REGULAR AND RESIDUAL EISENSTEIN SERIES AND THE AUTOMORPHIC COHOMOLOGY OF $Sp(2,2)$

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Abstract. Let G be the simple algebraic group $Sp(2,2)$, to be defined over $\mathbb{Q}$. It is a non-quasi-split, $\mathbb{Q}$-rank 2 inner form of the split symplectic group Sp_8 of rank 4. The cohomology of the space of automorphic forms on G has a natural subspace, which is spanned by classes represented by residues and derivatives of cuspidal Eisenstein series. It is called Eisenstein cohomology. In this paper we give a detailed description of the Eisenstein cohomology $H^q_{Eis}(G,E)$ of G in case of regular coefficients E. It is spanned only by holomorphic Eisenstein series. For non-regular coefficients E we really have to detect the poles of our Eisenstein series. Since G is not quasi-split, we are out of the scope of the so-called 'Langlands-Shahidi method', cf. [Shd81, Shd88]. We apply recent results of N. Grbac in order to find the double poles of Eisenstein series attached to the minimal parabolic P_0 of G. Having collected this information, we determine the square-integrable Eisenstein cohomology supported by P_0 with respect to arbitrary coefficients and prove of a vanishing-result. This will exemplify a general theorem we prove in this paper on the distribution of maximally residual Eisenstein cohomology classes.

Introduction

Let G be a connected, semisimple algebraic group defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank $rk_{\mathbb{Q}}(G) \geq 1$, E a finite-dimensional, irreducible complex representation of the Lie group $G(\mathbb{R})$ of real points of G, and $\Gamma \subset G(\mathbb{Q})$ an arithmetic congruence subgroup. The study of the cohomology spaces $H^*(\Gamma, E)$ has been carried out over the last 40 years from various points of view and motivations, using and comparing several techniques. Beside others, the cohomology of arithmetic groups has major applications within the Langlands Program, which itself is originated in the attempt to solve classical problems of algebraic and analytic number theory, such as giving a satisfactory non-abelian class field theory. This approach to cohomology of an arithmetically defined group indicates a close connection to the theory of automorphic forms, in particular to cusp forms and Eisenstein series.

The link between $H^*(\Gamma, E)$ and automorphic forms was first provided in a conceptual way by G. Harder in the case of groups of $\mathbb{Q}$-rank one, [Har75, Har73]. His method is of differential geometric nature and uses the fact that the cohomology of Γ is isomorphic to the cohomology of a certain compact space $\Gamma\backslash\overline{X}$, which is a orbifold with boundary $\partial(\Gamma\backslash\overline{X})$. In fact, $X = G(\mathbb{R})/K$ is the Riemannian symmetric space associated to the Lie group $G(\mathbb{R})$ and a maximal compact subgroup K and $\Gamma\backslash\overline{X}$ is the Borel-Serre-Compactification of the quotient $\Gamma\backslash X$ (locally symmetric, if Γ is torsionfree). With this framework at place, Harder showed that one can construct the "cohomology at infinity", i.e., (up to isomorphy) the image of the natural

2000 *Mathematics Subject Classification.* Primary: 11F75; Secondary: 11F70, 11F55, 22E55.

Key words and phrases. cohomology of arithmetic groups, Eisenstein cohomology, cuspidal automorphic representation, Eisenstein series, residual spectrum.

The author's work was supported in part by the "F124-N Forschungsstipendium der Universität Wien" and the Junior Research Fellowship of the ESI, Vienna

This is a slightly modified version of an article published in *Compos. Math.* **146** (2010) pp. 21-57.

restriction map $H^*(\Gamma\backslash\overline{X}, E) \to H^*(\partial(\Gamma\backslash\overline{X}), E)$ by means of Eisenstein series. The cohomology at infinity is complementary within $H^*(\Gamma, E)$ to the cohomology of a space of square-integrable automorphic forms, which contains the cusp forms.
In the early 90's, J. Franke finally proved in [Fra98] that such a decomposition can also be given in the general framework of an arbitrary connected, reductive algebraic group G. More precisely, Franke particularly showed that the cohomology of an arithmetic congruence subgroup $\Gamma \subset G(\mathbb{Q})$ decomposes as

$$H^*(\Gamma, E) = H_{cusp}(\Gamma, E) \oplus H_{Eis}(\Gamma, E)$$

into the cohomology-space of classes represented by cuspidal automorphic forms and a natural complement called Eisenstein cohomology of Γ. This is due to the adelic interpretation of $H^*(\Gamma, E)$ as a subspace of the space of K_Γ fixed vectors in

$$H^*(G, E) := H^*(\mathfrak{g}, K, \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E)$$

and an analogous decomposition of this cohomology as $H^*(G, E) = H_{cusp}(G, E) \oplus H_{Eis}(G, E)$. (Here K_Γ is an appropriate open, compact subgroup of the group of finite adelic points $G(\mathbb{A}_f)$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the usual space of (adelic) automorphic forms on G.)
In the case of regular coefficients E, i.e., the highest weight λ of E lies inside the open, positive Weyl chamber, the space of Eisenstein cohomology was investigated by J. Schwermer in [Sch94] and together with J.-S. Li in [LS04]. It was shown that under this assumption on E each class in $H^*_{Eis}(G, E)$ can be represented by a bunch of Eisenstein series evaluated at a certain point in the region of holomorphy ([Sch94], section 2 and section 6). This lead to a vanishing result in lower degrees of cohomology (cf. [LS04], Thm. 5.5).

Still, for non-regular coefficients E, little is known in general: In contrast to the regular case, residues of Eisenstein series really enter the game (i.e., can contribute non-trivially to cohomology), when regarding non-regular coefficient modules E. The results gained so far suggest that the poles of Eisenstein series are encoded by poles and zeros of automorphic L-functions. But even for square-integrable residues the situation is not fully understood, since e.g., a satisfactory theory of describing the residual spectrum of a non-quasi-split algebraic group G is not available, yet.
On the other hand, finding the poles of Eisenstein series is not the only difficulty one encounters in this case. One also has to understand *if* residual Eisenstein series contribute non-trivially to cohomology and - if they contribute - *in which degrees* of cohomology. Again, all these problems are entirely linked to deep questions of local and global representation theory and number theory, in particular the Langlands Program.

In the present work we consider the above questions and approaches to Eisenstein cohomology regarding the connected, simple algebraic group $Sp(2, 2)$ defined over $\mathbb{Q}$. It is a non-quasi-split $\mathbb{Q}$-rank two form of the split symplectic group $Sp_8/\mathbb{Q}$, the classical group of Cartan type C_4.

In **section 1** the necessary facts about the automorphic cohomology $H^*(G, E)$ for a connected, semisimple algebraic group $G/\mathbb{Q}$ are reviewed. We recall the decomposition of Eisenstein cohomology $H^*_{Eis}(G, E)$ along the cuspidal support of the Eisenstein series in question, see theorem 1.1 (resp. the original sources [FS98] or [MW95]): This is a decomposition along associate classes $\{P\}$ of proper, parabolic $\mathbb{Q}$-subgroup P of G and certain (collections φ of) cohomological, cuspidal automorphic representations $\widetilde{\pi}$ of the corresponding Levi subgroups L of P.

In **section 2**, still for an arbitrary connected, semisimple algebraic group $G/\mathbb{Q}$, we deal with the question how to breed the space of Eisenstein cohomology out of cohomological cuspidal automorphic representations $\widetilde{\pi}$ of the Levi subgroups L of parabolic $\mathbb{Q}$-subgroups $P \subsetneqq G$. Recall that P has a Levi- and a Langlands-decomposition $P = LN$ resp. $P = MAN$, N being a unipotent radical of P and A a maximal central $\mathbb{Q}$-torus of L. As in [FS98] we use the Eisenstein intertwining operator to get a map on the level of $(\mathfrak{g}, K)$-cohomology

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E^q_\pi} H^*_{Eis}(G, E). \tag{1}$$

Here, $W_{P,\widetilde{\pi}}$ is essentially the representation induced parabolically from $\widetilde{\pi}$. Further, $S_\chi(\mathfrak{a}^*)$ denotes the symmetric tensor algebra of the linear dual of $\mathfrak{a} = Lie(A(\mathbb{R}))$. (The symbol "$\chi$" shall indicate an action of $\mathfrak{a}$ onto $S_\chi(\mathfrak{a}^*)$ by means of a character χ of $A(\mathbb{R})^\circ$. See section 2.2 for details.) This construction procedure of Eisenstein cohomology is explained in detail. In particular we recall the notion of a class of type (π, w) (w a so-called Kostant representative with respect to the right action of the Weyl group of $L(\mathbb{C})$ on the Weyl group of $G(\mathbb{C})$): This is a non-trivial class in the right hand side of (1). It follows from general results on $(\mathfrak{g}, K)$-cohomology that the derivative of χ must satisfy $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$. We may also suppose that it lies inside the closed, positive Weyl chamber C defined by P and A.
In 2.3 we explain how the behaviour of holomorphy of an Eisenstein series $E_P(f, \Lambda)$, $f \in W_{P,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}_\mathbb{C}$, interacts with the degree(s) of cohomology in which the image of E^q_π lies. The case of holomorphic Eisenstein series was already solved by J. Schwermer in [Sch83] and is summarized shortly in section 2.3.1.
Again, the residual case is most difficult and in its full generality unsolved. We know by Langlands [Lan76] that the poles of the Eisenstein series $E_P(f, \Lambda)$ are the ones of its constant terms along parabolic subgroups. Assume that P is a self-associate, standard parabolic, then it suffices to consider the constant term along P itself. Putting $W(A) = N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$ (which is a subgroup of the Weyl group attached to the $\mathbb{Q}$-roots of G) we arrive at a decomposition of this constant term as a finite sum

$$E_P(f, \Lambda)_P = \sum_{w \in W(A)} M(\Lambda, \widetilde{\pi}, w)(f e^{\langle \Lambda + \rho_P, H_P(.)\rangle}), \tag{2}$$

where $M(\Lambda, \widetilde{\pi}, w)$ are certain well-known meromorphic functions associated to Λ, $\widetilde{\pi}$ and $w \in W(A)$ (cf. section 2.3.2 for their precise definition respectively for the other symbols not explain here). So the behaviour of holomorphy of $E_P(f, \Lambda)$ is given by the interplay of the poles and zeros of the finitely many functions $M(\Lambda, \widetilde{\pi}, w)$. If $M(\Lambda, \widetilde{\pi}, w)$ is residual at $\Lambda = \Lambda_0$, then we assume to have normalized it to a holomorphic and non-vanishing function $N(\Lambda_0, \widetilde{\pi}, w)$. Put

$$W(A)_{\text{res}} = \{w \in W(A) | M(\Lambda, \widetilde{\pi}, w) \text{ has a pole of order } \ell = \dim \mathfrak{a}_\mathbb{C} \text{ at } \Lambda = d\chi\}.$$

This means that the order of the pole is maximal and implies that the longest element w_0 of $W(A)$ (as a reduced word in the simple reflections generating $W(A)$) will be inside $W(A)_{\text{res}}$. We prove the following new theorem in section 2.3.2 (cf. theorem 2.1) on the degree of residual Eisenstein cohomology classes:

Theorem. *In the notation used above, let* $0 \neq [\omega] \in H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$. *If all Eisenstein series* $E_P(f, \Lambda)$, $f \otimes 1$ *in the image of* ω, *have a pole of maximal possible order* $\ell = \dim \mathfrak{a}_\mathbb{C}$ *at* $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}} \in C$ *and if* $ImN(d\chi, \widetilde{\pi}, w_0)$ *is a direct summand of* $\sum_{w \in W(A)_{res}} ImN(d\chi, \widetilde{\pi}, w)$, *then* $E^q_{\widetilde{\pi}}([\omega])$ *contributes at least in degree* $q' := q + \dim N(\mathbb{R}) - 2l(w)$, $l(w)$ *the length of* w.

From **section 3** on, we concentrate on the case $G = Sp(2,2)/\mathbb{Q}$. As mentioned earlier, G is a simple, connected, simply connected algebraic group over $\mathbb{Q}$, which is an non-quasi-split, $\mathbb{Q}$-rank 2 inner form of Sp_8 – the classical split group over $\mathbb{Q}$ of Cartan type C_4. Hence, the classes of associate and conjugate parabolic $\mathbb{Q}$-subgroups of G coincide and can be represented by the choice of three standard parabolic subgroups P_0 (a minimal one) and P_1 and P_2 (two maximal ones). In order to construct Eisenstein cohomology, we need to get some knowledge on cohomology classes of type (π, w) as remarked before: $\pi = \chi\widetilde{\pi}$ with χ a certain character of $A_i(\mathbb{R})^\circ$ and $\widetilde{\pi}$ a cohomological cuspidal automorphic representation of $L_i(\mathbb{A})$ $(i = 0,1,2)$; and w is a Kostant representative of a coset with respect to the right action of the Weyl group of $L(\mathbb{C})$ on the Weyl group of $G(\mathbb{C})$. In **section 4** the possible archimedean components $\widetilde{\pi}_\infty$ of cohomological cuspidal automorphic representations $\widetilde{\pi}$ are classified (cf. lemma 4.1 and proposition 4.2). These are irreducible unitary cohomological representations of the semisimple part $M_i(\mathbb{R})$ of the reductive Lie groups $L_i(\mathbb{R})$. We use the well known Vogan-Zuckerman classification of such representations, cf. [VZ84].

Having gained this knowledge, **section 5** then gives a complete description of the $G(\mathbb{A}_f)$-module structure of the Eisenstein cohomology spaces $H^q_{Eis}(G, E)$, under the assumption that the coefficient module E is regular. The case of each parabolic $\mathbb{Q}$-subgroup P_i, $i = 0,1,2$ is treated separately in three subsections. The main theorems describing the internal nature of Eisenstein cohomology classes with respect to regular coefficients are theorems 5.3, 5.4 and 5.5. The general phenomenon that each Eisenstein class can be represented by (a finite number of) regular values of Eisenstein series ([Sch94]) and the vanishing of $H^q_{Eis}(G, E)$ below the half of $\dim X = 16$ ([LS04]) is verified concretely in this case.

The much more difficult case of a general - meaning, not necessarily regular - coefficient system E is dealt with in **section 6**. We concentrate on the contribution of the minimal parabolic P_0. The analysis of (residual) Eisenstein cohomology supported in P_0 might be viewed as a case-study, which sources its interest in absence of a good general theory from the following questions, which have been stated already above: (a) How to find the poles of an Eisenstein series $E_{P_0}(f, \Lambda)$? To work on this question is particularly interesting in our concrete case, since $G = Sp(2,2)$ (and so L_0) is not quasi-split, whence we are out of scope of the 'Langlands-Shahidi-method', [Shd81, Shd88]; and (b): How to control the contribution of the various resulting residues to Eisenstein cohomology?

In order to answer (a), i.e., calculate the poles of $E_{P_0}(f, \Lambda)$ we have to normalize the operators $M(\Lambda, \widetilde{\pi}, w)$ of (2), meaning we have to find a function $r(\Lambda, \widetilde{\pi}, w)$ "whose behaviour of holomorphy we understand" such that $N(\Lambda, \widetilde{\pi}, w) = r(\Lambda, \widetilde{\pi}, w)^{-1} M(\Lambda, \widetilde{\pi}, w)$ – to be called the *normalized* intertwining operator – is holomorphic and non-vanishing in the region we need it. For quasi-split groups a suggestion for such a normalization is provided by the Langlands-Shahidi-method. But as remarked before, our group is not quasi-split. We apply a little trick (cf. proposition 6.2 resp. our original paper [Gro10], Prop. 3.1), which allows us to get a good normalization of $M(\Lambda, \widetilde{\pi}, w)$ by only normalizing the local operators $M(\Lambda, \widetilde{\pi}_p, w)$, p a place where $G(\mathbb{Q}_p) = Sp_8(\mathbb{Q}_p)$ (i.e., G splits). Then we use proposition 6.1, which tells us that we can reduce the problem of normalizing $M(\Lambda, \widetilde{\pi}_p, w)$ at such places to the $\mathbb{Q}$-rank one case. Still, we need some extra information, since we have to normalize also cuspidal representations of $L_0(\mathbb{A})$ which are locally not generic. At this point, we use the recent work of N. Grbac ([Grb07, Grb09]), which solves the question of how

to normalize our operators for such non-generic local representations. The candidates for double poles of Eisenstein series are finally listed in our propositions 6.6 and 6.8.
Question (b) is the most subtle matter. Here we confine ourselves to consider the space of square-integrable Eisenstein cohomology, (supported by P_0), denoted $H^q(\mathfrak{g}, K, L_{E,P_0} \otimes E)$. Its coefficient system L_{E,P_0} is a subspace of the residual spectrum of G and decomposes hence as a direct Hilbert sum over unitary residual automorphic representations of $G(\mathbb{A})$, each of which is generated by two-times iterated residues of Eisenstein series. By our propositions 6.6 and 6.8 we can therefrom determine the internal nature of a representative of a square-integrable Eisenstein cohomology class. This is contained in theorem 6.9:

Theorem. *Let P_0 be the minimal standard parabolic $\mathbb{Q}$-subgroup of $G = Sp(2,2)$ and E any irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R})$. Then the square-integrable Eisenstein cohomology supported by P_0, $H^*(\mathfrak{g}, K, L_{E,P_0}\otimes E)$, is spanned by cohomology classes which are Eisenstein lifts of a class of type (π, w), $\pi = \chi\widetilde{\pi} \in \varphi_{P_0} \in \varphi \in \Psi_{E,P_0}$, $\widetilde{\pi} = \theta\hat{\otimes}\tau$ and $w \in W^{P_0}$, such that necessarily one of the following conditions holds:*
If $d\chi$ is inside the open, positive Weyl chamber defined by P_0 and A_0:

- (A) *If $\dim\theta > 1$ and $\dim\tau > 1$:*
  $\widetilde{\pi} = \tau\hat{\otimes}\tau$, $\chi_\tau = \mathbf{1}$, $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
- (B) *If $\dim\theta = 1$, $\dim\tau > 1$:*
  $\widetilde{\pi} = \mathbf{1} \otimes \tau$, $\chi_\tau = \mathbf{1}$, $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
- (C) *If $\dim\theta = \dim\tau = 1$:*
  1) $\widetilde{\pi} = \mathbf{1}\hat{\otimes}\tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
  2) $\widetilde{\pi} = \tau\hat{\otimes}\tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{5}{2}, \frac{1}{2})$.
  3) $\widetilde{\pi} = \mathbf{1} \otimes \mathbf{1}$ *and* $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_{P_0}$.

If $d\chi$ is on the boundary of the closed, positive Weyl chamber defined by P_0 and A_0:

- (A) *If $\dim\theta > 1$ and $\dim\tau > 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ *or* $(1, 0)$
- (B) *If $\dim\theta = 1$, $\dim\tau > 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$ *or* $(\frac{3}{2}, 0)$
- (B') *If $\dim\theta > 1$, $\dim\tau = 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$ *or* $(\frac{1}{2}, 0)$
- (C) *If $\dim\theta = \dim\tau = 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ *or* $(2, 0)$

Our general theorem 2.1 on the other hand gives a partial answer on how square-integrable Eisenstein cohomology classes are distributed in the various degrees. In addition, the classification of cohomological, irreducible, unitary representations of $G(\mathbb{R})$ given by [VZ84], essentially implies the following vanishing-result (see theorem 6.10).

Theorem. *If $E \neq \mathbb{C}$, then square-integrable Eisenstein cohomology supported by P_0 vanishes below degree* 3

$$H^q(\mathfrak{g}, K, L_{E,P_0} \otimes E) = 0 \quad \text{for } q \leq 3.$$

If $E = \mathbb{C}$, then there is an epimorphism

$$H^0(\mathfrak{g}, K, L_{\mathbb{C},P_0}) \twoheadrightarrow H^0(G, \mathbb{C}) = \mathbb{C}$$

and

$$H^q(\mathfrak{g}, K, L_{\mathbb{C},P_0}) = 0 \quad \text{for } 1 \leq q \leq 3.$$

In fact, $q = 3$ is a sharp bound for the vanishing of $(\mathfrak{g}, K)$-cohomology in low degrees, so $H^4(\mathfrak{g}, K, L_{E,P_0} \otimes E)$ should not vanish. But this should also follow from our theorem 2.1, as we point out in section 6.4.

Finally, we give all necessary computational data (e.g., the sets of Kostant representatives w) in eight tables put in our small appendix.

Acknowledgments:

This paper is partly an outgrowth of my Ph. D. thesis. I am grateful to my adviser Joachim Schwermer for his varied kind support. Also, I want to express my gratitude to Neven Grbac, Goran Muić and Katharina Neusser for many helpful and inspiring conversations. I am also grateful to the referee for his/her comments and questions concerning the first version of this paper. Besides, I want to thank Jakub Orbán, who developed a computer program that calculated the huge tables in this paper. I also profited from the kind hospitality of the Faculty of Mathematics of the University of Vienna, Austria.

The author's work was supported in part by the "F124-N Forschungsstipendium der Universität Wien" and the Junior Research Fellowship of the ESI, Vienna.

Notation and Conventions. Throughout this paper G will be a connected, simply connected, semisimple algebraic group over $\mathbb{Q}$[1] of rank $rk_{\mathbb{Q}}(G) \geq 1$ with finite center. Lie algebras of groups of real points of algebraic groups will be denoted by the same but fractional letter, e.g., $\mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}$. The complexification of a Lie algebra will be denoted by subscript "$\mathbb{C}$", e.g., $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$. If $U(\mathfrak{g})$ is the universal enveloping algebra of the complex algebra $\mathfrak{g}_{\mathbb{C}}$, $Z(\mathfrak{g})$ stands for its center.

We use the standard terminology and hypotheses concerning algebraic groups and their subgroups to be found in [MW95] I.1.4-I.1.12. In particular we assume that a minimal parabolic subgroup P_0 has been fixed and that $K_{\mathbb{A}} = K_{\mathbb{R}} \times K_{\mathbb{A}_f}$ is a maximal compact subgroup of the group $G(\mathbb{A})$ of adelic points of G which is in *good position* with respect to P_0 ([MW95], I.1.4). Then $K = K_{\mathbb{R}}$ is maximal compact in $G(\mathbb{R})$, hence comes with a Cartan involution ϑ. If H is a subgroup of G, we let $K_H = K \cap H(\mathbb{R})$.

Assume that L_0 is a Levi subgroup of P_0 which is invariant under ϑ and N_0 is the unipotent radical of P_0. Then we have the Levi decomposition $P_0 = L_0 N_0$ and if we additionally denote by A_0 a maximal, central $\mathbb{Q}$-split torus in L_0 then we also get a Langlands decomposition $P_0 = M_0 A_0 N_0$. Let P be a standard parabolic $\mathbb{Q}$-subgroup of G. It has a unique Levi decomposition $P = L_P N_P$, with $L_P \supseteq L_0$ and also a unique Langlands decomposition $P = M_P A_P N_P$ with unique ϑ-stable split component $A_P \subseteq A_0$. If it is clear from the context we will also omit the subscript "P". We write $\Delta(P, A)$ for the set of weights of the adjoint action of P with respect to A_P. ρ_P denotes the half-sum of these weights. In particular, $\rho = \rho_{P_0}$ is the half sum of positive $\mathbb{Q}$-roots of G with respect to A_0.
Extend the Lie algebra $\mathfrak{a}$ of $A(\mathbb{R})$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by adding a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. The absolute root system of $\mathfrak{g}$ is denoted $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, a simple subsystem (given by the obstruction that positivity on the system of absolute roots shall be compatible with the positivity on the set $\Delta_{\mathbb{Q}}$ of $\mathbb{Q}$-roots implied by the choice of the minimal pair (P_0, A_0)) is denoted Δ°. We also write Δ°_M for

[1] Which for us includes the (only technical) assumption that G is not obtained from restriction of scalars $Res_{F/\mathbb{Q}}$ with $F \neq \mathbb{Q}$.

the set of absolute simple roots of $\mathfrak{m}$ with respect to $\mathfrak{b}$ (so $\Delta^\circ = \Delta^\circ_G$). The Weyl groups associated to Δ and $\Delta_\mathbb{Q}$ are denoted $W = W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $W_\mathbb{Q}$. We let $W^P = \{w \in W | w^{-1}(\alpha) > 0 \quad \forall \alpha \in \Delta^\circ_M\}$. The elements of W^P are called *Kostant representatives*, cf. [BW80].
Using the fact that $K_\mathbb{A}$ is in good position, we can extend the standard *Harish-Chandra height-function* $H_P : P(\mathbb{A}) \to \mathfrak{a}^*$ given by $\prod_p |\chi(p)|_p = e^{\langle \chi, H_P(p)\rangle}$, for all $\mathbb{Q}$-characters χ of L (viewed as an element of $\mathfrak{a}^*_\mathbb{C}$), to a function on all of $G(\mathbb{A})$ by setting $H_P(g) := H_P(p)$, $g = kp$.

Let G be a connected, *reductive* group over $\mathbb{Q}$ and $\widetilde{\chi}$ a central character. As usual $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) denotes the discrete spectrum of G (resp. the part of it consisting of functions with central character $\widetilde{\chi}$). It can be written as the direct sum of the cuspidal spectrum $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$) and the residual spectrum $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$). By [GGPS69] the space $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}), \widetilde{\chi})$, decomposes as a direct Hilbert sum over all irreducible, admissible representations π of $G(\mathbb{A})$ with central character $\widetilde{\chi}$, each of which occurring with finite multiplicity $m_{dis}(\pi)$. The same is therefore true for the cuspidal (resp. residual) spectrum, if we replace the multiplicity by $m(\pi)$ (resp. $m_{res}(\pi)$). Every π can be written as a restricted tensor product $\pi = \otimes'_p \pi_p$, where p is a place of $\mathbb{Q}$. i.e., either a prime or ∞ and π_p is a local irreducible, admissible representation π_p of $G(\mathbb{Q}_p)$, [Fla79]. Further, π (and so all π_p) is unitary if and only if $\widetilde{\chi}$ is. Then π is the completed restricted tensor product $\pi = \hat{\otimes}'_p \pi_p$.

For any $G(\mathbb{A})$-representation σ, we will write σ^∞ for the space of its smooth vectors and $\sigma_{(K)}$ for the space of K-finite vectors. Clearly, if σ is unitary, then $\sigma^\infty_{(K)}$ is a unitary $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module.

1. Automorphic Cohomology

1.1. Let E be a finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})$ characterized by its highest weight λ. A starting point of our interest is the $G(\mathbb{A}_f)$-module structure of the $(\mathfrak{g}, K)$-cohomology of the space of (E-valued) adelic automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E$:

$$H^*(G, E) := H^*(\mathfrak{g}, K, \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E).$$

As it is well-known, and as we shall also see again later, in order to understand this cohomology space, one should understand the cohomological contribution of those automorphic representations $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{A})$ which have a cohomological infinite component π_∞.

By [Lan79], Prop. 2, a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module π is automorphic if and only if it is isomorphic to an irreducible subquotient of a parabolically induced representation $\pi' = \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)}[\sigma_{(K_L)}]$, σ being a cuspidal automorphic representation of a Levi subgroup L of a parabolic $\mathbb{Q}$-subgroup of G. This can be proved by use of the so-called Eisenstein intertwining operator (cf. section 2.2), which assigns, very roughly, to each function $f \in \pi'$ a regular value, a residue or a derivative of an Eisenstein series at a certain point (cf. [Fra98], Cor. 1, p. 236 for this more subtle approach).

Clearly, if π is cuspidal itself then we can take $P = G$ and this Eisenstein summation process degenerates essentially to the identity function. Therefore, as a

$(\mathfrak{g}, K, G(\mathbb{A}_f))$-module, $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes as the space of cuspidal automorphic forms $\mathcal{A}_{cusp}$ and the subrepresentation $\mathcal{A}_{Eis}$, which is spanned as a representation by all subquotients of parabolically induced representations $\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\mathrm{Ind}_{(\mathfrak{l},K_L)}^{(\mathfrak{g},K)}[\sigma]$, with $P \neq G$. By the very definition of the Eisenstein intertwining operator, this subspace is spanned by Eisenstein series, residues and derivatives of such. We get the decomposition as $G(\mathbb{A}_f)$-modules

$$H^*(G,E) = H^*(\mathfrak{g}, K, \mathcal{A}_{cusp} \otimes E) \oplus H^*(\mathfrak{g}, K, \mathcal{A}_{Eis} \otimes E).$$

The first space is called *cuspidal cohomology* and denoted $H^*_{cusp}(G,E)$, the second *Eisenstein cohomology*, to be denoted by $H^*_{Eis}(G,E)$. Now, what we are interested in is the space $H^*_{Eis}(G,E)$ of Eisenstein cohomology, on which we will focus in this paper. Since $(\mathfrak{g}, K)$-cohomology only takes into account representations which have a certain infinitesimal character, see [BW80], one can replace the space of all automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by the space $\mathcal{A}_E$ consisting of those automorphic forms which are annihilated by a power of the ideal $\mathcal{Z}$ of $Z(\mathfrak{g})$, which annihilates the dual representation of E: $\mathcal{Z} \cdot \check{E} = 0$,

$$\mathcal{A}_E = \{f \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) | \mathcal{Z}^n f = 0 \quad \text{for some } n\}$$

and

$$H^*(G,E) = H^*(\mathfrak{g}, K, \mathcal{A}_E \otimes E)$$

1.2. The spaces $\mathcal{A}_{E,P}$. In [FS98], J. Franke and J. Schwermer (and also in [MW95], C. Mœglin and J.-L. Waldspurger) were able to give a much more detailed decomposition of the space $\mathcal{A}_E$, taking into account the cuspidal support along Levi subgroups of the Eisenstein series involved.

First of all, the space $\mathcal{A}_E$ admits a certain decomposition as a direct sum with respect to the classes $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups $P \subseteq G$. This relies on such a decomposition of the space V_G of K-finite, left $G(\mathbb{Q})$-invariant, smooth functions $f : G(\mathbb{A}) \to \mathbb{C}$ of uniform moderate growth, first proved by Langlands in a letter to Borel, [Lan72]. See also [BLS96] Thm. 2.4: $V_G = \bigoplus_{\{P\}} V_G(\{P\})$, where $V_G(\{P\})$ denotes the space of elements f in V_G which are negligible along Q for every parabolic $\mathbb{Q}$-subgroup $Q \subseteq G$, $Q \notin \{P\}$. Putting $\mathcal{A}_{E,P} = V_G(\{P\}) \cap \mathcal{A}_E$ we get the desired decomposition of $\mathcal{A}_E$ as $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module

$$\mathcal{A}_E = \bigoplus_{\{P\}} \mathcal{A}_{E,P}.$$

Observe that $\mathcal{A}_{E,G} \subset V_G(\{G\}) = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$. Hence,

$$H^q_{cusp}(G,E) = H^q(\mathfrak{g}, K, \mathcal{A}_{E,G} \otimes E),$$

and

$$H^q_{Eis}(G,E) = \bigoplus_{\{P\}, P \neq G} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P} \otimes E).$$

Since $V_G(\{G\}) = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$ decomposes as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module as a direct sum over all cuspidal automorphic representations of $G(\mathbb{A})$, each of which occurring with finite multiplicity $m(\pi)$, we get by [BW80], XIII, a finite direct sum decomposition

$$H^*_{cusp}(G,E) = \bigoplus_\pi H^*(\mathfrak{g}, K, \pi \otimes E)^{m(\pi)} = \bigoplus_\pi (H^*(\mathfrak{g}, K, (\pi_\infty)_{(K)} \otimes E) \otimes \pi_f^{\infty_f})^{m(\pi)},$$

the sum ranging over all cuspidal automorphic representations π of $G(\mathbb{A})$.

1.3. Eisenstein series. Also the summands $\mathcal{A}_{E,P}$ giving Eisenstein cohomology have a decomposition as $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module. We refer the reader for details to the original paper [FS98].

Some technical assumptions and notations have to be fixed:
For $Q = LN = MAN$ associate to the standard parabolic P, φ_Q is a finite set of irreducible representations $\pi = \chi\widetilde{\pi}$ of $L(\mathbb{A})$, with $\chi : A(\mathbb{R})^\circ \to \mathbb{C}^*$ a continuous character and $\widetilde{\pi}$ an irreducible, unitary subrepresentation of $L^2_{cusp}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$ whose central character induces a continuous morphism $A(\mathbb{Q})A(\mathbb{R})^\circ\backslash A(\mathbb{A}) \to U(1)$ and whose infinitesimal character matches the one of the dual of an irreducible subrepresentation of $H^*(\mathfrak{n}, E)$. This means that $\widetilde{\pi}$ is a unitary, cuspidal automorphic representation of $L(\mathbb{A})$ whose central and infinitesimal character satisfy the above conditions. Finally, three further "compatibility conditions" have to be satisfied between these sets φ_Q, skipped here and written down in [FS98], 1.2. The family of all collections $\varphi = \{\varphi_Q\}$ of such finite sets is denoted $\Psi_{E,P}$.

Now, let $W_{Q,\widetilde{\pi}}$ be the space of all smooth, K-finite functions

$$f : L(\mathbb{Q})N(\mathbb{A})A(\mathbb{R})^\circ\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $l \mapsto f(lg)$ on $L(\mathbb{A})$ is contained in the $\widetilde{\pi}$-isotypic component $\widetilde{\pi}^{m(\widetilde{\pi})}$ of $L^2_{cusp}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$. For a function $f \in W_{Q,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}^*_\mathbb{C}$ and $g \in G(\mathbb{A})$ an *Eisenstein series* is formally defined as

$$E_Q(f,\Lambda)(g) := \sum_{\gamma \in Q(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) e^{\langle \Lambda + \rho_Q, H_Q(\gamma g)\rangle}.$$

If we set $(\mathfrak{a}^*)^+ := \{\Lambda \in \mathfrak{a}^*_\mathbb{C} | \Re e(\Lambda) \in \rho_Q + C\}$, where C equals the open, positive Weyl-chamber with respect to $\Delta(Q, A)$, the series converges absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times (\mathfrak{a}^*)^+$. It is known that $E_Q(f, \Lambda)$ is an automorphic form there and that the map $\Lambda \mapsto E_Q(f,\Lambda)(g)$ can be analytically continued to a meromorphic function on all of $\mathfrak{a}^*_\mathbb{C}$, cf. [MW95] or [Lan76] §7. It is known that the singularities Λ_0 (i.e poles) of $E_Q(f, \Lambda)$ lie along certain affine hyperplanes of the form $R_{\alpha,t} := \{\xi \in \mathfrak{a}^*_\mathbb{C} | \langle \xi, \alpha\rangle = t\}$ for some constant t and some root $\alpha \in \Delta(Q, A)$, called "root-hyperplanes" ([MW95] Prop. IV.1.11 (a) or [Lan76] p.131). Choose a normalized vector $\nu \in \mathfrak{a}^*_\mathbb{C}$ orthogonal to $R_{\alpha,t}$ and assume that Λ_0 is on no other singular hyperplane of $E_Q(f, \Lambda)$. Then define $\Lambda_0(u) := \Lambda_0 + u\nu$ for $u \in \mathbb{C}$. If c is a positively oriented circle in the complex plane around zero which is so small that $E_Q(f, \Lambda_0(.))(g)$ has as no singularities on the interior of the circle with double radius, then

$$\mathrm{Res}_{\Lambda_0}(E_Q(f,\Lambda)(g)) := \frac{1}{2\pi i}\int_c E_Q(f,\Lambda_0(u))(g)du$$

is a meromorphic function on $R_{\alpha,t}$, called the *residue* of $E_Q(f, \Lambda)$ at Λ_0. Its poles lie on the intersections of $R_{\alpha,t}$ with the other singular hyperplanes of $E_Q(f, \Lambda)$. So one gets a function holomorphic at Λ_0 in finitely many steps by taking successive residues as explained above.

1.4. The spaces $\mathcal{A}_{E,P,\varphi}$. Now we are able to turn to the desired decomposition of $\mathcal{A}_{E,P}$: For $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$ let $\mathcal{A}_{E,P,\varphi}$ be the space of functions, spanned by all possible residues and derivatives of Eisenstein series defined via all $f \in W_{P,\widetilde{\pi}}$, at the value $d\chi$ inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. It is a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module. Thanks to the functional equations (see [MW95] IV.1.10) satisfied by the Eisenstein series considered, this is well defined, i.e., independent of the choice of a representative for the class of P (whence we took P itself) and the choice of a representation $\pi \in \varphi_P$. Finally, we get

Theorem 1.1 ([FS98] Thm. 1.4. + Thm. 2.3.; see also [MW95] III, Thm. 2.6.). *There is direct sum decomposition as* $(\mathfrak{g}, K, G(\mathbb{A}_f))$*-module*

$$\mathcal{A}_{E,P} = \bigoplus_{\varphi \in \Psi_{E,P}} \mathcal{A}_{E,P,\varphi}$$

giving rise to

$$H^*_{Eis}(G,E) = \bigoplus_{\{P\}, P \neq G} \bigoplus_{\varphi \in \Psi_{E,P}} H^*(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

2. Construction of Eisenstein cohomology

We review now a method to construct Eisenstein cohomology, using the notion of so-called "(π, w)-types".

2.1. Classes of type (π, w)**.** Take $\pi = \chi\widetilde{\pi} \in \varphi_P$ and consider the symmetric tensor algebra

$$S_\chi(\mathfrak{a}^*) = \bigoplus_{n \geq 0} \bigodot^n \mathfrak{a}^*_{\mathbb{C}},$$

$\bigodot^n \mathfrak{a}^*_{\mathbb{C}}$ being the symmetric tensor product of n copies of $\mathfrak{a}^*_{\mathbb{C}}$, as module under $\mathfrak{a}$: Since $S_\chi(\mathfrak{a}^*)$ can be viewed as the space of polynomials on $\mathfrak{a}_{\mathbb{C}}$, we let $\xi \in \mathfrak{a}$ act via translation followed by multiplication with $\langle \xi, \rho_P + d\chi \rangle$. This explains the subscript "χ". We extend this action trivially on $\mathfrak{l}$ and $\mathfrak{n}$ to get an action of the Lie algebra $\mathfrak{p}$ on the space $S_\chi(\mathfrak{a}^*)$. We may also define a $P(\mathbb{A}_f)$-module structure via the rule

$$q \cdot X = e^{\langle d\chi + \rho_P, H_P(q) \rangle} X,$$

for $q \in P(\mathbb{A}_f)$ and $X \in S_\chi(\mathfrak{a}^*)$. There is a linear isomorphism

$$\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_L)} \left[\widetilde{\pi}^\infty_{(K_L)} \otimes S_\chi(\mathfrak{a}^*) \right]^{m(\widetilde{\pi})} \xrightarrow{\sim} W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*),$$

induced by the tensor map $\otimes$ and the evaluation of functions $f \in C^\infty(G(\mathbb{A}), (\widetilde{\pi}^\infty)^{m(\widetilde{\pi})})$ at the identity, $f \mapsto ev_{id}(f) : g \mapsto f(g)(id)$, so in particular one can view the right hand side as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module by transport of structure. Doing this, [Fra98] pp. 256-257 show

$$H^q(\mathfrak{g}, K, W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \cong$$

(3)

$$\bigoplus_{\substack{w \in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}} = d\chi}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \left[H^{q-l(w)}(\mathfrak{m}, K_M, (\widetilde{\pi}_\infty)_{(K_M)} \otimes {}^\circ F_w) \otimes \mathbb{C}_{d\chi + \rho_P} \otimes \widetilde{\pi}_f^{\infty_f} \right]^{m(\widetilde{\pi})}.$$

Here ${}^\circ F_w$ is the finite dimensional representation of $M(\mathbb{C})$ with highest weight $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{\mathbb{C}}}$ and $\mathbb{C}_{d\chi+\rho_P}$ the one-dimensional, complex $P(\mathbb{A}_f)$-module on which $q \in P(\mathbb{A}_f)$ acts by multiplication by $e^{\langle d\chi + \rho_P, H_P(q) \rangle}$. A non-trivial class in a summand of the right hand side is called a cohomology class *of type* (π, w), $\pi \in \varphi_P$, $w \in W^P$. (This notion was first introduced in [Sch83].)
Further, as $L(\mathbb{R}) \cong M(\mathbb{R}) \times A(\mathbb{R})^\circ$, $\widetilde{\pi}_\infty$ can be regarded as an irreducible, unitary representation of $M(\mathbb{R})$. Therefore, a (π, w) type consists out of an irreducible representation $\pi = \chi\widetilde{\pi}$ whose unitary part $\widetilde{\pi} = \widetilde{\pi}_\infty \hat{\otimes} \widetilde{\pi}_f$ has at the infinite place an irreducible, unitary representation $\widetilde{\pi}_\infty$ of the semisimple group $M(\mathbb{R})$ with non-trivial $(\mathfrak{m}, K_M)$-cohomology with respect to ${}^\circ F_w$.

2.2. The Eisenstein map. In order to construct Eisenstein cohomology classes, we start from a class of type (π, w). Since we are interested in cohomology, we can by (3) assume without loss of generality that $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ lies inside the closed Weyl chamber defined by $\Delta(P,A)$.

We reinterpret $S_\chi(\mathfrak{a}^*)$ as the space of formal, finite $\mathbb{C}$-linear combinations of differential operators $\frac{\partial^\alpha}{\partial\Lambda^\alpha}$ on the complex, l-dimensional vector space $\mathfrak{a}^*_\mathbb{C}$. It is understood that some choice of Cartesian coordinates $z_1(\Lambda), ..., z_l(\Lambda)$ on $\mathfrak{a}^*_\mathbb{C}$ has been fixed and $\alpha = (n_1, ..., n_l) \in \mathbb{N}_0^l$ denotes a multi-index with respect to these. As a consequence of [MW95] Prop. IV.1.11, there exists a polynomial $0 \neq q(\Lambda)$ on $\mathfrak{a}^*_\mathbb{C}$ such that for every $f \in W_{P,\tilde{\pi}}$ the function

$$\Lambda \mapsto q(\Lambda) E_P(f,\Lambda)$$

is holomorphic at $d\chi$. Since $\mathcal{A}_{E,P,\varphi}$ can be written as the space which is generated by the coefficient functions in the Taylor series expansion of $q(\Lambda)E_P(f,\Lambda)$ at $d\chi$, f running through $W_{P,\tilde{\pi}}$, we are able to define a surjective homomorphism of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules $E_{P,\pi}$

$$W_{P,\tilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \xrightarrow{E_{P,\pi}} \mathcal{A}_{E,P,\varphi}$$
$$f \otimes \frac{\partial^\alpha}{\partial\Lambda^\alpha} \mapsto \frac{\partial^\alpha}{\partial\Lambda^\alpha}\left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}.$$

and get a well-defined map in cohomology

$$H^q(\mathfrak{g}, K, W_{P,\tilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E^q_\pi} H^*(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E). \tag{4}$$

2.3. Degrees of Eisenstein cohomology classes.

2.3.1. *Regular Eisenstein series.* Suppose $[\omega] \in H^q(\mathfrak{g}, K, W_{P,\tilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class of type (π, w), represented by a morphism ω, such that for all elements $f \otimes \frac{\partial^\alpha}{\partial\Lambda^\alpha}$ in its image, $E_{P,\pi}(f \otimes \frac{\partial^\alpha}{\partial\Lambda^\alpha}) = \frac{\partial^\alpha}{\partial\Lambda^\alpha}\left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}$ is just the regular value $E_P(f, d\chi)$ of the Eisenstein series $E_P(f,\Lambda)$, which is assumed to be holomorphic at the point $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ inside the closed, positive Weyl chamber defined by $\Delta(P,A)$. Then $E^q_\pi([\omega])$ is a non-trivial Eisenstein cohomology class

$$E^q_\pi([\omega]) \in H^q(\mathfrak{g}, K, \mathcal{A}_{E,P,\varphi} \otimes E).$$

This is a consequence of [Sch83], Thm. 4.11.

2.3.2. *Residual Eisenstein series.* In the residual case, there might no longer be an unique degree, in which the image of E^q_π contributes. But we can still single out a certain degree in which residual Eisenstein series contribute, if they have a pole of *maximal possible order* at $\Lambda = d\chi$ and satisfy some extra condition to be introduced below. (Observe that this maximum is precisely the dimension of $\mathfrak{a}_\mathbb{C}$.)

Let us explain this. As a matter of fact, the poles of the Eisenstein series $E_P(f,\Lambda)$ are the ones of its constant terms, [Lan76]. Further more, it is enough to consider the constant term along associate parabolic subgroups. Indeed, due to the functional equation satisfied by Eisenstein series (cf. [MW95], IV.1.10) it suffices to consider the constant term along the standard parabolic subgroup $P' \in \{P\}$, which is conjugate to $\overline{P}$, the parabolic opposite to P. For sake of simplicity we assume that P is self-associate, i.e., $P = P'$. Put

$$\mathrm{I}_{P,\widetilde{\pi},\Lambda} := \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\mathrm{Ind}_{(\mathfrak{l},K_M)}^{(\mathfrak{g},K)}\left[\widetilde{\pi}^{\infty}_{(K_M)} \otimes \mathbb{C}_{\Lambda+\rho_P}\right]^{m(\widetilde{\pi})},$$

where we assume that $q \in P(\mathbb{A}_f)$ acts on $\mathbb{C}_{\Lambda+\rho_P}$ by multiplication with $e^{\langle \Lambda+\rho_P, H_P(q)\rangle}$. Then the constant term along P can be written as a finite sum over certain Weyl group elements $w \in W(A) := N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$

$$E_P(f,\Lambda)_P = \sum_{w\in W(A)} M(\Lambda,\widetilde{\pi},w)(fe^{\langle \Lambda+\rho_P, H_P(.)\rangle}), \tag{5}$$

see e.g., [MW95], prop. II.1.7 and the poles of the Eisenstein series are determined by the mutual influence of the poles of the $(\mathfrak{g},K,G(\mathbb{A}_f))$-intertwining operators

$$M(\Lambda,\widetilde{\pi},w) : \mathrm{I}_{P,\widetilde{\pi},\Lambda} \to \mathrm{I}_{P,w(\widetilde{\pi}),w(\Lambda)}$$

$$M(\Lambda,\widetilde{\pi},w)\psi(g) = \int_{N(\mathbb{Q})\cap wN(\mathbb{Q})w^{-1}\backslash N(\mathbb{A})} \psi(w^{-1}ng)dn.$$

Let us assume that $E_P(f,\Lambda)$ has got a pole of order ℓ at the point $d\chi$ for an $f \in W_{P,\widetilde{\pi}}$. Then the residue of the Eisenstein series $\mathrm{Res}_{d\chi}E_P(f,\Lambda)$ will be via the constant term map in the sum of the images $J(d\chi,\widetilde{\pi},w)$ of those normalized intertwining operators $N(d\chi,\widetilde{\pi},w)$ for which $M(\Lambda,\widetilde{\pi},w)$ has a pole of at least order ℓ at $\Lambda = d\chi$. (By a normalization we mean a function which results out of $M(\Lambda,\widetilde{\pi},w)$ when dividing out the poles, i.e., more precisely, we assume to have found a meromorphic function $r(\Lambda,\widetilde{\pi},w)$ such that $N(\Lambda,\widetilde{\pi},w) = r(\Lambda,\widetilde{\pi},w)^{-1}M(\Lambda,\widetilde{\pi},w)$ is holomorphic and non-vanishing in a region containing $d\chi$.) This set of operators therefore defines a subset $W(A)_{\mathrm{res}} \subseteq W(A)$, given by

$$W(A)_{\mathrm{res}} = \{w \in W(A) | M(\Lambda,\widetilde{\pi},w) \text{ has a pole of at least order } \ell \text{ at } \Lambda = d\chi\}.$$

If we particularly assume that ℓ is maximal, then $M(\Lambda,\widetilde{\pi},w_0)$, with w_0 the longest element of $W(A)$, will be among these operators, i.e., $w_0 \in W(A)_{\mathrm{res}}$. Now, let $[\omega] \in H^q(\mathfrak{g},K,W_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ be a class represented by a morphism ω having only functions $f \otimes 1$ in its image whose associated Eisenstein series $E_P(f,\Lambda)$ have a pole of maximal possible order at the uniquely determined point $\Lambda = d\chi$. We recall that the class $[E_{P,\pi}(\omega)_P]$ which is represented by the constant term of the residues $\mathrm{Res}_{d\chi}E_P(f,\Lambda)$ along P equals the natural restriction $res^q_P(E^q_\pi([\omega]))$ of the class $E^q_\pi([\omega])$ to the face $e'(P)_\mathbb{A} := P(\mathbb{Q})\backslash P(\mathbb{A})/K_PA(\mathbb{R})^\circ$ of the adelic Borel-Serre-compactification of $S := G(\mathbb{Q})\backslash G(\mathbb{A})/K$. As this will not play a big role here, we refere the reader to [Sch83], Satz 1.10 and [Roh96] for details. Having observed this, we see that

$$res^q_P(E^q_\pi([\omega])) \in H^q(\mathfrak{g},K,\sum_{w\in W(A)_{\mathrm{res}}} J(d\chi,\widetilde{\pi},w) \otimes E).$$

The reader should observe that the sum $\sum_{w\in W(A)_{\mathrm{res}}} J(d\chi,\widetilde{\pi},w)$ will not be direct in general. This is the point where we introduce the extra condition mentioned already at the beginning of this subsection: We will from now on assume that $J(d\chi,\widetilde{\pi},w_0)$ is a *direct* summand of our coefficient space, i.e., there is a $(\mathfrak{g},K,G(\mathbb{A}_f))$-module N such that

$$J(d\chi,\widetilde{\pi},w_0)\oplus N=\sum_{w\in W(A)_{\mathrm{res}}}J(d\chi,\widetilde{\pi},w). \tag{6}$$

This assumption is not too strong. But it enables us to write $res^q_P(E^q_\pi([\omega]))$ as $res^q_P(E^q_\pi([\omega]))=[\Omega_{w_0}]\oplus[\Omega_N]$, where clearly $[\Omega_{w_0}]\in H^q(\mathfrak{g},K,J(d\chi,\widetilde{\pi},w_0)\otimes E)$ and $[\Omega_N]\in H^q(\mathfrak{g},K,N\otimes E)$. We will now show that $[\Omega_{w_0}]$ might be viewed as a cohomology class in a certain degree q'.

As P is self-associate, we have $L(\mathbb{A})=w_0L(\mathbb{A})w_0^{-1}=\overline{L}(\mathbb{A})$ and $N(\mathbb{A})=w_0\overline{N}(\mathbb{A})w_0^{-1}$. This implies that we can rewrite the intertwining operator $M(\Lambda,\widetilde{\pi},w_0)$ as

$$M(\Lambda,\widetilde{\pi},w_0)\psi(g)=\int_{\overline{N}(\mathbb{A})}\psi(nw_0^{-1}g)dn.$$

and hence $M(\Lambda,\widetilde{\pi},w_0)\psi\in \mathrm{I}_{\overline{P},\widetilde{\pi},\Lambda}$, the representation induced from the *opposite* parabolic $\overline{P}$. Therefore it is justified to look at the image $J(d\chi,\widetilde{\pi},w_0)$ of $N(d\chi,\widetilde{\pi},w_0)$ as a subspace of $\mathrm{I}_{\overline{P},\widetilde{\pi},d\chi}$. But this implies further that $[\Omega_{w_0}]$ can be viewed as a cohomology class in $H^q(\mathfrak{g},K,\mathrm{I}_{\overline{P},\widetilde{\pi},d\chi}\otimes E)$. Clearly, the degrees in which $J(d\chi,\widetilde{\pi},w_0)$ has cohomology with respect to E are determined by its infinite component

$$J(d\chi,\widetilde{\pi}_\infty,w_0)\hookrightarrow \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{l},K_M)}\left[(\widetilde{\pi}_\infty)_{(K_M)}\otimes\mathbb{C}_{d\chi+\rho_{\overline{P}}}\right]^{m(\widetilde{\pi})}.$$

As a consequence of the first half of the proof of [BW80] V, Prop. 1.5, $[\Omega_{w_0}]$ defines in this case, (i.e., if all Eisenstein series $E_P(f,\Lambda)$, $f\otimes 1$ in the image of ω, have a pole of maximal possible order ℓ at $d\chi=-w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ and if (6) holds), a cohomology class in degree $q':=q+\dim N(\mathbb{R})-2l(w)$.

By (3) $r=q-l(w)$ is a degree, in which $\widetilde{\pi}_\infty$ has $(\mathfrak{m},K_M)$-cohomology. So we have proved:

Theorem 2.1. *Let $[\omega]\in H^q(\mathfrak{g},K,W_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*)\otimes E)$ be a non-trivial class of type (π,w), $\pi=\chi\widetilde{\pi}$, $w\in W^P$ such that $\widetilde{\pi}_\infty$ has non-zero $(\mathfrak{m},K_M)$-cohomology in degree $r=q-l(w)$ with respect to ${}^\circ F_w$. Suppose that all Eisenstein series $E_P(f,\Lambda)$, $f\otimes 1$ in the image of ω, have a pole of maximal possible order $\ell=\dim\mathfrak{a}_\mathbb{C}$ at the uniquely determined point $d\chi=-w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ and that $J(d\chi,\widetilde{\pi},w_0)$ is a direct summand of $\sum_{w\in W(A)_{res}}J(d\chi,\widetilde{\pi},w)$. Then the restriction of $E^q_\pi([\omega])$ to the face $e'(P)_\mathbb{A}$ has a summand which defines an Eisenstein cohomology class in degree $r+\dim N(\mathbb{R})-l(w)$.*

Remark (Maximal parabolic P)**.** If P maximal, then P will automatically be self-associate if G is not of type A_n $(n\geq 2)$, D_n (n odd) or E_6. Assume that P is self-associate. Then only the longest (since it is the only non-trivial) Weyl group element $w\in W(A)$ can contribute a pole to an Eisenstein series and we are in the situation considered above. We recall further that if P is not self-associate, then $E_P(f,\Lambda)$ will be holomorphic for $\Re e(\Lambda)\geq 0$.
Clearly, if $rk_\mathbb{Q}(G)=1$ then any proper parabolic P will be self-associate and hence the above said always applies to these groups.

3. The group $Sp(2,2)$

3.1. We collect now necessary, basic facts concerning the group $G=Sp(2,2)$. Therefore let B be a quaternion algebra over $\mathbb{Q}$ with canonical involution $x\mapsto\overline{x}$, s.t. $B\otimes_\mathbb{Q}\mathbb{R}\cong\mathbb{H}$ where $\mathbb{H}$ equals the real Hamilton quaternions. We denote by $S(B)$ the finite set of places p where B does not split, i.e., $B\otimes_\mathbb{Q}\mathbb{Q}_p$ is a division algebra. Suppose $f:B^n\times B^n\to B$ is a Hermitian form of signature (p,q), where $0\leq q\leq p$

with $n = p + q$ and B^n is being regarded as a B-right module. We suppose that f is equivalent to $(x, y) \mapsto \sum_{i=1}^{p} x_i \overline{y}_i - \sum_{j=1}^{q} x_{j+p}\overline{y}_{j+p}$. Then we define $Sp(p, q)$ to be the group of all B-linear automorphisms of B^n leaving f invariant:

$$Sp(p, q) = \{g \in M_n(B) | g^* K_{p,q} g = K_{p,q}\}.$$

Here, $g^* = (\overline{g}_{ji})_{i,j} = \overline{g}^t$ and

$$K_{p,q} := \begin{pmatrix} id_{p\times p} & 0 \\ 0 & -id_{q\times q} \end{pmatrix}.$$

$Sp(p, q)$ is a connected, simply connected, simple algebraic group over $\mathbb{Q}$ of ranks $rk_{\mathbb{Q}}(G) = rk_{\mathbb{R}}(G) = \min(p, q)$. It is a non-quasisplit inner form of Sp_{2n}, the split group of type C_n. From now on let $G = Sp(2, 2)$. A maximal compact subgroup K of $G(\mathbb{R})$ is isomorphic to $K = Sp(2) \times Sp(2)$.

3.2. Parabolic groups. We fix a minimal parabolic $P_0 = L_0N_0 = M_0A_0N_0$ as in the introduction. We see that

$$L_0 \cong GL_1(B) \times GL_1(B)$$

and so

$$M_0 = SL_1(B) \times SL_1(B).$$

Further, A_0 can be chosen such that $Lie(A_0(\mathbb{R})) = \mathfrak{a}_0$, with

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, a = \mathrm{diag}(a_1, a_2) \in M_2(\mathbb{R}) \right\}$$

and we can identify the set of $\mathbb{Q}$- and $\mathbb{R}$-roots of G with

$$\Delta_{\mathbb{Q}} = \Delta(\mathfrak{g}, \mathfrak{a}_0) = \{\pm\beta_i \pm \beta_j, 1 \le i < j \le 2\} \cup \{\pm 2\beta_1, \pm 2\beta_2\},$$

β_i being the linear functional on $\mathfrak{a}_0$ extracting the value a_i. The simple $\mathbb{Q}$-roots are $\Delta^\circ_{\mathbb{Q}} = \{\beta_1 - \beta_2, 2\beta_2\}$. The unipotent radical N_0 of P_0 is of dimension 14.

There are two standard, maximal parabolic $\mathbb{Q}$-subgroups P_1, P_2 (the latter being the Siegel parabolic). Explicitly we get

$$\begin{aligned} L_1 &\cong GL_1(B) \times Sp(1,1) \\ M_1 &= SL_1(B) \times Sp(1,1) \\ A_1 &= \{g \in A_0 | a_2 = 1\} \\ \dim N_1(\mathbb{R}) &= 11 \\ K_{M_1} &\cong SL_1(\mathbb{H}) \times Sp(1) \times Sp(1) \end{aligned}$$

and

$$\begin{aligned} L_2 &\cong GL_2(B) \\ M_2 &= SL_2(B) \\ A_2 &= \{g \in A_0 | a_1 = a_2\} \\ \dim N_2(\mathbb{R}) &= 10 \\ K_{M_2} &\cong Sp(2) \end{aligned}$$

3.3. Root data. For $i=0,1,2$, extend $\mathfrak{a}_i$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by adding a Cartan subalgebra $\mathfrak{b}_i$ of $\mathfrak{m}_i$. We may take

$$\mathfrak{b}_0=\left\{\left(\begin{array}{cc} b & 0\\ 0 & b\end{array}\right)\middle| b=\operatorname{diag}(b_1,b_2)\in iM_2(\mathbb{R})\right\}.$$

Then the absolute root system of G is given as

$$\Delta(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})=\{\pm\lambda_i\pm\lambda_j, 1\leq i<j\leq 4\}\cup\{\pm 2\lambda_i, 1\leq i\leq 4\}$$

where λ_i equals the functional sending $H\in\mathfrak{h}_{\mathbb{C}}$ to

$$\lambda_i(H)=\begin{cases} b_i+a_i & 1\leq i\leq 2,\\ b_{i-2}-a_{i-2} & 3\leq i\leq 4.\end{cases}$$

A simple subsystem which is compatible with the choice of positivity on $\mathfrak{a}_0^*$ is hence

$$\Delta^\circ=\{\underbrace{\lambda_1+\lambda_3}_{=:\alpha_1},\underbrace{-\lambda_2-\lambda_3}_{=:\alpha_2},\underbrace{\lambda_2+\lambda_4}_{=:\alpha_3},\underbrace{-2\lambda_4}_{=:\alpha_4}\}.$$

The highest weight λ of an irreducible, finite-dimensional representation E of $G(\mathbb{R})$ may be written as $\lambda=\sum_{i=1}^4 c_i\alpha_i$, where c_i are non-negative half-integers.
The corresponding systems of simple roots for the three standard parabolics are

$$\begin{aligned}\Delta^\circ_{M_0} &= \{\alpha_1,\alpha_3\}\\ \Delta^\circ_{M_1} &= \{\alpha_1,\alpha_3,\alpha_4\}\\ \Delta^\circ_{M_2} &= \{\alpha_1,\alpha_2,\alpha_3\}.\end{aligned}$$

Clearly, the restrictions of the roots $\alpha_j\in\Delta^\circ\backslash\Delta^\circ_{M_i}$ to $\mathfrak{a}_i$ gives the set of simple roots within $\Delta(P_i,A_i)$.
For later purpose we also fix the following notation for the corresponding fundamental weights: ω_{ij}, $j=1,2,3$, denotes the j-th fundamental weight of $M_i(\mathbb{C})$, $i=1,2$. The fundamental weights of $M_0(\mathbb{C})$ are denoted by ω_{01} and ω_{02}.
We list the tables of values $w(\lambda+\rho)-\rho|_{\mathfrak{b}_{i_{\mathbb{C}}}}$, $w\in W^{P_i}$, and $\langle -w(\lambda+\rho)|_{\mathfrak{a}_{i_{\mathbb{C}}}},\alpha_j\rangle$, $\alpha_j\in\Delta(P_i,A_i)$, $w\in W^{P_i}$, (and therefore also the sets W^{P_i}) in the appendix of this paper.

4. Cohomological representations for the three standard Levi subgroups

4.1. Recall the notion of (π,w)-types and the construction process of Eisenstein cohomology described in section 2. We need to find the cohomological, irreducible, unitary representations of $M_i(\mathbb{R})$. Denote the set of irreducible, unitary representations by $\widehat{M_i(\mathbb{R})}$, the cohomological ones among them by $\widehat{M_i(\mathbb{R})}_{\text{coh}}$. Connected, semisimple Lie groups are of "type I" (or "tame" in the sense of Kirillow and Bernstein), so by the Künneth rule (cf. [BW80], I.1.3)

$$\begin{aligned}\widehat{M_0(\mathbb{R})}_{\text{coh}} &= \widehat{SL_1(\mathbb{H})}_{\text{coh}}\hat{\otimes}\widehat{SL_1(\mathbb{H})}_{\text{coh}},\\ \widehat{M_1(\mathbb{R})}_{\text{coh}} &= \widehat{SL_1(\mathbb{H})}_{\text{coh}}\hat{\otimes}\widehat{Sp(1,1)}_{\text{coh}},\\ \widehat{M_2(\mathbb{R})}_{\text{coh}} &= \widehat{SL_2(\mathbb{H})}_{\text{coh}}.\end{aligned}$$

4.2. Compact factors. The cohomological representations of a $SL_1(\mathbb{H})$-factor of $M_i(\mathbb{R})$, $i=0,1$ are easily determined in the next lemma. For sake of simplicity we identify ${}^\circ F_w$ with its restriction to this factor:

Lemma 4.1. *Let $w \in W^P$ ($P = P_0$ or P_1) and V be an irreducible, unitary representation of $SL_1(\mathbb{H})$. Then*

$$H^q(\mathfrak{sl}_1(\mathbb{H}), SL_1(\mathbb{H}), V \otimes {}^\circ F_w) = \begin{cases} \mathbb{C} & \text{if } q=0 \text{ and } V = {}^\circ F_w \\ 0 & \text{else} \end{cases}$$

Proof. Since $SL_1(\mathbb{H})$ is compact, relative Lie algebra-cohomology with respect to $V \otimes {}^\circ F_w$ is one-dimensional, if $V \cong {}^\circ \check{F}_w$ (the representation contragredient to ${}^\circ F_w$) and $q=0$ and vanishes otherwise. By [Sch94], prop. 4.13 and our tables 5 and 6, respectively our table 1 we see that ${}^\circ \check{F}_w \cong {}^\circ F_w$. □

4.3. Non-compact factors. The paper [VZ84] provides a full classification of irreducible, unitary, cohomological representations of a connected semisimple Lie group. In order to apply it to the simple Lie groups $Sp(1,1)$ and $SL_2(\mathbb{H})$, let us fix a maximal compact Cartan algebra $\mathfrak{t}_1 \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ of $\mathfrak{sp}(1,1)$ (resp. $\mathfrak{t}_2 \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathbb{R}$ of $\mathfrak{sl}_2(\mathbb{H})$). We can arrange that with respect to this Cartan algebra the system of positive roots looks like $\Delta_1^+ = \{\mu_1 \pm \mu_2, 2\mu_1, 2\mu_2\}$, (resp. $\Delta_2^+ = \{\mu_1 \pm \mu_2, \mu_1 \pm \mu_3, \mu_2 \pm \mu_3\}$). Take a finite-dimensional, irreducible, complex representation F of $Sp(1,1)$ (resp. $SL_2(\mathbb{H})$) with highest weight μ with respect to Δ_1^+ (resp. Δ_2^+). Skipping the details, we get

Proposition 4.2 ([VZ84]). *For each μ there is an integer $j_1(\mu)$, $0 \leq j_1(\mu) \leq 2$ such that the irreducible, unitary $(\mathfrak{sp}(1,1), Sp(1) \times Sp(1))$-modules with non-trivial cohomology with respect to F are the uniquely determined irreducible, unitary representations $A_j(\mu)$, $j_1(\mu) \leq j \leq 1$ having the property*

$$H^q(\mathfrak{sp}(1,1), Sp(1)\times Sp(1), A_j(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=j \text{ or } q=4-j \\ 0 & \text{otherwise} \end{cases}$$

together with the two irreducible, unitary $(\mathfrak{sp}(1,1), Sp(1) \times Sp(1))$-modules $A^+(\mu)$, $A^-(\mu)$ with

$$H^q(\mathfrak{sp}(1,1), Sp(1)\times Sp(1), A^\pm(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=2 \\ 0 & \text{otherwise} \end{cases}$$

This integer is given as

$$j_1(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu = k\mu_1,\ k=1,2,3,... \\ 2 & \text{otherwise} \end{cases}$$

Analogously, there is an integer $j_2(\mu)$, $0 \leq j_2(\mu) \leq 3$ such that the irreducible, unitary $(\mathfrak{sl}_2(\mathbb{H}), Sp(2))$-modules with non-trivial cohomology with respect to F are the uniquely determined irreducible, unitary representations $B_j(\mu)$, $j_2(\mu) \leq j \leq 2$ having the property

$$H^q(\mathfrak{sl}_2(\mathbb{H}), Sp(2), B_j(\mu) \otimes F) = \begin{cases} \mathbb{C} & \text{if } q=j \text{ or } q=5-j \\ 0 & \text{otherwise} \end{cases}$$

This integer is given as

$$j_2(\mu) = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu = k\mu_1,\ k=1,2,3,... \\ 2 & \text{if } \mu \circ \vartheta = \mu \\ 3 & \text{otherwise} \end{cases}$$

Remark. One can see this also by use of the isomorphisms $SO(4,1)^\circ \cong PSp(1,1)$ and $SO(5,1)^\circ \cong PSL_2(\mathbb{H})$ of real Lie groups and the classification of $\widehat{SO(n,1)}_{\mathrm{coh}}$ as given essentially in [BW80] and later on completely in [RS87].
The condition $j_2(\mu) = 3$ can be interpreted as $F \ncong \check{F}$, see [BC83], Cor. 1.6.(a).

4.4. We will have to compare weights with respect to maximally non-compact Cartans to weights in $\mathfrak{t}^*_{i_\mathbb{C}}$. Therefore, let $\varpi_{ij} \in \mathfrak{t}^*_{i_\mathbb{C}}$, be the fundamental weights corresponding to the simple roots in Δ_i^+ and consider the linear maps given by

$$\varphi_1 : (\mathfrak{sp}(1,1) \cap \mathfrak{b}_1)^*_\mathbb{C} \to \mathfrak{t}^*_{1_\mathbb{C}}, \quad \omega_{12} \mapsto \varpi_{12}, \quad \omega_{13} \mapsto \varpi_{11}$$

and

$$\varphi_2 : \mathfrak{b}^*_{2_\mathbb{C}} \to \mathfrak{t}^*_{2_\mathbb{C}}, \quad \omega_{21} \mapsto \varpi_{22}, \quad \omega_{22} \mapsto \varpi_{21}, \quad \omega_{23} \mapsto \varpi_{23}.$$

These are isomorphisms respecting the choices of positivity on each side and transferring fundamental representations to fundamental representations.
In particular, we can compare highest weights of irreducible representations of $Sp(1,1)$ and $SL_2(\mathbb{H})$ with respect to the two Cartan subalgebras and their choices of positivity by applying the corresponding map φ_i.

5. Eisenstein Cohomology of $Sp(2,2)$ with respect to regular coefficients

5.1. Having listed the sets W^{P_i}, $i = 0, 1, 2$, in our appendix, and the cohomological representations of the groups $M_i(\mathbb{R})$ in the last section we are now ready to attack the problem of determining the Eisenstein cohomology of G. In view of our section 2, we need to construct the spaces $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P} \otimes E)$ for each class $\{P\}$ of proper, associate parabolic $\mathbb{Q}$-subgroups of G. We remark that for $G = Sp(2,2)$ the associate classes and conjugacy classes of parabolic $\mathbb{Q}$-subgroups coincide, hence we can suppose that P is one of the groups P_0, P_1 or P_2.
This section deals with the case of *regular* coefficients E. That means the highest weight λ of E has strictly positive integer coefficients with respect to a decomposition according to the fundamental weights. Recall the following crucial result on Eisenstein cohomology with respect to regular coefficients E, which reads in our particular case as

Theorem 5.1 ([Sch94]; see also [Fra98], Thm. 19.II)**.** *Residual Eisenstein series do not contribute to the Eisenstein cohomology of* G *with respect to regular* E*. More precisely, if* Π *is a set of representatives of irreducible representations* $\pi = \chi\widetilde{\pi}$ *of the Levi components* $L(\mathbb{A})$ *of standard parabolic* $\mathbb{Q}$*-subgroups of* G*, which give rise to non-trivial maps* E^*_π*. Then* E^q_π *is an isomorphism and we get*

$$H^q_{Eis}(G,E) \cong$$

$$\bigoplus_{\pi\in\Pi} \bigoplus_{\substack{w\in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}=d\chi}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)} \left[H^{q-l(w)}(\mathfrak{m}, K_M, (\widetilde{\pi}_\infty)_{(K_M)} \otimes {}^\circ F_w) \otimes \mathbb{C}_{d\chi+\rho_P} \otimes \widetilde{\pi}_f^{\infty_f}\right]^{m(\widetilde{\pi})}.$$

5.2. The minimal parabolic subgroup. In order to perform the construction via (π, w)-types, we need to know for which $w \in W^{P_0}$, $\Lambda_w = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}}$ lies inside the closed, positive Weyl chamber. This is achieved explicitly in tables 7 and 8 and we see that only very few elements in W^{P_0} can actually satisfy this condition. These are underlined in table 8. Among them, only six elements satisfy it for sure, i.e., for all coefficient systems E (even non-regular ones). The others need some extra condition on the highest weight λ which might also be not satisfied by a regular representation E. It is given in table 8. We will denote by $W^+(\lambda)$ the set of $w \in W^{P_0}$ giving rise to $\Lambda_w \in \overline{C}$.

Remark 5.2. General theory, as developed in [Sch94], tells us that Λ_w to make part of the closed, positive Weyl chamber must at least satisfy $l(w) \geq \frac{1}{2}\dim N_0(\mathbb{R}) = 7$. However, instead of looking at all $w \in W^{P_0}$ having $l(w) \geq 7$, it would have been enough to consider those $w \in W^{P_0}$ giving rise to the inequality

$$l(w^{P_i/P_0}) \geq \frac{\dim N_0(\mathbb{R})}{2\dim N_i(\mathbb{R})}, \quad i = 1, 2. \tag{7}$$

This follows from [Sch94], Thm. 6.4. Here, the Weyl group element w^{P_i/P_0} is defined as follows: Let W^{P_i/P_0} be the set of representatives of minimal length for the right cosets of $W(\mathfrak{m}_{0_\mathbb{C}}, \mathfrak{b}_{0_\mathbb{C}})$ in $W(\mathfrak{m}_{i_\mathbb{C}}, \mathfrak{b}_{i_\mathbb{C}})$. Such representatives are unique by [Kos61], Prop. 5.13. Now, for a given $w \in W^{P_0}$ there are uniquely determined elements $w^{P_i/P_0} \in W^{P_i/P_0}$, $w^{P_i} \in W^{P_i}$ satisfying $w = w^{P_i/P_0} \circ w^{P_i}$ and $l(w) = l(w^{P_i/P_0}) + l(w^{P_i})$, see [Sch94], Prop. 4.7.

In our cases, (7) reads as

$$l(w^{P_1/P_0}) \geq \frac{7}{11} \quad \text{and} \quad l(w^{P_2/P_0}) \geq \frac{7}{10},$$

meaning that we only have to consider those $w \in W^{P_0}$ which are neither in W^{P_1} nor in W^{P_2}. In fact, these elements can be excluded by direct means as tables 7 and 8 in our appendix show.

Collecting this information we get the following theorem:

Theorem 5.3. *Let E be an irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P_0}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{Eis}(G,E)$ is given as a $G(\mathbb{A}_f)$-module by

$$\begin{aligned}
H^q(\mathfrak{g}_0, K, \mathcal{A}_{E,P_0} \otimes E) &= \bigoplus_{\substack{w \in W^+(\lambda) \\ l(w)=q}} \bigoplus_{\substack{\pi = \chi\widetilde{\pi} \\ \widetilde{\pi}_\infty = {}^\circ F_w, \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}} \\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_0(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_{P_0}} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})} \\
&\qquad \textit{for } 8 \leq q \leq 13 \\
H^{14}(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) &= \bigoplus_{\substack{\pi = \chi\widetilde{\pi} \\ \widetilde{\pi}_\infty = E|_{M_0(\mathbb{R})}, \\ d\chi = \lambda+\rho|_{\mathfrak{a}_{0_\mathbb{C}}}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_0(\mathbb{A}_f)}[\mathbb{C}_{d\chi+\rho_{P_0}} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})} \\
H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) &= 0 \quad \textit{else}
\end{aligned}$$

All these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. Recalling the construction process via (π, w)-types, and the result on cohomological, irreducible, unitary representations in lemma 4.1, it is clear that $\widetilde{\pi}_\infty$ and $d\chi$ must satisfy the above conditions. By theorem 5.1, E^q_π is already an isomorphism, if it is not identically zero. Looking up in tables 7 and 8 the possible $w \in W^{P_0}$, that can give rise to values $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{0_\mathbb{C}}}$ inside the closed, positive Weyl chamber defined by the positive restricted roots $\Delta(P_0, A_0)$ or recalling remark 5.2, proves that $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_0} \otimes E) = 0$ if $q \leq 7$. Our table 8 shows that $W^+(\lambda)$ can actually contain representatives w having $l(w)$ equal to 8, 9, 10, 11, 12, 13 and 14, whence we have to list cohomology in all these degrees. Again by our table 8 there is a unique Kostant representative of length 14 in $W^+(\lambda)$ for all λ and its corresponding evaluation point $d\chi = \lambda + \rho|_{\mathfrak{a}_{0_\mathbb{C}}}$ lies in the region $C + \rho_{P_0}$ of absolute convergence of the Eisenstein series $E_{P_0}(f, \Lambda)$, since λ is regular. Hence, we can omit the condition $E_\pi^{14} \neq 0$. This is not true for the other degrees $8 \leq q \leq 13$, see table 8. This proves the theorem. $\square$

5.3. The first maximal parabolic subgroup. We explain now which classes of type (π, w), $\pi \in \varphi_{P_1} \in \varphi \in \Psi_{E,P_1}$ and $w \in W^{P_1}$ contribute to the Eisenstein cohomology of G.

Since the highest weight λ of E is supposed to be regular, each irreducible module ${}^\circ F_w$ is a regular as well ([Sch94], Lemma 4.9). Therefore, $\widetilde{\pi}_\infty$ must equal the tensor product of the representation $V = {}^\circ F_w|_{SL_1(\mathbb{H})}$ as in lemma 4.1 with one of the two discrete series representations $A^\pm(\mu_w)$, $\mu_w = \varphi_1(w(\lambda+\rho) - \rho|_{(\mathfrak{sp}(1,1)\cap\mathfrak{b}_1)_\mathbb{C}})$, see 4.4, having non-trivial $(\mathfrak{sp}(1,1), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$-cohomology only in degree 2, as it is proved in proposition 4.2. The actual contribution of the first maximal parabolic $\mathbb{Q}$-subgroup to Eisenstein cohomology is given in the next

Theorem 5.4. *Let E be an irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P_1}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{Eis}(G, E)$ is given as a $G(\mathbb{A}_f)$-module by

$$\begin{aligned} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E) &= \bigoplus_{\substack{w \in W^{P_1} \\ l(w) = q-2}} \bigoplus_{\substack{\pi = \chi\widetilde{\pi} \\ \widetilde{\pi}_\infty = V \otimes A^\pm(\mu_w), \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{1_\mathbb{C}}} \\ E_\pi^q \neq 0}} \mathrm{Ind}_{P_1(\mathbb{A}_f)}^{G(\mathbb{A}_f)} [\mathbb{C}_{d\chi + \rho_{P_1}} \otimes \widetilde{\pi}_f^{\infty_f}]^{m(\widetilde{\pi})} \\ &\quad \textit{for } 8 \leq q \leq 13 \\ &= 0 \quad \textit{else} \end{aligned}$$

All these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. The assertions on $d\chi$ and $\widetilde{\pi}_\infty$ are already explained. By theorem 5.1, we only need to sum over those π, which satisfy $E_\pi^q \neq 0$ and for which E_π^q is therefore an isomorphism. Now proposition lemma 4.1 and 4.2 imply that we must have $l(w) = q-2$, since $H^r(\mathfrak{sl}_1(\mathbb{H}) \oplus \mathfrak{sp}(1,1), SL_1(\mathbb{H}) \times Sp(1) \times Sp(1), V \otimes A^\pm(\mu_w) \otimes {}^\circ F_w) = 0$ for $r \neq 2$. By table 3 there is no element $w \in W^{P_1}$ of length $l(w) \geq 12$ but also $d\chi = \Lambda_w$ does not lie inside the closed, positive Weyl chamber for $l(w) \leq 5$. This proves the vanishing of $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_1} \otimes E)$ in the degrees $q \leq 7$ and $q \geq 14$. $\square$

Remark. In fact, table 3 also shows that all $w \in W^{P_1}$ with $l(w) \geq 9$ give rise to evaluation points $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{1_\mathbb{C}}}$ which lie in the region $C + \rho_{P_1}$ of absolute convergence of the Eisenstein series $E_{P_1}(f, \Lambda)$. Hence, we could have omitted the condition $E_\pi^{l(w)+2} \neq 0$ for these w.

5.4. The second maximal parabolic subgroup. We conclude the analysis of Eisenstein cohomology of G with respect to regular coefficients E describing the remaining summand $H^*(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E)$. Again, since E is supposed to be regular, each representation ${}^\circ F_w$, $w \in W^{P_2}$, of the group $M_2(\mathbb{C})$ is regular, too. Recalling proposition 4.2, there can only be one single cohomological, irreducible, unitary representation of $M_2(\mathbb{R})$ with respect to ${}^\circ F_w$, namely $B_2(\mu_w)$ with $\mu_w = \varphi_2(w(\lambda +$

$\rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}})$, see 4.4. Proposition 4.2 now gives us the appropriate tool to decide, when $j_2(\mu_w) = 2$, i.e., when $B_2(\mu_w)$ exists. This is the case if and only if the first and the third coefficient of $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}}$ in its decomposition according to the basis of fundamental weights ω_{21}, ω_{22} and ω_{23} coincide. Our table 2 answers in details the question, when exactly this happens. Observe that the two conditions $c_1 - c_4 = 1$ and $c_3 - c_4 = c_1$ from table 2 contradict each other, so they cannot be satisfied at the same time. It can very well happen that they are both not satisfied, e.g., if $c_1 < c_3 - c_4$, or equivalently if the first coefficient of $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{2\mathbb{C}}}$ in its decomposition according to the basis of fundamental weights ω_{21}, ω_{22} and ω_{23} is strictly smaller than the third coefficient. Clearly, there are even regular representations E satisfying $c_1 < c_3 - c_4$. In this case $j_2(\mu_w) = 3$ for *all* $w \in W^{P_2}$, implying that P_2 does not give any contribution to Eisenstein cohomology with respect to such E. Generally, this contribution is described in the following

Theorem 5.5. *Let E be an irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with regular highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. Let us write $W^{P_2}(\lambda) := \{w \in W^{P_2} | j(\mu_w) = 2\}$. The summand*

$$H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P_2}} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2,\varphi} \otimes E)$$

in the Eisenstein cohomology $H^q_{Eis}(G, E)$ is given as a $G(\mathbb{A}_f)$-module by

$$\begin{aligned} H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E) \;&=\; \bigoplus_{\substack{w \in W^{P_2}(\lambda) \\ l(w) = q-3}} \;\bigoplus_{\substack{\pi = \chi\tilde{\pi} \\ \text{with } \tilde{\pi}_\infty = B_2(\mu_w), \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{2\mathbb{C}}} \\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_2(\mathbb{A}_f)}[\mathbb{C}_{d\chi + \rho_{P_2}} \otimes \tilde{\pi}_f^{\infty_f}]^{m(\tilde{\pi})} \\ &\oplus \bigoplus_{\substack{w \in W^{P_2}(\lambda) \\ l(w) = q-2}} \;\bigoplus_{\substack{\pi = \chi\tilde{\pi} \\ \text{with } \tilde{\pi}_\infty = B_2(\mu_w), \\ d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{2\mathbb{C}}} \\ E^q_\pi \neq 0}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_2(\mathbb{A}_f)}[\mathbb{C}_{d\chi + \rho_{P_2}} \otimes \tilde{\pi}_f^{\infty_f}]^{m(\tilde{\pi})} \\ &\quad \textit{for } 8 \leq q \leq 13 \\ &= \; 0 \quad \textit{else} \end{aligned}$$

All these spaces are entirely built up by cohomology classes representable by regular values of Eisenstein series.

Proof. This is proved similarly to Theorem 5.3 and 5.4, so we will be very brief. Recall from proposition 4.2 that $B_2(\mu_w)$ has non-trivial $(\mathfrak{sl}_2(\mathbb{H}), \mathrm{Sp}(2))$-cohomology with respect to ${}^\circ F_w$ only in degrees 2 and 3. Therefore $H^q(\mathfrak{g}, K, \mathcal{A}_{E,P_2} \otimes E)$ is built up by classes of type (π, w), having $l(w) = q-2$ or $l(w) = q-3$. The rest follows from table 4. $\square$

Remark. The vanishing of $H^q_{Eis}(G, E)$ for $q \leq 7$ is also a consequence of [LS04], Thm. 5.5.

6. Residual Eisenstein cohomology classes supported by the minimal parabolic

6.1. In section 5 we discussed the contribution of the various standard parabolic $\mathbb{Q}$-subgroups to the Eisenstein cohomology $H^q_{Eis}(G, E)$, for finite-dimensional irreducible representations E of G with regular highest weight. The regularity condition assured that residual Eisenstein series would not contribute to cohomology, so we did not really have to check the analytic behaviour of Eisenstein series at the various points of evaluation in question.

However, in principle it is possible to give a complete description of Eisenstein cohomology even if the regularity condition is dropped, but we first have to understand the analytic behaviour of the Eisenstein series $E_P(f,\Lambda)$ at the points $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$. As our parabolics are all self-associate, we can reduce this problem by section 2.3.2 to the following task: *Understand the interplay of the various poles of the intertwining operators $M(\Lambda,\widetilde{\pi},w)$, $w \in W(A)$.*

In order to exemplify the difficulties and some general phenomena that occur during the analysis of residual Eisenstein cohomology, we will now consider the space of square-integrable Eisenstein cohomology supported by the minimal parabolic subgroup P_0. We enforce square-integrability because then we only need to consider Eisenstein series which have poles of maximal possible order $\ell = 2$. This allows us to use the results of section 2.3.2, which give a partial answer to the question in which degrees of cohomology maximally residual Eisenstein series contribute.

6.2. When trying to find out the various poles of the intertwining operators $M(\Lambda,\widetilde{\pi},w)$, $w \in W(A)$, the actual problem is to give a suitable normalization, i.e., to find a function $r(\Lambda,\widetilde{\pi},w)$ such that $N(\Lambda,\widetilde{\pi},w) = r(\Lambda,\widetilde{\pi},w)^{-1}M(\Lambda,\widetilde{\pi},w)$ – to be called the *normalized* intertwining operator – is holomorphic and non-vanishing on the open, positive Weyl chamber defined by the pair (P,A). The difficulty relies on the fact that each standard Levi group L of G is a non-quasi-split algebraic group, whence one cannot apply the Langlands-Shahidi-method, as developed in [Shd81], [Shd88] in order to normalize the local intertwining operators at the non-split places. However, if $L(\mathbb{Q}_p)$ is compact modulo its center, we can use the same trick as in [Gro10] (see Proposition 3.1) and show that the local intertwining operator at the place p is itself holomorphic and non-vanishing inside the open, positive Weyl chamber defined by $\Delta(P,A)$. Clearly, only the minimal parabolic $P = P_0$ gives a Levi subgroup $L = L_0$ which satisfies the condition to be compact modulo its center at all non-split places.

For the rest of this section let P be the standard minimal parabolic $\mathbb{Q}$-subgroup P_0 of $G = Sp(2,2)$ with decompositions $P = LN = MAN$. As already remarked, $L(\mathbb{Q}_p)$ is compact modulo its center at all places $p \in S(B)$, since $M = SL_1(B) \times SL_1(B)$. We have $W(A) = W_\mathbb{Q}$.

Let $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$, $f \in W_{P,\widetilde{\pi}}$ and identify $\Lambda = x\alpha_2 + y\alpha_4 \in \mathfrak{a}^*_\mathbb{C}$ with $\underline{s} = (s_1,s_2) \in \mathbb{C}^2$ via $s_1 = \frac{x}{2}$ and $s_2 = y - \frac{x}{2}$. As in the sequel, we assume here for sake of simplicity that all roots α_j mean their restriction to $\mathfrak{a}_\mathbb{C}$. Further, observe that since $L = GL_1(B) \times GL_1(B)$, each $\widetilde{\pi}$ factors as $\widetilde{\pi} = \theta\hat{\otimes}\tau$, where θ and τ are cuspidal automorphic representations of $GL_1(B)$.
Now, as mentioned in section 2.3.2, the holomorphic behaviour of the Eisenstein series $E_P(f,\Lambda)$ is the same as of its constant term along P, which can be rewritten as

$$E_P(f,\Lambda)_P = \sum_{w \in W_\mathbb{Q}} M(\underline{s},\widetilde{\pi},w)(fe^{\langle \Lambda+\rho_P, H_P(.)\rangle}). \tag{8}$$

Therefore the poles of $E_P(f,\Lambda)$ are determined by the poles of $M(\underline{s},\widetilde{\pi},w)$, $w \in W_\mathbb{Q}$. We recall the following fact:

Proposition 6.1 (Section 2.1 of [Shd81] + Section 2 of [MS00])**.** *Let $w \in W_\mathbb{Q}$ be an element of the Weyl group with decomposition $w = w_{n_1}...w_{n_k}$ according to the reflections w_{n_i} corresponding to the simple $\mathbb{Q}$-roots α_{n_i}, $n_i \in \{2,4\}$. Then the local* *intertwining operator* $M(\underline{s}, \widetilde{\pi}_p, w)$ *decomposes as*

$$M(\underline{s}, \widetilde{\pi}_p, w) = M\left(s_k, \widetilde{\pi}_{k,p}, w_{n_k}\right) ... M\left(s_1, \widetilde{\pi}_{1,p}, w_{n_1}\right)$$

where we put recursively $s_i = \frac{2\langle \underline{s}_i, \alpha_{n_i}\rangle}{\langle \alpha_{n_i}, \alpha_{n_i}\rangle}$, $\underline{s}_i = w_{n_{i-1}}(\underline{s}_{i-1})$ *with* $\underline{s}_1 = \underline{s}$ *and* $\widetilde{\pi}_{i,p} = w_{n_{i-1}}(\widetilde{\pi}_{i-1,p})$ *with* $\widetilde{\pi}_{1,p} = \widetilde{\pi}_p$. *The action of a Weyl group element on a representation* $\widetilde{\pi}_p = \theta_p \hat{\otimes} \tau_p$ *is given by* $w_2(\widetilde{\pi}_p) = \tau_p \hat{\otimes} \theta_p$ *and* $w_4(\widetilde{\pi}_p) = \theta_p \hat{\otimes} \check{\tau}_p$.

The point of this proposition is that for each $w \in W_{\mathbb{Q}}$ we can write the local intertwining operator $M(\underline{s}, \widetilde{\pi}_p, w)$ as a finite product of the analogous local intertwining operators $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ attached to the two standard maximal Levi subgroups of G: If $n_i = 4$, the maximal Levi is $Sp(1,1)$, while if $n_i = 2$, it is $GL_2(B)$. Hence, on the one hand, we can apply the following proposition

Proposition 6.2 ([Gro10], Prop. 3.1)**.** *Let* $p \in S(B)$. *Then* $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ *is holomorphic and non-vanishing for* $\Re e(s_i) > 0$.

and get

Corollary 6.3. *The poles of* $M(\underline{s}, \widetilde{\pi}, w)$ *in the region* $\Re e(s_1) > \Re e(s_2) > 0$ *are the poles of* $\hat{\otimes}'_{p \notin S(B)} M(\underline{s}, \widetilde{\pi}_p, w)$.

One the other hand, we can *normalize each local operator* $M(\underline{s}, \widetilde{\pi}_p, w)$ *by normalizing the factors* $M(s_i, \widetilde{\pi}_{i,p}, w_{n_i})$ *and get a global normalization*

$$r(\underline{s}, \widetilde{\pi}, w) = \prod_{i=1}^{k} \hat{\otimes}'_{p \notin S(B)} r(s_{k-i+1}, \widetilde{\pi}_{k-i+1,p}, w_{n_{k-i+1}}). \tag{9}$$

6.2.1. $Sp(1,1)$. The corresponding normalizing factors for $n_i = 4$, i.e., our maximal Levi looks like $Sp(1,1)$, can be found in [Gro10], section 5, where the whole residual spectrum of $Sp(1,1)$ was calculated[2]. For the convenience of the reader, we review these results shortly: Recall that we may write $\widetilde{\pi}_i = \theta_i \hat{\otimes} \tau_i$, with θ_i and τ_i being cuspidal automorphic representations of $GL_1(B)$. The only proper parabolic $\mathbb{Q}$-subgroup inside $Sp(1,1)$ has a Levi subgroup isomorphic to $GL_1(B)$, which is actually the second $GL_1(B)$-factor of L_0. Hence we will always identify $\widetilde{\pi}_i$ with its second $GL_1(B)$-factor τ_i, when it comes to $n_i = 4$. Now suppose $p \notin S(B)$. If τ_i is not one-dimensional, then the needed normalization follows from the Gindikin-Karpelevich-integral-formula, as shown in [Lan71], p. 27 (see also [Shd88], p.554) and had been already given in [Kim95] and [MW89]:

$$r(s_i, \widetilde{\pi}_{i,p}, w_{n_i}) = \frac{L(s_i, \tau_{i,p})}{L(1 + s_i, \tau_{i,p})\varepsilon(s_i, \tau_{i,p})} \frac{L(2s_i, \widetilde{\chi}_{i,p})}{L(1 + 2s_i, \widetilde{\chi}_{i,p})\varepsilon(2s_i, \widetilde{\chi}_{i,p})}. \tag{10}$$

Here we wrote $\widetilde{\chi}_{i,p}$ for the central character of $\tau_{i,p}$. The L- and ε-functions are the standard Jacquet-Langlands- resp. Hecke- L- and ε-functions of the second $GL_1(B)$-factor $\tau_{i,p}$ of $\widetilde{\pi}_{i,p}$ resp. its central character $\widetilde{\chi}_{i,p}$.

If $\tau_i = \widetilde{\chi}_i$ is one-dimensional, then we used the concrete normalization of [Grb07], who himself had applied the idea of [MW89], Lemme I.8, i.e., induction from generic representations of smaller parabolic subgroups:

[2](added November 2008) The residual spectrum of $Sp(1,1)$ was – even in greater generality – calculated independently by T. Yasuda in The Residual Spectrum of Inner Forms of Sp(2), *Pac. J. Math.* **232** (2007) 471–490

(11)
$$r(s_i,\widetilde{\pi}_{i,p},w_{n_i})=\frac{L(s_i-\frac{1}{2},\widetilde{\chi}_{i,p})L(2s_i,\widetilde{\chi}^2_{i,p})}{L(s_i+\frac{3}{2},\widetilde{\chi}_{i,p})\varepsilon(s_i-\frac{1}{2},\widetilde{\chi}_{i,p})\varepsilon(s_i+\frac{1}{2},\widetilde{\chi}_{i,p})L(1+2s_i,\widetilde{\chi}^2_{i,p})\varepsilon(2s_i,\widetilde{\chi}^2_{i,p})}$$

6.2.2. $GL_2(B)$. Here we have to distinguish three cases: Suppose $\widetilde{\pi}_i=\theta_i\hat{\otimes}\tau_i$, with θ_i and τ_i being cuspidal automorphic representations of $GL_1(B)$, satisfies $\dim\theta_i>1$ and $\dim\tau_i>1$. Then again after having used the Gindikin-Karpelevich-integral-formula our local normalizing factor at $p\notin S(B)$ looks like

$$r(s_i,\widetilde{\pi}_{i,p},w_{n_i})=\frac{L(s_i,\theta_{i,p}\times\check{\tau}_{i,p})}{L(1+s_i,\theta_{i,p}\times\check{\tau}_{i,p})\varepsilon(s_i,\theta_{i,p}\times\check{\tau}_{i,p})}. \tag{12}$$

Here, the L-functions and the ε-factor are of Rankin-Selberg-type. See again [MW89].

Suppose now that one factor of $\widetilde{\pi}_i$ is one-dimensional, without loss of generality say $\dim\theta_i=1$. We can use [Grb09] to normalize $M(s_i,\widetilde{\pi}_{i,p},w_{n_i})$ and get

$$r(s_i,\widetilde{\pi}_{i,p},w_{n_i})=\frac{L(s_i-\frac{1}{2},\theta_{i,p}\check{\tau}_{i,p})}{L(s_i+\frac{3}{2},\theta_{i,p}\check{\tau}_{i,p})\varepsilon(s_i-\frac{1}{2},\theta_{i,p}\check{\tau}_{i,p})\varepsilon(s_i+\frac{1}{2},\theta_{i,p}\check{\tau}_{i,p})}. \tag{13}$$

In the third case, i.e., both factors θ_i and τ_i are one-dimensional again [Grb09] provides a normalization by

$$r(s_i,\widetilde{\pi}_{i,p},w_{n_i})=$$

(14)
$$\frac{L(s_i,\theta_{i,p}\tau^{-1}_{i,p})L(s_i-1,\theta_{i,p}\tau^{-1}_{i,p})}{L(s_i+2,\theta_{i,p}\tau^{-1}_{i,p})L(s_i+1,\theta_{i,p}\tau^{-1}_{i,p})\varepsilon(s_i,\theta_{i,p}\tau^{-1}_{i,p})^2\varepsilon(s_i-1,\theta_{i,p}\tau^{-1}_{i,p})\varepsilon(s_i+1,\theta_{i,p}\tau^{-1}_{i,p})}$$

Therefore, we have defined recursively the global normalization factor $r(\underline{s},\widetilde{\pi},w)$ as in (9) for each $w\in W_{\mathbb{Q}}$. We conclude finally

Proposition 6.4. *Let $\underline{s}$ be inside the open region $\Re e(s_1)>\Re e(s_2)>0$. Then there is an $f\in W_{P,\widetilde{\pi}}$ such that the Eisenstein series $E_P(f,\Lambda)$ has a double pole at $\underline{s}$ if and only if*

$$r(\underline{s},\widetilde{\pi})=\sum_{w\in W_{\mathbb{Q}}}r(\underline{s},\widetilde{\pi},w)$$

has a double pole at $\underline{s}$.

Proof. Suppose $r(\underline{s},\widetilde{\pi})$ has a double pole at $\underline{s}$, $\Re e(s_1)>\Re e(s_2)>0$. Then there is a $w\in W_{\mathbb{Q}}$ such that $r(\underline{s},\widetilde{\pi},w)$ has a double pole at $\underline{s}$. By our discussion of the normalizing factors we know that $N(\underline{s},\widetilde{\pi},w)=r(\underline{s},\widetilde{\pi},w)^{-1}M(\underline{s},\widetilde{\pi},w)$ is holomorphic and non-vanishing at $\underline{s}$. So there is an $f\in W_{P,\widetilde{\pi}}$, which is not sent to 0 by $N(\underline{s},\widetilde{\pi},w)$ and therefore $M(\underline{s},\widetilde{\pi},w)f=r(\underline{s},\widetilde{\pi},w)N(\underline{s},\widetilde{\pi},w)f$ really has a double pole at $\underline{s}$. By the decomposition (8), the constant term $E_P(f,\Lambda)_P$ has a double pole at $\underline{s}$, wherefrom it finally follows that $E_P(f,\Lambda)$ has a double pole at $\underline{s}$. □

6.3. Double-Poles of normalizing factors. Recall the well known facts on the analytic behaviour of Jacquet-Langlands-, Hecke- and Rankin-Selberg-L-functions, summarized in our next

Lemma 6.5 ([JL70],[Tat67],[Jac72]). (i) *Let $\sigma = \hat{\otimes}'_p \sigma_p$ be a cuspidal automorphic representation of $GL_1(B)$ with central character $\chi_\sigma = \hat{\otimes}'_p \chi_{\sigma_p}$, assuming that $\dim \sigma > 1$. Then: The local Jacquet-Langlands-L-function $L(s, \sigma_p)$ is holomorphic and non-zero on $\Re e(s) > 1$ at each place p. For the infinite place we particularly get $L(s, \sigma_\infty) = 2(2\pi)^{-s-n-\frac{1}{2}}\Gamma(s+n+\frac{1}{2})$ if σ_∞ is the n-th symmetric power $\bigodot^n \mathbb{C}^2$ and hence this local L-factor is holomorphic and non-vanishing for $\Re e(s) \geq 0$. The global Jacquet-Langlands L-function $L(s, \sigma)$ is an entire function and has no zeros for $\Re e(s) \geq 1$.*
(ii) *The local Hecke-L-function $L(s, \chi_{\sigma_p})$ has a simple pole at $s = 0$ if $\chi_{\sigma_p} = \mathbf{1}_p$ and is entire otherwise. It vanishes nowhere. The global Hecke-L-function $L(s, \chi_\sigma)$ has simple poles at $s = 0$ and $s = 1$ if $\chi_\sigma = \mathbf{1}$ (and $L(s, \mathbf{1}) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$) and is entire otherwise. It is non-zero for $\Re e(s) \geq 1$.*
(iii) *Let $\rho = \hat{\otimes}'_p \rho_p$, $\eta = \hat{\otimes}'_p \eta_p$ be two cuspidal automorphic representations of $GL_2(\mathbb{A})$. Then: The local Rankin-Selberg-L-function $L(s, \rho_p \times \eta_p)$ is holomorphic and non-vanishing for $\Re e(s) \geq 1$. If ρ_p and η_p are both square-integrable, then $L(s, \rho_p \times \eta_p)$ is holomorphic and non-zero in $\Re e(s) > 0$. The global Rankin-Selberg-L-function $L(s, \rho \times \eta)$ has simple poles at $s = 0$ and $s = 1$ if and only if $\rho \cong \check{\eta}$ and is entire otherwise. It has no zeros in $\Re e(s) \geq 1$.*

Proposition 6.6. *For an Eisenstein series $E_P(f, \Lambda)$ to have a double pole at $\underline{s} = (s_1, s_2)$ inside the region $\Re e(s_1) > \Re e(s_2) > 0$ it is necessary that one of the following three conditions holds:*

(A) $\dim \theta > 1$ *and* $\dim \tau > 1$*:*
$\underline{s} = A := (\frac{3}{2}, \frac{1}{2})$, $\widetilde{\pi} = \tau \hat{\otimes} \tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$.
(B) $\dim \theta = 1$, $\dim \tau > 1$*:*
$\underline{s} = B := (\frac{3}{2}, \frac{1}{2})$, $\chi_\tau = \mathbf{1}$, $\theta = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$.
(C) $\dim \theta = \dim \tau = 1$*:*
1) $\underline{s} = C_1 := (\frac{3}{2}, \frac{1}{2})$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$, $\theta = \check{\tau}$ *or* $\theta = \mathbf{1}$
2) $\underline{s} = C_2 := (\frac{5}{2}, \frac{1}{2})$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$, $\theta = \tau$
3) $\underline{s} = C_3 := (\frac{7}{2}, \frac{3}{2}) = \rho_P$, $\widetilde{\pi} = \mathbf{1} \otimes \mathbf{1}$.

Sketch of a proof: As the determination of these necessary conditions is easy (by the concrete form of our normalizing factors $r(\underline{s}, \widetilde{\pi})$ and lemma 6.5) but rather cumbersome, we will confine ourselves in exemplifying the general procedure in the case (A). There is no loss of generality if we assume that $\underline{s} \in \mathbb{R}^2$, since this can be achieved by just twisting a cuspidal automorphic representation of $L(\mathbb{A})$ with an appropriate imaginary power of the absolute value of the reduced norm of the determinant.
So let $\widetilde{\pi}$ be a cuspidal automorphic representation whose two cuspidal factors θ and τ are both not one-dimensional. We need to regard the global function $r(\underline{s}, \widetilde{\pi})$. Each of its summands $r(\underline{s}, \widetilde{\pi}, w)$, $w \in W_{\mathbb{Q}}$, is according to (9) a finite product of some of the following five functions $r_1(\underline{s}, \widetilde{\pi}) = r(s_1 - s_2, \widetilde{\pi}, w_2)$, $r_2(\underline{s}, \widetilde{\pi}) = r(s_2, \tau, w_4)$, $r_3(\underline{s}, \widetilde{\pi}) = r(s_1, \theta, w_4)$, $r_4(\underline{s}, \widetilde{\pi}) = r(s_1 + s_2, \tau \hat{\otimes} \check{\theta}, w_2)$ or $r_5(\underline{s}, \widetilde{\pi}) = r(s_1 + s_2, \theta \hat{\otimes} \check{\tau}, w_2)$. Here we already calculated the various infinite products $\hat{\otimes}'_{p \notin S(B)} r(s_{k-i+1}, \widetilde{\pi}_{k-i+1,p}, w_{n_{k-i+1}})$ according to the rule of proposition 6.1. By the concrete form of $r_1(\underline{s}, \widetilde{\pi})$, given by (12), lemma 6.5 now gives that the poles of $r_1(\underline{s}, \widetilde{\pi})$ are the ones of $L(s_1 - s_2, JL(\theta) \times JL(\check{\tau}))$, where $JL(\sigma)$ denotes the global Jacquet-Langlands-lift of the cuspidal automorphic representation σ of $GL_1(B)$ to a cuspidal automorphic representation $JL(\sigma)$ of GL_2, see [GJ79], Thm. (8.3). Therefore, again by the above lemma, $r_1(\underline{s}, \widetilde{\pi})$ has simple poles in the region $\Re e(s_1) > \Re e(s_2) > 0$ if and only if $s_1 - s_2 = 1$ and $\theta = \tau$. Analogously, the poles of $r_2(\underline{s}, \widetilde{\pi})$ are by its concrete form given in (10) the ones of $L(s_2, \tau)L(2s_2, \chi_\tau)$. We apply lemma 6.5 and see that $r_2(\underline{s}, \widetilde{\pi})$ has simple poles in the region $\Re e(s_1) > \Re e(s_2) > 0$ if and only if

$2s_2 = 1$, $\chi_\tau = \mathbf{1}$ and $L(\frac{1}{2},\tau) \neq 0$. An analog and easy observation shows that $r_3(\underline{s},\widetilde{\pi})$ has simple poles along $2s_1 = 1$ if $\chi_\theta = \mathbf{1}$ and $L(\frac{1}{2},\theta) \neq 0$ and that the poles of $r_j(\underline{s},\widetilde{\pi})$, $j = 4,5$ lie along $s_1 + s_2 = 1$ for $\theta \cong \check{\tau}$. Only the singular hyperplanes of $r_1(\underline{s},\widetilde{\pi})$ and $r_2(\underline{s},\widetilde{\pi})$ intersect in the region $\Re e(s_1) > \Re e(s_2) > 0$ and they intersect in $\underline{s} = (\frac{3}{2},\frac{1}{2})$. □

Remark 6.7. Only the longest element $w_0 = w_2w_4w_2w_4$ in $W_\mathbb{Q}$ gives rise to a normalizing factor $r(\underline{s},\widetilde{\pi},w)$ which carries $r_1(\underline{s},\widetilde{\pi})$ and $r_2(\underline{s},\widetilde{\pi})$. The remaining other two factors $r_3(\underline{s},\widetilde{\pi})$ and $r_4(\underline{s},\widetilde{\pi})$ showing up in the decomposition of $r(\underline{s},\widetilde{\pi},w_0)$ have no zero at $A = (\frac{3}{2},\frac{1}{2})$. So for any cuspidal automorphic representation $\widetilde{\pi}$ of $L(\mathbb{A})$, satisfying $\widetilde{\pi} = \tau\hat{\otimes}\tau$, $\chi_\tau = \mathbf{1}$ and $L(\frac{1}{2},\tau) \neq 0$, the necessary condition given above is also sufficient to ensure that there will be an $f \in W_{P,\widetilde{\pi}}$ such that the Eisenstein series $E_P(f,\Lambda)$ has a double pole at A.
In fact, by the same argument the points C_2 and C_3 will really give rise to double poles of Eisenstein series attached to cuspidal automorphic representations $\widetilde{\pi}$ of the Levi L that satisfy the indicated condition. So also for these points the given conditions on $\widetilde{\pi}$ will be sufficient for an appropriate choice of $f \in W_{P,\widetilde{\pi}}$.

Proposition 6.8. *For an Eisenstein series $E_P(f,\Lambda)$ to have a double pole at $\underline{s} = (s_1,s_2)$ on the boundary of the closed, positive Weyl chamber, i.e., either $\Re e(s_1) - \Re e(s_2) = 0$ or $\Re e(s_1) = 0$, it is necessary that*

(A) $\dim\theta > 1$ *and* $\dim\tau > 1$*:*
$\underline{s} = (\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0)$ *or* $(1,0)$
(B) $\dim\theta = 1$, $\dim\tau > 1$*:*
$\underline{s} = (\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, $(\frac{1}{2},0)$ *or* $(\frac{3}{2},0)$
(B') $\dim\theta > 1$, $\dim\tau = 1$*:*
$\underline{s} = (\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, *or* $(\frac{1}{2},0)$
(C) $\dim\theta = \dim\tau = 1$*:*
$\underline{s} = (\frac{1}{2},\frac{1}{2})$, $(1,1)$, $(\frac{3}{2},\frac{3}{2})$, $(\frac{1}{2},0)$, $(\frac{3}{2},0)$ *or* $(2,0)$

Sketch of a proof: Again this is easy, but not very instructive, so we will again confine ourselves to case (A). We may also assume that $\underline{s} \in \mathbb{R}^2$. We cannot decide by our means chosen here if the root hyperplanes $R_1 := \{\underline{s} \in \mathbb{R}^2 | s_1 - s_2 = 0\}$ and $R_2 := \{\underline{s} \in \mathbb{R}^2 | s_1 = 0\}$ forming the boundary of the closed, positive Weyl chamber are actually singular hyperplanes for Eisenstein series attached to cuspidal automorphic representations $\widetilde{\pi}$ of $L(\mathbb{A})$. But in order to have a double pole at a point $\underline{s}$ on this boundary, we need to have one of our singular root hyperplanes, given by the five factors $r_i(\underline{s},\widetilde{\pi})$, $1 \le i \le 5$, to cross R_1 or R_2 in $\underline{s}$. From the proof above we know that these singular hyperplanes are $s_1 - s_2 = 1$, $2s_2 = 1$, $2s_1 = 1$ and $s_1 + s_2 = 1$. Their intersection points with one of the boundary hyperplanes R_1 and R_2 are precisely the points we claim to be the only candidates for double poles of Eisenstein series on the boundary of the positive Weyl chamber in case (A). □

6.4. Square-integrable Eisenstein cohomology. We will now determine the square-integrable Eisenstein cohomology supported by P.
Therefore, let

$$L_{E,P,\varphi} \subseteq \mathcal{A}_{E,P,\varphi}$$

be the subspace of $\mathcal{A}_{E,P,\varphi}$ which consists of square-integrable automorphic forms. By [Lan76] or [MW95] it is spanned by all two-times iterated, square-integrable residues of Eisenstein series $E_P(f,\Lambda)$, $f \in W_{P,\widetilde{\pi}}$, $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$ at the point $d\chi$ inside the closed, positive Weyl chamber defined by $\Delta(P,A)$. Hence, it is spanned by the square-integrable residues at $d\chi$ of those Eisenstein series attached to a cuspidal automorphic representation $\widetilde{\pi}$ of $L(\mathbb{A})$ which have a double pole there.

This is, since simple poles integrate to zero. Put

$$L_{E,P} := \bigoplus_{\varphi \in \Psi_{E,P}} L_{E,P,\varphi}.$$

We define the space of *square-integrable Eisenstein cohomology* (supported by P) by

$$H^q(\mathfrak{g}, K, L_{E,P} \otimes E) = \bigoplus_{\varphi \in \Psi_{E,P}} H^q(\mathfrak{g}, K, L_{E,P,\varphi} \otimes E).$$

Combining our previous results, propositions 6.6 and 6.8, with Langlands' "Square Integrability Criterion" (cf., [MW95], Lemma I.4.11.) we conclude:

Theorem 6.9. *Let $P = LN$ be the minimal standard parabolic $\mathbb{Q}$-subgroup of $G = Sp(2,2)$ and E any irreducible, finite-dimensional complex-rational representation of $G(\mathbb{R})$. Then the square-integrable Eisenstein cohomology supported by P, $H^*(\mathfrak{g}, K, L_{E,P} \otimes E)$, is spanned by cohomology classes which are Eisenstein lifts of a class of type (π, w), $\pi = \chi\widetilde{\pi} \in \varphi_P \in \varphi \in \Psi_{E,P}$, $w \in W^P$, such that necessarily one of the following conditions holds:*
If $d\chi$ is inside the open, positive Weyl chamber defined by $\Delta(P, A)$:

- (A) *If $\dim\theta > 1$ and $\dim\tau > 1$:*
  $\widetilde{\pi} = \tau\hat{\otimes}\tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
- (B) *If $\dim\theta = 1$, $\dim\tau > 1$:*
  $\widetilde{\pi} = \mathbf{1} \otimes \tau$, $\chi_\tau = \mathbf{1}$ *and* $L(\frac{1}{2}, \tau) \neq 0$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
- (C) *If $\dim\theta = \dim\tau = 1$:*
  1) $\widetilde{\pi} = \mathbf{1}\hat{\otimes}\tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{3}{2}, \frac{1}{2})$.
  2) $\widetilde{\pi} = \tau\hat{\otimes}\tau$, $\tau \neq \mathbf{1}$, $\tau^2 = \mathbf{1}$, $\tau_p \neq \mathbf{1}_p$ $\forall p \in S(B)$ *and* $d\chi = (\frac{5}{2}, \frac{1}{2})$.
  3) $\widetilde{\pi} = \mathbf{1} \otimes \mathbf{1}$ *and* $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_P$.

If $d\chi$ is on the boundary of the closed, positive Weyl chamber defined by $\Delta(P, A)$:

- (A) *If $\dim\theta > 1$ and $\dim\tau > 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ *or* $(1, 0)$
- (B) *If $\dim\theta = 1$, $\dim\tau > 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$ *or* $(\frac{3}{2}, 0)$
- (B') *If $\dim\theta > 1$, $\dim\tau = 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$, *or* $(\frac{1}{2}, 0)$
- (C) *If $\dim\theta = \dim\tau = 1$:*
  $d\chi = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ *or* $(2, 0)$

Remark. By the Square Integrability Criterion, i.e., [MW95], Lemma I.4.11, (non-zero) iterated residues of Eisenstein series at $C_1 = (\frac{3}{2}, \frac{1}{2})$ cannot be square integrable, if they come from a representation $\widetilde{\pi} = \check{\tau}\hat{\otimes}\tau$ of $L(\mathbb{A})$. Hence, we excluded them from the list in theorem 6.9.

We also have the following vanishing theorem:

Theorem 6.10. *If $E \neq \mathbb{C}$, then square-integrable Eisenstein cohomology supported by P vanishes below degree* 3

$$H^q(\mathfrak{g}, K, L_{E,P} \otimes E) = 0 \quad \textit{for } q \leq 3.$$

If $E = \mathbb{C}$, then there is an epimorphism

$$H^0(\mathfrak{g}, K, L_{\mathbb{C},P}) \twoheadrightarrow H^0(G, \mathbb{C}) = \mathbb{C}$$

and

$$H^q(\mathfrak{g}, K, L_{\mathbb{C},P}) = 0 \quad \textit{for } 1 \leq q \leq 3.$$

Proof. For any E, $L_{E,P}$ is a direct summand of the residual spectrum of $G(\mathbb{A})$, so $L_{E,P}$ is the direct Hilbert sum of certain residual automorphic representations. So in order to give a non-trivial cohomological contribution in the degrees $0 \leq q \leq 3$, it is necessary that there is an irreducible, unitary representation $\pi = \hat{\otimes}'_p \pi_p$ of $G(\mathbb{A})$ with π_∞ cohomological with respect to E. However, if $\pi_\infty \neq \mathbb{C}$ then $H^q(\mathfrak{g}, K, \pi_\infty \otimes E) = 0$ for $q \leq 3$, as it follows from Thm. 8.1 of [VZ84]. Conversely, we can only have $\pi_\infty = \mathbb{C}$ if $E = \mathbb{C}$ itself, and then we know that

$$H^q(\mathfrak{g}, K, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } q = 0, 4, 12, 16 \\ \mathbb{C}^2 & \text{if } q = 8 \end{cases} \tag{15}$$

and vanishes in all other degrees. This happens, since $H^q(\mathfrak{g}, K, \mathbb{C})$ equals the de Rham cohomology of the quaternionic Grassmannian $G_2(\mathbb{H}^4)$ of 2-dimensional $\mathbb{H}$-subspaces in $\mathbb{H}^4$. Further, identifying $H^0(G, \mathbb{C})$ with the de Rham cohomology of $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ proves $H^0(G, \mathbb{C}) = \mathbb{C}$. Observe that we have now shown every assertion except that there is a surjection $H^0(\mathfrak{g}, K, L_{\mathbb{C},P}) \twoheadrightarrow H^0(G, \mathbb{C})$. This is certainly well-known and follows from general theory, but for the convenience of the reader we give a direct argument here.
Therefore, recall that for $E = \mathbb{C}$ the longest element in W^P will give the evaluation point $d\chi = (\frac{7}{2}, \frac{3}{2}) = \rho_P$ and $\widetilde{\pi}_\infty = \mathbf{1}_\infty \otimes \mathbf{1}_\infty$ and consider the image of the local normalized intertwining operator $N((\frac{7}{2}, \frac{3}{2}), \mathbf{1}_\infty \otimes \mathbf{1}_\infty, w_0)$. As $\infty \in S(B)$, $\mathbf{1}_\infty \otimes \mathbf{1}_\infty$ is compactly supported modulo the center, whence tempered, and so the image of the local normalized operator is the Langlands quotient of the local trivial representation. As $C_3 = (\frac{7}{2}, \frac{3}{2}) = \rho_P$, this quotient is the local trivial representation of $G(\mathbb{R})$. A two-times iterated residue at $d\chi = \rho_P$ of a singular Eisenstein series will be square-integrable, by [MW95], Lemma I.4.11. Therefore, there is a global residual automorphic representation $\pi \subset L_{\mathbb{C},P}$, namely the image of the global normalized intertwining operator $N((\frac{7}{2}, \frac{3}{2}), \mathbf{1} \otimes \mathbf{1}, w_0)$, such that $\pi_\infty = \mathbb{C}$. (In fact, by the above local argument, one can also easily see that the image of the global operator is the global trivial representation $\mathbf{1}$ of $G(\mathbb{A})$.) By (15) we are done. □

Remark. Of course we could have gained the result in degrees $q = 0, 1$ also by the following: In degree $q = 0$ we could have used the equality $H^0(G, E) = \varinjlim_{K_\Gamma} E^\Gamma$ and A. Borel's "Density Theorem" (cf. e.g., [PR93]) or in degree $q = 1$ we could have referred to the well-known vanishing results of G. A. Margulis and M. S. Raghunathan, which give $H^1(G, E) = 0$ (see [Mar91, Rag67]). Especially for $E = \mathbb{C}$ we could also have used [Bor74], which shows that $H^q(\mathfrak{g}, K, \mathbb{C}) \to H^q(G, \mathbb{C})$ is an isomorphism in low degrees together with our computations of line (15).

Remark. As $H^4(\mathfrak{g}, K, \mathbb{C}) = \mathbb{C}$, $q = 3$ is in fact a sharp upper bound for the vanishing of $(\mathfrak{g}, K)$-cohomology of $G = Sp(2,2)$ in low degrees. But there is also another residual representation which has $(\mathfrak{g}, K)$-cohomology in degree 4. This is a consequence of our theorem 2.1: In fact, if we consider $q = 4$, then for all λ the element $w = w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1$ of length 10 is in $W^+(\lambda)$ as indicated in our table 8. We get $\widetilde{\pi}_\infty = \bigodot^{4+c_1-c_3+2c_4} \mathbb{C}^2 \otimes \bigodot^{4-c_1+c_3} \mathbb{C}^2$, whence $\dim \theta \geq 5$ and $\dim \tau \geq 5$ and we are in case (A). The corresponding evaluation point $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}}$ reads as $\underline{s} = (\frac{3+c_1+c_3-2c_4}{2}, \frac{1+2c_2-c_1-c_3}{2})$, satisfies $s_1 \geq \frac{3}{2}$, $s_2 \geq \frac{1}{2}$ and is always inside the open, positive Weyl chamber. By proposition 6.6 and remark 6.7 there will be really an Eisenstein series $E_P(f, \Lambda)$ which has a double pole at $A = (\frac{3}{2}, \frac{1}{2})$ for all cuspidal automorphic representations $\widetilde{\pi} = \theta \hat{\otimes} \tau$ of $L(\mathbb{A})$ subject to the condition $\theta = \tau$, $\chi_\tau = \mathbf{1}$ and $L(\frac{1}{2}, \tau) \neq 0$. According to [MW95], lemma I.4.11, the space of two-times iterated residues at $A = (\frac{3}{2}, \frac{1}{2})$ of such Eisenstein series consists of square-integrable automorphic forms. Playing around with the concrete form of $\widetilde{\pi}_\infty$ and $\underline{s}$ given above yields $\lambda = k\omega_4$, $k = 0, 1, 2, 3...$ and ω_4 the fundamental weight of $\mathfrak{g}_\mathbb{C}$ which corresponds to the fourth simple root α_4. By theorem 2.1 the two-times iterated residue of $E_P(f, \Lambda)$ will therefore contribute to square-integrable Eisenstein cohomology with respect to $E = E_{k\omega_4}$ in degree $\dim N(\mathbb{R}) - l(w) = 14 - 10 = 4$.

Appendix: Tables for the three standard parabolic $\mathbb{Q}$-subgroups

As before, E denotes a finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R}) = Sp(2,2)$ with highest weight $\lambda = \sum_{i=1}^4 c_i\alpha_i$. As λ is algebraically integral and dominant, we can easily see that we get the following relations among the coefficients:

$$c_4 \geq \frac{c_3}{2} \geq \frac{c_2}{2} \geq \frac{c_3}{3} \geq 0 \quad \text{and} \quad c_1 \geq \frac{c_2}{2}. \tag{16}$$

Let ω_{01} and ω_{02} be the two fundamental weights of $M_0(\mathbb{C})$. Analogously, ω_{ij}, $j = 1, 2, 3$, denotes the j-th fundamental weight of $M_i(\mathbb{C})$, $i = 1, 2$.

The first two tables give the values $w(\lambda + \rho) - \rho|_{\mathfrak{b}_{i_\mathbb{C}}}$, $w \in W^{P_i}$, $i = 1, 2$ in terms of the fundamental weights ω_{ij}. Recall that $\rho = 4\alpha_1 + 7\alpha_2 + 9\alpha_3 + 5\alpha_4$. Table 2 additionally says for which $w \in W^{P_2}$ the principal series representation $B_2(\mu_w)$ exists. If some condition is added, then it is necessary and sufficient for the existence of $B_2(\mu_w)$: "$*$" means "$c_3 - c_4 = c_1$" and "$**$" means "$c_1 - c_4 = 1$", while "$\nexists$" indicates that for these w the representation $B_2(\mu_w)$ never exists. The next two tables give the values of the inner product of the point $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_{i_\mathbb{C}}}$ of evaluation of Eisenstein series and the only simple root within $\Delta(P_i, A_i)$. Using (16), we can therefrom read off which points $d\chi$ lie inside the closed, positive Weyl chamber defined by the above system.

The last four tables give the previous data for the standard minimal parabolic $\mathbb{Q}$-subgroup P_0. For lack of space we divided the set of Kostant representatives W^{P_0} in a "lower" and in an "upper" half, according to the length of the elements $w \in W^{P_0}$. In table 8 the Kostant representatives w which *can* give rise to values $\Lambda_w = -w(\lambda + \rho)|_{\mathfrak{a}_{0_\mathbb{C}}} \in \overline{C}$ are underlined. No point Λ_w in table 7 will lie inside $\overline{C}$.

All lists were compiled by a computer program, developed by Jakub Orbán.

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TABLE 1

	$\boldsymbol{w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{1_{\mathbb{C}}}}}$
$\boldsymbol{id}$	$(2c_1-c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2}$	$(1+c_1+c_2-c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_1}$	$(-c_1+2c_2-c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3}$	$(2+c_1+c_3-2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_1}$	$(1-c_1+c_2+c_3-2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_4}$	$(4+c_1-c_3+2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{W_2W_3W_4W_1}$	$(3-c_1+c_2-c_3+2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_1w_2}$	$(-c_2+2c_3-2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3}$	$(5+c_1-c_2+c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3w_1}$	$(4-c_1+c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2}$	$(2-c_2+2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3w_2}$	$(6+c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3w_2w_1}$	$(6+c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3w_1w_2}$	$(4-c_1+c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3}$	$(2-c_2+2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_3w_1w_2w_1}$	$(5+c_1-c_2+c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_2}$	$(3-c_1+c_2-c_3+2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4}$	$(-c_2+2c_3-2c_4)\omega_{11}+(2c_1-c_2)\omega_{12}+(2-c_1+c_2)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_2w_1}$	$(4+c_1-c_3+2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2}$	$(1-c_1+c_2+c_3-2c_4)\omega_{11}+(1+c_1+c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_1}$	$(2+c_1+c_3-2c_4)\omega_{11}+(-c_1+2c_2-c_3)\omega_{12}+(1-c_2+c_3)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3}$	$(-c_1+2c_2-c_3)\omega_{11}+(2+c_1+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1}$	$(1+c_1+c_2-c_3)\omega_{11}+(1-c_1+c_2+c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2}$	$(2c_1-c_2)\omega_{11}+(-c_2+2c_3-2c_4)\omega_{12}+(-c_3+2c_4)\omega_{13}$

TABLE 2

	$\boldsymbol{w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{2_{\mathbb{C}}}}}$	
$\boldsymbol{id}$	$(2c_1-c_2)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(-c_2+2c_3-2c_4)\omega_{23}$	
$\boldsymbol{w_4}$	$(2c_1-c_2)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2-c_2+2c_4)\omega_{23}$	
$\boldsymbol{w_4w_3}$	$(2c_1-c_2)\omega_{21}+(1-c_1+c_2+c_3-2c_4)\omega_{22}+(2-c_2+2c_4)\omega_{23}$	
$\boldsymbol{w_4w_3w_2}$	$(1+c_1+c_2-c_3)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(3-c_1+c_2-c_3+2c_4)\omega_{23}$	
$\boldsymbol{w_4w_3w_4}$	$(2c_1-c_2)\omega_{21}+(3-c_1+c_2-c_3+2c_4)\omega_{22}+(-c_2+2c_3-2c_4)\omega_{23}$	
$\boldsymbol{w_4w_3w_4w_2}$	$(1+c_1+c_2-c_3)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(1-c_1+c_2+c_3-2c_4)\omega_{23}$	
$\boldsymbol{w_4w_3w_2w_1}$	$(-c_1+2c_2-c_3)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(4+c_1-c_3+2c_4)\omega_{23}$	$\nexists$
$\boldsymbol{w_4w_3w_4w_2w_1}$	$(-c_1+2c_2-c_3)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(2+c_1+c_3-2c_4)\omega_{23}$	$\nexists$
$\boldsymbol{w_4w_3w_4w_2w_3}$	$(2+c_1+c_3-2c_4)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(-c_1+2c_2-c_3)\omega_{23}$	$\nexists$
$\boldsymbol{w_4w_3w_4w_2w_3w_4}$	$(4+c_1-c_3+2c_4)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(-c_1+2c_2-c_3)\omega_{23}$	$\nexists$
$\boldsymbol{w_4w_3w_4w_2w_3w_1}$	$(1-c_1+c_2+c_3-2c_4)\omega_{21}+(2-c_2+2c_4)\omega_{22}+(1+c_1+c_2-c_3)\omega_{23}$	*
$\boldsymbol{w_4w_3w_4w_2w_3w_4w_1}$	$(3-c_1+c_2-c_3+2c_4)\omega_{21}+(-c_2+2c_3-2c_4)\omega_{22}+(1+c_1+c_2-c_3)\omega_{23}$	**
$\boldsymbol{w_4w_3w_4w_2w_3w_1w_2}$	$(-c_2+2c_3-2c_4)\omega_{21}+(3-c_1+c_2-c_3+2c_4)\omega_{22}+(2c_1-c_2)\omega_{23}$	*
$\boldsymbol{w_4w_3w_4w_2w_3w_4w_1w_2}$	$(2-c_2+2c_4)\omega_{21}+(1-c_1+c_2+c_3-2c_4)\omega_{22}+(2c_1-c_2)\omega_{23}$	**
$\boldsymbol{w_4w_3w_4w_2w_3w_4w_1w_2w_3}$	$(2-c_2+2c_4)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2c_1-c_2)\omega_{23}$	**
$\boldsymbol{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4}$	$(-c_2+2c_3-2c_4)\omega_{21}+(-c_1+2c_2-c_3)\omega_{22}+(2c_1-c_2)\omega_{23}$	*

TABLE 3

	$\boldsymbol{\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{1_{\mathbb{C}}}}, \alpha_2\vert_{\mathfrak{a}_{1_{\mathbb{C}}}}\rangle}$
$\boldsymbol{id}$	$2(-7-c_2)\leq -14$
$\boldsymbol{w_2}$	$2(-6-c_1+c_2-c_3)\leq -12$
$\boldsymbol{w_2w_1}$	$2(-5+c_1-c_3)\leq -10$
$\boldsymbol{w_2w_3}$	$2(-5-c_1+c_3-2c_4)\leq -10$
$\boldsymbol{w_2w_3w_1}$	$2(-4+c_1-c_2+c_3-2c_4)\leq -8$
$\boldsymbol{w_2w_3w_4}$	$2(-3-c_1-c_3+2c_4)\leq -6$
$\boldsymbol{w_2w_3w_1w_2}$	$2(-3+c_2-2c_4)\leq -6$
$\boldsymbol{w_2w_3w_4w_1}$	$2(-2+c_1-c_2-c_3+2c_4)\leq -4$
$\boldsymbol{w_2w_3w_4w_3}$	$2(-2-c_1-c_2+c_3)\leq -4$
$\boldsymbol{w_2w_3w_4w_3w_1}$	$2(-1+c_1-2c_2+c_3)\leq -2$
$\boldsymbol{w_2w_3w_4w_1w_2}$	$2(-1+c_2-2c_3+2c_4)\leq -2$
$\boldsymbol{w_2w_3w_4w_3w_2}$	$2(-1-2c_1+c_2)\leq -2$
$\boldsymbol{w_2w_3w_4w_3w_2w_1}$	$2(1+2c_1-c_2)\geq 2$
$\boldsymbol{w_2w_3w_4w_3w_1w_2}$	$2(1-c_1+2c_2-c_3)\geq 2$
$\boldsymbol{w_2w_3w_4w_1w_2w_3}$	$2(1-c_2+2c_3-2c_4)\geq 2$
$\boldsymbol{w_2w_3w_4w_3w_1w_2w_1}$	$2(2+c_1+c_2-c_3)\geq 4$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_2}$	$2(2-c_1+c_2+c_3-2c_4)\geq 4$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4}$	$2(3-c_2+2c_4)\geq 6$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_2w_1}$	$2(3+c_1+c_3-2c_4)\geq 6$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2}$	$2(4-c_1+c_2-c_3+2c_4)\geq 8$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_1}$	$2(5+c_1-c_3+2c_4)\geq 10$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3}$	$2(5-c_1+c_3)\geq 10$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1}$	$2(6+c_1-c_2+c_3)\geq 12$
$\boldsymbol{w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2}$	$2(7+c_2)\geq 14$

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TABLE 4

	$\langle -w(\lambda+\rho)\|_{\mathfrak{a}_{2_{\mathbb{C}}}}, \alpha_4\|_{\mathfrak{a}_{2_{\mathbb{C}}}}\rangle$
id	$4(-5-c_4) \leq 20$
w_4	$4(-4-c_3+c_4) \leq -16$
w_4w_3	$4(-3-c_2+c_3-c_4) \leq -12$
$w_4w_3w_2$	$4(-2-c_1+c_2-c_4) \leq -8$
$w_4w_3w_4$	$4(-2-c_2+c_4) \leq -8$
$w_4w_3w_4w_2$	$4(-1-c_1+c_2-c_3+c_4) \leq -4$
$w_4w_3w_2w_1$	$4(-1+c_1-c_4)$?
$w_4w_3w_4w_2w_1$	$4(c_1-c_3+c_4)$?
$w_4w_3w_4w_2w_3$	$4(-c_1+c_3-c_4)$?
$w_4w_3w_4w_2w_3w_4$	$4(1-c_1+c_4)$?
$w_4w_3w_4w_2w_3w_1$	$4(1+c_1-c_2+c_3-c_4) \geq 4$
$w_4w_3w_4w_2w_3w_4w_1$	$4(2+c_1-c_2+c_4) \geq 8$
$w_4w_3w_4w_2w_3w_1w_2$	$4(2+c_2-c_4) \geq 8$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$4(3+c_2-c_3+c_4) \geq 12$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$4(4+c_3-c_4) \geq 16$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$4(5+c_4) \geq 20$

TABLE 5. ${}^\circ F_w$ for lower-half representatives

	$w(\lambda+\rho)-\rho\|_{\mathfrak{b}_{0_{\mathbb{C}}}}$
id	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
w_2	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
w_4	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
w_2w_1	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
w_4w_2	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
w_2w_3	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
w_4w_3	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_3w_2$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_3w_4$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_3w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_2$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_2w_3w_2w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_1$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_1w_2w_1$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_1$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_2w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_1$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_1w_2$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$

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TABLE 6. $^{\circ}F_w$ for upper half representatives

	$w(\lambda+\rho)-\rho\vert_{\mathfrak{b}_{0_{\mathbb{C}}}}$
$w_4w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_2$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_2w_1$	$(6+c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1$	$(2-c_2+2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_2$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_2w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_2w_1$	$(6+c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1$	$(-c_2+2c_3-2c_4)\omega_{01}+(6+c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2$	$(4-c_1+c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$(2-c_2+2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2w_1$	$(5+c_1-c_2+c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(5+c_1-c_2+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2$	$(3-c_1+c_2-c_3+2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$(-c_2+2c_3-2c_4)\omega_{01}+(2c_1-c_2)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(4-c_1+c_3)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$(4+c_1-c_3+2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$(1-c_1+c_2+c_3-2c_4)\omega_{01}+(1+c_1+c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(4+c_1-c_3+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$(2+c_1+c_3-2c_4)\omega_{01}+(-c_1+2c_2-c_3)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(3-c_1+c_2-c_3+2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$(-c_1+2c_2-c_3)\omega_{01}+(2+c_1+c_3-2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$(1+c_1+c_2-c_3)\omega_{01}+(1-c_1+c_2+c_3-2c_4)\omega_{02}$
$w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(2-c_2+2c_4)\omega_{02}$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$(2c_1-c_2)\omega_{01}+(-c_2+2c_3-2c_4)\omega_{02}$

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TABLE 7. Λ_w for lower-half representatives

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_\mathbb{C}}}, \alpha_2\vert_{\mathfrak{a}_{0_\mathbb{C}}}\rangle$	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_\mathbb{C}}}, \alpha_4\vert_{\mathfrak{a}_{0_\mathbb{C}}}\rangle$
id	$2(-2-c_2+c_4) \leq -4$	$2(-3+c_2-2c_4) \leq -6$
w_2	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(-4+c_1-c_2+c_3-2c_4) \leq -8$
w_4	$2(-3-c_2+c_3-c_4) \leq -6$	$2(-1+c_2-2c_3+2c_4) \leq -2$
w_2w_1	$2(c_1-c_3+c_4)$?	$2(-5-c_1+c_3-2c_4) \leq -10$
w_4w_2	$2(-2-c_1+c_2-c_4) \leq -4$	$2(-2+c_1-c_2-c_3+2c_4) \leq -4$
w_2w_3	$2(-c_1+c_3-c_4)$?	$2(-5+c_1-c_3) \leq -10$
w_4w_3	$2(-4-c_3+c_4) \leq -8$	$2(1-c_2+2c_3-2c_4) \geq 2$
$w_4w_2w_1$	$2(-1+c_1-c_4)$?	$2(-3-c_1-c_3+2c_4) \leq -6$
$w_2w_3w_1$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(-6-c_1+c_2-c_3) \leq -12$
$w_4w_3w_2$	$2(-4-c_3+c_4) \leq -8$	$2(2-c_1+c_2+c_3-2c_4) \geq 4$
$w_4w_2w_3$	$2(-2-c_1+c_2-c_4) \leq -4$	$2(-1+c_1-2c_2+c_3) \leq -2$
$w_2w_3w_4$	$2(1-c_1+c_4)$?	$2(-5+c_1-c_3) \leq -10$
$w_4w_3w_4$	$2(-5-c_4) \leq 10$	$2(3-c_2+2c_4) \geq 6$
$w_4w_3w_2w_1$	$2(-4-c_3+c_4) \leq -8$	$2(3+c_1+c_3-2c_4) \geq 6$
$w_4w_2w_3w_1$	$2(-1+c_1-c_4)$?	$2(-2-c_1-c_2+c_3) \leq -4$
$w_2w_3w_4w_1$	$2(2+c_1-c_2+c_4) \geq 4$	$2(-6-c_1+c_2-c_3) \leq -12$
$w_2w_3w_1w_2$	$2(2+c_2-c_4) \geq 4$	$2(-7-c_2) \leq -14$
$w_4w_2w_3w_2$	$2(-3-c_2+c_3-c_4) \leq -6$	$2(1-c_1+2c_2-c_3) \geq 2$
$w_4w_3w_4w_2$	$2(-5-c_4) \leq -10$	$2(4-c_1+c_2-c_3+2c_4) \geq 8$
$w_2w_3w_4w_3$	$2(1-c_1+c_4)$?	$2(-4+c_1-c_2+c_3-2c_4) \leq -8$
$w_4w_2w_3w_4$	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(-1+c_1-2c_2+c_3) \leq -2$
$w_4w_2w_3w_2w_1$	$2(-3-c_2+c_3-c_4) \leq -6$	$2(2+c_1+c_2-c_3) \geq 4$
$w_4w_3w_4w_2w_1$	$2(-5-c_4) \leq -10$	$2(5+c_1-c_3+2c_4) \geq 10$
$w_2w_3w_4w_3w_1$	$2(2+c_1-c_2+c_4) \geq 4$	$2(-5-c_1+c_3-2c_4) \leq -10$
$w_4w_2w_3w_4w_1$	$2(c_1-c_3+c_4)$?	$2(-2-c_1-c_2+c_3) \leq -4$
$w_4w_2w_3w_1w_2$	$2(-1+c_1-c_4)$?	$2(-1-2c_1+c_2) \leq -2$
$w_2w_3w_4w_1w_2$	$2(3+c_2-c_3+c_4) \geq 6$	$2(-7-c_2) \leq -14$
$w_2w_3w_4w_3w_2$	$2(1-c_1+c_4)$?	$2(-3+c_2-2c_4) \leq -6$
$w_4w_2w_3w_4w_2$	$2(-2-c_2+c_4) \leq -4$	$2(1-c_1+2c_2-c_3) \geq 2$
$w_4w_3w_4w_2w_3$	$2(-5-c_4) \leq -10$	$2(5-c_1+c_3) \geq 10$
$w_4w_2w_3w_4w_3$	$2(-c_1+c_3-c_4)$?	$2(-2+c_1-c_2-c_3+2c_4) \leq -4$
$w_4w_2w_3w_1w_2w_1$	$2(-2-c_1+c_2-c_4) \leq -4$	$2(1+2c_1-c_2) \geq 2$
$w_2w_3w_4w_3w_2w_1$	$2(2+c_1-c_2+c_4) \geq 4$	$2(-3+c_2-2c_4) \leq -6$
$w_4w_2w_3w_4w_2w_1$	$2(-2-c_2+c_4) \leq -4$	$2(2+c_1+c_2-c_3) \geq 4$
$w_4w_3w_4w_2w_3w_1$	$2(-5-c_4) \leq -10$	$2(6+c_1-c_2+c_3) \geq 12$
$w_4w_2w_3w_4w_3w_1$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(-3-c_1-c_3+2c_4) \leq -6$
$w_2w_3w_4w_3w_1w_2$	$2(3+c_2-c_3+c_4) \geq 6$	$2(-5-c_1+c_3-2c_4) \leq -10$
$w_4w_2w_3w_4w_1w_2$	$2(c_1-c_3+c_4)$?	$2(-1-2c_1+c_2) \leq -2$
$w_4w_2w_3w_4w_3w_2$	$2(-c_1+c_3-c_4)$?	$2(-1+c_2-2c_3+2c_4) \leq -2$
$w_2w_3w_4w_1w_2w_3$	$2(4+c_3-c_4) \geq 8$	$2(-7-c_2) \leq -14$
$w_4w_2w_3w_4w_2w_3$	$2(-2-c_2+c_4) \leq -4$	$2(2-c_1+c_2+c_3-2c_4) \geq 4$
$w_4w_3w_4w_2w_3w_4$	$2(-4-c_3+c_4) \leq -8$	$2(5-c_1+c_3) \geq 10$
$w_2w_3w_4w_3w_1w_2w_1$	$2(3+c_2-c_3+c_4) \geq 6$	$2(-4+c_1-c_2+c_3-2c_4) \leq -8$
$w_4w_2w_3w_4w_1w_2w_1$	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(1+2c_1-c_2) \geq 2$
$w_4w_2w_3w_4w_3w_2w_1$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(-1+c_2-2c_3+2c_4) \leq -2$
$w_4w_2w_3w_4w_2w_3w_1$	$2(-2-c_2+c_4) \leq -4$	$2(3+c_1+c_3-2c_4) \geq 6$
$w_4w_3w_4w_2w_3w_4w_1$	$2(-4-c_3+c_4) \leq -8$	$2(6+c_1-c_2+c_3) \geq 12$
$w_4w_3w_4w_2w_3w_1w_2$	$2(-5-c_4) \leq -10$	$2(7+c_2) \geq 14$

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TABLE 8. Λ_w for upper half representatives

	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_{\mathbb{C}}}}, \alpha_2\vert_{\mathfrak{a}_{0_{\mathbb{C}}}} \rangle$	$\langle -w(\lambda+\rho)\vert_{\mathfrak{a}_{0_{\mathbb{C}}}}, \alpha_4\vert_{\mathfrak{a}_{0_{\mathbb{C}}}} \rangle$
$w_4w_2w_3w_4w_3w_1w_2$	$2(2+c_2-c_4) \geq 4$	$2(-3-c_1-c_3+2c_4) \leq -6$
$w_2w_3w_4w_1w_2w_3w_2$	$2(4+c_3-c_4) \geq 8$	$2(-6-c_1+c_2-c_3) \leq -12$
$w_4w_2w_3w_4w_2w_3w_2$	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(1-c_2+2c_3-2c_4) \geq 2$
$w_4w_2w_3w_4w_1w_2w_3$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(-1-2c_1+c_2) \leq -2$
$w_4w_3w_4w_2w_3w_4w_3$	$2(-3-c_2+c_3-c_4) \leq -6$	$2(4-c_1+c_2-c_3+2c_4) \geq 8$
$w_2w_3w_4w_1w_2w_3w_4$	$2(5+c_4) \geq 10$	$2(-7-c_2) \leq -14$
$w_4w_2w_3w_4w_3w_1w_2w_1$	$2(2+c_2-c_4) \geq 4$	$2(-2+c_1-c_2-c_3+2c_4) \leq -4$
$w_2w_3w_4w_1w_2w_3w_2w_1$	$2(4+c_3-c_4) \geq 8$	$2(-5+c_1-c_3) \leq -10$
$\underline{w_4w_2w_3w_4w_2w_3w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(1-c_2+2c_3-2c_4) \geq 2$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1}$	$2(-c_1+c_3-c_4)$?	$2(1+2c_1-c_2) \geq 2$
$w_4w_3w_4w_2w_3w_4w_3w_1$	$2(-3-c_2+c_3-c_4) \leq -6$	$2(5+c_1-c_3+2c_4) \geq 10$
$w_4w_2w_3w_4w_2w_3w_1w_2$	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(3+c_1+c_3-2c_4) \geq 6$
$w_4w_3w_4w_2w_3w_4w_1w_2$	$2(-4-c_3+c_4) \leq -8$	$2(7+c_2) \geq 14$
$w_4w_2w_3w_4w_1w_2w_3w_2$	$2(2+c_2-c_4) \geq 4$	$2(-2-c_1-c_2+c_3) \leq -4$
$w_4w_3w_4w_2w_3w_4w_3w_2$	$2(-2-c_1+c_2-c_4) \leq -4$	$2(3-c_2+2c_4) \geq 6$
$w_2w_3w_4w_1w_2w_3w_4w_2$	$2(5+c_4) \geq 10$	$2(-6-c_1+c_2-c_3) \leq -12$
$w_4w_2w_3w_4w_1w_2w_3w_4$	$2(2+c_1-c_2+c_4) \geq 4$	$2(-1-2c_1+c_2) \leq -2$
$\underline{w_4w_2w_3w_4w_2w_3w_1w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(2-c_1+c_2+c_3-2c_4) \geq 4$
$w_4w_2w_3w_4w_1w_2w_3w_2w_1$	$2(2+c_2-c_4) \geq 4$	$2(-1+c_1-2c_2+c_3) \leq -2$
$\underline{w_4w_3w_4w_2w_3w_4w_3w_2w_1}$	$2(-1+c_1-c_4)$?	$2(3-c_2+2c_4) \geq 6$
$w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$2(5+c_4) \geq 10$	$2(-5+c_1-c_3) \leq -10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1}$	$2(1-c_1+c_4)$?	$2(1+2c_1-c_2) \geq 2$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1w_2}$	$2(-c_1+c_3-c_4)$?	$2(2+c_1+c_2-c_3) \geq 4$
$w_4w_3w_4w_2w_3w_4w_3w_1w_2$	$2(-2-c_1+c_2-c_4) \leq -4$	$2(5+c_1-c_3+2c_4) \geq 10$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$2(3+c_2-c_3+c_4) \geq 6$	$2(-2-c_1-c_2+c_3) \leq -4$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3$	$2(-3-c_2+c_3-c_4) \leq -6$	$2(7+c_2) \geq 14$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$2(5+c_4) \geq 10$	$2(-5-c_1+c_3-2c_4) \leq -10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_1w_2w_1}$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(1-c_1+2c_2-c_3) \geq 2$
$\underline{w_4w_3w_4w_2w_3w_4w_3w_1w_2w_1}$	$2(-1+c_1-c_4)$?	$2(4-c_1+c_2-c_3+2c_4) \geq 8$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1$	$2(3+c_2-c_3+c_4) \geq 6$	$2(-1+c_1-2c_2+c_3) \leq -2$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$2(5+c_4) \geq 10$	$2(-4+c_1-c_2+c_3-2c_4) \leq -8$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2}$	$2(1-c_1+c_4)$?	$2(2+c_1+c_2-c_3) \geq 4$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2$	$2(-2-c_1+c_2-c_4) \leq -4$	$2(6+c_1-c_2+c_3) \geq 12$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3$	$2(4+c_3-c_4) \geq 8$	$2(-3-c_1-c_3+2c_4) \leq -6$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4$	$2(-2-c_2+c_4) \leq -4$	$2(7+c_2) \geq 14$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_1}$	$2(2+c_1-c_2+c_4) \geq 4$	$2(1-c_1+2c_2-c_3) \geq 2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_2w_1}$	$2(-1+c_1-c_4)$?	$2(5-c_1+c_3) \geq 10$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1$	$2(4+c_3-c_4) \geq 8$	$2(-2+c_1-c_2-c_3+2c_4) \leq -4$
$w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$2(5+c_4) \geq 10$	$2(-3+c_2-2c_4) \leq -6$
$w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2$	$2(-1-c_1+c_2-c_3+c_4) \leq -2$	$2(6+c_1-c_2+c_3) \geq 12$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3}$	$2(1-c_1+c_4)$?	$2(3+c_1+c_3-2c_4) \geq 6$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_1}$	$2(c_1-c_3+c_4)$?	$2(5-c_1+c_3) \geq 10$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1}$	$2(2+c_1-c_2+c_4) \geq 4$	$2(2-c_1+c_2+c_3-2c_4) \geq 4$
$w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2$	$2(4+c_3-c_4) \geq 8$	$2(-1+c_2-2c_3+2c_4) \leq -2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3}$	$2(-c_1+c_3-c_4)$?	$2(5+c_1-c_3+2c_4) \geq 10$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1}$	$2(1+c_1-c_2+c_3-c_4) \geq 2$	$2(4-c_1+c_2-c_3+2c_4) \geq 8$
$\underline{w_4w_2w_3w_4w_1w_2w_3w_4w_1w_2w_3w_1w_2}$	$2(3+c_2-c_3+c_4) \geq 6$	$2(1-c_2+2c_3-2c_4) \geq 2$
$\underline{w_4w_3w_4w_2w_3w_4w_1w_2w_3w_4w_2w_3w_1w_2}$	$2(2+c_2-c_4) \geq 4$	$2(3-c_2+2c_4) \geq 6$

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