Quantum coordinates of an event

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(based on joint work with Romeo Brunetti and Marc Hoge)

Introduction

Main conceptual problem for the quantization of gravity:

Spacetime should be observable in the sense of quantum physics,

but spacetime in quantum field theory is merely a tool for the parametrization of observables (local fields)
(a priori structure)

Analogous problem in quantum mechanics:

Parameter time versus observable time

A new interpretation of the Schrödinger equation

Schrödinger equation:

$$i\frac{d}{dt}\psi(t) = H_0\psi(t) \ ,$$

H_0 selfadjoint operator on Hilbert space $\mathfrak{H}_0$ with domain $D(H_0)$, ψ differentiable function on $\mathbb{R}$ with values in $D(H_0)$.

Reinterpretation as a constraint

$$H\psi = 0$$

where $H = -i\frac{d}{dt} + H_0$ is a selfadjoint operator on $\mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$.

Problem: H has continuous spectrum, hence $\psi \notin \mathfrak{H}$.

Traditional description of eigenfunctions associated to points in the continuous spectrum : realization of ψ as a linear functional on a dense subspace $D \subset \mathfrak{H}$.

Buchholz-Porrmann: improper eigenfunctions give rise to weights on a suitable subset of observables ($\Longrightarrow$ new approach to the infrared problem ("charged particles without photon cloud"))

Weights: positive linear functionals on the algebra of observables which are not necessarily normalizable

Classical analogue: unbounded positive measures

Standard example: Trace on an infinite dimensional Hilbert space

Example in physics: scattering cross sections

Construction of a weight w_ψ, $\psi : D \to \mathbb{C}$ linear, D dense in $\mathfrak{H}$:

$$R_D := \{A \in \mathcal{B}(\mathfrak{H}) | A\mathfrak{H} \subset D\}$$

$$A \in R_D \Longrightarrow A^*\psi \in \mathfrak{H}$$

$$w_\psi(\sum A_i B_i^*) = \sum \langle A_i^*\psi, B_i^*\psi\rangle \ , \ A_i, B_i \in R_D$$

Extension to all positive bounded operators C:

$$w_\psi(C) = \sup_{0 \leq B \leq C, \ B \in R_D R_D^*} w_\psi(B)$$

Define the left ideal

$$L_\psi := \{A \in \mathcal{B}(\mathfrak{H}) | w_\psi(A^* A) < \infty\} \ .$$

and extend the weight to $L_\psi^* L_\psi$ by linearity and the polarization equality.

Positive semidefinite scalar product on L_ψ:

$$\langle A, B \rangle := w_\psi(A^* B) \ .$$

$\Longrightarrow$ GNS-representation $(\mathfrak{H}_\psi, \pi_\psi)$ by left multiplication and dividing out the null space of the scalar product.

Interpretation of the state induced by $A \in L_\psi$:

$$\omega_{A\psi}(B) := \frac{\langle A, \pi_\psi(B)A\rangle}{\langle A, A\rangle} = \frac{w_\psi(A^*BA)}{w_\psi(A^*A)}$$

is the expectation value of B under the condition that the event A^*A took place. (Note the dependence on the phase of A.)

Application to solutions of the Schrödinger equation

$$\psi : \mathbb{R} \in \mathfrak{H}_0 \ , \ \psi(t) = e^{-iH_0 t}\psi(0)$$

Domain of ψ as a linear functional on $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}, \mathfrak{H}_0)$:

$$D = \{\varphi : \mathbb{R} \to \mathfrak{H}_0 \text{ continuous} \ , \int dt ||\varphi(t)|| < \infty\}$$

Let $C : \mathbb{R} \to \mathcal{B}(\mathfrak{H}_0)$ be strongly continuous, bounded and positive operator valued. The weight associated to ψ is defined on C by

$$w_\psi(C) := \int dt \langle \psi(t), C(t)\psi(t) \rangle \in \mathbb{R}_+ \cup \{\infty\}$$

The left ideal L_ψ contains e.g. multiplication operators by test functions $g(t)$. For $A \in \mathcal{B}(\mathfrak{H}_0)$ we find

$$\omega_{g\psi}(A) = \frac{w_\psi(g^* A g)}{w_\psi(g^* g)} = \frac{\int dt\, |g(t)|^2 \langle \psi(t), A\psi(t) \rangle}{\int dt\, |g(t)|^2 \langle \psi(t), \psi(t) \rangle}$$

If $|g(t)|^2 \to \delta_{t_0}$, we obtain the state induced by $\psi(t_0)$. Hence standard quantum mechanics on $\mathfrak{H}_0$ is covered by the enlarged formalism.

Additional elements of L_ψ: Operators $A \in \mathcal{B}(\mathfrak{H}_0)$ with

$$w_\psi(A^*A) \equiv \int dt \langle \psi(t), A^*A\psi(t)\rangle < \infty$$

exist for suitable ψ iff the spectrum of H_0 is absolutely continuous.

Let $B = \int dt e^{iH_0 t} A^*A e^{-iH_0 t}$. B can be interpreted as the

dwell time

of the event A^*A and is in general unbounded.

Assumption: The kernel of B is trivial (otherwise replace $\mathfrak{H}_0$ by the orthogonal complement of the kernel).

Relation between the states $\omega_{A\psi}$ on $\mathcal{B}(\mathfrak{H})$ and $\omega_{\sqrt{B}\psi(0)}$ on $\mathcal{B}(\mathfrak{H}_0)$:

$$\omega_{A\psi} = \omega_{\sqrt{B}\psi(0)} \circ \Phi_A$$

where Φ_A is the completely positive mapping

$$\Phi_A(C) = V_A^* C V_A$$

and $V_A : \mathfrak{H}_0 \to \mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$ is the isometry

$$(V_A \psi_0)(t) = A e^{-itH_0} B^{-\frac{1}{2}} \psi_0 \ .$$

The time parameter of the Schrödinger equation is a selfadjoint multiplication operator on $\mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$. Its spectral projections $E(I)$ can be mapped to positive operators

$$P_A(I) = \Phi_A(E(I))$$

on $\mathfrak{H}_0$.

One obtains the positive operator valued measure

$$I \to P_A(I) = B^{-\frac{1}{2}} \int_I dt\, e^{iH_0 t} A^* A e^{-iH_0 t} B^{-\frac{1}{2}}$$

interpreted as time of occurence of the event A^*A in [Brunetti,F 2002]

Example: Particle moving freely in 1 dimension.
Event: Particle stays in a neighbourhood of the origin.
Event represented by the projection

$$A_a\Phi(x) \equiv \chi_a(x)\Phi(x) = \begin{cases} \Phi(x) & , \quad |x| \leq a/2 \\ 0 & , \quad \text{else} \end{cases}$$

Time, the particle spends inside the interval $[-a/2, a/2]$:

$$B_a = \frac{ma}{|p|}\left(1 + \frac{\sin pa}{pa}\Pi\right)$$

with the parity operator Π
(vanishes on the antisymmetric subspace in the limit $a \to 0$).
Isometry V_A:

$$V_A = \frac{\chi_a(x)}{\sqrt{a}} e^{-it\frac{p^2}{2m}} \sqrt{\frac{|p|}{m}} \left(1 + \frac{\sin pa}{pa}\Pi\right)^{-\frac{1}{2}}$$

POVM in the limit $a \to 0$:

$$P(I)(p,q) = V_A^* \chi_I V_A(p,q) = \begin{cases} \frac{\sqrt{pq}}{2\pi m} \int_I dt\, e^{it\frac{p^2-q^2}{2m}} & , \quad pq > 0 \\ 0 & , \quad \text{else} \end{cases}$$

First moment

$$T = \int tP(t, dt)$$

yields Aharanov's time operator

$$T = -\frac{m}{2}(p^{-1}x + xp^{-1})$$

Warning: T is not selfadjoint, but maximally symmetric with deficiency indices $(2, 0)$.
$(P(I))_I$ generates Töplitz quantization of $\mathbb{R}$ ($\Longrightarrow$ nontrivial uncertainty relation for time measurements (Brunetti, F 2002))

Event localization on Minkowski space

U_0 representation of the translation group on $\mathfrak{H}_0$

$\mathfrak{H} := L^2(\mathbb{M}, \mathfrak{H}_0)$

$$(U(x)\psi)(y) := U_0(x)\psi(y-x)\ ,\ \ \psi \in \mathfrak{H}$$

Constraint: $U(x)\psi = \psi$

Associated weight w_ψ, with left ideal L_ψ.

$A \in L_\psi \cap \mathcal{B}(\mathfrak{H}_0) \iff \int d^4x \langle \psi(0), U_0(x)A^*AU_0(-x)\psi(0)\rangle < \infty$

$B := \int d^4x U_0(x)A^*AU_0(-x)$ spacetime volume of the event

Restriction of $\mathfrak{H}_0$ to $ker(B)^\perp$

Construction of a positive operator valued measure on Minkowski space:

$$P(G) = V_A^* \chi_G V_A$$

with $V_A : \mathfrak{H}_0 \to L^2(\mathbb{M}, \mathfrak{H}_0)$

$$(V_A \psi_0)(x) = A U_0(x) B^{-\frac{1}{2}} \psi_0$$

Example: Free scalar field on Fock space with mass m

$$U_0(x)A^*AU_0(-x) = a^*(x)^2 a(x)^2$$

a, a^* annihilation and creation operators

Interpretation: 2 particles collide at the spacetime point x

Restriction to the 2 particle subspace: $\mathrm{Ker} B^\perp$ is the space of s waves
(collisions occur only if the relative angular momentum vanishes)

$$\mathcal{H}_0 \simeq L^2(H^+_{>2m})$$

(as representations of the translation group)
$H^+_{>2m} = \{p \in \mathbb{M}^*, p^2 > 4m^2, p_0 > 0\}$ 2 particle momentum spectrum

$$(V_A\Phi)(x,k) = (2\pi)^{-2}g(k)\int d^4p e^{-ipx}\Phi(p)\ ,\ ||g||_2^2 = 1\ ,\ g \geq 0$$

"Coordinate operators":

$$\hat{x}^\mu := V_A^* x^\mu V_A = \frac{1}{i}\frac{\partial}{\partial p_\mu}$$

(with Dirichlet boundary conditions on the boundary of $H^+_{>2m}$)

("Töplitz quantization" of Minkowski space)

Localization in spacetime

$\mathcal{M} = \Sigma_0 \times \mathbb{R}$ globally hyperbolic d-dimensional spacetime.
$\chi_x : \Sigma_0 \to \Sigma_x$ Cauchy surface in $\mathcal{M}$, $x \in \mathcal{U}$, $0 \in \mathcal{U} \subset \mathbb{R}^d$
Time slice axiom: $\alpha_{\chi_x} : \mathfrak{A}(\Sigma_0) \to \mathfrak{A}(\mathcal{M})$ isomorphism

$$\mathfrak{B} := \mathcal{C}_0(\mathcal{U}, \mathfrak{A}(\mathcal{M}))$$

ω_0 state on $\mathfrak{A}(\mathcal{M}) \Longrightarrow$

$$w(B) = \int_{\mathcal{U}} d^d x \omega_0(\alpha_{\chi_x} \alpha_{\chi_0}^{-1}(B(x)))$$

weight on $\mathfrak{B}$.

$$\frac{w(A^* CA)}{w(A^* A)}$$

conditional expectation value of $C \in \mathfrak{B}$.

$p \in \Sigma_0 \subset \mathcal{M}$, $\mathcal{O}$ neighbourhood of p in $\mathcal{M}$.
$A \in \mathfrak{A}(\mathcal{O}) \Longrightarrow$

$$\alpha_{\chi_x}\alpha_{\chi_0}^{-1}(A) \in \mathfrak{A}(\chi_x \circ \chi_o^{-1}(\mathcal{O}))$$

$0 \leq A^*A \leq 1 \Longrightarrow w(A^*A)$ integral over the (not mutually exclusive) probabilities of the effects $\alpha_{\chi_x}\alpha_{\chi_0}^{-1}(A^*A)$

$$B = \int_{\mathcal{U}} d^d x \alpha_{\chi_x}\alpha_{\chi_0}^{-1}(A^*A)$$

$$\phi_A(C) = B^{-\frac{1}{2}} \int_{\mathcal{U}} d^d x \alpha_{\chi_x}\alpha_{\chi_0}^{-1}(A^*C(x)A)B^{-\frac{1}{2}}$$

completely positive mapping.

Let χ_G be the characteristic function of $G \subset \mathcal{U}$:

Interpretation of expectation values of the effect operator $\phi_A(\chi_G)$:

(generalized Schrödinger picture)

Probability, that that the event A^*A took place in $\mathcal{O}_G$, provided it took place within $\mathcal{O}_\mathcal{U}$

$$(\mathcal{O}_G = \bigcup_{x \in G} \chi_x \chi_0^{-1}(\mathcal{O}))$$

Conclusions and Outlook

- Observables (in the sense of positive operator valued measures) of time of occurence and of spacetime localization of events can be given.
- They typically yield noncommutative spaces. For instance in the case $\mathrm{sp}(H) = \mathbb{R}_+$ one obtains the Töplitz quantization of $\mathbb{R}$ as the quantized time axis. This implies new uncertainty relations for time measurements alone,

$$\Delta T \geq \frac{d}{\langle H \rangle}$$

with $d = 1.376$.

- There exists an interacting model (the Grosse-Lechner-Buchholz-Summers model) which delivers coordinate operators with commutation relations

$$[q^\mu, q^\nu] = \theta^{\mu\nu}$$

- In analogy to renormalization theory one may interpret parametric spacetime as bare spacetime and the observable spacetime as the physical spacetime.