NON-VANSISHING OF SHALIKA NEWVECTORS AT THE IDENTITY

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Abstract. The aim of these notes is to show the non-vanishing of local, non-archimedean Shalika newvectors at the identity matrix $g = id$. This is also meant as some sort of addendum to our paper [Gro-Rag14], where this result was used in Lem. 3.8.1 without comment. It has to be pointed out that the proof presented here is due to very generous explanations of Nadir Matringe, leaving basically only editorial work to us. We are grateful for Nadir's permission to present these arguments in these notes. We are also grateful to Binyong Sun for pointing out the need for this local result and hinting us to [Jia-Sun-Tia19], where an alternative proof of our [Gro-Rag14, Lem. 3.8.1] was given.

Contents

1. The setup

1.1. **Fields and chatacters.** Throughout these notes, F will denote a local, non-archimedean field with ring of integers $\mathcal{O}$, valuation v and normalized absolute value $|\ .\ | = |\ .\ |_{\mathsf{v}}$. We fix a non-trivial, unramified additive continuous character $\psi : F \to \mathbb{C}$ and let η be a continuous character $\eta : F^* \to \mathbb{C}^*$. The trivial character is denoted $\mathbf{1}$. By M_k we shall denote the algebra of $k \times k$-matrices with entires in F and by N_k the subset of upper triangular elements in M_k. If X is any subset of M_k, we will denote $\mathbf{1}_X$ the characteristic function of X.

1.2. **Local groups.** For any $k \geq 1$, we abbreviate $G_k := \mathrm{GL}_k(F)$, the F-points of the split general linear group over F. The group G_k has some well-known subgroups:

$$\mathcal{P}_k := \left\{ p = \begin{pmatrix} g_{k-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} id_{k-1} & x \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{l} g_{k-1} \in G_k \\ x \in F^{k-1} \end{array} \right\}.$$

the so-called mirabolic subgroup, and U_k, the subgroup of upper triangular unipotent matrices. If $k = 2n$ is even, then there are also the following subgroups:

$$L_{2n} := \left\{ \ell = \begin{pmatrix} g_n & 0 \\ 0 & h_n \end{pmatrix} \middle| \ g_n, h_n \in G_n \right\},$$

the Levi subgroup of the "Siegel parabolic", and

$$\mathcal{S}_{2n} := \left\{ s = \begin{pmatrix} g_n & 0 \\ 0 & g_n \end{pmatrix} \begin{pmatrix} id_n & X \\ 0 & id_n \end{pmatrix} \middle| \begin{array}{l} g_n \in G_n \\ X \in \mathrm{M}_n \end{array} \right\},$$

called the *Shalika subgroup*.

All measures in this note are normalized to give volume 1 to the respective maximal compact subgroups, consisting of $\mathcal{O}$-points.

1.3. **Non-archimedean Shalika models.** The characters η and ψ can be extended to a character of $\mathcal{S}_{2n}$,

$$s = \begin{pmatrix} g_n & 0 \\ 0 & g_n \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto (\eta \otimes \psi)(s) := \eta(\det(g_n))\psi(Tr(X)).$$

We will also write $\eta(s) = \eta(\det(g_n))$ and $\psi(s) = \psi(Tr(X))$.

Definition 1. Let π be an irreducible admissible representation of G_{2n}. We say that π has a (η, ψ)-*Shalika model* $\mathcal{S}_\psi^\eta(\pi)$, if there is a G-submodule

$$\mathcal{S}_\psi^\eta(\pi) \subseteq \mathrm{Ind}_{\mathcal{S}_{2n}}^{G_{2n}}[\eta \otimes \psi],$$

which is isomorphic to π.

Shalika models, if they exist, are unique: This has been established by Jacquet–Rallis [Jac-Ral96] for $\eta = \mathbf{1}$ and in general by Chen–Sun in [Che-Sun19], Thm. A.

In this note we will first concentrate on the case when $\eta = \mathbf{1}$. Doing so, we will abbreviate

$$\mathcal{S}_\psi(\pi) := \mathcal{S}_\psi^{\mathbf{1}}(\pi).$$

2. Whittaker models vs. Shalika models

2.1. **Model comparision.** Let π be an irreducible admissible representation of $G_{2n} = \mathrm{GL}_{2n}(F)$, which admits a $(\mathbf{1}, \psi)$-Shalika model $\mathcal{S}_\psi(\pi)$. As for the final purpose of these notes we shall only be interested in local components of cuspidal automorphic representations, it is no harm, if we additionally assume that π is unitary[1] and that is has a (unique) Whittaker model

$$\mathcal{W}_\psi(\pi) \subseteq \mathrm{Ind}_{U_{2n}}^{G_{2n}}[\psi],$$

where, as usual, ψ is extended to U_{2n} by the rule

$$\begin{pmatrix} 1 & u_{1,2} & & & * \\ & 1 & u_{2,3} & & \\ & & 1 & \ddots & \\ & & & \ddots & u_{2n-1,2n} \\ 0 & & & & 1 \end{pmatrix} \mapsto \psi(u_{1,2} + u_{2,3} + \cdots + u_{2n-1,2n}).$$

By uniqueness of Whittaker– and Shalika–models, $\mathrm{Hom}_{G_{2n}}(\mathcal{W}_\psi(\pi), \mathcal{S}_\psi(\pi))$ is of dimension 1. We will now exhibit an explicit, non-trivial intertwining operator

$$\Theta : \mathcal{W}_\psi(\pi) \to \mathcal{S}_\psi(\pi).$$

We start off with

Proposition 2. *We have*

$$\mathrm{Hom}_{\mathcal{S}_{2n} \cap \mathcal{P}_{2n}}(\pi, \psi) = \mathrm{Hom}_{\mathcal{S}_{2n}}(\pi, \psi).$$

[1] Local components of cuspidal automorphic representations are unitary times a twist by a potentially non-unitary character. The assumption that our representation admits a $(\mathbf{1}, \psi)$-Shalika model hence implies that it is itself unitary.

Proof. By Frobenius reciprocity and uniqueness of local Shalika models, we obviously have

$$\mathbb{C} \cong \mathrm{Hom}_{\mathcal{S}_{2n}}(\pi, \psi) \subseteq \mathrm{Hom}_{\mathcal{S}_{2n} \cap \mathcal{P}_{2n}}(\pi, \psi).$$

The result then follows from a combination of results of Matringe: By [Mat14], Prop. 4.3, the latter space $\mathrm{Hom}_{\mathcal{S}_{2n} \cap \mathcal{P}_{2n}}(\pi, \psi)$ embeds into $\mathrm{Hom}_{L_{2n} \cap \mathcal{P}_{2n}}(\pi, \mathbf{1})$, whereas it follows from the proof of Cor. 4.18 in [Mat15], that the – again latter – space, $\mathrm{Hom}_{L_{2n} \cap \mathcal{P}_{2n}}(\pi, \mathbf{1})$, has dimension at most 1. This implies the assertion. □

Because of Prop. 2, if λ is a non-zero element of $\mathrm{Hom}_{\mathcal{S}_{2n} \cap \mathcal{P}_{2n}}(\mathcal{W}_\psi(\pi), \psi)$, then

$$\Theta : W \mapsto (g \mapsto \lambda(g \cdot W)),$$

is a non-zero element of $\mathrm{Hom}_{G_{2n}}(W_\psi(\pi), \mathcal{S}_\psi(\pi))$. In what follows we will determine such a λ.

To this end, consider the Weyl group representative-matrix w_{2n}, corresponding to the permutation of $\{1, \dots, 2n\}$:

$$\begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & 4 & \dots & 2n \end{pmatrix}$$

and put

$$\lambda(W) := \int_{U_n \backslash \mathcal{P}_n} \int_{\mathrm{N}_n \backslash \mathrm{M}_n} W \left(w_{2n} \begin{pmatrix} id_n & X \\ & id_n \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \right) \psi^{-1}(Tr(X))\ dX\ dp.$$

If well-defined, i.e., absolutely convergent, this integral is in $\mathrm{Hom}_{\mathcal{S}_{2n} \cap \mathcal{P}_{2n}}(\mathcal{W}_\psi(\pi), \psi)$. Moreover, λ will be non-zero as $W_\psi(\pi)|_{\mathcal{P}_{2n}}$ contains $\mathrm{Ind}_{U_{2n}}^{\mathcal{P}_{2n}}(\psi)$, cf. [Ber-Zel77]. It hence remains to prove the following

Proposition 3. *For any $W \in W_\psi(\pi)$ the integral defining $\lambda(W)$ is absolutely convergent.*

Proof. Following Matringe, let us introduce some notation: Let A_n denote the standard maximal torus of G_n and recall that the map $a \in A_n \mapsto \mu_n(a) \in A_n$,

$$\mu_n(a) := \mathrm{diag}(a_1 \cdots a_n, a_2 \cdots a_n, \dots, a_{n-1}a_n, a_n),$$

is a group isomorphism. It will be sometimes convenient to use the notation

$$\mu_n(a_1, \dots, a_n) := \mu_n(a)$$

for $a \in A_n$. Put

$$\mathfrak{J}_k(W, a) = \int_{\mathrm{N}_k \backslash \mathrm{M}_k} W \left(w_{2n} \begin{pmatrix} id_k & 0 & X & 0 \\ & id_{n-k} & 0 & 0 \\ & & id_k & 0 \\ & & & id_{n-k} \end{pmatrix} \begin{pmatrix} a & \\ & a \end{pmatrix} \right) \psi^{-1}(Tr(X)) dX.$$

This definition is inspired by [Jo19, Lemma 3.14]: In fact, as our first observation, by the Iwasawa decomposition, the convergence of $\lambda(W)$ for any W is reduced to that of

$$\int_{a \in A_{n-1}} \mathfrak{J}_n(W, a) \delta_{B_{n-1}}(a)^{-1} da,$$

where $\delta_{B_{n-1}}$ denotes as usual the modulus character of the standard Borel B_{n-1} of G_{n-1}. Now, by the proof of [Jo19, Lemma 3.14] the convergence of the latter integral is reduced to that of

$$\int_{A_{n-1}} \prod_{k=0}^{n-1} |a_k|^{-k(n-k)} W \left(w_{2n} \begin{pmatrix} \mu_{n-1}(a) & \\ & \mu_{n-1}(a) \end{pmatrix} \right) \delta_{B_{n-1}}(\mu_{n-1}(a))^{-1} da$$

Note that

$$w_{2n}\,\mathrm{diag}(\mu_{n-1}(a),1,\mu_{n-1}(a),1)w_{2n}^{-1}=\mu_{2n}(1,a_1,1,a_2,\dots,1,a_{n-1},1,1). \tag{4}$$

Hence applying the asymptotic expansion of [Mat11, Proposition 2.9] or – as pointed out by Matringe – rather its corrected version in https://arxiv.org/abs/1004.1315v2 (see also [Jo19, Proposition 3.1]), it is sufficient to check that integrals of the form converge:

$$\int_{A_{n-1}}\prod_{k=1}^{n-1}|a_k|^{-k(n-k)}\prod_{k=1}^{n-1}|a_k|^{2k(n-k)}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)\delta_{B_{n-1}}(\mu_{n-1}(a))^{-1}da.$$

Here, ω_k is the central character of an irreducible subquotient of the derivative $\pi^{(2n-2k)}$ of π (see [Ber-Zel77]), $m_k\in\mathbb{N}$ and Φ is a Schwartz function on F. However, this integral equals

$$\begin{aligned}&\textstyle\int_{A_{n-1}}\prod_{k=1}^{n-1}|a_k|^{k(n-k)}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)\delta_{B_{n-1}}(\mu_{n-1}(a))^{-1}da\\ =\ &\textstyle\int_{A_{n-1}}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)|\det(\mu_{n-1}(a))|da\\ =\ &\textstyle\prod_{k=1}^{n-1}\int_{F^\times}|a_k|^k\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)da_k.\end{aligned}$$

whence, we are reduced to the one-dimensional case of integrating characters against Schwartz functions. Now, recalling that π is unitary, one has $|a_k|^k\omega_k(a_k)=|a_k|^{r_k}u_k(a_k)$ for some $r_k>0$ and a u_k unitary character, due to [Ber84, Section 7.3]. The convergence of the last integral now follows from classical results of Tate. □

3. Non-vanishing of $\lambda(W^\circ)$

3.1. **Preparatory results.** According to our definition of Θ it remains to show that $\lambda(W^\circ)\neq 0$ for one (and hence, by [Jac-PS-Sha81, Thm. 5.1], every) non-trivial Whittaker newvector $W^\circ\in W_\psi(\pi)$. We may hence work with a choice of a Whittaker newvector: Fix W_π° defined by $W_\pi^\circ(id)=1$, cf. [Miy14, Cor. 4.4] for its existence.

We will now follow [Ana-Mat17] closely, where the analog computation is performed for local Flicker periods, but the computation of the present note is more involved. To start, we recall a form of [Mat13, Theorem 3.1]:

Theorem 5. *There is an integer r, $0\le r\le 2n-1$ and an unramified standard module π_u of G_r such that the spherical Whittaker function $W_{\pi_u}^\circ\in W_\psi(\pi_u)$, normalized by $W_{\pi_u}^\circ(id)=1$ satisfies*

$$\begin{aligned}&W_\pi^\circ(\mu_{2n}(a_1,\dots,a_{2n-1},1))\\ &=W_{\pi_u}^\circ(\mu_r(a_1,\dots,a_r))\ |\det(\mu_r(a_1,\dots,a_r))|^{(2n-r)/2}\mathbf{1}_{\mathcal{O}}(a_r)\prod_{j=r+1}^{2n-1}\mathbf{1}_{\mathcal{O}^\times}(a_j)\end{aligned}$$

.

Now, set $w_{2n+1}=\mathrm{diag}(w_{2n},1)\in G_{2n+1}$ and let

$$\mathcal{P}_m(\mathcal{O}):=\left\{p=\begin{pmatrix}g_{m-1}&0\\0&1\end{pmatrix}\begin{pmatrix}id_{m-1}&x\\0&1\end{pmatrix}\ \middle|\ \begin{matrix}g_{m-1}\in\mathrm{GL}_m(\mathcal{O})\\ x\in\mathcal{O}^{m-1}\end{matrix}\right\}.$$

The integrals $\mathfrak{J}_k$ and their "odd" cousins

$$\mathfrak{J}'_k(W,a):=\int_{\mathrm{N}_k\backslash\mathrm{M}_k}W\left(w_{2n+1}\begin{pmatrix}id_k&0&X&0&0\\&id_{n-k}&0&0&0\\&&id_k&0&0\\&&&id_{n-k}&0\\&&&&1\end{pmatrix}\begin{pmatrix}a&&\\&a&\\&&1\end{pmatrix}\right)\psi^{-1}(Tr(X))dX$$

for $a \in A_n$ will naturally appear in our computation. We record the following useful relations staisfied by them as a lemma.

Lemma 6. *Let π be a standard module of G_m and $W \in W_\psi(\pi)$ which is fixed by $\mathcal{P}_m(\mathcal{O})$. If $m = 2n$ is even, then*

$$\begin{aligned}\mathfrak{J}_n(W, \mu_n(a)) &= \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} W\left(w_{2n}\begin{pmatrix}\mu_n(a) & \\ & \mu_n(a)\end{pmatrix}\right)\\ &= \prod_{k=1}^{n} |a_k|^{-k(n-k)} W\left(w_{2n}\begin{pmatrix}\mu_n(a) & \\ & \mu_n(a)\end{pmatrix}\right).\end{aligned}$$

If $m = 2n+1$ is odd, then

$$\begin{aligned}\mathfrak{J}'_n(W, \mu_n(a)) &= \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} W\left(w_{2n+1}\begin{pmatrix}\mu_n(a) & & \\ & \mu_n(a) & \\ & & 1\end{pmatrix}\right)\\ &= \prod_{k=1}^{n} |a_k|^{-k(n-k)} W\left(w_{2n+1}\begin{pmatrix}\mu_n(a) & & \\ & \mu_n(a) & \\ & & 1\end{pmatrix}\right)\end{aligned}$$

Proof. We provide the argument for $\mathfrak{J}_n(W, \mu_n(z))$ (the proof of the second assertion being completely analogous): Right $\mathcal{P}_n(\mathcal{O})$-invariance of W implies that

$$\mathfrak{J}_{k+1}(W, \mu_n(a)) = \prod_{i=1}^{k} |a_i|^{-i} \mathfrak{J}_k(W, \mu_n(a)).$$

as it follows from the end of the proof of [Jo19, Lemma 3.14] (put $\phi = \mathbf{1}_{\mathcal{O}^k}$ there). Note that the proof in question only deals with $\mu_n(a)$ with $a_n = 1$, but it remains valid for any $\mu_n(a)$. This already shows the claim. □

3.2. **The main result.** We are now ready to compute $\lambda(W_\pi^\circ)$. To this end, note that

$$\lambda(W_\pi^\circ) = \int_{U_n \backslash \mathcal{P}_n} \int_{\mathrm{N}_n \backslash \mathrm{M}_n} W_\pi^\circ\left(w_{2n}\begin{pmatrix}id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix}p & \\ & p\end{pmatrix}\right) \psi^{-1}(Tr(X))\, dX\, dp$$

$$= \int_{U_{n-1} \backslash G_{n-1}} \int_{\mathrm{N}_n \backslash \mathrm{M}_n} W_\pi^\circ\left(w_{2n}\begin{pmatrix}id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix}\mathrm{diag}(g,1) & \\ & \mathrm{diag}(g,1)\end{pmatrix}\right) \psi^{-1}(Tr(X))\, dX\, dg.$$

By the Iwasawa decomposition and because W_π° is right $P_{2n}(\mathcal{O})$-invariant, this simplifies to

$$\lambda(W_\pi^\circ) = \int_{A_{n-1}} \mathfrak{J}_n(W_\pi^\circ, \mathrm{diag}(a,1)) \delta_{B_{n-1}}^{-1}(a) da.$$

Applying first our Lem. 6, a simple coordinate change and finally Eq. (4), this integral becomes

$$\int_{A_{n-1}} W_\pi^\circ\left(w_{2n}\begin{pmatrix}\mathrm{diag}(\mu_{n-1}(a),1) & \\ & \mathrm{diag}(\mu_{n-1}(a),1)\end{pmatrix}\right) \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} \delta_{B_{n-1}}^{-1}(\mu_{n-1}(a)) da$$

$$= \int_{A_{n-1}} W_\pi^\circ\left(w_{2n}\begin{pmatrix}\mathrm{diag}(\mu_{n-1}(a),1) & \\ & \mathrm{diag}(\mu_{n-1}(a),1)\end{pmatrix} w_{2n}^{-1}\right) \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} \delta_{B_{n-1}}^{-1}(\mu_{n-1}(a)) da$$

$$= \int_{A_{n-1}} W_\pi^\circ(\mu_{2n}(1, a_1, \ldots, 1, a_{n-1}, 1, 1) \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} \delta_{B_{n-1}}^{-1}(\mu_{n-1}(a)) da.$$

We will now use Matringe's Thm. 5, distinguishing two cases: First, suppose that $r = 2r'$ is even. Then we get

$$\lambda(W_\pi^\circ) = \int_{A_{r'}} W_{\pi_u}^\circ(\mu_r(1,a_1,\dots,1,a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})|\det(\mu_r(1,a_1,\dots,1,a_{r'}))|^{(n-r')}$$

$$\times \prod_{k=1}^{r'} |a_k|^{-k(n-k)}\delta_{B_{n-1}}^{-1}(\mathrm{diag}(\mu_{r'}(a), id_{n-1-r'}))da$$

$$= \int_{A_{r'}} W_{\pi_u}^\circ(\mu_r(1,a_1,\dots,1,a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})|\det(\mu_{r'}(a_1,\dots,a_{r'}))|^{2(n-r')}$$

$$\times \prod_{k=1}^{r'} |a_k|^{-k(n-k)}\delta_{B_{n-1}}^{-1}(\mathrm{diag}(\mu_{r'}(a), id_{n-1-r'}))da$$

$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(n-k)} W_{\pi_u}^\circ(\mu_{r'}(1,a_1,\dots,1,a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{n-r'+1}da$$

$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(n-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{n-r'+1}da$$

$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{k^2} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{-r'+1}da$$

$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(r'-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|da$$

$$= \int_{A_{r'-1}} \prod_{k=1}^{r'-1} |a_k|^{-k(r'-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mathrm{diag}(\mu_{r'-1}(a),1) & \\ & \mathrm{diag}(\mu_{r'-1}(a),1)\end{pmatrix}\right)$$

$$\times\delta_{B_{r'}}^{-1}(\mathrm{diag}(\mu_{r'-1}(a),1))|\det(\mu_{r'-1}(a))|da\int_{F^\times}\omega_{\pi_u}(a_{r'})\mathbf{1}_{\mathcal{O}}(a_{r'})|a_{r'}|^{r'}da_{r'}$$

which, according to Lemma 6 again, becomes

$$= \int_{A_{r'-1}} \mathfrak{J}_{r'}(W_{\pi_u}^\circ, \mathrm{diag}(\mu_{r'-1}(a),1))\delta_{B_{r'}}^{-1}(\mu_{r'-1}(a))|\det(\mu_{r'-1}(a))|da\int_{F^\times}\omega_{\pi_u}(a_{r'})\mathbf{1}_{\mathcal{O}}(a_{r'})|a_{r'}|^{r'}da_{r'}.$$

Now, set $e_{r'} := (0,\dots,0,1)$ (a $1\times r'$ unit-vector) and denote by $J(s,W,\Phi)$ the Jacquet-Shalika integral

$$J(s,W,\Phi) := \int_{U_{r'}\backslash G_{r'}}\int_{\mathrm{N}_{r'}\backslash \mathrm{M}_{r'}} W\left(w_{2r'}\begin{pmatrix}id_{r'} & X\\ & id_{r'}\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi^{-1}(\mathrm{Tr}(X))\Phi(e_{r'}g)|\det(g)|^s dXdg.$$

Once more by the Iwasawa decomposition we get that $\lambda(W_\pi^\circ) = J(1, W_{\pi_u}^\circ, \mathbf{1}_{\mathcal{O}^{r'}})$, whence, applying the unramfied computation of the exterior square L-function from [Jac-Sha90, Section 7.2], we finally obtain

$$\lambda(W_\pi^\circ) = L(1,\pi_u,\Lambda^2) \neq 0.$$

Now assume $r = 2r'+1$ is odd. As the variable $b_{2r'+1} = b_{2r'+2}$ of $b = \mu_{2n}(a)$ in the integral must vary in $\mathcal{O}^\times$, whereas $W_{\pi_u}^\circ$ is $\mathrm{GL}_r(\mathcal{O})$-invariant, we obtain:

$$\lambda(W_\pi^\circ) =$$

$$\int_{A_{r'}} W^\circ_{\pi_u}(\mu_r(1,a_1,\dots,1,a_{r'},1)|\det(\mu_{r'}(a))|^{2(n-r')-1}\prod_{k=1}^{r'}|a_k|^{-k(n-k)}\delta^{-1}_{B_{n-1}}(\mathrm{diag}(\mu_{r'}(a),id_{n-1-r'}))da.$$

Following the steps of the even computation we arrive at

$$\begin{aligned}\lambda(W^\circ_\pi) &= \int_{A_{r'}}\prod_{k=1}^{r'}|a_k|^{k^2}W^\circ_{\pi_u}\left(w_{2r'+1}\begin{pmatrix}\mu_{r'}(a) & & \\ & \mu_{r'}(a) & \\ & & 1\end{pmatrix}\right)\delta^{-1}_{B_{r'}}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{-r'}da\\ &= \int_{A_{r'}}\prod_{k=1}^{r'}|a_k|^{-k(r'-k)}W^\circ_{\pi_u}\left(w_{2r'+1}\begin{pmatrix}\mu_{r'}(a) & & \\ & \mu_{r'}(a) & \\ & & 1\end{pmatrix}\right)\delta^{-1}_{B_{r'}}(\mu_{r'}(a))da\\ &= \int_{A_{r'}}\mathfrak{J}'_{r'}(W^\circ_{\pi_u},\mu_{r'}(a))\delta^{-1}_{B_{r'}}(\mu_{r'}(a))da\end{aligned}$$

where the last equality follows from Lemma 6. Denoting by $J(s,W)$ the Jacquet-Shalika integral

$$J(s,W) := \int_{U_{r'}\backslash G_{r'}}\int_{\mathrm{N}_{r'}\backslash \mathrm{M}_{r'}} W\left(w_{2r'+1}\begin{pmatrix}id_{r'} & X & \\ & id_{r'} & \\ & & 1\end{pmatrix}\begin{pmatrix}g & & \\ & g & \\ & & 1\end{pmatrix}\right)\psi^{-1}(Tr(X))|\det(g)|^{s-1}dg,$$

we obtain thanks to Iwasawa decomposition $\lambda(W^\circ_\pi) = J(1,W^\circ_{\pi_u})$, and so, applying the unramified computation of [Jac-Sha90, Section 9.4], finally,

$$\lambda(W^\circ_\pi) = L(1,\pi_u,\Lambda^2)\neq 0.$$

We summarize this computation in the following

Theorem 7 (Matringe)**.** *Let $\Theta:\mathcal{W}_\psi(\pi)\to\mathcal{S}_\psi(\pi)$ be the nonzero intertwining operator given by*

$$\Theta(W)(g) = \int_{U_n\backslash\mathcal{P}_n}\int_{\mathrm{N}_n\backslash\mathrm{M}_n} W\left(w_{2n}\begin{pmatrix}id_n & X\\ & id_n\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}g\right)\psi^{-1}(Tr(X))\ dX\ dp.$$

and let $S^\circ_\pi = \Theta(W^\circ_\pi)$, where W°_π is the unique new vector in $\mathcal{W}_\psi(\pi)$, satisfying $W^\circ_\pi(id)=1$. Then

$$S^\circ_\pi(id) = L(1,\pi_u,\Lambda^2)\neq 0.$$

As a consequence, any (non-zero) Shalika newvector in $\mathcal{S}_\psi(\pi)$ does not vanish at $g=id$.

4. Twisting characters

4.1. As this small note is also meant as an addendum to our paper [Gro-Rag14], where other twisting characters than $\eta=\mathbf{1}$ we used, we will now consider some more general twisting characters.

4.2. First, assume that η, as defined in 1.1, is unitary and unramified. If η has these two properties, then, as a character of $\mathcal{S}_{2n}$ it extends to G_{2n}. We shall denote this new character by the same letter.

Let π an irreducible generic unitary representation of G_{2n}. If π has a (η,ψ)-Shalika model $\mathcal{S}^\eta_\psi(\pi)$, then $\eta^{-1}\pi$ is a unitary representation, which has a $(\mathbf{1},\psi)$-Shalika model $\mathcal{S}_\psi(\pi)$ and one is in the situation considered above. The vector $\eta^{-1}\cdot W^\circ_\pi$ is then the unique newvector in $\mathcal{W}_\psi(\eta^{-1}\pi)$, which is 1 at the identity and so $S^\circ_{\eta^{-1}\pi} := \Theta(\eta^{-1}\cdot W^\circ_\pi)$ comes under the purview of Thm. 7: Observing that $(\eta^{-1}\pi)_u = \eta^{-1}\pi_u$, since η is unramified, we get

$$S^\circ_{\eta^{-1}\pi}(id) = L(1,\eta^{-1}\pi_u,\Lambda^2) = L(1,\pi_u,\Lambda^2\otimes\eta)\neq 0.$$

However, $\eta\cdot S^\circ_{\eta^{-1}\pi}$ is a non-zero newvector in $\mathcal{S}^\eta_\psi(\pi)$, so these newvectors do not vanish at $g=id$, either.

4.3. Now, let $\eta = \tilde{\eta}|\cdot|^w$, with $w \in \mathbb{Z}$ and $\tilde{\eta}$ a unitary unramified character. Let π be an irreducible generic representation of G_{2n}, which has a (η, ψ)-Shalika model $\mathcal{S}^{\eta}_{\psi}(\pi)$. Then, every newvector S in $\mathcal{S}^{\eta}_{\psi}(\pi)$ is of the form $S = |\det(\cdot)|^{w/2} \cdot \tilde{S}$, with $\tilde{S}$ a newvector in $\mathcal{S}^{\tilde{\eta}}_{\psi}(\pi)$. See also [Gro-Rag14], Lem. 5.1.1. As a consequence, every non-zero newvector in $\mathcal{S}^{\eta}_{\psi}(\pi)$ does not vanish at $g = id$.

We summarize the latter observations in the following

Corollary 8. *Let $\eta = \tilde{\eta}|\cdot|^w$, with $w \in \mathbb{Z}$ and $\tilde{\eta}$ a unitary unramified character. Let π be an irreducible generic representation of G_{2n}, which has a (η, ψ)-Shalika model $\mathcal{S}^{\eta}_{\psi}(\pi)$. Then, for every newvector S in $\mathcal{S}^{\eta}_{\psi}(\pi)$,*

$$S(id) \neq 0.$$

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