AN EXPLICIT GROWTH CONDITION FOR MIDDLE-DEGREE CUSPIDAL COHOMOLOGY OF ARITHMETICALLY DEFINED QUATERNIONIC HYPERBOLIC n-MANIFOLDS

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ABSTRACT. Let $G/\mathbb{Q}$ be the simple algebraic group $Sp(n,1)$ and $\Gamma = \Gamma(N)$ a principal congruence subgroup of level $N \geq 3$. Denote by K a maximal compact subgroup of the real Lie group $G(\mathbb{R})$. Then a double quotient $\Gamma\backslash G(\mathbb{R})/K$ is called an arithmetically defined, quaternionic hyperbolic n-manifold. In this paper we give an explicit growth condition for the dimension of cuspidal cohomology $H^{2n}_{cusp}(\Gamma\backslash G(\mathbb{R})/K, E)$ in terms of the underlying arithmetic structure of G and certain values of zeta-functions. These results rely on the work of T. Arakawa, [2].

1. INTRODUCTION AND GENERALITIES

1.1. Let G be a semisimple algebraic group over $\mathbb{Q}$, K a maximal compact subgroup of the real Lie group $G(\mathbb{R})$ and $\mathfrak{g} = Lie(G(\mathbb{R}))$. Then for each irreducible, finite-dimensional, complex representation E of $G(\mathbb{R})$ and for each torsionfree arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ the space of *cuspidal cohomology* of $S(\Gamma) = \Gamma\backslash G(\mathbb{R})/K$ is defined as the $(\mathfrak{g}, K)$-cohomology of the space of (smooth and K-finite) cuspidal L^2-functions on $\Gamma\backslash G(\mathbb{R})$:

$$H^q_{cusp}(S(\Gamma), E) := H^q(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E).$$

Recall that a square-integrable function $f : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$ is called cuspidal, if all of its constant Fourier coefficients along proper parabolic $\mathbb{Q}$-subgroups vanish. Let $Z(\mathfrak{g})$ be the centre of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. It acts naturally on the $(\mathfrak{g}, K)$-module $V(\Gamma) = L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty_{(K)}$ of smooth and K-finite functions inside $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$. The subspace determined by the $Z(\mathfrak{g})$-finite functions in $V(\Gamma)$ equals the space of cuspidal automorphic forms $\mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R}))$ for the arithmetic group Γ.

It is well-known that one can construct each automorphic form on G out of cuspidal automorphic forms on smaller groups $L \subseteq G$. On the other hand, the theory of unitary representations shows that the natural inclusion $\mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R})) \hookrightarrow L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ gives rise to an isomorphism

$$H^q_{cusp}(S(\Gamma), E) \cong H^q(\mathfrak{g}, K, \mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R})) \otimes E),$$

pointing out deep arithmetic connections between cuspidal cohomology and automorphic forms in general. In particular, a solid and comprehensive understanding of cohomological automorphic forms should be preceded by a thorough analysis of cuspidal cohomology.

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1.2. One can show that

$$H^q_{cusp}(S(\Gamma), E) = \bigoplus_{\pi \in \widehat{G}_{\mathrm{coh}}} H^q(\mathfrak{g}, K, \pi_{(K)} \otimes E)^{m(\pi,\Gamma)},$$

the sum ranging over the set $\widehat{G}_{\mathrm{coh}}$ of all equivalence classes of irreducible, unitary, cohomological representations π of $G(\mathbb{R})$, all of which having finite multiplicity $m(\pi, \Gamma)$ inside $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$. As $\widehat{G}_{\mathrm{coh}}$ is known by Vogan and Zuckerman, [13], the real problem in analyzing cuspidal cohomology consists in an effective description of the multiplicities $m(\pi, \Gamma)$. This question is far from being answered in this generality, though, e.g., for discrete series representations π, non-vanishing results have already been established by a wide range of people. See, e.g., [12]. Recall that $(\mathfrak{g}, K)$-cohomology of a (cohomological) discrete series representation is one-dimensional in the middle degree $q = \frac{1}{2}(\dim_{\mathbb{R}} G(\mathbb{R}) - \dim_{\mathbb{R}} K)$ and vanishes in all other degrees. In this paper we deal with the multiplicities of such discrete series representations in the setup of arithmetically defined quaternionic hyperbolic n-manifolds.

2. Quaternionic hyperbolic space

2.1. Let G be the simple algebraic group $Sp(n, 1)$ to be defined as the group of isometries of a non-degenerate Hermitian form of signature $(n, 1)$ on a definite quaternion algebra B over $\mathbb{Q}$. It is a connected, non-quasisplit inner form of $Sp_{2(n+1)}$ (the $\mathbb{Q}$-split group of Cartan-type C_{n+1}) of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = 1$. The associate real Lie group $G(\mathbb{R})$ has discrete series. A maximal compact subgroup K of $G(\mathbb{R})$ is isomorphic to $K = Sp(1) \times Sp(n)$. The Riemannian symmetric quotient

$$\mathcal{H}_n := G(\mathbb{R})/K$$

is called the *quaternionic hyperbolic n-space*, which is of dimension $\dim_{\mathbb{R}} \mathcal{H}_n = 4n$. Fix a maximal order $\mathfrak{O}$ of B. Then the principal congruence subgroup of G of level $N \geq 3$ is given explicitly as $\Gamma = \Gamma(N) = \{g = (g_{ij}) \in G(\mathbb{Q}) | g_{ij} - \delta_{ij} \in N\mathfrak{O}\}$, where δ_{ij} is the Kronecker delta-function. The condition $N \geq 3$ ensures that it is "nice", meaning that for each $\gamma \in \Gamma$ for which there is a positive integer ℓ such that γ^ℓ is unipotent, γ itself must be unipotent. This particularly implies that Γ is torsionfree. We call the locally symmetric space $S(\Gamma) = \Gamma\backslash\mathcal{H}_n$ an *arithmetically defined* quaternionic hyperbolic n-manifold.

2.2. The aim of this paper is to give a concrete formula which expresses the multiplicity $m(\pi, \Gamma)$ of certain cohomological discrete series representations π of $G(\mathbb{R})$ in terms of the arithmetic of the underlying quaternion algebra B and values of the classical Riemannian ζ-function. In particular this will give rise to a growth condition of the dimension of the space of cuspidal cohomology of Γ in the middle degree $q = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{H}_n = 2n$, cf. 1.2. Similar results have already been obtained for arithmetically defined, real and complex hyperbolic manifolds, see, e.g., [10] for $SO(n, 1)$.

We will achieve this using Selberg's Trace Formula for groups of $\mathbb{R}$-rank one together with T. Arakawa's explicit calculations for certain integrable discrete series of $G = Sp(n, 1)$. To be more precise, let λ be the highest weight of our finite-dimensional representation E of $G(\mathbb{R})$. Suppose that $\lambda = (\nu - 2n)\varpi_1$, where $\nu > 4n$ is an integer and ϖ_1 is the first fundamental weight of $\mathfrak{g}_{\mathbb{C}}$. Then Arakawa showed in [2], Lemma 2.10.(ii), that the discrete series representation $A(\lambda)$ which is of Harish-Chandra parameter $\lambda + \rho$ (ρ being the sum of all fundamental weights) is integrable. Whence, by [8], p. 305, we can compute the multiplicity $m(A(\lambda), \Gamma)$

as a sum of terms showing up in the Selberg Trace Formula. The fact that Γ is "nice" simplifies this sum ([2], Lemma 5.5). Observing Harish-Chandra's "Selberg principle" ([6], Thm. 11)) we end up in calculating the contribution of the central and of the unipotent elements of Γ to the Selberg Trace Formula. If $n \geq 2$, this last contribution was shown to vanish by Arakawa in [2], Prop. 5.4. Therefore we arrive at the following formula

$$m(A(\lambda), \Gamma) = d_\lambda vol(\Gamma \backslash G(\mathbb{R})).$$

Here, d_λ is the formal degree of $A(\lambda)$ and $vol(\Gamma\backslash G(\mathbb{R}))$ the covolume of Γ in $G(\mathbb{R})$. Both numbers are defined with respect to a re-scaled version of the *standard Haar measure* on $G(\mathbb{R})$, given as follows: We assume that a minimal parabolic subgroup P has been fixed. Assume that L is a Levi subgroup of P with maximal, central $\mathbb{Q}$-split torus A and N is an unipotent radical of P. As usual we write ρ_P for the half sum of positive restricted roots of $G(\mathbb{R})$ with respect to $A(\mathbb{R})$, counted with multiplicity. Fix the Lebesgue measures on the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, which are normalized relative to the Euclidean structure associated with the Killing form. Exponentiating it, we get a Haar measure da on $A(\mathbb{R})^\circ$ and dn on $N(\mathbb{R})$. Let us also denote by dk the unique Haar measure on K, which gives it total volume one. Then our Haar measure on $G(\mathbb{R})$ is defined as

$$dg(g) = \tfrac{1}{\sqrt{2}} e^{\rho_P \log(a)} dk(k) da(a) dn(n),$$

where $g = kan$ according to the Iwasawa decomposition $G(\mathbb{R}) = KA(\mathbb{R})^\circ N(\mathbb{R})$.

3. The central contribution

3.1. The covolume. In order to calculate $vol(\Gamma\backslash G(\mathbb{R}))$ we use the following notation: If $R = \mathbb{Z}$ (resp. $\mathbb{Z}_p$) and $\mathbb{K} = \mathbb{Q}$ (resp. $\mathbb{Q}_p$) we write for short $G(R) := G(\mathbb{K}) \cap GL_{n+1}(\mathfrak{O} \otimes_{\mathbb{Z}} R)$. Further, we denote by $S(B)$ the set of finite primes p, where B ramifies. Now we observe that $\Gamma = \Gamma(N)$ has finite index $h_N = |\Gamma\backslash G(\mathbb{Z})|$ in $G(\mathbb{Z})$. Hence, $vol(\Gamma\backslash G(\mathbb{R})) = h_N vol(G(\mathbb{Z})\backslash G(\mathbb{R}))$ and we can concentrate on calculating the covolume of $G(\mathbb{Z})$ in $G(\mathbb{R})$. This is done in the next proposition.

Proposition 3.1. *We have*

$$vol(G(\mathbb{Z})\backslash G(\mathbb{R})) = \frac{2^{4n}\pi^{2n}}{(2n+1)!} \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} (p^j + (-1)^j) \right).$$

Proof. Let $\mathscr{G}_p$ (resp. G_p) be the smooth affine $\mathbb{Z}_p$-group scheme associated with $Sp_{2(n+1)}(\mathbb{Z}_p)$ (resp. $G(\mathbb{Z}_p)$) and denote by $\overline{\mathscr{G}}_p$ (resp. $\overline{G}_p$) the group $\mathscr{G}_p \times_{\mathbb{Z}_p} \mathbb{F}_p$ (resp. $G_p \times_{\mathbb{Z}_p} \mathbb{F}_p$). Then both $\overline{\mathscr{G}}_p$ and $\overline{G}_p$ admit a Levi decomposition over $\mathbb{F}_p$ with Levi $\mathbb{F}_p$-subgroups $\overline{\mathscr{L}}_p$ and $\overline{L}_p$, say. Using Prasad's volume formula, cf. [9], Thm. 3.7, the volume for $G(\mathbb{Z})\backslash G(\mathbb{R})$ with respect to the Euler-Poincaré-measure equals

$$vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \chi_{EP}(U/K) \cdot \prod_{j=1}^{n+1} \frac{m_j!}{(2\pi)^{m_j+1}} \mathrm{Tam(G)} \prod_p \frac{p^{\frac{1}{2}(\dim_{\mathbb{F}_p} \overline{L}_p + \dim_{\mathbb{F}_p} \overline{\mathscr{L}}_p)}}{|\overline{L}_p(\mathbb{F}_p)|},$$

where $U = Sp(n+1)$ is the compact dual of G and $\chi_{EP}(U/K)$ denotes the Euler-Poincaré-characteristic of the compact symmetric space U/K (which is nothing but the quaternionic projective space $\mathbb{P}(\mathbb{H}^n)$). Also, $\mathrm{Tam}(G)$ denotes the Tamagawa measure of G, which equals $\mathrm{Tam}(G) = 1$, G being simply connected.
The integers m_j can be found, e.g., in [9], 1.5: $m_j = 2j - 1$.

For a prime $p \notin S(B)$, G splits over $\mathbb{Q}_p$. Thus, $\dim_{\mathbb{F}_p} \overline{L}_p = 2(n+1)^2 + n + 1$ and

$$|\overline{L}(\mathbb{F}_p)| = |Sp_{2(n+1)}(\mathbb{F}_p)| = p^{(n+1)^2} \prod_{i=1}^{n+1} (p^{2i} - 1).$$

The last equation is proved in [7], 1.2. Having collected this information, we can rewrite the covolume with respect to the Euler-Poincaré-measure as

$$vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \chi_{EP}(U/K) \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} \lambda_p \right),$$

if we set for $p \in S(B)$

$$\lambda_p = p^{\frac{1}{2}(\dim_{\mathbb{F}_p} \overline{L}_p - \dim_{\mathbb{F}_p} \overline{\mathscr{L}}_p)} \frac{|Sp_{2(n+1)}(\mathbb{F}_p)|}{|\overline{L}_p(\mathbb{F}_p)|}.$$

The exact values for λ_p can be found in [4], 8, (8.10): $\lambda_p = \prod_{j=1}^{n+1}(p^j + (-1)^j)$. It is well-know (cf. [11], the proof of Prop. 4.5) that

$$vol = \frac{vol_{g_0}(U/K)}{\chi_{EP}(U/K)} vol_{EP},$$

where vol_{g_0} denotes the volume uniquely determined by the canonical Riemannian metric g_0 on $U/K = \mathbb{P}(\mathbb{H}^n)$ which gives it geodesic diameter π. For details, see [3]. *Ibidem*, (3.10), we can find the volume-formula

$$vol_{\frac{1}{2}g_0}(\mathbb{P}(\mathbb{H}^n)) = \frac{\pi^{2n}}{(2n+1)!}.$$

Since stretching the metric $\frac{1}{2}g_0$ to its double g_0 multiplies the corresponding volume by $2^{\dim_{\mathbb{R}} \mathbb{P}(\mathbb{H}^n)} = 2^{4n}$, the proposition is proved. $\square$

3.2. The formal degree. We need to calculate the formal degree d_λ of $A(\lambda)$. One can do this by inserting in the general formula mentioned in [8] or one uses the concrete calculations in [2], Proposition 2.9, to get

Proposition 3.2. *The formal degree d_λ of $A(\lambda)$, $\lambda = (\nu - 2n)\varpi_1$, $\nu > 4n$, is given as*

$$d_\lambda = \frac{(\nu+1)!}{2^{4n}\pi^{2n}(\nu-2n)!}.$$

4. The main result

4.1. We summarize the results of the previous section in the following

Theorem 4.1. *Let G be the simple algebraic group $Sp(n,1)$, $n \geq 2$, defined via a quaternion algebra B (as in 2.2) and $\Gamma = \Gamma(N)$ a principal congruence subgroup of level $N \geq 3$. Let $E = E_\lambda$ be a finite-dimensional, irreducible, complex representation of the Lie group $G(\mathbb{R})$ of highest weight $\lambda = (\nu - 2n)\varpi_1$ and suppose $\nu > 4n$. Then the multiplicity of the discrete series representation $A(\lambda)$ of Harish-Chandra parameter $\lambda + \rho$ within the space $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ of classical cusp forms is given by*

$$m(A(\lambda), \Gamma) = \frac{h_N(\nu+1)!}{(2n+1)!(\nu-2n)!} \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} (p^j + (-1)^j) \right).$$

In particular, this establishes a growth condition for the dimension of the space of cuspidal cohomology $H^{2n}_{cusp}(\Gamma\backslash\mathcal{H}_n, E)$.

The (in fact more difficult) case $n = 1$ was already treated by Arakawa in details in [1], Thm. 2 and transferred to the cohomological setting by us in [5], Prop. 7.1. The resulting growth condition of $\dim_{\mathbb{C}} H^2_{cusp}(\Gamma\backslash\mathcal{H}_1, E)$ is in fact identical to the one given in our above theorem (for $n = 1$), except that now the contribution of the unipotent elements of Γ does not vanish.

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