Covariant Lyapunov vectors and local exponents

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Abstract. Using a doubly-thermostated heat-conducting oscillator as an example, we demonstrate how time-reversal invariance affects the perturbation vectors in tangent space and the associated local Lyapunov exponents. We also find that the local covariant exponents vary discontinuously along directions transverse to the phase flow.

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For classical chaotic systems, the set of Lyapunov exponents, $\{\lambda_\ell\}, \ell = 1, \ldots, D$, measures the exponential growth, or decay, of small (infinitesimal) perturbations of the phase-space trajectory. Here, D is the dimension of the phase space. The standard algorithms for the computation of the exponents (see Ref. [1] for a review) probe the tangent-space dynamics by a set of orthonormal Gram-Schmidt (GS) vectors $\{\mathbf{g}_\ell\}$. Since their orthonormality is not preserved, it must be periodically restored by a GS procedure, or continuously kept up by Lagrange-multiplier constraints. These schemes are all based on the volume changes of d-dimensional volume elements in phase space, $d \le D$, but they simultaneously destroy any information concerning the angles between the perturbation vectors. Thus, symmetries concerning phase-space volumes generate symmetries of the associated local GS Lyapunov exponents[1], whereas time-reversal symmetry does not. For example, for a phase-space conserving *symplectic system* the equalities ${}^{(+)}\Lambda_\ell^{\mathrm{GS}}(t) = -{}^{(+)}\Lambda_{D+1-\ell}^{\mathrm{GS}}(t)$ and ${}^{(-)}\Lambda_\ell^{\mathrm{GS}}(t) = -{}^{(-)}\Lambda_{D+1-\ell}^{\mathrm{GS}}(t)$ hold, if the trajectory is followed forward or backward in time as indicated by the upper indices ${}^{(+)}$ and ${}^{(-)}$, respectively. However, time reversal invariance of the motion equations is not reflected by the GS local exponents: ${}^{(-)}\Lambda_\ell^{\mathrm{GS}}(t) \ne -{}^{(+)}\Lambda_{D+1-\ell}^{\mathrm{GS}}(t)$.

The multiplicative ergodic theorem of Oseledec [2, 3] asserts that there exists another spanning set of normalized vectors $\mathbf{v}^\ell(\Gamma(0))$ in tangent space. These vectors evolve (co-rotate) with the natural tangent flow, $\mathbf{v}^\ell(\Gamma(t)) = D\phi^t|_{\Gamma(0)}\, \mathbf{v}^\ell(\Gamma(0))$, (where $D\phi^t|_{\Gamma(0)}$ is the propagator), and directly generate the Lyapunov exponents, $\pm\lambda_\ell = \lim_{t\to\pm\infty}(1/|t|)\ln\left\| D\phi^t|_{\Gamma(0)}\ \mathbf{v}^\ell(\Gamma(0))\right\|$, $\ell \in \{1, \cdots, D\}$, along the way. They are referred to as covariant vectors. Generally, they are not pairwise orthogonal and span invariant manifolds, for which the local expansion (contraction) rates are given by the local covariant Lyapunov exponents ${}^{(\pm)}\Lambda_\ell^{\mathrm{cov}}$. In contrast to the GS exponents, they respect

[1] Local Lyapunov exponents give the local (time-dependent) exponential rate of growth (shrinkage) of the norm for GS or covariant vectors at a phase point $\Gamma(t)$ along the trajectory. The global exponents are time averages of the local exponents.

the time-reversal invariance of the motion equations, such that

$$^{(-)}\Lambda_\ell^{\mathrm{cov}}(\Gamma(t)) = -^{(+)}\Lambda_{D+1-\ell}^{\mathrm{cov}}(\Gamma(t))\,; \quad \ell = 1,\cdots,D. \tag{1}$$

Local expansion forward in time implies local contraction backward in time and *vice versa*. However, they do not reflect (possible) phase-volume conservation.

Recently, reasonably efficient algorithms for the computation of covariant vectors have become available [4, 5], which were applied to a variety of systems [1, 6, 7]. In the panel on the left of the figure we demonstrate the time-reversal symmetry displayed by the local covariant exponents for a one-dimensional harmonic oscillator coupled to

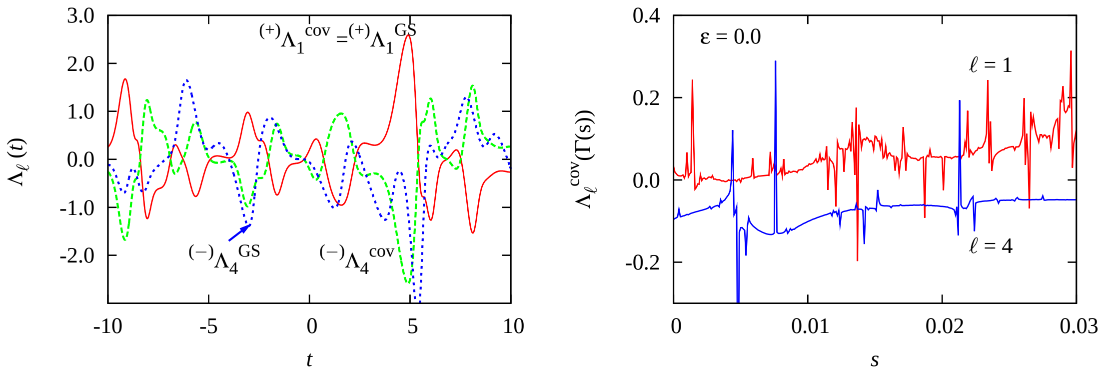

FIGURE 1. Panel on the left: Time-dependent local Lyapunov exponents $\Lambda(t)$ (as identified by the labels) for the doubly-thermostated oscillator in a nonequilibrium stationary state ($\varepsilon = 0.25$). Panel on the right: Fractal behavior of the local covariant exponents $^{(+)}\Lambda_i^{\mathrm{cov}}$ for $\ell \in \{1,4\}$ along a parametric straight line transverse to the phase flow. The parameter s specifies the location $\Gamma(s)$ in phase space. The data are for the doubly-thermostated oscillator in thermal equilbrium ($\varepsilon = 0$).

a position-dependent temperature $T(q) = 1 + \varepsilon \tanh(q)$ with a two-stage Nosé-Hoover thermostat, which makes use of two thermostat variables [1]. The equations of motion are time reversible and not symplectic, and $D = 4$. The control parameter ε denotes the temperature gradient at the oscillator position $q = 0$. As the figure shows, the time-reversal symmetry of Eq. (1) is clearly obeyed for the maximum ($\ell = 1$) and minimum ($\ell = 4$) covariant exponents. Since, by construction, $^{(+)}\Lambda_1^{\mathrm{GS}} \equiv ^{(+)}\Lambda_1^{\mathrm{cov}}$ [1], one observes that no analogous relation holds for the GS exponents.

In the panel on the right of the figure it is demonstrated that the local covariant exponents show a fractal-like structure along a straight line transverse to the phase-space flow [1]. This is to be expected in view of the different past (and future) histories for trajectories passing through adjacent phase-space points transverse to the flow.

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