A GROWTH CONDITION FOR CUSPIDAL COHOMOLOGY OF ARITHMETICALLY DEFINED QUATERNIONIC HYPERBOLIC n-MANIFOLDS

HARALD GROBNER

ABSTRACT. Let $G/\mathbb{Q}$ be the simple algebraic group $Sp(n,1)$ and $\Gamma = \Gamma(N)$ a principal congruence subgroup of level $N \geq 3$. Denote by K a maximal compact subgroup of the real Lie group $G(\mathbb{R})$. Then a double quotient $\Gamma\backslash G(\mathbb{R})/K$ is called an arithmetically defined, quaternionic hyperbolic n-manifold. In this paper we give an explicit growth condition for the dimension of cuspidal cohomology $H^*_{cusp}(\Gamma\backslash G(\mathbb{R})/K)$ in terms of the underlying arithmetic structure of G and certain values of zeta-functions. These results rely on the work of T. Arakawa, [2, 1].

INTRODUCTION

Let G be a semisimple algebraic group over the rationals $\mathbb{Q}$, K a maximal compact subgroup of the real Lie group $G(\mathbb{R})$ and $\mathfrak{g} = Lie(G(\mathbb{R}))$. Then for each irreducible, finite-dimensional representation E of $G(\mathbb{R})$ and for each torsionfree arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ the space of *cuspidal cohomology* of $S(\Gamma) = \Gamma\backslash G(\mathbb{R})/K$ is defined as the $(\mathfrak{g}, K)$-cohomology of the space of (smooth and K-finite) cuspidal L^2-functions on $\Gamma\backslash G(\mathbb{R})$:

$$H^q(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E).$$

Recall that a square-integrable function $f : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$ is called cuspidal, if all of its constant Fourier coefficients along proper parabolic $\mathbb{Q}$-subgroups vanish. Let $Z(\mathfrak{g})$ be the centre of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. It acts naturally on the $(\mathfrak{g}, K)$-module $V(\Gamma) = L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty_{(K)}$ of smooth and K-finite functions inside $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$. The subspace determined by the $Z(\mathfrak{g})$-finite functions in $V(\Gamma)$ equals the space of cuspidal automorphic forms $\mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R}))$ for the arithmetic group Γ.

We recall that by a fundamental work [8] of J. Franke (for the number field case) it is now known that one can - at least theoretically - construct each automorphic form out of cuspidal automorphic forms on smaller algebraic groups $L \subseteq G$. On the other hand, the theory of unitary representations shows that the natural inclusion $\mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R})) \hookrightarrow L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ gives rise to an isomorphism

$$H^q(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E) = H^q(\mathfrak{g}, K, \mathcal{A}_{cusp}(\Gamma\backslash G(\mathbb{R})) \otimes E),$$

pointing out deep arithmetic connections between cuspidal cohomology and automorphic forms in general. In particular, a solid and comprehensive understanding of cohomological automorphic forms should be preceded by a thorough analysis of cuspidal cohomology.

Date: March 12, 2010.

2000 *Mathematics Subject Classification.* Primary: 11F75, 11F72; Secondary: 20G20, 20H05.

Key words and phrases. cohomology of arithmetic groups, cuspidal cohomology, cuspidal automorphic representation, trace formula, multiplicities.

Using the work of Gelfand, Graev and Piatetski-Shapiro ([10]), one can show that

$$H^q(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E) = \bigoplus_{\pi \in \widehat{G}_{\mathrm{coh}}} H^q(\mathfrak{g}, K, \pi_{(K)} \otimes E)^{m(\pi,\Gamma)},$$

the sum ranging over the set $\widehat{G}_{\mathrm{coh}}$ of all equivalence classes of irreducible, unitary, cohomological representations π of $G(\mathbb{R})$, all of which having finite multiplicity $m(\pi, \Gamma)$ inside $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$. As $\widehat{G}_{\mathrm{coh}}$ is known by Vogan and Zuckerman, [22], the real problem in analyzing cuspidal cohomology consists in an effective description of the multiplicities $m(\pi, \Gamma)$. This question is far from being answered in this generality, though e. g. for discrete series representations π, non-vanishing results have already been established by a wide range of people.

Let G be the simple algebraic group $Sp(n, 1)$ to be defined as the group of isometries of a non-degenerate Hermitian form of signature $(n, 1)$ on a definite quaternion algebra B over $\mathbb{Q}$. It is a connected $\mathbb{Q}$-group of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = 1$ whose associate real Lie group $G(\mathbb{R})$ has discrete series. Assume that $\Gamma = \Gamma(N)$ is a principal congruence subgroup of G of level $N \geq 3$. The aim of this paper is to give a concrete formula which expresses the multiplicity $m(\pi, \Gamma)$ for certain cohomological discrete series representations π in terms of the arithmetic of the underlying quaternion algebra B and values of the classical Riemannian ζ-function. In particular this will give rise to a growth condition of the dimension of the space of cuspidal cohomology of Γ.

We will achieve this using Selberg's Trace Formula for groups of $\mathbb{R}$-rank one together with T. Arakawa's explicit calculations for certain integrable discrete series of $G = Sp(n, 1)$. To be more precise, let λ be the highest weight of our finite-dimensional representation E of $G(\mathbb{R})$. Suppose, that $\lambda = (\nu - 2n)\varpi_1$ where $\nu > 4n$ is an integer and ϖ_1 is the first fundamental weight of $\mathfrak{g}$. Then Arakawa showed in [2] that the discrete series representation $A(\lambda)$ which is of Harish-Chandra parameter $\lambda + \rho$ (ρ being the sum of all fundamental weights) is integrable. Whence, by the work of Osborne and Warner ([16], [23]), we can compute the multiplicity $m(A(\lambda), \Gamma)$ as a sum of terms showing up in the Selberg Trace Formula. The fact that Γ is "nice" if $N \geq 3$ (i.e. any element $\gamma \in \Gamma$ for which there is an integer ℓ such that γ^ℓ is unipotent must be itself unipotent) will simplify this sum and we end up in calculating the formal degree d_λ of $A(\lambda)$, the covolume $vol(\Gamma\backslash G(\mathbb{R}))$ and the contribution of the unipotent elements of Γ to the Selberg Trace Formula. If $n \geq 2$, this last contribution was shown to vanish by Arakawa. Therefore we arrive at the following formula $m(A(\lambda), \Gamma) = d_\lambda vol(\Gamma\backslash G(\mathbb{R}))$. Now suppose $S(B)$ is the set of prime numbers p, for which B does not split and say h_N is the finite index of Γ in $G(\mathbb{Z})$. Then an explicit calculation (cf. proposition 4.1) will show our main result here (cf. theorem 5.1):

Theorem. *Let G be the simple algebraic group $Sp(n, 1)$, $n \geq 2$, defined via a quaternion algebra $B/\mathbb{Q}$ and $\Gamma = \Gamma(N)$ a principal congruence subgroup of level $N \geq 3$. Let E be a finite-dimensional, irreducible, complex representation of the Lie group $G(\mathbb{R})$ of highest weight $\lambda = (\nu - 2n)\varpi_1$ and suppose $\nu > 4n$. Then the multiplicity of the discrete series representation $A(\lambda)$ of Harish-Chandra parameter $\lambda + \rho$ within the space $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ of classical cusp forms is given by*

$$m(A(\lambda), \Gamma) = h_N \frac{(\nu-1)(\nu+2n-1)\nu!}{(2n+1)!(\nu-2n+1)!} \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} (p^j + (-1)^j) \right).$$

Recall that the symmetric space $\mathcal{H}_n = G(\mathbb{R})/K$ is the quaternionic hyperbolic n-space. Thus the above theorem will give the following corollary on the dimension of cuspidal cohomology of arithmetically defined quaternionic hyperbolic n-manifolds:

Corollary (Growth condition)**.** *Keeping the assumptions of the theorem, the dimension of the space of cuspidal cohomology of an arithmetically defined, quaternionic hyperbolic n-manifold $\Gamma\backslash\mathcal{H}_n$ grows at least as*

$$h_N \frac{(\nu-1)(\nu+2n-1)\nu!}{(2n+1)!(\nu-2n+1)!}\prod_{j=1}^{n+1}\left(\frac{(2j-1)!}{(2\pi)^{2j}}\zeta(2j)\prod_{p\in S(B)}(p^j+(-1)^j)\right).$$

The (in fact more difficult) case $n=1$ was already treated by Arakawa in details in [1], Thm. 2 and transferred to the cohomological setting by us in [11], Prop. 7.1. The resulting growth condition of cuspidal cohomology of principal congruence subgroups Γ of $Sp(1,1)$ is in fact identical to the one given in our above corollary (for $n=1$), except that now the contribution of the unipotent elements of Γ does not vanish (whence it has to be added to the formula of the corollary). Compare this also to our discussion in section 4.2.

Notation and Conventions. Throughout this paper G will be a connected, simple algebraic group over $\mathbb{Q}$ of ranks $rk_{\mathbb{Q}}(G)=rk_{\mathbb{R}}(G)=1$. Lie algebras of groups of real points of algebraic groups will be denoted by the same but fractional letter, e.g. $\mathrm{Lie}(G(\mathbb{R}))=\mathfrak{g}$.

We use the standard terminology and hypotheses concerning algebraic groups and their subgroups to be found in [14] I.1.4-I.1.12. In particular we assume that a minimal (and so because of $rk_{\mathbb{Q}}(G)=1$ also maximal) parabolic subgroup P has been fixed. Assume that L is a Levi subgroup of P and N is an unipotent radical of P so that we have the Levi decomposition $P=LN$. If we additionally denote by A a maximal, central $\mathbb{Q}$-split torus in L then we also get the Langlands decomposition $P=MAN$. As usual, $M=\bigcap_\chi \ker\chi$, χ ranging over the group $X(L)$ of all $\mathbb{Q}$-characters on L. We write $\Delta(P,A)$ for the set of weights of the adjoint action of P with respect to A. ρ_P denotes the half-sum of these weights, counted with multiplicity. That is the half sum of positive restricted roots of $G(\mathbb{R})$ with respect to $A(\mathbb{R})$, counted with multiplicity.

Fix the Lebesgue measures on the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, which are normalized relative to the Euclidean structure associated with the Killing form. Exponentiating it, we get a Haar measure da on $A(\mathbb{R})^\circ$ and dn on $N(\mathbb{R})$. Let us also denote by dk the unique Haar measure on K, which gives it total volume one. Then the standard Haar measure on $G(\mathbb{R})$ (with respect to which all volumes or integrability conditions will be defined in this article) is defined as $dg(g)=e^{2\rho_P\log(a)}dk(k)da(a)dn(n)$, where $g=kan$ according to the Iwasawa decomposition $G(\mathbb{R})=KA(\mathbb{R})^\circ N(\mathbb{R})$.

1. Cuspidal cohomology: Generalities

1.1. Let G be a simple algebraic group over $\mathbb{Q}$, which satisfies the assumptions of our section of conventions. These are imposed on G in order to avoid difficulties later on. Let K be a maximal compact subgroup of the real Lie group $G(\mathbb{R})$. Then $X=G(\mathbb{R})/K$ is a Riemannian symmetric space associated to $G(\mathbb{R})$ and K. We let Γ be any torsionfree arithmetic subgroup of $G(\mathbb{Q})$. The double quotient

$$S(\Gamma)=\Gamma\backslash X \tag{1}$$

is a locally symmetric space, which is by the present assumptions a smooth, non-compact manifold of dimension $\dim_{\mathbb{R}} S(\Gamma) = \dim_{\mathbb{R}} X$ and of finite volume. Let E be a finite-dimensional, irreducible, complex representation of $G(\mathbb{R})$. Then this representation gives rise to a locally constant sheaf $\widetilde{E}$ on (1). Hence, the sheaf-cohomology $H^*(S(\Gamma), \widetilde{E})$ is defined.

It is well-known (cf. [5], VII, Corollary 2.7) that there is an equality

$$H^*(S(\Gamma), \widetilde{E}) = H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes E).$$

1.2. Let P be the unique standard parabolic $\mathbb{Q}$-subgroup of G having Langlands decomposition $P = MAN$. Let $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ be the space of cuspidal, square-integrable functions $f : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$. Recall that cuspidality just means by the present assumptions that

$$\int_{\Gamma\cap N(\mathbb{R})\backslash N(\mathbb{R})} f(ng)dn = 0 \quad \forall g \in G(\mathbb{R}).$$

By [4], Corollary 5.5. the natural map

$$H^*(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E) \to H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes E)$$

given by the inclusion $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \hookrightarrow C^\infty(\Gamma\backslash G(\mathbb{R}))$ is injective. (We remark that this would not be true if E is infinite-dimensional, see [7], Cor. 5.2.) This motivates the definition of the subspace of *cuspidal cohomology* of $S(\Gamma)$:

$$H^*_{cusp}(S(\Gamma), \widetilde{E}) := H^*(\mathfrak{g}, K, L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))^\infty \otimes E).$$

1.3. The space $L^2_{cusp}(\Gamma\backslash G(\mathbb{R}))$ decomposes as a $G(\mathbb{R})$-module as a direct Hilbert sum over all irreducible, unitary representations of $G(\mathbb{R})$, each of which occurring with finite multiplicity $m(\pi, \Gamma) = \dim \operatorname{Hom}_{G(\mathbb{R})}(\pi, L^2_{cusp}(\Gamma\backslash G(\mathbb{R})))$, see [10], Theorem p. 97. Let $\widehat{G}_{\mathrm{coh}}$ be the cohomological, unitary dual of $G(\mathbb{R})$, i.e. the set of all (equivalence classes of) irreducible unitary representations π of $G(\mathbb{R})$, which have non-trivial $(\mathfrak{g}, K)$-cohomology when twisted by some finite dimensional representation of $G(\mathbb{R})$. By [5], VII, Lemma 3.3, we get a *finite* direct sum decomposition

$$H^*_{cusp}(S(\Gamma), \widetilde{E}) = \bigoplus_{\pi\in\widehat{G}_{\mathrm{coh}}} H^*(\mathfrak{g}, K, \pi_{(K)} \otimes E)^{m(\pi,\Gamma)}.$$

1.4. The set $\widehat{G}_{\mathrm{coh}}$ was classified by D. Vogan and G. Zuckerman in [22]. Fix a maximally compact Cartan subgroup T of $G(\mathbb{R})$ and some choice of positive roots Δ^+. Then our coefficient-module E has a highest weight λ with respect to this system. Let us write $E = E_\lambda$ to indicate this. In [22] is was shown that for each irreducible, finite-dimensional representations E_λ of $G(\mathbb{R})$, there exist finitely many irreducible, unitary representations - written $A_\mathfrak{q}(\lambda)$ - which are cohomological with respect to E_λ.

1.5. Discrete Series. Let us suppose from now on that $G(\mathbb{R})$ has discrete series representations, i.e. $rk_\mathbb{C}(G) = rk_\mathbb{C}(K)$ or again equivalently that our Cartan subgroup T is already compact: $T \subset K$. Let Δ^+ (resp. Δ^+_c) be a choice of positive roots of $G(\mathbb{R})$ (resp. K) with respect to T and suppose that these choices are compatible, i.e. $\Delta^+_c \subseteq \Delta^+$. The corresponding Weyl groups are denoted W_G resp. W_K. It is well known that for each discrete series representation π there is a character τ of T which is regular with respect to Δ^+ and unique up to the natural action of W_K and a discrete series representation π_τ such that $\pi_\tau \cong \pi$. Otherwise put, these regular characters, to be called *Harish-Chandra parameters* parameterize the discrete series. As a consequence, there are exactly $|W_G|/|W_K|$ many pairwisely inequivalent representations among the $\pi_{w(\tau)}$, w running through W_G. For all of this see (cf. [12]; [13], 9 + 12).

By the description of the $A_{\mathfrak{q}}(\lambda)$-modules in [22], Thm. 5.3, we conclude that the Harish-Chandra parameter of a discrete series representation which is cohomological with respect to E_λ looks like

$$\tau = w(\lambda + \rho), \quad w \in W_G.$$

Hence, for each E_λ there are exactly $b := |W_G|/|W_K| \geq 1$ many inequivalent discrete series representations, $A_i(\lambda)$, $1 \leq i \leq b$, which have non-trivial cohomology when twisted by E_λ. All of them contribute in the middle degree (cf. [5], II, Prop. 5.3 + Thm. 5.4):

$$H^q(\mathfrak{g}, K, A_i(\lambda) \otimes E) = \begin{cases} \mathbb{C} & \text{if } q = q(G) := \frac{1}{2}\dim_{\mathbb{R}} X \\ 0 & \text{else} \end{cases}$$

In [20] it was finally proved that for each infinite decreasing tower of torsionfree arithmetic groups

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset ...,$$

satisfying $\bigcap_i \Gamma_i = \{id\}$ and for each discrete series representation π, the multiplicity $m(\pi, \Gamma_i)$ will grow in i like the covolume $vol(\Gamma_i \backslash G(\mathbb{R}))$. So, indeed we know that

$$H^{q(G)}_{cusp}(S(\Gamma), \widetilde{E}) \neq 0$$

for Γ small enough.

2. Quaternionic hyperbolic space

2.1. The aim of this paper is to exploit calculations of T. Arakawa (see [2], [1]) to get hold of a *concrete formula* for the multiplicity $m(A(\lambda), \Gamma)$ of certain cohomological discrete series representations of $G = Sp(n,1)$ and principal congruence subgroups $\Gamma = \Gamma(N)$, $N \geq 3$; and so to determine an explicit growth condition for cuspidal cohomology of arithmetically defined, quaternionic hyperbolic manifolds $S(\Gamma)$.

Similar results have already been obtained for arithmetically defined, real and complex hyperbolic manifolds, see e.g. [18] for $SO(n,1)$.

2.2. Let us collect some basic information on quaternionic hyperbolic manifolds. Therefore let B be a quaternion algebra over $\mathbb{Q}$ with canonical involution $x \mapsto \overline{x}$, s.t. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$ where $\mathbb{H}$ equals the real Hamilton quaternions. We denote by $S(B)$ the finite set of non-archimedean places p where B does not split, i.e. $p \neq \infty$ and $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. Suppose $f : B^{n+1} \times B^{n+1} \to B$ is a Hermitian form of signature $(n, 1)$, where $n \geq 1$ and B^{n+1} is being regarded as a B-right module. We suppose that f is equivalent to $(x, y) \mapsto \sum_{i=1}^n x_i \overline{y}_i - x_{n+1}\overline{y}_{n+1}$. Then we define $Sp(n,1)$ to be the group of all B-linear automorphisms of B^{n+1} leaving invariant f:

$$G := Sp(n,1) = \{g \in GL_{n+1}(B) | g^* K_{n,1} g = K_{n,1}\}.$$

Here, $g^* = (\overline{g}_{ji})_{i,j} = \overline{g}^t$ and

$$K_{n,1} := \begin{pmatrix} id_{n\times n} & 0 \\ 0 & -1 \end{pmatrix}.$$

$Sp(n,1)$ is a connected, simply connected, simple algebraic group over $\mathbb{Q}$ of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = 1$ and $rk_{\mathbb{C}}(G) = n+1$. It is a non-quasisplit inner form of

$Sp_{2(n+1)}$, the $\mathbb{Q}$-split group of Cartan-type C_{n+1}. A maximal compact subgroup K of $G(\mathbb{R})$ is isomorphic to $K = Sp(1) \times Sp(n)$. The Riemannian symmetric quotient

$$\mathcal{H}_n := G(\mathbb{R})/K$$

is called the *quaternionic hyperbolic n-space*, which is of dimension $\dim_{\mathbb{R}} \mathcal{H}_n = 4n$.

2.3. Fix a maximal order $\mathfrak{O}$ of B. Then the principal congruence subgroup of G of level $N \geq 3$ is given explicitly as

$$\Gamma = \Gamma(N) = \{g = (g_{ij}) \in G(\mathbb{Q}) | g_{ij} - \delta_{ij} \in N\mathfrak{O}\},$$

where δ_{ij} is the Kronecker delta-function. The condition $N \geq 3$ ensures that it is "nice", meaning that for each $\gamma \in \Gamma$ for which there is a positive integer ℓ such that γ^ℓ is unipotent, γ itself must be unipotent. This particularly implies that Γ is torsionfree.

We call the locally symmetric space $S(\Gamma) = \Gamma \backslash \mathcal{H}_n$ an *arithmetically defined* quaternionic hyperbolic n-manifold.

2.4. As $rk_{\mathbb{C}}(K) = rk_{\mathbb{C}}(Sp(1)) + rk_{\mathbb{C}}(Sp(n)) = 1 + n$, there is a compact Cartan subgroup $T \subset K$ for $G(\mathbb{R})$. It is isomorphic to $T \cong U(1)^{n+1}$. As before, let Δ^+ (resp. Δ_c^+) be a choice of positive roots of $G(\mathbb{R})$ (resp. K) with respect to T. By a standard argument of Lie theory, we can arrange that these choices look like

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n+1\} \cup \{2\varepsilon_i, 1 \leq i \leq n+1\},$$

which is of type C_{n+1} and

$$\Delta_c^+ = \{\varepsilon_i \pm \varepsilon_j, 2 \leq i < j \leq n+1\} \cup \{2\varepsilon_i, 1 \leq i \leq n+1\},$$

being of type $A_1 \times C_n$. The orders of the corresponding Weyl groups W_G and W_K are readily computed as $|W_G| = 2^{n+1}(n+1)!$ and $|W_K| = 2^{n+1}n!$. So, by the considerations of section 1.5, we know that for each highest weight representation E_λ of $G(\mathbb{R})$, $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i$, there are precisely $b = n+1$ distinct discrete series representations $A_i(\lambda)$, $1 \leq i \leq n+1$, of $G(\mathbb{R})$, having non-trivial $(\mathfrak{g}, K)$-cohomology when tensorized by E_λ and

$$H^q(\mathfrak{g}, K, A_i(\lambda) \otimes E_\lambda) = \begin{cases} \mathbb{C} & \text{if } q = q(G) = 2n \\ 0 & \text{else} \end{cases}$$

2.5. The half-sums of positive roots $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, resp. $\rho_c = \frac{1}{2}\sum_{\alpha \in \Delta_c^+} \alpha$ look in the coordinates given by the ε_i, $1 \leq i \leq n+1$, like

$$\rho = (n+1, n, n-1, ..., 2, 1)$$

and

$$\rho_c = (1, n, n-1, ..., 2, 1).$$

Let us now consider highest weight representations E_λ with $\lambda = (\nu - 2n, 0, 0, ..., 0)$, $\nu > 4n$, the reason for that being clear in a second. The corresponding discrete series representation $A(\lambda) := A_1(\lambda)$ with Harish-Chandra parameter $\tau = \tau_1 = \lambda + \rho$ has minimal K-type $\sigma_\nu = S^\nu\mathbb{C}^2 \otimes \mathbb{C}$, where $S^j\mathbb{C}^2$ denotes the j-th symmetric power of the standard representation $\mathbb{C}^2$ of $Sp(1)$ and $\mathbb{C}$ the trivial representation of $Sp(n)$. This is clear, since the highest weight ξ of the minimal K-type of $A(\lambda)$ is connected to the Harish-Chandra parameter τ by

$$\xi = \tau + \rho - 2\rho_c = \lambda + 2(\rho - \rho_c) = (\nu, 0, 0, ..., 0).$$

T. Arakawa constructed in [2], 2.6, for each ν as above (in fact even for each $\nu \geq 2n$) a discrete series representation $U_{\nu,0}$ of $G(\mathbb{R})$ out of the minimal K-type σ_ν. Comparing the Harish-Chandra parameter of our $A(\lambda)$ to the Harish-Chandra parameter of the representation $U_{\nu,0}$, we see that they coincide, whence $A(\lambda) \cong U_{\nu,0}$ and we can use all computations made in [2] for $U_{\nu,0}$. This is also the reason for us to look at the representation $A(\lambda) = A_1(\lambda)$, given by the Harish-Chandra parameter corresponding to the representative of the trivial class in W_G/W_K. By the next lemma also the assumption $\nu > 4n$ becomes clear:

Lemma 2.1 ([2], Lemma 2.10.(ii))**.** *If $\nu > 4n$, the matrix coefficients ω_λ of $A(\lambda)$ satisfy*

$$\int_{G(\mathbb{R})} \|\omega_\lambda(g)\| dg < \infty.$$

In other words, the representations $A(\lambda)$ are integrable.

3. Selberg's Trace Formula

3.1. We keep the assumptions and notation of the previous section. We want to use Selberg's Trace Formula for $\mathbb{R}$-rank one groups, in order to get a formula for $m(A(\lambda), \Gamma)$. An introduction to this field of mathematics can be found in [23] and [16], whose results we will use freely.

Let us first recall that we can decompose any arithmetic group Γ as a finite union of disjoint sets C_Γ, E_Γ, H_Γ, U_Γ, L_Γ, called (in the order of appearance) the set of *central*, *elliptic*, *hyperbolic*, *unipotent* and *loxodromic* elements. A precise definition of these sets can be found in [23], 5. Recalling that the arithmetic congruence subgroups we are considering are all nice, we get the following simplification:

Proposition 3.1 ([2], Lemma 5.5)**.** *Let Γ be a nice arithmetic subgroup of G. Then $C_\Gamma = \{id\}$, $E_\Gamma = L_\Gamma = \emptyset$.*

Let ω be a K-finite function in the L^p-Schwartz space of $G(\mathbb{R})$, which is denoted $\mathscr{C}^p(G(\mathbb{R}))$ in [23] and [12]. It acts by convolution on the discrete spectrum of our nice group Γ. If $0 < p < 1$, the Selberg Trace Formula roughly asserts that one can compute the trace of this action $tr(\omega)$ as the sum of three terms, denoted $C(\omega)$, $H(\omega)$ and $U(\omega)$, which stand for the contribution of the sets $C_\Gamma = \{id\}$ of central, H_Γ of hyperbolic and U_Γ of unipotent elements in Γ. We will be more precise in the case we need it. For a thorough treatment see [23], Thm. 8.4.

3.2. Let ω_λ be the matrix coefficient of $A(\lambda)$ with respect to a K-finite unit vector. Then also ω_λ will be K-finite. In addition, let d_λ be the *formal degree* of $A(\lambda)$. It is the unique positive, real number such that

$$\int_{G(\mathbb{R})} \langle A(\lambda)(g)u_1, v_1\rangle \overline{\langle A(\lambda)(g)u_2, v_2\rangle} dg = d_\lambda^{-1} \langle u_1, u_2\rangle\langle v_1, v_2\rangle$$

for all $u_1, u_2, v_1, v_2 \in A(\lambda)$.

By a theorem of Trombi and Varadarajan, [21], there is a p, $0 < p < 1$, such that $d_\lambda\omega_\lambda$ is in the L^p-Schwartz space $\mathscr{C}^p(G(\mathbb{R}))$ of $G(\mathbb{R})$. But this means that we can insert $d_\lambda\omega_\lambda$ into the Selberg Trace formula. Observing Harish-Chandra's "Selberg principle" (cf. [12], Thm. 11), which gives the vanishing of $H(\omega_\lambda)$, we finally get:

Proposition 3.2 ([16], p. 305)**.** *If $\Gamma = \Gamma(N)$, $N \geq 3$ then*

$$m(A(\lambda), \Gamma) = d_\lambda vol(\Gamma\backslash G(\mathbb{R})) + U(\omega_\lambda)$$

4. Computation of contributions

4.1. The central contribution. We need to calculate $d_\lambda vol(\Gamma\backslash G(\mathbb{R}))$. We do this in two steps:

Step 1: We calculate the covolume of Γ. Therefore, we use the following notation: If $R=\mathbb{Z}$ (resp. $\mathbb{Z}_p$) and $\mathbb{K}=\mathbb{Q}$ (resp. $\mathbb{Q}_p$) we write for short $G(R):=G(\mathbb{K})\cap GL_{n+1}(\mathfrak{O}\otimes_{\mathbb{Z}} R)$. Now we observe that $\Gamma=\Gamma(N)$ has finite index $h_N=|\Gamma\backslash G(\mathbb{Z})|$ in $G(\mathbb{Z})$. Hence, $vol(\Gamma\backslash G(\mathbb{R}))=h_N vol(G(\mathbb{Z})\backslash G(\mathbb{R}))$ and we can concentrate on calculating the covolume of $G(\mathbb{Z})$ in $G(\mathbb{R})$. This is done in the next proposition.

Proposition 4.1. *We have*

$$vol(G(\mathbb{Z})\backslash G(\mathbb{R}))=\frac{2^{4n}\pi^{2n}}{(2n+1)!}\prod_{j=1}^{n+1}\left(\frac{(2j-1)!}{(2\pi)^{2j}}\zeta(2j)\prod_{p\in S(B)}(p^j+(-1)^j)\right).$$

Proof. Let $\mathscr{G}_p$ (resp. G_p) be the smooth affine $\mathbb{Z}_p$-group scheme associated with $Sp_{2(n+1)}(\mathbb{Z}_p)$ (resp. $G(\mathbb{Z}_p)$) and denote by $\overline{\mathscr{G}}_p$ (resp. $\overline{G}_p$) the group $\mathscr{G}_p\times_{\mathbb{Z}_p}\mathbb{F}_p$ (resp. $G_p\times_{\mathbb{Z}_p}\mathbb{F}_p$). Then both $\overline{\mathscr{G}}_p$ and $\overline{G}_p$ admit a Levi decomposition over $\mathbb{F}_p$ with Levi $\mathbb{F}_p$-subgroups $\overline{\mathscr{L}}_p$ and $\overline{L}_p$, say. Using Prasad's volume formula, cf. [17], Thm. 3.7. the volume for $G(\mathbb{Z})\backslash G(\mathbb{R})$ with respect to the Euler-Poincaré-measure equals

$$vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R}))=\chi_{EP}(U/K)\cdot\prod_{j=1}^{n+1}\frac{m_j!}{(2\pi)^{m_j+1}}\mathrm{Tam(G)}\prod_p\frac{p^{\frac12(\dim_{\mathbb{F}_p}\overline{L}_p+\dim_{\mathbb{F}_p}\overline{\mathscr{L}}_p)}}{|\overline{L}_p(\mathbb{F}_p)|},$$

where $U=Sp(n+1)$ is the compact dual of G and $\chi_{EP}(U/K)$ denotes the Euler-Poincaré-characteristic of the compact symmetric space U/K (which is nothing but the quaternionic projective space $\mathbb{P}(\mathbb{H}^n)$). Also, $\mathrm{Tam}(G)$ denotes the Tamagawa measure of G, which equals $\mathrm{Tam}(G)=1$, G being simply connected.
The integers m_j can be found in [17] 1.5 or - more originally - in [6], VI, "Planche III" (IX): $m_j=2j-1$.
For all prime numbers $p\notin S(B)$, G splits over $\mathbb{Q}_p$ and $\overline{L}_p\cong\overline{\mathscr{L}}_p\cong Sp_{2(n+1)}$ over $\mathbb{F}_p$. For these primes we have thus:

$$\dim_{\mathbb{F}_p}\overline{L}_p=2(n+1)^2+n+1$$

and

$$|\overline{L}(\mathbb{F}_p)|=|Sp_{2(n+1)}(\mathbb{F}_p)|=p^{(n+1)^2}\prod_{i=1}^{n+1}(p^{2i}-1).$$

The last equation is proved in [15] 1.2. Having collected this information, we can rewrite the covolume with respect to the Euler-Poincaré-measure as

$$vol_{EP}(G(\mathbb{Z})\backslash G(\mathbb{R}))=\chi_{EP}(U/K)\prod_{j=1}^{n+1}\left(\frac{(2j-1)!}{(2\pi)^{2j}}\zeta(2j)\prod_{p\in S(B)}\lambda_p\right),$$

if we set for $p\in S(B)$

$$\lambda_p=p^{\frac12(\dim_{\mathbb{F}_p}\overline{L}_p-\dim_{\mathbb{F}_p}\overline{\mathscr{L}}_p)}\frac{|Sp_{2(n+1)}(\mathbb{F}_p)|}{|\overline{L}_p(\mathbb{F}_p)|}.$$

The exact values for λ_p can be found in [9] 8, (8.10): $\lambda_p=\prod_{j=1}^{n+1}(p^j+(-1)^j)$. It is well-know (cf. [19], the proof of Prop. 4.5) that

$$vol=\frac{vol_{g_0}(U/K)}{\chi_{EP}(U/K)}vol_{EP},$$

where vol_{g_0} denotes the volume uniquely determined by the canonical Riemannian metric g_0 on $U/K = \mathbb{P}(\mathbb{H}^n)$ which gives it geodesic diameter π. For details, see [3]. *Ibidem*, (3.10), we can find the volume-formula

$$vol_{\frac{1}{2}g_0}(\mathbb{P}(\mathbb{H}^n)) = \frac{\pi^{2n}}{(2n+1)!}.$$

Since stretching the metric $\frac{1}{2}g_0$ to its double g_0 multiplies the corresponding volume by $2^{\dim_{\mathbb{R}} \mathbb{P}(\mathbb{H}^n)} = 2^{4n}$, the proposition is proved. □

2. Step: We need to calculate the formal degree d_λ of $A(\lambda)$. One can do this by the general formula mentioned in [16] or one uses the concrete calculations in [2], Proposition 2.9, to get

Proposition 4.2. *The formal degree d_λ of $A(\lambda)$, $\lambda = (\nu - 2n)\varepsilon_1$, $\nu > 4n$, is given as*

$$d_\lambda = \frac{(\nu-1)(\nu+2n-1)\nu!}{2^{4n}\pi^{2n}(\nu-2n+1)!}.$$

Thus the central contribution is calculated.

4.2. The unipotent contribution. Luckily, the summand $U(\omega_\lambda)$, standing for the contribution of the set of unipotent elements of Γ to the multiplicity of $A(\lambda)$ was calculated by T. Arakawa. Let us assume from now on that $n \geq 2$: In fact, the case $n = 1$ was investigated by Arakawa himself in details in [1], Thm. 2 and transferred to the cohomological setting by us in [11], Proposition 7.1. We get

Proposition 4.3 ([2], Prop. 5.4)**.** *If $n \geq 2$, then*

$$U(\omega_\lambda) = 0$$

Let us sketch the idea of the proof: Arakawa shows that $U(\omega_\lambda)$ can be written as a sum of special values of ζ-integrals. These special values can be themselves expressed in terms of values of the so-called Epstein-ζ-function $\zeta(\ :\ )$ attached to the quadratic form $Q(x) = x\overline{x}$ on $W = \{x \in \mathbb{H} | x + \overline{x} = 0\}$ for certain lattices $\Lambda \subset W$ at $1-n$. We recall that this Epstein zeta function is (at least formally) defined as

$$\zeta(\Lambda : s) := \sum_{0 \neq x \in \Lambda} Q(x)^{-s},$$

and absolutely convergent for $Re(s) > \frac{3}{2}$. It can be meromorphically continued to all of $\mathbb{C}$ with only and simple pole at $s = \frac{3}{2}$. It vanishes at negative integers s. As $s = 1-n$ will be such a negative number as soon as $n \geq 2$, the proposition can be therefrom deduced.

5. The main result

5.1. We summarize the results of the previous section in the following

Theorem 5.1. *Let G be the simple algebraic group $Sp(n,1)$, $n \geq 2$, defined via a quaternion algebra B (as in 2.2) and $\Gamma = \Gamma(N)$ a principal congruence subgroup of level $N \geq 3$. Let $E = E_\lambda$ be a finite-dimensional, irreducible, complex representation of the Lie group $G(\mathbb{R})$ of highest weight $\lambda = (\nu - 2n, 0, ..., 0)$ and suppose $\nu > 4n$. Then the multiplicity of the discrete series representation $A(\lambda)$ of Harish-Chandra parameter $\lambda + \rho = (\nu - n + 1, n, n-1, ..., 2, 1)$ within the space $L^2_{cusp}(\Gamma \backslash G(\mathbb{R}))$ of classical cusp forms is given by*

$$m(A(\lambda), \Gamma) = h_N \frac{(\nu-1)(\nu+2n-1)\nu!}{(2n+1)!(\nu-2n+1)!} \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} (p^j + (-1)^j) \right).$$

Therefrom we can immediately read of the following corollary on the dimension of cuspidal cohomology:

Corollary 5.1 (Growth condition). *Keeping the assumptions of theorem 5.1, the dimension of the space of cuspidal cohomology of an arithmetically defined, quaternionic hyperbolic n-manifold* $\Gamma\backslash\mathcal{H}_n$ *grows at least as*

$$\dim H^*_{cusp}(S(\Gamma), \widetilde{E}) \geq$$

$$h_N \frac{(\nu-1)(\nu+2n-1)\nu!}{(2n+1)!(\nu-2n+1)!} \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) \prod_{p \in S(B)} (p^j + (-1)^j) \right).$$

Acknowledgements

I am grateful to Joachim Schwermer for all of his kind support. I also profited while compiling this work from the hospitality of the Erwin Schrödinger Institute (ESI), Vienna, and the Faculty of Mathematics of the University of Vienna, Austria.

The author's work was supported by the "F124-N Forschungsstipendium der Universität Wien" and the Junior Research Fellowship of the ESI, Vienna.

References

[1] T. Arakawa, On automorphic forms of a quaternion unitary group of degree two, *J. Fac. Sci. Univ. Tokyo* Sect. IA **28** (1982) 547–566.
[2] T. Arakawa, On certain automorphic forms of Sp(1,q), in *Automorphic Forms of Several Variables: Taniguchi Symposium, Katata, 1983*, eds. I. Satake and Y. Morita (Birkhäuser, Boston, 1984), pp. 1–48.
[3] M. Berger, Du côte de chez Pu, *Ann. Sci. de l'ENS* 4^e série tome 5 **1** (1972) 1–44
[4] A. Borel, Stable real cohomology of arithmetic groups II, in *Manifolds an Liegroups*, J. Hano et al. eds., Progress in Maths. **14** (1981) 21–55
[5] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies, (Princeton Univ. Press, New Jersey, 1980)
[6] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4 à 6* (Hermann, Paris, 1968)
[7] A. Deitmar, J. Hilgert, Cohomology of Arithmetic Groups with Infinite Dimensional Coefficient Spaces, *Documenta Mathematica* **10** (2005) 199–216
[8] J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. de l'ENS* 4^e série tome 31 **2** (1998) 181–279
[9] W. T. Gan, J. P. Hanke, J.-K. Yu, On an exact mass formula of Shimura, *Duke Math. J.* **107** (2001) 103–133
[10] I. M. Gelfand, M. I. Graev, I. Piatetski-Shapiro, *Representation theory and automorphic functions* (W. B. Saunders Company, Philadelphia, 1969)
[11] H. Grobner, The Automorphic Cohomology and the Residual Spectrum of Hermitian Groups of Rank One, *Int. J. Math.* **21** (2010) 255–278
[12] B. Harish-Chandra, Discrete series for semisimple Lie Groups II, *Acta Math.* **116** (1966) 1–111
[13] A. W. Knapp, *Representation Theory of Semisimple Groups - An Overview Based On Examples* (Princeton Univ. Press, 1986)
[14] C. Mœglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series* (Cambridge Univ. Press, 1995)
[15] T. Ono, On algebraic groups and discontinuous groups, *Nogoya Math. J.* **27** (1966) 279–322
[16] M. S. Osborne, G. Warner, Multiplicities of the Integrable Discrete Series: The Case of a Nonuniform Lattice in an R-Rank One Semisimple Group, *Journal of Funct. Analysis* **30** (1978) 287–310
[17] G. Prasad, Volumes of S-arithmetic quotients of semi-simple groups, *Publ. math. de l'I.H.É.S.* **69** (1989) 91–114
[18] J. Rohlfs, B. Speh, Representations with cohomology in the discrete spectrum of subgroups of $SO(n,1)(\mathbb{Z})$ and Lefschetz numbers, *Ann. Sci. de l'ENS* **1** (1987) 89–136
[19] J. Rohlfs, The Lefschetz number of an involution on the space of classes of positive definite quadratic forms, *Comm. Math. Helv.* **56** (1981) 272–296
[20] G. Savin, Limit multiplicities of cusp forms, *Invent. Math.* **95** (1989) 149–159

[21] P. Trombi, V. S. Varadarajan, Asymptotic behavior of eigenfunctions of a semisimple Lie group, *Acta Math.* **129** (1972) 237–280

[22] D. A. Vogan Jr. and G. J. Zuckerman, Unitary representations with nonzero cohomology, *Comp. Math.* **53** (1984) 51–90

[23] G. Warner, Selberg's Trace Formula for Nonuniform Lattices: The R-Rank One Case, in *Studies in Algebra and Number Theory*, ed. Gian-Carlo Rota, Advances in Math. Supp. Studies **6** (1979) 1–142

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria

E-mail address: harald.grobner@univie.ac.at

URL: http://homepage.univie.ac.at/harald.grobner