Ultraviolet/Infrared Mixing and Translational Invariance

Fedele Lizzi

With S. Galluccio & P. Vitale

To appear shortly

Bayrischzell 2009

Among the original motivations to consider space(time) to be noncommutative there was the hope that the presence of a cutoff would cure some of the ultraviolet problems of field theory

The analogy is with the origin of quantum mechanics where the presence of $\boxed{\hbar}$ cures the ultraviolet catastrophe of black-body radiation, and introduces noncommutativity on phase space

Soon after the introduction of field theory on noncommutative space it was realised that the presence of a cutoff does not ensure the cancelation of infinities.

Moreover there was the appearance of the Ultraviolet/Infrared mixing

Consider a field theory on a noncommutative space (for example bosonic $\boxed{\varphi^{\star 4}}$), where the noncommutativity of the space is enforced by the Grönewold-Moyal product

Then it is true that some ultraviolet divergence disappear, But not all of them

The infinities which disappear from the ultraviolet, reappear as infrared divergences

To be more specific consider a field theory described by the action

$$\boxed{S = \int \mathrm{d}x^d \left(\frac{1}{2} \left(\partial_i \varphi \star \partial_i \varphi - m^2 \varphi \star \varphi \right) + \frac{g}{4!} \varphi \star \varphi \star \varphi \star \varphi \right)}$$

Where $\star$ is the usual Grönewold-Moyal product and we consider the theory to be Euclidean (or equivalently we do not consider timespace noncommutativity)

Let me give two expressione for this product

$$(f \star_{\mathsf{M}} g)(x) = \mathrm{e}^{\frac{i}{2}\theta^{ij}\partial_{y_i}\partial_{z_j}} f(y)g(z)\Big|_{x=y=z}$$

$$(f \star_{\mathsf{M}} g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int \mathrm{d}p^d \mathrm{d}q^d \tilde{f}(q)\tilde{g}(p-q)\mathrm{e}^{ip\cdot x}\mathrm{e}^{ip_i\theta^{ij}q_j}$$

where $\tilde{f}$ and $\tilde{g}$ are the usual Fourier transforms of f and g

The two definitions have different (dense) domains of definition, but coincide on their interesction

Since for this product $\boxed{\int \mathrm{d}x^d f \star_{\mathrm{M}} g = \int \mathrm{d}x^d f g}$ the quadratic (free) theroy is the same as in the commutative case

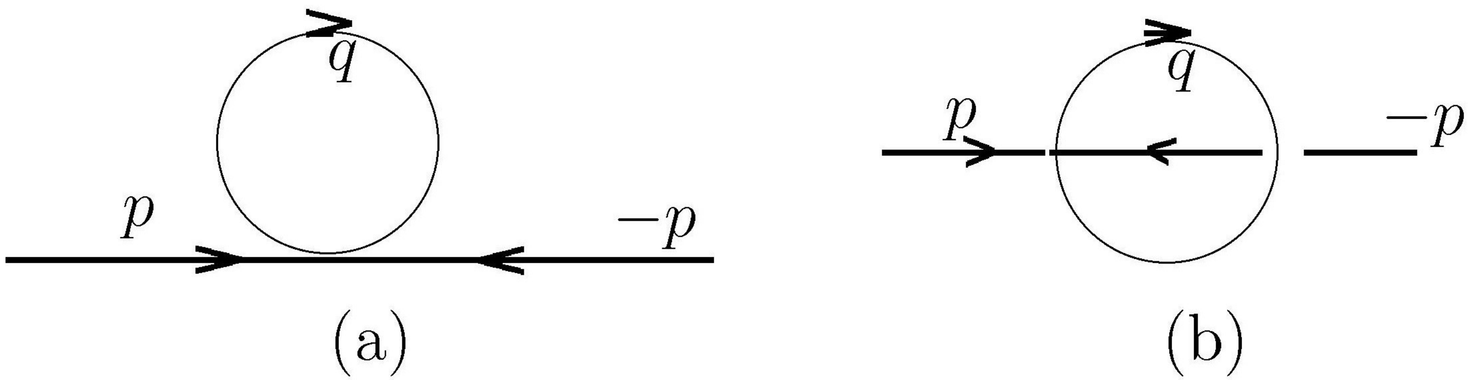
The vertex changes. For four incoming momenta we have

$$\boxed{V_{\mathrm{Moyal}} = V_0 \mathrm{e}^{-\frac{i}{2}\sum_{a\leq b}\theta^{ij}k_{a_i}k_{b_j}}}$$

where the ordinary vertex is

$$\boxed{V_0 = -i\frac{g}{4!}(2\pi)^d \delta^d\left(\sum_{a=1}^{4} k_a\right)}$$

This vertex is not invariant for arbitrary permutation of the momenta, therefore planar and nonplanar diagrams have different behaviour



The planar diagram is unchanged, because the momenta in the exponential cancel once conservation of momentum is imposed

$$G_{\mathrm{P}}^{(2)} = -i\frac{g}{3}\int \frac{\mathrm{d}q^d}{(2\pi)^d}\frac{1}{(p^2-m^2)^2(q^2-m^2)}$$

The nonplanar one gains an oscillating term which at high momentum is convergent

$$G_{\mathrm{N}P}^{(2)} = -i\frac{g}{6}\int \frac{\mathrm{d}q^d}{(2\pi)^d}\frac{\mathrm{e}^{ip_i\theta^{ij}q_j}}{(p^2-m^2)^2(q^2-m^2)}$$

The exponential in the nonplanar term makes the integral convergent for high momentum, but in the infrared the divergence comes back

$$\lim_{p_i\theta^{ij}\to 0} G_{\mathrm{N}P}^{(2)} = \frac{1}{2}G_{\mathrm{P}}^{(2)}$$

The heuristic explanation for this phenomenon is that the presence of the nontrivial commutator $[x^i, x^j]_{\star_\mathsf{M}} = i\theta^{ij}$ causes a generalization of Heisenberg's uncertainty principle, so that the short distance behaviour (high p_i) is linked to the large distance behaviour (low $\theta^{ij}p_j$)

The Grönewold-Moyal product has not succeeded in eliminating the divergences

One can ask if other products can solve the problem

It is known that of course fuzzy models, being finite, are free of ultraviolet, that the phenomenon persists in the nonrelativistic case (Grosse and Wohlgeannt), it may be absent for timespace noncommutativity (Bahns). Modifications of the action (Grosse Wulkenhaar, Rivasseau group) can also substantially alter the ultraviolet behaviour

Considering the Wick-Voros product also the situation is not changed, the loop behaviour is the same

We have investigated the issue for a general translationally invariant associative star products

Defining the translation by a vector $\boxed{a}$ by $\boxed{\mathcal{T}_a(f)(x) = f(x+a)}$, by translation invariant product we mean

$$\boxed{\mathcal{T}_a(f) \star \mathcal{T}_a(g) = \mathcal{T}_a(f \star g)}$$

Which for Fourier transform becomes

$$\boxed{\widetilde{\mathcal{T}_a f}(q) = \mathrm{e}^{iap} \tilde{f}(q)}$$

Such a product can be expressed, in integral form as

$$f \star g = \frac{1}{(2\pi)^{\frac{d}{2}}} \int \mathrm{d}p^d \mathrm{d}q^d \mathrm{e}^{ip\cdot x} \tilde{f}(q)\tilde{g}(p-q)\mathrm{e}^{\alpha(p,q)}$$

Associativity and the requirement that the integral is a trace impose severe constraints on $\boxed{\alpha}$ which has to satisfy

$$\alpha(p,q) + \alpha(q,r) = \alpha(p,r) + \alpha(p-r,q-r)$$

A product of this form can be commutative, in this case

$$\alpha(p,q) = \alpha(p,p-q)$$

The quadratic action changes because

$$\int \mathrm{d}x^d f \star g = \int \mathrm{d}x^d g \star f \neq \int \mathrm{d}x^d f g$$

Therefore also the free propagator is altered

$$\tilde{G}_0^2(p) = \frac{e^{-\alpha(0,p)}}{p^2 - m^2}$$

The difference in the propagator can be absorbed in the S-matrix if the product is coming from a twist.

In this talk we are not interested in the propagator and consider only at the changes in the vertex

The vertex is given by (using associativity)

$$V_\star = V_0 e^{\alpha(k_1+k_2,k_1)+\alpha(k_3+k_4,k_3)+\alpha(0,k_1+k_2)}$$

The Green's function for the planar cases is, after several transformations using associativity:

$$G_P^{(2)} = \int \mathrm{d}q^d \frac{\mathrm{e}^{-\alpha(0,p)}}{(p^2-m^2)^2(q^2-m^2)}$$

We see that with respect to the commutative case the only correction is in the factor $\mathrm{e}^{-\alpha(0,p)}$ which is the correction of the free propagator.

The ultraviolet divergences of the loop are the same and therefore the short distance physics is unaffected (in this respect) by the star product. The momentum q in the internal loop appears in an unchanged form. The loop correction is the same and will have the same properties of the commutative theory.

For the nonplanar case we obtain

$$G_{NP}^{(2)} = \int \mathrm{d}q^d \, \frac{\mathrm{e}^{-\alpha(0,p)+\omega(p,q)}}{(p^2 - m^2)^2 (q^2 - m^2)}$$

with

$$\omega(p,q) = \alpha(p+q,p) - \alpha(p+q,q)$$

For the Groönewold-Moyal product (for which $\alpha(0,p) = 0$) we have $\omega(p,q) = i p_i \theta^{ij} q_j$

Using the associativity properties one can show that

$$\omega(p,q) = \alpha(p,p-q) - \alpha(p,q)$$

Which was the quantity that had to be zero in the case of commutative product

Therefore the nontrivial contribution to the loop is exactly the failure of the product to be commutative

We have therefore shown that a commutative star product does not change the ultraviolet behaviour of the theory.

This is non trivial, fudging with the (commutative) product the vertex and the Green's function are different. Only imposing twisted symmetry one can probably showthat the S-matrix is unchanged

We can say more. Let us express the function ω with a multi-index notation as

$$\omega(p,q) = \sum_{\vec{i}\,\vec{j}} a_{\vec{i}\,\vec{j}}\, p^{\vec{i}} q^{\vec{j}}$$

with $\vec{i} = (i_1, \ldots i_d)$ and $p^{\vec{i}} = p_1^{i_1} p_2^{i_2} \ldots p_d^{i_d}$

Using identities coming from associativity and antisymmetry we have

$$\boxed{\sum_{\vec{i}\,\vec{j}} a_{\vec{i}\,\vec{j}}\, q^{\vec{i}}(p^{\vec{j}} - (p-q)^{\vec{j}}) = 0}$$

This condition implies that the coefficient $\boxed{a}$ must vanish except in the case in which all of the $\boxed{j_a}$'s but one vanish. In this case the antisymmetry of the $\boxed{a}$'s ensures vanishing of the expression without further constraints. Using antisymmetry the same reasoning can be done for the first multiindex

This shows that the term appearing in the one loop amplitude for the nonplanar graphs is necessarily quadratic and therefore it must be

$$\boxed{\omega(p,q) = i\theta^{ij} p_i q_j}$$

From associativity it is easy to prove that $\boxed{\alpha}$ cannot have linear terms, we can therefore express it as a series starting from quadratic terms

$$\boxed{\alpha(p,q) = \alpha_{ij} p^i q^j + \ldots}$$

And therefore

$$\boxed{\omega(p,q) = i\theta^{ij} p_i q_j = \alpha^{ij}(p_i + q_i)p_j - \alpha^{ij}(p_i + q_i)q_j = \alpha^{ij}(p_i + q_i)(p_j - q_j)}$$

which means, calculating the product among the coordinates

$$\boxed{[x^i, x^j]_\star = i\theta^{ij}}$$

In conclusion, the correction in the loop is only dependent on the commutator among the coordinates

A classical result of Kontsevich says that two products, $\boxed{\star}$ and $\boxed{\star'}$ with the same Poisson structure, i.e. the same commutator among coordinates in the flat case, are equivalent. There exists a map $\boxed{T}$ with the property

$$\boxed{T(f) \star T(g) = T(f \star' g)}$$

And this in turn means that the ultraviolet/infrared mixing (in the translationally invariant case) is the same for all noncommutative products (apart from the choice of $\boxed{\theta}$)

Conclusions

We have seen how the Ultraviolet/Infrared mixing is generic for translationally invariant products and that it depends on an antisymmetric matrix, the commutator of the coordinate

We have done this not for for products defined as formal series, but operatively defined (for functions with a Fourier transform)

Equivalent products have the same Ultraviolet/Infrared mixing, but can have different Green's functions

It would be nice to connect this with Drinfeld twists (easy) and to generalize to more general classes of products