S}{r}$

We have in second case

$$[p_a, p_b] = i\epsilon\epsilon_{abc}p_0 p^c,$$

$$[p_a, p_0] = 0,$$

$$[p_4, p_a] = -(c_1 + c_2 i\epsilon p_0)p_a,$$

$$[p_4, p_0] = -(c_1 + c_2 i\epsilon p_0)p_0.$$

These relations define the quadratic algebra

Rescaled momenta

$$i\bar{k}\bar{p}_a = p_0^{-1}p_a, \qquad \bar{p}_4 = p_4, \qquad i\bar{k}\bar{p}_0 = p_0^{-1}.$$

We find that

$$[\bar{p}_a, \bar{p}_b] = i\epsilon\epsilon_{abc}\bar{p}^c, \quad [\bar{p}_a, p_0] = 0,$$

$$[\bar{p}_a, \bar{p}_4] = 0, \qquad [\bar{p}_4, \bar{p}_0] = \mu^{-2}.$$

Tensor product of rotation and Heisenberg.

A priori structure

The momenta p_α bracket

$$[p_\alpha p_\beta] = K_{\alpha\beta} + F^\gamma{}_{\alpha\beta} p_\gamma - 2i\epsilon Q^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta.$$

Possible non-vanishing coefficients $(u, v = 4, 0)$

$$Q^u{}_{abc} = q_1{}^u \epsilon_{abc} \quad Q^{ua}{}_{bv} = q_2{}^u_v \delta^a_b$$

$$Q^{uv}{}_{04} = q_3{}^{uv} \qquad Q^{ab}{}_{04} = g^{ab} q_4$$

$$F_{abc} = f_1 \epsilon_{abc} \qquad F^a{}_{bu} = f_{2u} \delta^a_b$$

$$F^u{}_{04} = f^u_3 \qquad K_{uv} = k_{uv}$$

The system of equations reduces therefore to

$$[p_a, p_b] = (f_1 - 2i\epsilon {q_1}^u p_u)\epsilon_{abc}p^c$$

$$[p_a, p_u] = (f_{2u} - 2i\epsilon q_{2\,u}^{\,v} p_v)p_a$$

$$[p_0, p_4] = k_{04} + f_3^v p_v - 2i\epsilon({q_3}^{uv} p_u p_v + q_4 p^2)$$

But, we recall :

$$i\hbar p_a = x_a \qquad i\hbar p_0 = f(r) \qquad i\hbar p_4 = t$$

Thus :

$$r_S^{-2} = (f_1 - 2i\epsilon {q_1}^0 p_0)$$

$$[x^a, t] = (f_{20} - 2i\epsilon q_{2\,0}^{\,0} p_0)p_a$$

$$f' = k_{04} + f_3^0 f - 2i\epsilon({q_3}^{00} f^2 + q_4 c_2)$$

5 The calculus