THE AUTOMORPHIC COHOMOLOGY AND THE RESIDUAL SPECTRUM OF HERMITIAN GROUPS OF RANK ONE

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Abstract. Let $G/\mathbb{Q}$ be an inner form of $Sp_4/\mathbb{Q}$ which does not split over $\mathbb{R}$. Consequently, it is not quasisplit. In this paper we determine completely the automorphic cohomology of G. That is, we describe the Eisenstein and the cuspidal cohomology of congruence subgroups Γ of G with respect to arbitrary coefficient systems. In particular we establish precise non-vanishing results for cuspidal cohomology. In addition, we calculate the residual spectrum $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of G.

Introduction

In one of his early papers, [24], G. Harder developed a technique to describe the cohomology of a torsionfree, arithmetic subgroup Γ of a semisimple, algebraic $\mathbb{Q}$-group G of $\mathbb{Q}$-rank 1 by means of Eisenstein series and cusp forms. This method was of geometric nature and used the fact that the cohomology of Γ is isomorphic to the cohomology of a certain compact space $\Gamma\backslash\overline{X}$, which is a manifold with boundary $\partial(\Gamma\backslash\overline{X})$. In fact, X is the Riemannian symmetric space associated to the Lie group $G(\mathbb{R})$ of real points of G and $\Gamma\backslash\overline{X}$ is the Borel-Serre compactification of the locally symmetric quotient $\Gamma\backslash X$. With this framework at place, Harder showed that one can construct the "cohomology at infinity", i.e., the image of the natural restriction map $H^*(\Gamma\backslash\overline{X}) \to H^*(\partial(\Gamma\backslash\overline{X}))$ by means of Eisenstein series.
This description is explicit only up to the knowledge of the various poles of the Eisenstein series in question, that is, by the work of R. P. Langlands, [33], up to the knowledge of the residual spectrum $L^2_{res}(\Gamma\backslash G(\mathbb{R}))$. Even for the rank-one case, this problem is not solved yet, since for example a satisfactory description of the residual spectrum of non-quasisplit groups is not available presently.

Put in a more modern language, in this paper we first explicitly determine the residual spectrum $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of an inner $\mathbb{Q}$-form G of the split group Sp_4, which does not split over $\mathbb{R}$, i.e., $G(\mathbb{R}) \cong Sp(1,1)$. It is a *non*-quasisplit, simple group of $\mathbb{Q}$-rank one and hence we are out of reach of the Langlands-Shahidi method. Still we can give an entire description of the whole residual spectrum of G.
To that end, let L be a Levi subgroup of the only proper standard parabolic $\mathbb{Q}$-subgroup P of G and let π be a cuspidal automorphic representation of $L(\mathbb{A})$. We can form the parabolically induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\pi \otimes e^{(s+\rho_P)H_P(.)}]$, where as usual ρ_P is the half-sum of positive $\mathbb{Q}$-roots and $H_P(.)$ is the Harish-Chandra height function (see our section of conventions for a precise definition). The poles

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of the Eisenstein series $E_P(f,s)$, $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\pi]$, in the one-dimensional complex parameter s, are determined by the ones of the usual intertwining operator

$$M(s,\pi) : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\pi \otimes e^{(s+\rho_P)H_P(.)}] \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\tilde{\pi} \otimes e^{(-s+\rho_P)H_P(.)}],$$

(cf. section 3.2), $\tilde{\pi}$ denoting the representation contragredient to π. It splits as a restricted tensor product $M(s,\pi) = \hat{\otimes}'_p A(s,\pi_p)$ over all places p of $\mathbb{Q}$. We show in a general setting in our proposition 3.1 that in order to find the poles of $M(s,\pi)$ we just need to consider those places p, where L splits, i.e., $L(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$. This makes it possible to use the normalization of $A(s,\pi_p)$ given by N. Grbac in [19] for non-generic representations π_p of $GL_2(\mathbb{Q}_p)$. For generic representations π_p, we can use Kim's normalization. A careful analysis of the holomorphic behavior of the resulting products of normalizing factors finally gives the classification of the residual spectrum of G. This is contained in our theorem 5.1.

In the sequel we determine the automorphic cohomology of G. That is, we completely describe the Eisenstein and the cuspidal cohomology of congruence subgroups Γ of G with respect to arbitrary coefficient systems.
Section 6 deals with the Eisenstein cohomology of G, the main theorem is theorem 6.1, which gives a comprehensive structural description of the internal nature of a representative of an Eisenstein cohomology class. Using the gained complete knowledge of the residual spectrum of G, we can single out those Eisenstein cohomology classes, which are represented by regular values of Eisenstein series and give necessary and sufficient conditions for the existence of non-trivial residual Eisenstein cohomology classes for each possible cohomology-degree q separately. It turns out that only for a certain branch of (non-regular) coefficient systems E and only in degree $q = 0, 1$, residual Eisenstein cohomology classes will exist. These will form the whole Eisenstein cohomology in degrees $q = 0, 1$.
Section 7 closes our study of the automorphic cohomology of G by analyzing cuspidal cohomology. We give a precise formula, see proposition 7.1, for the local multiplicity $m(\pi_\infty, \Gamma(N))$, with π_∞ a certain representation in the discrete series using the works of T. Arakawa, [1], [2]. In particular we establish non-vanishing in our corollary 7.1 and a growth-condition for the cuspidal cohomology for arbitrary principal congruence subgroups $\Gamma(N)$, $N \geq 3$. The paper concludes with a short remark on the possible extension of such results to the groups $Sp(n,1)$, $n \geq 1$, which is contained in our paper [21].

Notation and Conventions. Throughout this paper G will be a connected, simply connected, simple algebraic group over $\mathbb{Q}$[1], of rank $rk_{\mathbb{Q}}(G) = 1$. In this case G satisfies Strong Approximation. Lie algebras of groups of real points of algebraic groups will be denoted by the same but fractional letter, e.g., $\mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}$. The complexification of a Lie algebra will be denoted by the subscript "$\mathbb{C}$", e.g., $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$. If $U(\mathfrak{g})$ is the universal enveloping algebra of the complex algebra $\mathfrak{g}_{\mathbb{C}}$, $Z(\mathfrak{g})$ stands for its center.

We use the standard terminology and hypotheses concerning algebraic groups and their subgroups to be found in [36], I.1.4-I.1.12. In particular we assume that a minimal (and so, because $rk_{\mathbb{Q}}(G) = 1$ also maximal) parabolic $\mathbb{Q}$-subgroup P has been fixed and that $K_{\mathbb{A}} = K_{\mathbb{R}} \times K_{\mathbb{A}_f}$ is a maximal compact subgroup of the group $G(\mathbb{A})$ of adelic points of G which is in *good position* with respect to P ([36], I.1.4). Then $K = K_{\mathbb{R}}$ is maximal compact in $G(\mathbb{R})$, hence we have an associated Cartan

[1] Which for us includes the (only technical) assumption that G is not obtained from restriction of scalars $Res_{F/\mathbb{Q}}$ with $F \neq \mathbb{Q}$.

involution ϑ. If H is a subgroup of G, we let $K_H = K \cap H(\mathbb{R})$.

Assume that L is a Levi subgroup of P which is invariant under ϑ and N is the unipotent radical of P. Then we have the Levi decomposition $P = LN$ and if we additionally denote by A a maximal, central $\mathbb{Q}$-split torus in L, then we also get a Langlands decomposition $P = MAN$. We write $\Delta(P, A)$ for the set of weights of the adjoint action of P with respect to A. ρ_P denotes the half-sum of these weights, that is, the half sum of positive $\mathbb{Q}$-roots of G with respect to A. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{m}$ and $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ a maximally non-compact Cartan subalgebra of $\mathfrak{g}$. Then $\Delta^+ = \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ denotes some choice of positive roots, extending the choice of positivity on the set of $\mathbb{Q}$-roots $\Delta_\mathbb{Q}$ given by $\Delta(P, A)$ and Δ° a simple subset of Δ^+. The Weyl group corresponding to $\Delta_\mathbb{Q}$, resp. Δ, is denoted $W_\mathbb{Q}$, resp. $W = W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We write $\Delta^\circ_M := \Delta^\circ \cap \Delta(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ and call $W^P = \{w \in W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) | w^{-1}(\Delta^\circ_M) \subset \Delta^+\}$ the set of *Kostant representatives.*

As $K_\mathbb{A}$ is in good position, the *Harish-Chandra height-function* $H_P : P(\mathbb{A}) \to \mathfrak{a}^*$, given by the condition $\prod_p |\chi(p)|_p = e^{\langle \chi, H_P(p)\rangle}$, for all $\mathbb{Q}$-characters χ of L (viewed as elements of $\mathfrak{a}^*_\mathbb{C} \cong \mathbb{C}$), extends to a function on all of $G(\mathbb{A})$ by setting $H_P(g) := H_P(p)$, $g = kp$.

Let G be a connected, *reductive* group over $\mathbb{Q}$ and $\widetilde{\chi}$ a central character. As usual $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) denotes the discrete spectrum of G (resp. the part of it consisting of functions with central character $\widetilde{\chi}$). It can be written as the direct sum of the cuspidal spectrum $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) and the residual spectrum $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$). By [17], the space of cusp forms $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$ decomposes as direct Hilbert sum over all irreducible, admissible representations π of $G(\mathbb{A})$ with central character $\widetilde{\chi}$, each of which occurring with finite multiplicity $m(\pi)$. Every π can be written as a restricted tensor product $\pi = \otimes'_p \pi_p$, where p is a place of $\mathbb{Q}$, i.e., either a prime or ∞ and π_p is a local irreducible, admissible representation π_p of $G(\mathbb{Q}_p)$, [13]. Further, π (and so every π_p) is unitary if and only if $\widetilde{\chi}$ is. Then π is the completed restricted tensor product $\pi = \hat{\otimes}'_p \pi_p$.
For any $G(\mathbb{A})$-representation σ, we will write σ^∞ for the space of its smooth vectors and $\sigma_{(K)}$ for the space of K-finite vectors. Clearly, if σ is unitary, then $\sigma^\infty_{(K)}$ is a unitary $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module.

1. Automorphic cohomology

1.1. For details concerning this section see, e.g., [42], [47], [15]. As G satisfies Strong Approximation there will be for any open compact subgroup K_f of $G(\mathbb{A}_f)$ a homeomorphism

$$S(K_f) := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\mathbb{R} K_f \cong \Gamma\backslash X \tag{1}$$

where $X = G(\mathbb{R})/K$ is the symmetric space associated to $G(\mathbb{R})$ and its maximal compact subgroup $K = K_\mathbb{R}$ and $\Gamma = G(\mathbb{Q}) \cap K_f$ an arithmetic congruence subgroup of G.

Let E be a finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})$. Then this representation gives rise to a certain sheaf $\widetilde{E}$ on (1) (locally constant, if K_f is small enough). What we are interested in is the $G(\mathbb{A}_f)$-module structure of

$$H^*(G,E) := \varinjlim_{K_f} H^*(S(K_f),\widetilde{E}) = \varinjlim_{\Gamma} H^*(\Gamma\backslash X,\widetilde{E}).$$

In the classical setting of cohomology of arithmetic groups this means that we look at the cohomology of all arithmetic congruence subgroups "at the same time". As is well known, we have several isomorphisms:

$$\begin{aligned} H^*(G,E) &= H^*(\mathfrak{g},K,C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))\otimes E) \\ &= H^*(\mathfrak{g},K,V_G\otimes E) \\ &= H^*(\mathfrak{g},K,\mathcal{A}_E\otimes E). \end{aligned}$$

The second line, where V_G denotes the space of K-finite, smooth functions $f: G(\mathbb{Q})\backslash G(\mathbb{A})\to\mathbb{C}$ of uniform moderate growth, was proved by A. Borel, [4]. The last line was shown by J. Franke in [14], Thm. 18. Here $\mathcal{A}_E$ denotes the space of those automorphic forms which are annihilated by a power of the ideal $\mathcal{Z}$ of $Z(\mathfrak{g})$, which annihilates the dual representation of E:

$$\mathcal{A}_E = \{f\in\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))|\mathcal{Z}^n f=0 \quad \text{for some } n\}.$$

1.2. Spaces $\mathcal{A}_{E,P}$. In [15], J. Franke and J. Schwermer (and also in [36], C. Mœglin and J.-L. Waldspurger) were able to give a fine decomposition of the space $\mathcal{A}_E$, taking into account the cuspidal support along Levi subgroups of the Eisenstein series involved.

First of all, the space $\mathcal{A}_E$ admits a certain decomposition as a direct sum with respect to the standard parabolic $\mathbb{Q}$-subgroups P and G. This relies on such a decomposition of the space V_G, first proved by Langlands in a letter to Borel, [32]. See also [6], Thm. 2.4: $V_G = V_G(G)\oplus V_G(P)$, where $V_G(Q)$ denotes the space of elements f in V_G which are negligible along every parabolic $\mathbb{Q}$-subgroup not conjugate to Q. Putting $\mathcal{A}_{E,Q} = V_G(Q)\cap\mathcal{A}_E$ we get the desired decomposition of $\mathcal{A}_E$ as $(\mathfrak{g},K,G(\mathbb{A}_f))$-module:

$$\mathcal{A}_E = \mathcal{A}_{E,G}\oplus\mathcal{A}_{E,P}.$$

Hence, we arrive at a decomposition of cohomology as

$$H(G,E) = H^*_{cusp}(G,E)\oplus H^*_{Eis}(G,E),$$

with

$$H^*_{cusp}(G,E) = H^*(\mathfrak{g},K,V_G(\{G\})\otimes E) = H^*(\mathfrak{g},K,\mathcal{A}_{E,G}\otimes E),$$

and

$$H^*_{Eis}(G,E) = H^*(\mathfrak{g},K,\mathcal{A}_{E,P}\otimes E).$$

Observe that $V_G(G) = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$, which decomposes as a $(\mathfrak{g},K,G(\mathbb{A}_f))$-module as a direct Hilbert sum over all cuspidal automorphic representations of $G(\mathbb{A})$, each of which occurring with finite multiplicity $m(\pi)$. By [8], XIII, we get a finite direct sum decomposition

$$H^*_{cusp}(G,E) = \bigoplus_\pi H^*(\mathfrak{g},K,\pi\otimes E)^{m(\pi)} = \bigoplus_\pi (H^*(\mathfrak{g},K,(\pi_\infty)_{(K)}\otimes E)\otimes\pi_f^{\infty_f})^{m(\pi)},$$

the sum ranging over all cuspidal automorphic representations π of $G(\mathbb{A})$.

1.3. Eisenstein series. As mentioned before, the summand $\mathcal{A}_{E,P}$ giving Eisenstein cohomology, also has a decomposition as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module. For details we refer the reader to the original paper [15].

Some technical assumptions and notations have to be fixed:
For $Q = LN = MAN$ conjugate to the standard parabolic P, φ_Q is a finite set of irreducible representations $\pi = \chi\widetilde{\pi}$ of $L(\mathbb{A})$, with $\chi : A(\mathbb{R})^\circ \to \mathbb{C}^*$ a continuous character and $\widetilde{\pi}$ an irreducible, unitary subrepresentation of $L^2_{cusp}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$ whose central character induces a continuous morphism $A(\mathbb{Q})A(\mathbb{R})^\circ\backslash A(\mathbb{A}) \to U(1)$ and whose infinitesimal character matches the one of the dual of an irreducible subrepresentation of $H^*(\mathfrak{n}, E)$. This means that $\widetilde{\pi}$ is a unitary, cuspidal automorphic representation of $L(\mathbb{A})$ whose central and infinitesimal character satisfy the above conditions. Finally, three further "compatibility conditions" have to be satisfied between these sets φ_Q, skipped here and written down in [15], 1.2. The family of all collections $\varphi = \{\varphi_Q\}$ of such finite sets is denoted $\Psi_{E,P}$.

Now, let $W_{Q,\widetilde{\pi}}$ be the space of all smooth, K-finite functions

$$f : L(\mathbb{Q})N(\mathbb{A})A(\mathbb{R})^\circ\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $l \mapsto f(lg)$ on $L(\mathbb{A})$ is contained in the $\widetilde{\pi}$-isotypic component $\widetilde{\pi}^{m(\widetilde{\pi})}$ of $L^2_{cusp}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$. For a function $f \in W_{Q,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}^*_\mathbb{C}$ and $g \in G(\mathbb{A})$ an Eisenstein series is formally defined as

$$E_Q(f,\Lambda)(g) := \sum_{\gamma\in Q(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g)e^{\langle H_Q(\gamma g),\Lambda+\rho_Q\rangle}.$$

If we set $(\mathfrak{a}^*)^+ := \{\Lambda \in \mathfrak{a}^*_\mathbb{C} | \Re e(\Lambda) \in \rho_Q + C\}$, where C equals the open, positive Weyl-chamber with respect to $\Delta(Q, A)$, the series converges absolutely and uniformly on compact subsets of $G(\mathbb{A})\times(\mathfrak{a}^*)^+$. It is known that $E_Q(f,\Lambda)$ is an automorphic form there and that the map $\Lambda \mapsto E_Q(f,\Lambda)(g)$ can be analytically continued to a meromorphic function on all of $\mathfrak{a}^*_\mathbb{C} \cong \mathbb{C}^*$ (cf., [36] or [33], §7). It has only finitely many, at most simple poles Λ_0 within $\mathfrak{a}^*_{\geq 0} := \{\Lambda \in \mathfrak{a}^*_\mathbb{C} | \Im m(\Lambda) = 0, \Re e(\Lambda) \geq 0\}$ with smooth residue

$$\mathrm{Res}_{\Lambda_0}(E_Q(f,\Lambda)(g)) = \lim_{\Lambda\to\Lambda_0}(\Lambda - \Lambda_0)E_Q(f,\Lambda)(g).$$

For each $\Lambda \in C$ the residue is square-integrable, by [36], Lemma I.4.11.

1.4. Spaces $\mathcal{A}_{E,P,\varphi}$. Now we are able to turn to the desired decomposition of $\mathcal{A}_{E,P}$: For $\pi = \chi\widetilde{\pi} \in \varphi_P$ let $\mathcal{A}_{E,P,\varphi}$ be the space of functions, spanned by all possible residues and derivatives of Eisenstein series defined via all $f \in W_{P,\widetilde{\pi}}$, at the value $d\chi$ inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. It is a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module. Thanks to the functional equations (see [36] IV.1.10) satisfied by the Eisenstein series considered, this is well defined, i.e., independent of the choice of a representative for the class of P (whence we took P itself) and the choice of a representation $\pi \in \varphi_P$. Finally, we get

Theorem 1.1 ([15], Thm. 1.4 & Thm. 2.3; see also [36] III, Thm. 2.6)**.** *There is a direct sum decomposition as* $(\mathfrak{g}, K, G(\mathbb{A}_f))$*-module*

$$\mathcal{A}_{E,P} = \bigoplus_{\varphi\in\Psi_{E,P}} \mathcal{A}_{E,P,\varphi}$$

giving rise to

$$H^*_{Eis}(G,E) = \bigoplus_{\varphi\in\Psi_{E,\{P\}}} H^*(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

2. Construction of Eisenstein cohomology

We review now a method to construct Eisenstein cohomology, using the notion of "(π, w)-types". The next subsection is devoted to the definition of the latter.

2.1. Classes of type (π, w). Take $\pi = \chi\widetilde{\pi} \in \varphi_P$ and consider the symmetric tensor algebra

$$S_\chi(\mathfrak{a}^*) = \bigoplus_{n\geq 0} \bigodot^n \mathfrak{a}^*_{\mathbb{C}},$$

$\bigodot^n \mathfrak{a}^*_{\mathbb{C}}$ being the symmetric tensor product of n copies of $\mathfrak{a}^*_{\mathbb{C}}$, as module under $\mathfrak{a}_{\mathbb{C}}$: Via the natural identification $\mathfrak{a}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{a}^*_{\mathbb{C}}$ it is an $\mathfrak{a}_{\mathbb{C}}$-module acted upon by $\xi \in \mathfrak{a}_{\mathbb{C}} \cong \mathfrak{a}^*_{\mathbb{C}}$ via multiplication with $\langle \xi, \rho_P + d\chi\rangle + \xi$ (within the symmetric tensor algebra). This explains the subscript "χ". We extend this action trivially on $\mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}$ to get an action of the Lie algebra $\mathfrak{p}_{\mathbb{C}}$ on the Banach space $S_\chi(\mathfrak{a}^*)$. We may also define a $P(\mathbb{A}_f)$-module structure via the rule

$$q \cdot X = e^{\langle d\chi + \rho_P, H_P(q)\rangle} X,$$

for $q \in P(\mathbb{A}_f)$ and $X \in S_\chi(\mathfrak{a}^*)$. There is a continuous linear isomorphism

$$\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)} \left[\widetilde{\pi}^\infty_{(K_L)} \otimes S_\chi(\mathfrak{a}^*)\right]^{m(\widetilde{\pi})} \xrightarrow{\sim} W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*),$$

induced by the tensor map $\otimes$ and the evaluation of functions $f \in C^\infty(G(\mathbb{A}), (\widetilde{\pi}^\infty)^{m(\widetilde{\pi})})$ at the identity, $f \mapsto ev_{id}(f) : g \mapsto f(g)(id)$, so in particular one can view the right hand side as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module by transport of structure. Doing this, it is shown in [14], pp. 256-257, that

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \cong$$

(2)

$$\bigoplus_{\substack{w\in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}} = d\chi}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \left[H^{q-l(w)}(\mathfrak{m}, K_M, (\widetilde{\pi}_\infty)_{(K_M)} \otimes {}^\circ F_w) \otimes \mathbb{C}_{d\chi+\rho_P} \otimes \widetilde{\pi}_f^{\infty_f}\right]^{m(\widetilde{\pi})}.$$

Here ${}^\circ F_w$ is the finite dimensional representation of $M(\mathbb{C})$ with highest weight $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{\mathbb{C}}}$ and $\mathbb{C}_{d\chi+\rho_P}$ the one-dimensional, complex $P(\mathbb{A}_f)$-module on which $q \in P(\mathbb{A}_f)$ acts by multiplication by $e^{\langle d\chi+\rho_P, H_P(q)\rangle}$. A non-trivial class in a summand of the right hand side is called a cohomology class *of type* (π, w), $\pi \in \varphi_P$, $w \in W^P$. Further, as $L(\mathbb{R}) \cong M(\mathbb{R}) \times A(\mathbb{R})^\circ$, $\widetilde{\pi}_\infty$ can be regarded as an irreducible, unitary representation of $M(\mathbb{R})$. Therefore, a (π, w) type consists out of an irreducible representation $\pi = \chi\widetilde{\pi}$ whose unitary part $\widetilde{\pi} = \widetilde{\pi}_\infty \hat{\otimes} \widetilde{\pi}_f$ has at the infinite place an irreducible, unitary representation $\widetilde{\pi}_\infty$ of the semisimple group $M(\mathbb{R})$ with non-trivial $(\mathfrak{m}, K_M)$-cohomology with respect to ${}^\circ F_w$.

2.2. The Eisenstein map. In order to construct Eisenstein cohomology classes, we start from a class of type (π, w). Since we are interested in cohomology, we can by (2) assume without loss of generality that $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_{\mathbb{C}}}$ lies inside the closed, positive Weyl chamber defined by $\Delta(P, A)$

We reinterpret $S_\chi(\mathfrak{a}^*)$ as the (Banach) space of formal, finite $\mathbb{C}$-linear combinations of differential operators $\frac{d^n}{d\Lambda^n}$ on $\mathfrak{a}_{\mathbb{C}} \cong \mathbb{C}$. Let $q(\Lambda) = \Lambda - d\chi$ if $E_P(f, \Lambda)$ has a pole at $d\chi$ for some $f \in W_{P,\widetilde{\pi}}$ and put $q(\Lambda) \equiv 1$ otherwise. Since $\mathcal{A}_{E,P,\varphi}$ can be written as the space which is generated by the coefficient functions in the Taylor series expansion of $q(\Lambda)E_P(f, \Lambda)$ at $d\chi$, f running through $W_{P,\widetilde{\pi}}$, we are able to define a surjective homomorphism $E_{P,\pi}$ of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules

$$W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \xrightarrow{E_{P,\pi}} \mathcal{A}_{E,P,\varphi}$$
$$f \otimes \frac{d^n}{d\Lambda^n} \mapsto \frac{d^n}{d\Lambda^n}\left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}.$$

and get a well-defined map in cohomology

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E^q_\pi} H^*(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

2.3. Holomorphic Case. Suppose $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class of type (π, w), represented by a morphism ω, such that for all elements $f \otimes \frac{d^n}{d\Lambda^n}$ in its image, $E_{P,\pi}(f \otimes \frac{d^n}{d\Lambda^n}) = \frac{d^n}{d\Lambda^n}\left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}$ is just the regular value $E_P(f, d\chi)$ of the Eisenstein series $E_P(f,\Lambda)$, which is assumed to be holomorphic at the point $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ inside the closed, positive Weyl chamber defined by $\Delta(P,A)$. Then $E^q_\pi([\omega])$ is a non-trivial Eisenstein cohomology class

$$E^q_\pi([\omega]) \in H^q(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

This is a consequence of [47], Thm. 4.11.

2.4. Residual Case. Suppose now that there is an $f \in W_{P,\widetilde{\pi}}$ such that the Eisenstein series $E_P(f,\Lambda)$ has a pole at $d\chi$ and notice that $E_P(f,\Lambda)$ is always holomorphic at 0, by [25], Lemma 100. If $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class represented by a morphism ω having only functions f as in the previous sentence in its image, then we see as in [22] that the residual Eisenstein cohomology class $E^q_\pi([\omega])$ might be interpreted as a class

$$E^q_\pi([\omega]) \in H^{q'}(\mathfrak{g}, K, L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E),$$

with $q' := q + \dim N(\mathbb{R}) - 2l(w)$. If even $rk_\mathbb{R}(G) = 1$, then $q' = \dim N(\mathbb{R}) - q$.

3. The Residual Spectrum

3.1. The discussion of the construction of Eisenstein cohomology in the last section showed that a complete understanding of $H^*_{Eis}(G,E)$ can only be given if one knows to control the poles of Eisenstein series $E_P(f,\Lambda)$ within the closed, positive Weyl chamber. We already mentioned that residues of Eisenstein series at such points are all square-integrable. By the description of the discrete spectrum given by Langlands (cf. [33] or [36]), we know that its residual part $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is nothing but the space spanned by all the residues of the Eisenstein series at points inside the positive Weyl chamber. In particular, we can attack the task of calculating $H^*_{Eis}(G,E)$ as follows: Describe the residual spectrum of G and calculate its cohomology.

3.2. Constant term of Eisenstein series. Let $\widetilde{\pi}$ be a cuspidal automorphic representation of $L(\mathbb{A})$ and $f \in \mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\widetilde{\pi}]_{(K)}$. By [33] the poles of an Eisenstein series $E_P(f,\Lambda)$ (defined as in 1.3) are the ones of its constant term along P. Let $\mathfrak{a}^*_\mathbb{C} \cong \mathbb{C}$ be generated by $\widetilde{\alpha} = \langle\rho_P, \alpha\rangle^{-1}\rho_P$ and identify $\Lambda = s\widetilde{\alpha} \in \mathfrak{a}^*_\mathbb{C}$ with $s \in \mathbb{C}$. Then the constant term can be written as

$$E_P(f,\Lambda)_P = fe^{\langle\Lambda+\rho_P, H_P(.)\rangle} + M(s,\widetilde{\pi},w)fe^{\langle w(\Lambda)+\rho_P, H_P(.)\rangle},$$

where $M(s,\widetilde{\pi},w)$ is the meromorphic function in the parameter s, given for $g \in G(\mathbb{A})$, w the only non-trivial element in $W_\mathbb{Q}$ and $\Re e(s) \gg 0$ by

$$M(s,\widetilde{\pi},w)f(g) = \int_{\vartheta(N)(\mathbb{A})} f(nw^{-1}g)e^{\langle\Lambda+\rho_P, H_P(nw^{-1}g) - wH_P(g)\rangle}dn.$$

Strictly speaking $M(s,\widetilde{\pi},w)f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[w(\widetilde{\pi})]_{(K)}$, but we will also interpret $M(s,\widetilde{\pi},w)$ as the usual intertwining operator

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\widetilde{\pi}\otimes e^{\langle \Lambda+\rho_P, H_P(.)\rangle}] \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[w(\widetilde{\pi})\otimes e^{\langle w(\Lambda)+\rho_P, H_P(.)\rangle}],$$

as in [36], II.1.6. (We trust this will not cause any confusion.)
Let S be a finite set of places containing the ones where G does not split, such that $\widetilde{\pi}_p$ has got a non-trivial $L(\mathbb{Z}_p)$-fixed vector for $p \notin S$: That is, outside S, L splits and $\widetilde{\pi}_p$ is spherical. Now, we can formally write $f = \hat{\otimes}'_p f_p$, where f_p is a suitably normalized, $L(\mathbb{Z}_p)$-fixed function for $p \notin S$. Therefore, $M(s,\widetilde{\pi},w)f$ factors as $M(s,\widetilde{\pi},w)f = \hat{\otimes}'_p A(s,\widetilde{\pi}_p,w)f_p$. Let us write ${}^L L$ for the L-group of L, [3]. If we regard L as a group over the field $\mathbb{Q}_p$, we denote its L-group by ${}^L L_p$. Clearly, this gives rise to a natural homomorphism $\beta_p : {}^L L_p \to {}^L L$ and each finite dimensional representation r of ${}^L L$ defines a finite-dimensional representation r_p of ${}^L L_p$ by setting $r_p = r\circ\beta_p$. In particular, if ${}^L\mathfrak{n}$ is the complex Lie algebra of the L-group ${}^L N$ of N (cf. [3], §3.4), then one can take r equal to the adjoint representation of ${}^L L$ on ${}^L\mathfrak{n}$. It decomposes into the direct sum $r = \bigoplus_{j=1}^m r_j$ of its irreducible subrepresentations r_j indexed according to an upper central series of ${}^L\mathfrak{n}$. We write $\check{r}_j$ for the contragredient representation of r_j.

Using this setup we can rewrite the local operators $A(s,\widetilde{\pi}_p,w)$ at $p \notin S$ using the concept of local Langlands L-functions: Let $p \notin S$ and τ_p be the (unique) Frobenius conjugacy class in $\mathrm{Gal}(F_v/\mathbb{Q}_p)$, for F some finite Galois extension of $\mathbb{Q}$ over which G splits and $v|p$. According to [3], there exists $A_p \in {}^L A^\circ$ such that the ${}^L L$-conjugacy class of $A_p \rtimes \tau_p$ determines $\widetilde{\pi}_p$ up to equivalence. One may assume that A_p is fixed by τ_p (see [3], §6.3 and §6.5). Now, the local Langlands L-function of $\widetilde{\pi}_p$ and r_p is defined by

$$L(s,\widetilde{\pi}_p,r_p) = \det(id - r_p(A_p \rtimes \tau_p)p^{-s})^{-1}.$$

It is a consequence of the Gindikin-Karpelevich integral formula, as shown in [31], p. 27 (see also [48], p.554) that for suitably normalized, $L(\mathbb{Z}_p)$-fixed functions $\widetilde{f}_p \in \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}[w(\widetilde{\pi}_p)]$ we can finally write

$$M(s,\widetilde{\pi},w)f = \hat{\bigotimes}_{p\in S} A(s,\widetilde{\pi}_p,w)f_p \hat{\otimes} \hat{\bigotimes}_{p\notin S} \prod_{j=1}^m \frac{L(js,\widetilde{\pi}_p,(\check{r}_j)_p)}{L(1+js,\widetilde{\pi}_p,(\check{r}_j)_p)}\widetilde{f}_p. \tag{3}$$

Now observe the following important fact:

Proposition 3.1. *Let p be a place where $rk_{\mathbb{Q}_p}(G) = 1$. Then $A(s,\widetilde{\pi}_p,w)$ is holomorphic and non-vanishing for $\Re e(s) > 0$. Suppose $G/\mathbb{Q}$ is not split. Then there is an $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\widetilde{\pi}]_{(K)}$ such that $E_P(f,\Lambda)$ has a pole at $\Lambda = s\widetilde{\alpha}$, $\Re e(s) > 0$ if and only if the product of the local factors $A(s,\widetilde{\pi}_p,w)$ with $rk_{\mathbb{Q}_p}(G) > 1$ has a pole at s, $\Re e(s) > 0$.*

Proof. Let $\Re e(s)$ be strictly greater than 0. If p is any place, where $rk_{\mathbb{Q}_p}(G) = 1$, then $L(\mathbb{Q}_p)$ is compact modulo its center. Hence, the representation $\widetilde{\pi}_p$ is cuspidal for finite places p and (up to a twist) a discrete series representation if p is the infinite place. So, for any such p, $A(s,\widetilde{\pi}_p,w)$ is nothing but the w-left-translation of functions composed with the intertwining operator whose image is the Langlands quotient associate to $P(\mathbb{Q}_p)$, the tempered representation $\widetilde{\pi}_p$ and $\Re e(s) > 0$. In particular, $A(s,\widetilde{\pi}_p,w)$ is holomorphic and non-vanishing for $\Re e(s) > 0$, see, e.g., [8], IV, Lemma 4.4 and XI, Cor. 2.7. If G is not split, then the number of these places p, $rk_{\mathbb{Q}_p}(G) = 1$ is finite. Hence the proposition. □

4. The Group $Sp(1,1)$

4.1. Let B be a quaternion algebra over $\mathbb{Q}$ with canonical involution $x \mapsto \overline{x}$, s.t. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$ where $\mathbb{H}$ equals the real Hamilton quaternions. We denote by $S(B)$ the finite set of places p where B does not split, i.e., $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra and let $d(B)$ the discriminant of B, i.e., the product over all finite places $p \in S(B)$. For $v = (v_1, v_2), w = (w_1, w_2) \in B^2$ define $f(v,w) = v_1\overline{w}_1 - v_2\overline{w}_2$. Then f is a non-degenerate Hermitian form on the right-B-module B^2 and its group of B-linear isometries is given by

$$G = Sp(1,1) = \left\{ g \in GL_2(B) | g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Clearly, $g^* = (\overline{g}_{ji})_{i,j} = \overline{g}^t$. G is a connected, simply connected, simple algebraic group over $\mathbb{Q}$ of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = 1$. It is a non-quasisplit inner form of Sp_4, the split group of type C_2.

4.2. We fix a minimal parabolic $P = LN = MAN$ as in the introduction. We can arrange that

$$L \cong GL_1(B)$$

and so

$$M = SL_1(B)$$

will be compact. Further, A can be chosen such that $Lie(A(\mathbb{R})) = \mathfrak{a}$, with

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, a \in \mathbb{R} \right\},$$

and we can identify the set of $\mathbb{Q}$- and $\mathbb{R}$-roots of G with

$$\Delta_{\mathbb{Q}} = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha\},$$

α being the linear functional on $\mathfrak{a}$ extracting the value $2a$. It is clearly the only simple $\mathbb{Q}$- resp. $\mathbb{R}$-root and $\alpha = \widetilde{\alpha}$.

4.3. Absolute roots. Extending $\mathfrak{a}$ to a maximal non-compact Cartan subalgebra $\mathfrak{h}$ by adding a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$, e.g.,

$$\mathfrak{b} = \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \middle| b \in i\mathbb{R} \right\},$$

we get as set of absolute roots

$$\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\},$$

with λ_i sending $H \in \mathfrak{h}_{\mathbb{C}}$ to $\lambda_i(H) = b + (-1)^{i-1}a$. Choosing positivity on $\mathfrak{h}_{\mathbb{C}}$ compatible with the choice on $\mathfrak{a}$, the set of simple roots Δ° consists of $\alpha_1 = \lambda_1 + \lambda_2$ and $\alpha_2 = -2\lambda_2$. The corresponding fundamental weights are denoted $\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2$ and $\omega_2 = \alpha_1 + \alpha_2$. Let w_i be the root reflection corresponding to α_i. Then the Weyl group $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is generated by w_1 and w_2.

4.4. Data for M. On the level of the Levi subgroup we get $\Delta_M^\circ = \Delta^\circ \cap \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{\alpha_1\}$. The corresponding fundamental weight is denoted ω with fundamental representation the standard representation on $\mathbb{C}^2$. Further, the Weyl group W_M of Δ_M is generated by $w : \alpha_1 \mapsto -\alpha_1$. By the very definition, the set of Kostant representatives is

$$W^P = \{id, w_2, w_2w_1, w_2w_1w_2\}.$$

Let $\lambda = c_1\alpha_1 + c_2\alpha_2$ be a dominant, algebraically integral weight. For later use we now list

	$w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{\mathbb{C}}}$
id	$2(c_1-c_2)\omega$
w_2	$2(c_2+1)\omega$
w_2w_1	$2(c_2+1)\omega$
$w_2w_1w_2$	$2(c_1-c_2)\omega$

TABLE 1. $^\circ F_w$

	$\Re e\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{\mathbb{C}}}, \alpha\rangle$
id	$-4(c_2+\frac{3}{2})<0$
w_2	$4(c_2-c_1-\frac{1}{2})<0$
w_2w_1	$-4(c_2-c_1-\frac{1}{2})>0$
$w_2w_1w_2$	$4(c_2+\frac{3}{2})>0$

TABLE 2

5. THE RESIDUAL SPECTRUM OF INNER FORMS OF Sp_4

We retain the notation of section 3. Note that we may assume that $\widetilde{\pi}$ is unitary, since the central character $\widetilde{\chi}$ of any non-unitary cuspidal automorphic representation of $L(\mathbb{A})$ differs from a unitary character just by a twist with the reduced norm of B. Recall from [27], Chp. 1, Thm. 2.18, the definition of the partial Jacquet-Langlands-L-function $L(s,\widetilde{\pi}_p)$ associated to the local representation $\widetilde{\pi}_p$, and the definition of the partial Hecke-L-function $L(s,\widetilde{\chi}_p)$ associated to the central character $\widetilde{\chi}_p$ of $\widetilde{\pi}_p$. For later use, recall also the definition of the global analogs $L(s,\widetilde{\pi})$ (resp. $L(s,\widetilde{\chi})$) of $L(s,\widetilde{\pi}_p)$ (resp. $L(s,\widetilde{\chi}_p)$). Their well-known analytic properties are summarized in the following

Lemma 5.1 ([27],[50]). (i) *Let $\widetilde{\pi}=\hat{\otimes}'_p\widetilde{\pi}_p$ be a cuspidal automorphic representation of $L(\mathbb{A})$ with central character $\widetilde{\chi}=\hat{\otimes}'_p\widetilde{\chi}_p$, assuming that* $\dim\widetilde{\pi}>1$. *Then: the local Jacquet-Langlands-L-function $L(s,\widetilde{\pi}_p)$ is holomorphic and non-zero on $\Re e(s)>1$ at each place p. For the infinite place we get $L(s,\widetilde{\pi}_\infty)=2(2\pi)^{-s-n-\frac{1}{2}}\Gamma(s+n+\frac{1}{2})$ if $\widetilde{\pi}_\infty$ is the n-th symmetric power $\bigodot^n\mathbb{C}^2$ and hence this local L-factor is holomorphic and non-vanishing for $\Re e(s)\geq 0$. The global Jacquet-Langlands L-function $L(s,\widetilde{\pi})$ is an entire function and has no zeros for $\Re e(s)\geq 1$.*
(ii) *The local Hecke-L-function $L(s,\widetilde{\chi}_p)$ has a simple pole at $s=0$ if $\widetilde{\chi}_p=\mathbf{1}_p$ and is entire otherwise. It vanishes nowhere. The global Hecke-L-function $L(s,\widetilde{\chi})$ has simple poles at $s=0$ and $s=1$ if $\widetilde{\chi}=\mathbf{1}$ (and $L(s,\mathbf{1})=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$) and is entire otherwise. It is non-vanishing for $\Re e(s)\geq 1$.*

Recall the integer m in the decomposition (3): In our concrete situation of $G=Sp(1,1)$ one calculates it to be $m=2$: In fact, we have $r=r_1\oplus r_2$ where r_1 is the standard representation of $GL_2(\mathbb{C})$ and $r_2=\wedge^2 r_1$ (cf. [31], case (vi)).

5.1. Generic representations. Suppose that $\widetilde{\pi}$ is not one-dimensional. Then each local factor $\widetilde{\pi}_p$, $p\notin S(B)$ is generic. This follows from the global Jacquet-Langlands correspondence ([18], Thm. (8.3)) and the well known fact that the local factors of cuspidal automorphic representations of $GL_2(\mathbb{A})$ are all generic (see, e.g., [49], corollary on p. 190). But then $\widetilde{\pi}_p$ is fully induced from two characters and therefore one easily gets, at $p\notin S$,

$$L(s,\widetilde{\pi}_p,(\check{r}_1)_p))=L(s,\widetilde{\pi}_p)$$

and

$$L(s,\widetilde{\pi}_p,(\check{r}_2)_p))=L(s,\widetilde{\chi}_p),$$

(see [29], pp. 133-134, for details). It is shown in [29], Prop. 3.1, that for $p\in S\backslash S(B)$,
$L(s,\widetilde{\pi}_p)^{-1}L(2s,\widetilde{\chi}_p)^{-1}A(s,\widetilde{\pi}_p,w)$ is holomorphic for $\Re e(s)>0$. By the argument of the proof of proposition 3.1 this normalized local operator is also non-vanishing for tempered representations $\widetilde{\pi}_p$. The same holds for non-tempered representations $\widetilde{\pi}_p$ as shown in [20], Prop. 2.1. So, recalling our proposition 3.1, the poles of $M(s,\widetilde{\pi},w)$ are the poles of

$$\prod_{p\in S\setminus S(B)} L(s,\widetilde{\pi}_p)L(2s,\widetilde{\chi}_p)\prod_{p\notin S}\frac{L(s,\widetilde{\pi}_p)L(2s,\widetilde{\chi}_p)}{L(1+s,\widetilde{\pi}_p)L(1+2s,\widetilde{\chi}_p)}. \tag{4}$$

By the analytic properties of Jacquet-Langlands- and Hecke-L-functions, summarized in our lemma 5.1 above, the poles of (4) in the region $\Re e(s)>0$ are the poles of the finite product

$$r(s,\widetilde{\pi})=\frac{L(s,\widetilde{\pi})L(2s,\widetilde{\chi})}{L(1+s,\widetilde{\pi})L(1+2s,\widetilde{\chi})}\prod_{p\in S(B)}\frac{L(1+s,\widetilde{\pi}_p)L(1+2s,\widetilde{\chi}_p)}{L(s,\widetilde{\pi}_p)L(2s,\widetilde{\chi}_p)}. \tag{5}$$

(See also [35]). This settles the generic case, i.e., $\dim\widetilde{\pi}>1$.

5.2. Non-generic representations. In the one-dimensional case $\widetilde{\pi}=\widetilde{\chi}$ the above argument does not work any more, since the global lift of $\widetilde{\chi}$ prescribed by the Jacquet-Langlands correspondence is contained in the residual spectrum of GL_2. However, using our proposition 3.1, the poles of $M(s,\widetilde{\pi},w)$ are the ones of

$$\prod_{p\notin S(B)}\frac{L(s-\frac{1}{2},\widetilde{\chi}_p)L(2s,\widetilde{\chi}_p^2)}{L(s+\frac{3}{2},\widetilde{\chi}_p)L(1+2s,\widetilde{\chi}_p^2)} \tag{6}$$

This was achieved in [20], 1.2.(8) and Prop. 1.3, (see also [19] 2.(4)) applying the idea of [35], Lemme I.8, i.e., via induction from representations of smaller parabolic subgroups. See also [52], Lemma 3.1. Again, using the well-known analytic properties of Hecke-L-functions, a short moment of thought convinces us that the poles of (6) are exactly the ones of the finite product

$$r(s,\widetilde{\chi})=\frac{L(s-\frac{1}{2},\widetilde{\chi})L(2s,\widetilde{\chi}^2)}{L(s+\frac{3}{2},\widetilde{\chi})L(1+2s,\widetilde{\chi}^2)}\prod_{\substack{p\in S(B)\\ \widetilde{\chi}_p=\mathbf{1}_p}} L(s-\frac{1}{2},\widetilde{\chi}_p)^{-1}. \tag{7}$$

5.3. We are ready to calculate the poles of Eisenstein series $E_P(f,\Lambda)$ within the positive Weyl chamber. This is a direct consequence of our proposition 3.1, the explicit form of the normalizing factors (5) and (7) and lemma 5.1:

Proposition 5.1. *Let $\widetilde{\pi}$ be a cuspidal automorphic representation of $L(\mathbb{A})$. Then there is an $f\in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\widetilde{\pi}]_{(K)}$, such that the Eisenstein series $E_P(f,\Lambda)$ has a simple pole at $\Lambda=s\alpha$ inside the open, positive Weyl chamber defined by $\Delta(P,A)$ if and only if one of the following conditions holds:*

(I) ($\dim\widetilde{\pi}>1$): *At $s=\frac{1}{2}$, if $L(\frac{1}{2},\widetilde{\pi})\neq 0$ and $\widetilde{\chi}=\mathbf{1}$*
(II) ($\widetilde{\pi}=\widetilde{\chi}\neq\mathbf{1}$): *At $s=\frac{1}{2}$, if $\widetilde{\chi}^2=\mathbf{1}$ and $\widetilde{\chi}_p\neq\mathbf{1}_p$ for all $p\in S(B)$*
(III) ($\widetilde{\pi}=\widetilde{\chi}=\mathbf{1}$): *At $s=\frac{3}{2}$*

Let us write $\mathcal{A}(s,\widetilde{\pi})$ for the image of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\widetilde{\pi}\otimes e^{(s+\rho_P)H_P(.)}]$ under the normalized intertwining operator $N(s,\widetilde{\pi})=r(s,\widetilde{\pi})^{-1}M(s,\widetilde{\pi},w)$. Then we see as in [19], p. 52, that for $(s,\widetilde{\pi})$ of type II the local images $\mathcal{A}(s,\widetilde{\pi}_p)=\mathcal{A}(\frac{1}{2},\widetilde{\chi}_p)$ at a split place p are the sum of two irreducible representations $\mathcal{A}(\frac{1}{2},\widetilde{\chi}_p)^{\pm}$. As a consequence of proposition 5.1 we get the following description of the residual spectrum of G:

Theorem 5.1. *The residual spectrum of $G = Sp(1,1)$ decomposes as a direct sum of representations as*

$$L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\substack{(s,\widetilde{\pi})\\ \text{of type I, II or III}}} \mathcal{A}(s,\widetilde{\pi}).$$

The spaces of types I *and* III *are irreducible, while the space of type* II *decomposes as a sum of irreducible representations $\sigma = \hat{\otimes}'_p \sigma_p$ with $\sigma_p = \mathcal{A}(\frac{1}{2}, \widetilde{\chi}_p)^+$ at almost all places $p \notin S(B)$.*

Proof. Via the map $f \mapsto f_P$ the spaces $\mathcal{A}(s,\widetilde{\pi})$ are just spans of the residues of the Eisenstein series attached to a representation $\widetilde{\pi}$ at a point s satisfying one of the conditions I, II or III. We just need to show the assertions on irreducibility.
If $\dim \widetilde{\pi} > 1$, then the global lift prescribed by the Jacquet-Langlands Correspondence is a cuspidal automorphic representations having square-integrable local representations at each place $p \in S(B)$. Since the infinite place is assumed to be in $S(B)$, *every* local factor of $\widetilde{\pi}$ is tempered by the version of the Ramanujan Conjecture for GL_2 proved by P. Deligne ([11], Thm. I.6, together with [10], Thm. 5.6). Hence, $\mathcal{A}(s,\widetilde{\pi})$ is irreducible by the Langlands Classification, [8], IV, Lemma 4.5.
If $\widetilde{\pi} = \widetilde{\chi}$ is one-dimensional with a non-tempered local factor $\widetilde{\chi}_p$, with necessarily $p \notin S(B)$, then the argument of [19], Thm. 3.2, applies proving irreducibility of $\mathcal{A}(\frac{3}{2}, \mathbf{1})$. As the local representation $\mathcal{A}(\frac{1}{2}, \widetilde{\chi}_p)^-$ of $Sp_4(\mathbb{Q}_p)$ is never spherical, the last assertion follows by the above remark. □

5.4. Previous results on residual spectra. We want to point out that for the $\mathbb{Q}$-split group Sp_4 H. Kim described the residual spectrum in [29] and that N. Grbac gave a complete classification of the residual spectrum of inner $\mathbb{Q}$-forms G' which are $\mathbb{R}$-split but not $\mathbb{Q}$-split (i.e., G' is our group G but defined over a quaternion division algebra B' satisfying $B' \otimes_{\mathbb{Q}} \mathbb{R} \neq \mathbb{H}$) in [19]. Recently, a comprehensive and detailed description of the residual spectrum of *any* $\mathbb{Q}$-rank one inner form of $Sp_4/\mathbb{Q}$ was given by T. Yasuda in [52], Thm 4.1 which covers the results of [19] and our theorem 5.1. He described the irreducible constituents of the spaces $\mathcal{A}(\frac{1}{2}, \widetilde{\chi})$ of type II in a more refined way as theta-lifts from the trivial representation of $O^*(1)$. We will see that in order to calculate the Eisenstein cohomology of G, we will not need to make use of Yasuda's more refined description of the spaces of type II, since the latter will turn out to give a trivial contribution to cohomology, cf. theorem 6.1.

6. Eisenstein Cohomology for $Sp(1,1)$

6.1. The (π, w)-types. As an application of our theorem 5.1, we get a nice and comprehensive description of the Eisenstein cohomology space of $G = Sp(1,1)$ with respect to an arbitrary coefficient space E. As explained in section 2, we are just interested in the holomorphic behavior of the various Eisenstein series $E_P(f,\Lambda)$ at $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}}$, $w \in W^P$, inside the closed, positive Weyl chamber. Our table 2 shows these evaluation points $d\chi$ and gives that necessarily $w = w_2w_1, w_2w_1w_2$. We also need

Lemma 6.1. *Let $w \in W^P$ and $\widetilde{\pi}_\infty$ be an irreducible, unitary representation of $M(\mathbb{R})$. Then*

$$H^q(\mathfrak{m}, K_M, \widetilde{\pi}_\infty \otimes {}^\circ F_w) = \begin{cases} \mathbb{C} & \textit{if } q = 0 \textit{ and } \widetilde{\pi}_\infty = {}^\circ F_w \\ 0 & \textit{else.} \end{cases}$$

Proof. Since $M(\mathbb{R}) = SL_1(\mathbb{H})$ is compact, $(\mathfrak{m}, K_M)$-cohomology with respect to $\widetilde{\pi}_\infty \otimes {}^\circ F_w$ is one-dimensional, if $\widetilde{\pi}_\infty \cong {}^\circ\check{F}_w$ and $q = 0$, and vanishes otherwise. By [46], Prop. 4.13, and our table 1 we see that ${}^\circ\check{F}_w \cong {}^\circ F_w$. □

6.2. We have the following

Theorem 6.1. *Let $G = Sp(1,1)$ and E be any finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})$ of highest weight λ. For any tuple (π, w), $\pi = \chi\widetilde{\pi}$ and $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}} = s\alpha$, let $\Omega_{hol}(s,\widetilde{\pi})$ (resp. $\Omega_{res}(s,\widetilde{\pi})$) be the span of those classes $[\omega]$ of type (π, w) such that the associated Eisenstein series are holomorphic (resp. have a pole) at the uniquely determined point $\Lambda = s\alpha$. Then the Eisenstein cohomology of G with respect to E is given as*

(1) $\underline{\lambda = k\omega_2,\ k \in \mathbb{Z}_{\geq 0}}$:

$$H^0_{Eis}(G,\mathbb{C}) = E^3_{\mathbf{1}}(\Omega_{res}(\tfrac{3}{2},\mathbf{1})) = \mathbb{C}, \quad \textit{if } k=0$$

$$H^1_{Eis}(G,E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=\frac{1}{2}\alpha \\ \widetilde{\pi}_\infty={}^\circ F_{w_2w_1}=\bigodot^{2c_2+2}\mathbb{C}^2 \\ L(\frac{1}{2},\widetilde{\pi})\neq 0 \text{ and } \widetilde{\chi}=\mathbf{1}}} E^2_\pi(\Omega_{res}(\tfrac{1}{2},\widetilde{\pi}))$$

$$H^2_{Eis}(G,E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=\frac{1}{2}\alpha \\ \widetilde{\pi}_\infty={}^\circ F_{w_2w_1}=\bigodot^{2c_2+2}\mathbb{C}^2 \\ L(\frac{1}{2},\widetilde{\pi})=0 \text{ or } \widetilde{\chi}\neq\mathbf{1}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}^{\infty_f}_f] \oplus \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=\frac{1}{2}\alpha \\ \widetilde{\pi}_\infty={}^\circ F_{w_2w_1}=\bigodot^{2c_2+2}\mathbb{C}^2 \\ L(\frac{1}{2},\widetilde{\pi})\neq 0 \text{ and } \widetilde{\chi}=\mathbf{1}}} \Omega_{hol}(\tfrac{1}{2},\widetilde{\pi})$$

$$H^3_{Eis}(G,\mathbb{C}) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=\frac{3}{2}\alpha \\ \widetilde{\pi}_\infty=\mathbb{C} \text{ but } \widetilde{\pi}\neq\mathbf{1}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}^{\infty_f}_f] \oplus \Omega_{hol}(\tfrac{3}{2},\mathbf{1}) \quad \textit{if } k=0$$

$$H^3_{Eis}(G,E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=(c_2+\frac{3}{2})\alpha \\ \widetilde{\pi}_\infty=\mathbb{C}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}^{\infty_f}_f] \quad \textit{if } k\neq 0$$

$$H^q_{Eis}(G,E) = 0 \quad \textit{else.}$$

Cohomology in degrees 2, 3 is entirely built up by values of holomorphic Eisenstein series. Cohomology in degree 0 and 1 consists of residual classes, which can be represented by square-integrable, residual automorphic forms.

(2) $\underline{\lambda \neq k\omega_2,\ k \in \mathbb{Z}_{\geq 0}}$:

$$H^2_{Eis}(G,E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=(c_1-c_2+\frac{1}{2})\alpha \\ \widetilde{\pi}_\infty={}^\circ F_{w_2w_1}=\bigodot^{2(c_2+1)}\mathbb{C}^2}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}^{\infty_f}_f]$$

$$H^3_{Eis}(G,E) = \bigoplus_{\substack{\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_{E,P} \\ \text{with } d\chi=(c_2+\frac{3}{2})\alpha \\ \widetilde{\pi}_\infty={}^\circ F_{w_2w_1w_2}=\bigodot^{2(c_1-c_2)}\mathbb{C}^2}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_P}\otimes\widetilde{\pi}^{\infty_f}_f]$$

$$H^q_{Eis}(G,E) \quad = \quad 0 \qquad \textit{for } q \neq 2,3.$$

Both spaces are entirely built up by values of holomorphic Eisenstein series, whence the are no residual Eisenstein cohomology classes in this case.
All summands showing up in case (1) and (2) are non-trivial. Hence, there are non-trivial residual Eisenstein cohomology classes in degree $q = 0$ and 1, only if $\lambda = k\omega_2$, $k \in \mathbb{Z}_{\geq 0}$.

Proof. Recall the properties of $E^q_{\widetilde{\pi}}$ and the construction procedure explained in section 2. Since λ is algebraically integral and dominant, both c_i are positive half-integers and $0 \leq \frac{1}{2}c_1 \leq c_2 \leq c_1$. Our evaluation points of Eisenstein series $E_P(f,\Lambda)$ are given in table 2 as $\Lambda = (c_1 - c_2 + \frac{1}{2})\alpha$ resp. $(c_2 + \frac{3}{2})\alpha$. Generally, a cuspidal automorphic representation $\widetilde{\pi}$ of $L(\mathbb{A})$ satisfies $m(\widetilde{\pi}) = 1$. Indeed, we even have Strong Multiplicity One for $L = GL_1(B)$ by the global Jacquet-Langlands Correspondence and Strong Multiplicity One for GL_2, cf. [28].
We first consider the case $\lambda \neq k\omega_2$, $k = 0, 1, 2, \ldots$, i.e., $c_1 \neq c_2$, hence $c_1 - c_2 \geq \frac{1}{2}$. It follows that we only need to consider cuspidal representations $\widetilde{\pi}$ which are not one-dimensional. Indeed, by lemma 6.1, $\dim \widetilde{\pi}_\infty = \dim {}^\circ F_w$ and by table 1 and the fact that $c_1 - c_2 \geq \frac{1}{2}$, $\dim {}^\circ F_w \geq 2$. Further, for $c_1 - c_2 \geq \frac{1}{2}$, the above evaluation points satisfy $\Re e(s) \geq 1$. So, as the case $\widetilde{\pi} = \mathbf{1}$ is excluded, proposition 5.1 shows that there are no poles of Eisenstein series at the above points, whence the assertion of the theorem in case (2).
Now, let $\lambda = k\omega_2$, $k = 0, 1, 2, \ldots$, i.e., $c_1 = c_2 = k$. Again, for degrees 1 and 2 we need to consider cuspidal representations of $L(\mathbb{A})$ with $\dim \widetilde{\pi} > 1$, since $\dim \widetilde{\pi} \geq \dim \widetilde{\pi}_\infty = 2c_2 + 3 \geq 3$. As shown in proposition 5.1, the evaluation point $d\chi = \frac{1}{2}\alpha$ gives a pole if and only if $L(\frac{1}{2}, \widetilde{\pi}) \neq 0$ and $\widetilde{\chi} = \mathbf{1}$. This implies the assertion in theses degrees. For degrees 0 and 3 we need to consider the second evaluation point $d\chi = (c_2 + \frac{3}{2})\alpha$. Again the assertion follows by proposition 5.1. Finally, the non-vanishing of $E^2_{\widetilde{\pi}}(\Omega_{res}(\frac{1}{2}, \widetilde{\pi}))$ is a consequence of [44], Thm. III. 1. □

Remark 6.1. Let $(s, \widetilde{\pi})$ be a pair which is of type I or III in the sense of proposition 5.1. The cohomology-spaces $E^q_{\widetilde{\pi}}(\Omega_{res}(s, \widetilde{\pi}))$ (resp. $\Omega_{hol}(s, \widetilde{\pi})$) showing up in the above theorem are the cohomological analogs of $\mathcal{A}(s, \widetilde{\pi}) = \mathrm{Im} N(s, \widetilde{\pi})$ (resp. $\ker N(s, \widetilde{\pi})$).

Remark 6.2. Recall the group G' of section 5.4 (i.e., our group G defined via a quaternion division algebra B', which splits over $\mathbb{R}$). Any $\mathbb{R}$-split but not $\mathbb{Q}$-split $\mathbb{Q}$-form of the split group Sp_4 is of this type. Now, the Eisenstein cohomology of G' with respect to an arbitrary coefficient system E could of course be calculated in a similar way using the classification of the residual spectrum of G' accomplished independently by Grbac (cf. [19]) and Yasuda (cf. [52]). We remark that $H^q_{Eis}(G', \mathbb{C})$ was already calculated by D. Osenberg in his thesis, [40], but never published. The interested reader may find these result in [37], §§6-7.

6.3. A Remark on Geometry. For a moment, let G be any connected, semisimple algebraic group over $\mathbb{Q}$. As in 1.1 we can define spaces $S(K_f)$ for any open, compact subgroup $K_f \subset G(\mathbb{A}_f)$, which are by the possible failure of Strong Approximation a finite disjoint union of locally symmetric spaces $\Gamma_i \backslash G(\mathbb{R})/K_\mathbb{R}$, cf. [5], Thm. 5.1. We define the Borel-Serre compactification of $S(K_f)$ as usual, cf. [7] (resp. [42] for the adelic setting). It is a compact manifold $\overline{S(K_f)}$ with boundary $\partial\overline{S(K_f)}$ and interior $S(K_f)$. The inclusion $S(K_f) \hookrightarrow \overline{S(K_f)}$ is a homotopy-equivalence. So we can study the natural restriction map

$$res^q : H^q(S(K_f), \widetilde{E}) = H^q(\overline{S(K_f)}, \widetilde{E}) \to H^q(\partial\overline{S(K_f)}, \widetilde{E}).$$

Observe that $\ker res^q \supseteq H^q_{cusp}(G,E)^{K_f}$ and so (up to isomorphy) Im $res^q \subseteq H^q_{Eis}(G,E)^{K_f}$. As shown in [46], there is equality, if E has regular highest weight. In general, it may very well happen that there is no equality, meaning that there may exist non-cuspidal cohomology classes which restrict trivially to the cohomology of the boundary, see, e.g., [23], where such classes are constructed for $G = Res_{k/\mathbb{Q}}(SL_2)$, $k \neq \mathbb{Q}$ being a number field and $Res_{k/\mathbb{Q}}$ is Weil's restriction of scalars.

Let us come back to the case $G = Sp(1,1)$. From now on suppose also $E = \mathbb{C}$, the trivial representation.
By the work of J. Millson, [34], there are congruence subgroups $\Gamma = G(\mathbb{Q}) \cap K_f$ of $G(\mathbb{Q})$ which have non-trivial cohomology $H^1(S(K_f),\mathbb{C})$. For these groups Γ both cases, namely $H^1_{Eis}(G,\mathbb{C})^{K_f} \neq 0$ and $H^1_{cusp}(G,\mathbb{C})^{K_f} = 0$, seem interesting. In the first case, one may get non-cuspidal cohomology classes, which restrict trivially to the boundary-cohomology. In the second, we would have an example of a cuspidal automorphic representation π of $G(\mathbb{A})$, which has a *non*-tempered, cohomological, infinite component π_∞, that is, a candidate for an example of a cohomological CAP-representation.

7. Cuspidal Cohomology for $Sp(1,1)$

7.1. Local Multiplicities of the Archimedean Component. Let us now consider cuspidal cohomology in the classical case, i.e., let us consider cuspidal cohomology of principal congruence subgroups $\Gamma(N) = G(\mathbb{Q}) \cap K_N$. If we fix a maximal order $\mathfrak{O}$ of B, then

$$\Gamma(N) = \{g = (g_{ij}) \in G(\mathbb{Q}) | g_{ij} - \delta_{ij} \in N\mathfrak{O}\},$$

where δ_{ij} is the Kronecker-function and

$$H^q_{cusp}(\Gamma(N),E) := H^q_{cusp}(G,E)^{K_N} = \bigoplus_{\pi_\infty} H^q(\mathfrak{g},K,(\pi_\infty)_{(K)} \otimes E)^{m(\pi_\infty,\Gamma(N))},$$

where π_∞ runs over all irreducible, unitary, cohomological representations of $G(\mathbb{R})$ and

$$m(\pi_\infty,\Gamma(N)) = \sum_{\substack{\sigma \\ \sigma_\infty = \pi_\infty}} m(\sigma) \dim \sigma_f^{K_N} < \infty$$

is the multiplicity of π_∞ within $L^2_{cusp}(\Gamma(N)\backslash G(\mathbb{R})) = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_N}$.

The paper [51] provides a full classification of irreducible, unitary, cohomological representations of a semisimple Lie group. In order to apply it to $G(\mathbb{R}) = Sp(1,1)$, let us fix a maximal compact Cartan algebra $\mathfrak{t} \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ of $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ with respect to which the system of positive roots looks like $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C}) = \{\mu_1 \pm \mu_2, 2\mu_1, 2\mu_2\}$, μ_i extracting the entry of the i-th summand of $\mathfrak{t}$ of the form $\mathfrak{u}(1)$ and our representation E has highest weight μ, say. Skipping the details, we get

Theorem 7.1 ([51]). *For each μ there is an integer $j(\mu)$, $0 \leq j(\mu) \leq 2$, such that the irreducible, unitary $(\mathfrak{g},K)$-modules with non-trivial cohomology with respect to E are the uniquely determined representations $A_j(\mu)$, $j(\mu) \leq j \leq 1$, having the property*

$$H^q(\mathfrak{g},K,A_j(\mu) \otimes E) = \begin{cases} \mathbb{C} & \textit{if } q = j \textit{ or } q = 4-j \\ 0 & \textit{otherwise} \end{cases}$$

together with the two irreducible, unitary $(\mathfrak{g}, K)$*-modules* $A^+(\mu), A^-(\mu)$ *with*

$$H^q(\mathfrak{g}, K, A^{\pm}(\mu) \otimes E) = \begin{cases} \mathbb{C} & \text{if } q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This integer is given by

$$j(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu = k(\mu_1 + \mu_2),\ k = 1, 2, 3, ... \\ 2 & \text{otherwise.} \end{cases}$$

Since $rk_{\mathbb{C}}(G) = rk_{\mathbb{C}}(K)$, there are discrete series representations of $G(\mathbb{R})$, parameterized by their Harish-Chandra parameter. By [51], each cohomological discrete series representation has Harish-Chandra parameter $w(\mu+\rho)$, $w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})/W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Hence, by the classification of discrete series representations of semisimple Lie groups as done by Harish-Chandra (cf. [26]; [30], Thm. 9.20 and Thm. 12.21) for each highest weight μ, there are exactly two cohomological discrete series representations, which are precisely the modules $A^+(\mu)$ and $A^-(\mu)$. These are the only tempered representations in theorem 7.1.

It is known that $m(\pi_\infty, \Gamma(N))$ will grow like the covolume $vol(\Gamma(N)\backslash G(\mathbb{R}))$ in N, [45], so for sufficiently small congruence subgroups the multiplicity – and so by the above said also $H^*_{cusp}(\Gamma(N), E)$ – will be non-zero. However, applying the results of T. Arakawa given in [2], we can actually establish a concrete formula for the multiplicity of $A^+(\mu)$, $\mu = (\nu - 2)\mu_1$, $\nu > 4$, in $L^2_{cusp}(\Gamma(N)\backslash G(\mathbb{R}))$, $N \geq 3$. Hence we get a growth condition for the dimension of $H^2_{cusp}(\Gamma(N), E)$, $N \geq 3$.

7.2. Let $\mu = (\nu - 2)\mu_1$, $\nu > 4$ and $N \geq 3$. By the formula for the Harish-Chandra parameter given above, $A^+(\mu)$ is just the representation $U_{\nu,0}$ in [2]. In particular, the inequality $\nu > 4$ actually implies that $A^+(\mu)$ is integrable, i.e., sufficiently far away from the "tip of the closed, positive Weyl chamber at zero", cf. [2], Lemma 2.10.(ii). Hence, if we denote by $c(\mu)$ any matrix coefficient of $A^+(\mu)$ defined by a K-finite unit vector and by $d(\mu)$ the formal degree of $A^+(\mu)$, then we can insert $d(\mu)c(\mu)$ into the Selberg trace formula (cf. [39], p. 305). Observing Harish-Chandra's "Selberg Principle" ([26], Thm. 11) and the non-existence of elliptic and loxodromic elements in neat arithmetic groups (cf. [2], Lemma 5.5), we conclude that

$$m(A^+(\mu), \Gamma(N)) = d(\mu)vol(\Gamma(N)\backslash G(\mathbb{R})) + d(\mu)u,$$

for a certain number u standing for the contribution of the unipotent elements of $\Gamma(N)$ to the Selberg trace formula. Recall $h(\Gamma(N))$, the number of *cusps* of $\Gamma(N)$, i.e., the (finite) number of connected components of $\partial(\Gamma(N)\backslash\overline{X})$. Then Arakawa actually calculated

$$d(\mu)u = -h(\Gamma(N)),$$

(see [2], Prop. 5.4). One knows that $d(\mu) = \frac{\nu^3-\nu}{2^4\pi^2}$ by general theory, cf. [39].

Proposition 7.1. *Let* $\Gamma = \Gamma(N)$ *be a principal congruence subgroup of* G *of level* $N \geq 3$. *If* E *has highest weight* $\mu = (\nu - 2)\mu_1$, *satisfying* $\nu > 4$, *then*

$$m(A^+(\mu), \Gamma) = h(\Gamma)\Big(N^3(\nu^3 - \nu)\frac{\prod_{p|d(B)}(p^2+1)}{2^5 3^2 5} - 1\Big).$$

In particular, $m(A^+(\mu), \Gamma)$ *grows with* N *like the volume of* $\Gamma(N)\backslash G$, *for* $N \geq 3$

Proof. By the above discussion, all we have to do is calculate the covolume of Γ. This was done by Arakawa in [1], but no proof was provided there. So for sake of completeness we sketch a proof here.
We use the following notation: If $R = \mathbb{Z}$ (resp. $\mathbb{Z}_p$) and $\mathbb{K} = \mathbb{Q}$ (resp. $\mathbb{Q}_p$) we write

for brevity $G(R) := G(\mathbb{K}) \cap GL_2(\mathfrak{O} \otimes_{\mathbb{Z}} R)$. Let $\mathscr{G}_p$ (resp. G_p) be the smooth affine $\mathbb{Z}_p$-group scheme associated with $Sp_4(\mathbb{Z}_p)$ (resp. $G(\mathbb{Z}_p)$) and denote by $\overline{\mathscr{G}}_p$ (resp. $\overline{G}_p$) the group $\mathscr{G}_p \times_{\mathbb{Z}_p} \mathbb{F}_p$ (resp. $G_p \times_{\mathbb{Z}_p} \mathbb{F}_p$). Then both $\overline{\mathscr{G}}_p$ and $\overline{G}_p$ admit a Levi decomposition over $\mathbb{F}_p$ with Levi $\mathbb{F}_p$-subgroups $\overline{\mathscr{L}}_p$ and $\overline{L}_p$, say. Using Prasad's volume formula, cf. [41], Thm. 3.7, the volume for $G(\mathbb{Z})\backslash G(\mathbb{R})$ with respect to the Euler-Poincaré-measure equals

$$vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R})) = 2 \cdot \frac{m_1!}{(2\pi)^{m_1+1}} \frac{m_2!}{(2\pi)^{m_2+1}} \tau(G) \prod_p \frac{p^{\frac{1}{2}(\dim_{\mathbb{F}_p} \overline{L}_p + \dim_{\mathbb{F}_p} \overline{\mathscr{L}}_p)}}{|\overline{L}_p(\mathbb{F}_p)|}.$$

The integers m_1 and m_2 can be found in [41], 1.5, or – more originally – in [9], VI, "Planche III" (IX): $m_1 = 1$, $m_2 = 3$. The Tamagawa measure $\tau(G)$ equals one, G being simply connected. Whenever $p \nmid d(B)$, G splits over $\mathbb{Q}_p$ and $\overline{L}_p \cong \overline{\mathscr{L}}_p \cong Sp_4$ over $\mathbb{F}_p$. For these finite primes we have thus: $\dim_{\mathbb{F}_p} \overline{L}_p = 10$ and $|\overline{L}(\mathbb{F}_p)| = |Sp_4(\mathbb{F}_p)| = p^4(p^4-1)(p^2-1)$ as is proved in [38], 1.2. Hence, we can rewrite the last equation as

$$\begin{aligned} vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R})) &= \frac{12}{(2\pi)^6} \prod_p \frac{p^6}{(p^2-1)(p^4-1)} \prod_{p|d(B)} \frac{p^{\frac{1}{2}\dim_{\mathbb{F}_p} \overline{L}_p - 5} |Sp_4(\mathbb{F}_p)|}{|\overline{L}_p(\mathbb{F}_p)|} \\ &= \frac{12}{(2\pi)^6} \zeta(2)\zeta(4) \prod_{p|d(B)} \underbrace{\frac{p^{\frac{1}{2}\dim_{\mathbb{F}_p} \overline{L}_p - 1}(p^4-1)(p^2-1)}{|\overline{L}_p(\mathbb{F}_p)|}}_{=:\lambda_p} \\ &= \frac{1}{2^6 3^2 5} \prod_{p|d(B)} \lambda_p. \end{aligned}$$

The exact values for λ_p can be found in [16] 8, (8.10): $\lambda_p = (p^2+1)(p-1)$. As in [43], Prop. 4.5 and 4.6, one can prove that $vol = \frac{4\pi^2}{3} vol_{EP}$. This gives

$$vol(G(\mathbb{Z})\backslash G(\mathbb{R})) = \frac{\pi^2}{2^4 3^3 5} \prod_{p|d(B)} (p-1)(p^2+1).$$

So, in order to finish the proof, we need to find the finite index h_N of $\Gamma(N)$ in $G(\mathbb{Z})$. Using [1], 3 (in particular line (3.6)) and [2], Prop. 5.4 we can do this harking back to an old mass formula of Eichler, [12]:

$$h_N = 24N^3 h(\Gamma) \prod_{p|d(B)} (p-1)^{-1}.$$

Putting the pieces together gives the desired volume formula, hence the proposition. □

Corollary 7.1 (Non-vanishing of cuspidal cohomology). *Keeping the assumptions of proposition 7.1,* $m(A^+(\mu), \Gamma) > \frac{N^3}{12} - 1$. *In particular, the cuspidal cohomology* $H^2_{cusp}(\Gamma, E)$ *has dimension at least* 2 *for each* $N \geq 3$.

Remark 7.1. One can use Arakawa's paper [2] to show a much more general result on non-vanishing of cuspidal cohomology for $Sp(n,1)$, $n > 1$. We did this in [21].

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