WHITTAKER RATIONAL STRUCTURES AND SPECIAL VALUES OF THE ASAI L-FUNCTION

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Dedicated to Jim Cogdell on the occasion of his 60th birthday

Abstract. Let F be a totally real number field and E/F a totally imaginary quadratic extension of F. Let Π be a cohomological, conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Under a certain non-vanishing condition we relate the residue and the value of the Asai L-functions at $s = 1$ with rational structures obtained from the cohomologies in top and bottom degrees via the Whittaker coefficient map. This generalizes a result in Eric Urban's thesis when $n = 2$, as well as a result of the first two named authors, both in the case $F = \mathbb{Q}$.

Contents

1. Introduction 1
2. Notation and conventions 3
3. Instances of algebraicity 5
4. The Whittaker CRSs 6
5. A cohomological interpretation of $\mathrm{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$ 8
6. A cohomological interpretation of $\mathrm{Res}_{s=1} L(s, \Pi, \mathrm{As}^{(-1)^{n-1}})$ 11
7. A relation between the bottom Whittaker CRS and $L(1, \Pi, \mathrm{As}^{(-1)^n})$ 14
References 14

1. Introduction

Let F be a totally real number field and E/F a totally imaginary quadratic extension of F with non-trivial Galois involution τ. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. One can associate two Asai L-functions over F, denoted $L(s, \Pi, \mathrm{As}^+)$ and $L(s, \Pi, \mathrm{As}^-)$. These are Langlands L-functions attached to representations of the L-group

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of GL_n/E, and the Rankin-Selberg product of Π with Π^τ factors as

$$L(s,\Pi\times\Pi^\tau)=L(s,\Pi,\mathrm{As}^+)\cdot L(s,\Pi,\mathrm{As}^-). \tag{1.1}$$

In this paper we consider representations Π that arise by stable quadratic base change from an automorphic representation π of the unitary group H over F. In particular, Π is *conjugate self-dual*:

$$\Pi^\vee\cong\Pi^\tau.$$

We have an equality of partial L-functions

$$L^S(s,\Pi,\mathrm{As}^{(-1)^n})=L^S(s,\pi,\mathrm{Ad}),$$

where Ad is the adjoint representation of the L-group of H. (The other Asai L-function equals the L-function of π with respect to the twist of Ad by the character corresponding to E/F.)

Since $\Pi^\tau\cong\Pi^\vee$, the Rankin-Selberg L-function on the left-hand side of (1.1) has a simple pole at $s=1$ and the assumption that Π is a base change from a unitary group implies that this pole arises as the pole of $L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})$ at $s=1$. Moreover, $L(s,\Pi,\mathrm{As}^{(-1)^n})$ is holomorphic and non-vanishing at $s=1$. This applies in particular if Π is cohomological and conjugate-dual: Then it is known that Π is automatically a base change from some unitary group H, and moreover $s=1$ is a critical value of $L(s,\Pi,\mathrm{As}^{(-1)^n})$.

Hypothetically, $L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})=\zeta(s)L(s,M^\flat(\Pi))$ for some motive $M^\flat(\Pi)$ (which we do not specify here), where $\zeta(s)$ is the Riemann zeta function. One of the main goals of this note is to relate the residue at $s=1$ of $L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})$, which under the above hypothesis can be interpreted as a *non-critical* special value of the L-function of $M^\flat(\Pi)$, to a certain cohomology class attached to Π, of the adelic "locally symmetric" space $\mathcal{S}_E=\mathrm{GL}_n(E)\backslash\mathrm{GL}_n(\mathbb{A}_E)/A_GK_\infty$.

In fact, Π contributes to the cohomology of $\mathcal{S}_E$ (with suitable coefficients) in several degrees. For each degree q, where it contributes, one can define a rational structure on the q-th $(\mathfrak{m}_G,K_\infty)$-cohomology of Π_∞, which measures the difference between the global cohomological rational structure and the one defined using the Whittaker-Fourier coefficient. We call it the *Whittaker comparison rational structure* (CRS) of degree q and denote it by $\mathbb{S}_q$. Let b and t be the minimal and maximal degrees, respectively, where the $(\mathfrak{m}_G,K_\infty)$-cohomology of Π_∞ is non-zero. (See §2 for notation.) Roughly speaking, our main result is that under a suitable local non-vanishing assumption, $\mathrm{Res}_{s=1}L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})$ (resp., $L(1,\Pi,\mathrm{As}^{(-1)^n})$) spans, under suitable normalization, the one-dimensional spaces $\mathbb{S}_t$ (resp., $\mathbb{S}_b$) over the field of definition of Π. The precise results are stated in Theorems 6.4 and 7.1 in the body of the paper. In the case $n=2$ and $F=\mathbb{Q}$ such results had been proved in the theses of Eric Urban and Eknath Ghate, respectively [Urb95, Gha99]. The results here sharpen some of the main results of [GH16] (for $F=\mathbb{Q}$). We are hopeful that the pertinent non-vanishing assumption will be settled in the near future using the method recently developed by Binyong Sun.

The result for the top degree cohomology turns out to be a rather direct consequence of the well-known relation between the residue of the Asai L-function at $s=1$ and the period integral over $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})/A_G$ [Fli88]. This is a twisted analogue of the realization of the Petersson inner product in the Whittaker model, due to Jacquet–Shalika [JS81].

The two results for the top and bottom degrees are linked by Poincaré duality and the relation (1.1). More precisely, one can relate $\mathrm{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$ to a suitable pairing between $\mathbb{S}_q$ and $\mathbb{S}_{d-q}$ in any degree q (where d is the dimension of $\mathcal{S}_E$). Once again, this is a simple consequence of the aforementioned result of Jacquet–Shalika. In principle, the present methodology is applicable to any cohomological cuspidal automorphic representation over any number field. However, in this generality, the presence of real places creates specific difficulties that we have not examined. Also, we have confined ourselves to conjugate self-dual representations of $\mathrm{GL}_n(\mathbb{A}_E)$. The reader is referred to Balasubramanyam–Raghuram [BR14], who recently proved such a result in the aforementioned generality.

We also note that A. Venkatesh has recently proposed a conjecture relating the rational structure of the contribution of Π to cohomology (in all degrees) to the K-theoretic regulator map of a hypothetical motive attached to Π. We hope that the precise statement of the conjecture will be available soon. At any rate, our results seem to be compatible with this conjecture.

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2. Notation and conventions

2.1. **Number fields and the groups under consideration.** Throughout the paper, let F be a totally real number field and E/F denotes a totally imaginary quadratic extension of F with non-trivial Galois involution τ. The discriminant of F (resp., E) is denoted D_F (resp., D_E). We let I_E be the set of field embeddings $E \hookrightarrow \mathbb{C}$ and the ring of integers (resp., adeles) of E by $\mathcal{O}_E$ and $\mathbb{A}_E$. (Similar notation is used for the field F.) We fix a non-trivial, continuous, additive character $\psi : E\backslash\mathbb{A}_E \to \mathbb{C}^\times$.

We fix an integer $n \geq 1$ and use G to denote the general linear group GL_n viewed as a group scheme over $\mathbb{Z}$.

We denote by T the diagonal torus in G, by $\mathcal{P}$ the subgroup of G consisting of matrices whose last row is $(0, \ldots, 0, 1)$, and by U the unipotent subgroup of upper triangular matrices in G.

For brevity we write $G_\infty = R_{E/\mathbb{Q}}(G)(\mathbb{R})$, where $R_{E/\mathbb{Q}}$ stands for the restriction of scalars from E to $\mathbb{Q}$. We write A_G for the group of positive reals $\mathbb{R}_+$ embedded diagonally in the center of G_∞. Thus, $G(\mathbb{A}_E) \cong G(\mathbb{A}_E)^1 \times A_G$ where $G(\mathbb{A}_E)^1 := \{g \in G(\mathbb{A}_E) : |\det g|_{\mathbb{A}_E^*} = 1\}$.

Let K_∞ be the standard maximal compact subgroup of G_∞ isomorphic to $U(n)^{[F:\mathbb{Q}]}$. We set $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$, $\mathfrak{k}_\infty = \mathrm{Lie}(K_\infty)$, $\mathfrak{a}_G = \mathrm{Lie}(A_G)$, $\mathfrak{p} = \mathfrak{g}_\infty/\mathfrak{k}_\infty$, $\mathfrak{m}_G = \mathfrak{g}_\infty/\mathfrak{a}_G$ and $\tilde{\mathfrak{p}} = \mathfrak{p}/\mathfrak{a}_G$. Let $d := \dim_{\mathbb{R}} \mathfrak{m}_G - \dim_{\mathbb{R}} \mathfrak{k}_\infty$. The choice of measures is all-important in all results of this kind. Our choices are specified in Sections 5.2 and 6.3.

2.2. **Coefficient systems.** We fix an irreducible, finite-dimensional, complex, continuous algebraic representation E_μ of G_∞. It is determined by its highest weight $\mu = (\mu_\iota)_{\iota \in I_E}$ where for each ι, $\mu_\iota = (\mu_{1,\iota}, ..., \mu_{n,\iota}) \in \mathbb{Z}^n$ with $\mu_{1,\iota} \geq \mu_{2,\iota} \geq ... \geq \mu_{n,\iota}$. We assume that E_μ is conjugate self-dual, i.e., $E_\mu^\tau \cong E_\mu^\vee$, or, in other words, that

$$\mu_{j,\tau(\iota)} + \mu_{n+1-j,\iota} = 0, \qquad \iota \in I_E,\ 1 \leq j \leq n.$$

2.3. **Cuspidal automorphic representations.** Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_E)$. We shall assume that Π is conjugate self-dual, i.e., $\Pi^\tau \cong \Pi^\vee$. We say that Π is *cohomological* with respect to E_μ, if $H^*(\mathfrak{g}_\infty, K_\infty, \Pi \otimes E_\mu) \neq 0$. We refer to Borel–Wallach [BW00], I.5, for details concerning $(\mathfrak{g}_\infty, K_\infty)$-cohomology.

Throughout the paper we assume that Π *is a conjugate self-dual, cuspidal automorphic representation which is cohomological with respect to* E_μ. For convenience we will not distinguish between a cuspidal automorphic representation, its smooth limit-Fréchet-space completion of moderate growth and its (non-smooth) Hilbert space completion in the L^2-spectrum. Denote the Petersson inner product on $\Pi \times \Pi^\vee$ by

$$(\varphi, \varphi^\vee)_{\mathrm{Pet}} := \int_{G(E)\backslash G(\mathbb{A}_E)^1} \varphi(g)\varphi^\vee(g)\, dg.$$

We write $\mathcal{W}^\psi(\Pi)$ for the Whittaker model of Π with respect to the character

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

Similarly for $\mathcal{W}^{\psi^{-1}}(\Pi^\vee)$. Let

$$W^\psi : \Pi \to \mathcal{W}^\psi(\Pi)$$

be the realization of Π in the Whittaker model via the ψ-Fourier coefficient, namely

$$W^\psi(\varphi) = (\mathrm{vol}(U(E)\backslash U(\mathbb{A}_E)))^{-1} \int_{U(E)\backslash U(\mathbb{A}_E)} \varphi(ug)\psi(u)^{-1}\, du.$$

Analogous notation is used locally.

2.4. **Pairings of $(\mathfrak{m}_G, K_\infty)$-cohomology spaces.** Suppose that ρ and ρ^* are two irreducible $(\mathfrak{g}_\infty, K_\infty)$-modules which are in duality and let $(\cdot,\cdot)$ be a non-degenerate invariant pairing on $\rho \times \rho^*$. For $p + q = d$, let us define a pairing

$$\mathbb{K}^{\mathrm{coh},p}_{(\rho,\rho^*,(\cdot,\cdot))} : H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee) \to (\wedge^d \tilde{\mathfrak{p}})^*$$

as follows: Recall that

$$H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) \xrightarrow{\sim} \mathrm{Hom}_{K_\infty}(\wedge^p \tilde{\mathfrak{p}}, \rho \otimes E_\mu),$$
$$H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee) \xrightarrow{\sim} \mathrm{Hom}_{K_\infty}(\wedge^q \tilde{\mathfrak{p}}, \rho^* \otimes E_\mu^\vee).$$

Suppose that $\tilde{\omega} \in \mathrm{Hom}_{K_\infty}(\wedge^p \tilde{\mathfrak{p}}, \rho \otimes E_\mu)$ and $\tilde{\eta} \in \mathrm{Hom}_{K_\infty}(\wedge^q \tilde{\mathfrak{p}}, \rho^* \otimes E_\mu^\vee)$ represent $\omega \in H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ and $\eta \in H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee)$ respectively. The cap product

$$\tilde{\omega} \wedge \tilde{\eta} \in \mathrm{Hom}_{K_\infty}(\wedge^d \tilde{\mathfrak{p}}, \rho \otimes \rho^* \otimes E_\mu \otimes E_\mu^\vee),$$

together with the pairing on $\rho \times \rho^*$ and the canonical pairing on $E_\mu \otimes E_\mu^\vee$, defines an element

$$\mathbb{K}^{\mathrm{coh},p}_{(\rho,\rho^*,(\cdot,\cdot))}(\omega, \eta) \in (\wedge^d \tilde{\mathfrak{p}})^*.$$

Note that $(\wedge^d \mathfrak{p})^*$ is canonically isomorphic to the space of invariant measures on $G_\infty / A_G K_\infty$.

2.5. **Locally symmetric spaces over E.** Recall the adelic quotient

$$\mathcal{S}_E := G(E)\backslash G(\mathbb{A}_E)^1/K_\infty.$$

We can view $\mathcal{S}_E$ as the projective limit $\mathcal{S}_E = \varprojlim_{K_f} \mathcal{S}_{E,K_f}$ where

$$\mathcal{S}_{E,K_f} = G(E)\backslash G(\mathbb{A}_E)^1/K_\infty K_f$$

and K_f varies over the directed set of compact open subgroups of $G(\mathbb{A}_{E,f})$ ordered by opposite inclusion. Note that each $\mathcal{S}_{E,K_f}$ is a orbifold of dimension $d = n^2[F:\mathbb{Q}] - 1$.

A representation E_μ as in §2.2 defines a locally constant sheaf $\mathcal{E}_\mu$ on $\mathcal{S}_E$ whose espace étalé is $G(\mathbb{A}_E)^1/K_\infty \times_{G(E)} E_\mu$ with the discrete topology on E_μ. We denote by $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$ and $H^q_c(\mathcal{S}_E, \mathcal{E}_\mu)$ the corresponding spaces of sheaf cohomology and sheaf cohomology with compact support, respectively. They are $G(\mathbb{A}_{E,f})$-modules. We have

$$H^q(\mathcal{S}_E, \mathcal{E}_\mu) \cong \varinjlim_{K_f} H^q(\mathcal{S}_{E,K_f}, \mathcal{E}_\mu)$$

and

$$H^q_c(\mathcal{S}_E, \mathcal{E}_\mu) \cong \varinjlim_{K_f} H^q_c(\mathcal{S}_{E,K_f}, \mathcal{E}_\mu),$$

where the maps in the inductive systems are the pull-backs (Rohlfs [Roh96] Cor. 2.12 and Cor. 2.13). For our purposes we will only use this result to save notation (or to avoid an abuse of notation): we could have simply worked throughout with the inductive limits of cohomologies. (In fact, in [Clo90] $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$ is simply defined as $\varinjlim_{K_f} H^q(\mathcal{S}_{E,K_f}, \mathcal{E}_\mu)$.)

Let $H^q_{\mathrm{cusp}}(\mathcal{S}_E, \mathcal{E}_\mu)$ be the $G(\mathbb{A}_{E,f})$-module of cuspidal cohomology, being defined as the $(\mathfrak{m}_G, K_\infty)$-cohomology of the space of cuspidal automorphic forms. As cusp forms are rapidly decreasing, we obtain an injection

$$\Delta^q : H^q_{\mathrm{cusp}}(\mathcal{S}_E, \mathcal{E}_\mu) \hookrightarrow H^q_c(\mathcal{S}_E, \mathcal{E}_\mu)$$

(cf. [Clo90]).

3. Instances of algebraicity

3.1. **An action of** $\mathrm{Aut}(\mathbb{C})$**.** Let ν be a smooth representation of either $G(\mathbb{A}_{E,f})$ or $G(E_w)$ for a non-archimedean place w of E, on a complex vector space W. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define the *$\sigma$-twist* ${}^\sigma\nu$ following Waldspurger [Wal85], I.1: If W' is a $\mathbb{C}$-vector space which admits a σ-linear isomorphism $\phi : W \to W'$ then we set

$${}^\sigma\nu := \phi \circ \nu \circ \phi^{-1}.$$

This definition is independent of ϕ and W' up to equivalence of representations. One may hence always take $W' := W \otimes_\sigma \mathbb{C}$.

On the other hand, let $\nu = E_\mu$ be a highest weight representation of G_∞ as in §2.2. The group $\mathrm{Aut}(\mathbb{C})$ acts on I_E by composition. Hence, we may define ${}^\sigma E_\mu$ to be the irreducible representation of G_∞, whose local factor at the embedding ι is $E_{\mu_{\sigma^{-1}\iota}}$, i.e., has highest weight $\mu_{\sigma^{-1}\iota}$. As a representation of the diagonally embedded group $G(E) \hookrightarrow G_\infty$, ${}^\sigma E_\mu$ is isomorphic to $E_\mu \otimes_\sigma \mathbb{C}$, cf. Clozel [Clo90], p. 128. Moreover, we obtain

Proposition 3.1. *For all $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^{\sigma}\Pi_f$ is the finite part of a cuspidal automorphic representation ${}^{\sigma}\Pi$ which is cohomological with respect to ${}^{\sigma}E_\mu$. The representation ${}^{\sigma}\Pi$ is conjugate self-dual.*

Proof. See [Clo90], Thm. 3.13. (Note that an irreducible $(\mathfrak{g}_\infty, K_\infty)$-module is cohomological if and only if it is regular algebraic in the sense of [loc. cit.].) The last statement is obvious. □

3.2. **Rationality fields and rational structures.** Recall also the definition of the rationality field of a representation (e.g., [Wal85], I.1). If ν is any of the representations considered above, let $\mathfrak{S}(\nu)$ be the group of all automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^{\sigma}\nu \cong \nu$:

$$\mathfrak{S}(\nu) := \{\sigma \in \mathrm{Aut}(\mathbb{C}) | {}^{\sigma}\nu \cong \nu\}.$$

Then the *rationality field* $\mathbb{Q}(\nu)$ is defined as the fixed field of $\mathfrak{S}(\nu)$,

$$\mathbb{Q}(\nu) := \{z \in \mathbb{C} | \sigma(z) = z \text{ for all } \sigma \in \mathfrak{S}(\nu)\}.$$

As a last ingredient we recall that a group representation ν on a $\mathbb{C}$-vector space W is said to be *defined over a subfield* $\mathbb{F} \subset \mathbb{C}$, if there exists an $\mathbb{F}$-vector subspace $W_{\mathbb{F}} \subset W$, stable under the group action, and such that the canonical map $W_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C} \to W$ is an isomorphism. In this case, we say that $W_{\mathbb{F}}$ is an $\mathbb{F}$-*structure* for (ν, W).

Remark 3.2. If (ν, W) is irreducible, then a rational structure is unique up to homothety, if it exists. Moreover, if $W_{\mathbb{F}}$ is an $\mathbb{F}$-structure for (ν, W), with (ν, W) irreducible, and if V is a complex vector space with a trivial group action then any $\mathbb{F}$-structure for $(\nu \otimes 1, W \otimes V)$ is of the form $W_{\mathbb{F}} \otimes V_{\mathbb{F}}$ for a unique $\mathbb{F}$-structure $V_{\mathbb{F}}$ of V (as a complex vector space).

It is easy to see that as a representation of $G(E)$, E_μ has a $\mathbb{Q}(E_\mu)$-structure, whence, so does $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$, cf. [Clo90], p. 122.

Proposition 3.3. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E)$. Then Π_f has a $\mathbb{Q}(\Pi_f)$-structure, which is unique up to homotheties. If Π is cohomological with respect to E_μ, then $\mathbb{Q}(\Pi_f)$ is a number field. Similarly, $H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$ has a $\mathbb{Q}(\Pi_f)$-structure coming from the natural $\mathbb{Q}(E_\mu)$-structure of $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$.*

Proof. This is contained in [Clo90], Prop. 3.1, Thm. 3.13 and Prop. 3.16 (the Drinfeld-Manin principle). The reader may also have a look at [GR14] Thm. 8.1 and Thm. 8.6. For the last statement, one observes that $\mathbb{Q}(E_\mu) \subseteq \mathbb{Q}(\Pi_f)$ by Strong Multiplicity One (Cf. [GR14], proof of Cor. 8.7.). □

4. The Whittaker CRSs

4.1. **Rational structures on Whittaker models.** We recall the discussion of [RS08], §3.2, resp., [Mah05], §3.3. Fix a non-archimedean place w of E. Given a Whittaker function ξ on $G(E_w)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ we define the Whittaker function ${}^{\sigma}\xi$ by

$$ {}^{\sigma}\xi(g) := \sigma(\xi(t_\sigma \cdot g)), \tag{4.1}$$

where t_σ is the (unique) element in $T(E_w) \cap \mathcal{P}(E_w)$ that conjugates ψ_U to $\sigma\psi_U$. Note that t_σ does not depend on ψ. We have, $t_{\sigma_1\sigma_2} = t_{\sigma_1} t_{\sigma_2}$ and hence ${}^{\sigma_1\sigma_2}\xi \equiv {}^{\sigma_1}({}^{\sigma_2}\xi)$ for all $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathbb{C})$. Thus, if π is any irreducible admissible generic representation of $G(E_w)$,

then we obtain a σ-linear intertwining operator $\mathcal{T}_\sigma^\psi : \mathcal{W}^\psi(\pi) \to \mathcal{W}^\psi({}^\sigma\pi)$. In particular, we get a $\mathbb{Q}(\pi)$ structure on $\mathcal{W}^\psi(\pi)$ by taking invariant vectors. A similar discussion applies to irreducible admissible generic representations of $G(\mathbb{A}_{E,f})$.

4.2. The map $W^\psi : \Pi \to \mathcal{W}^\psi(\Pi)$ gives rise to an isomorphism

$$H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \xrightarrow{\sim} \tag{4.2}$$
$$H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^\psi(\Pi) \otimes E_\mu) \cong H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu) \otimes \mathcal{W}^{\psi_f}(\Pi_f).$$

Recall the $\mathbb{Q}(\Pi_f)$-structure on $H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$, (respectively on $\mathcal{W}^{\psi_f}(\Pi_f)$) from Prop. 3.3 (respectively from §4.1). Thus, by Rem. (3.2) we obtain a $\mathbb{Q}(\Pi_f)$-structure on the cohomology space $H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$ (as a $\mathbb{C}$-vector space) which we denote by $\mathbb{S}_\Pi^{\psi,q}$ and call it the *q-th Whittaker comparison rational structure* (CRS) of Π. In particular, $\mathbb{S}_\Pi^{\psi,q}$ is a $\mathbb{Q}(\Pi_f)$-vector subspace of $H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$ and $\dim_{\mathbb{Q}(\Pi_f)} \mathbb{S}_\Pi^{\psi,q} = \dim_{\mathbb{C}} H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$.

4.3. **Equivariance of local Rankin-Selberg integrals.** [1]

For this subsection only let F be a local non-archimedean field. Let π_1 and π_2 be two generic irreducible representations of $\mathrm{GL}_n(F)$ with Whittaker models $\mathcal{W}^\psi(\pi_1)$ and $\mathcal{W}^{\psi^{-1}}(\pi_2)$ respectively. Given $W_1 \in \mathcal{W}^\psi(\pi_1)$, $W_2 \in \mathcal{W}^{\psi^{-1}}(\pi_2)$ and $\Phi \in \mathcal{S}(F^n)$ a Schwartz-Bruhat function, consider the zeta integral

$$Z^\psi(W_1, W_2, \Phi, s) = \int_{U(F)\backslash \mathrm{GL}_n(F)} W_1(g) W_2(g) \Phi(e_n g) \, |\det g|^s \; dg,$$

where e_n is the row vector $(0, \ldots, 0, 1)$ and the invariant measure dg on $U(F)\backslash \mathrm{GL}_n(F)$ is *rational*, i.e., it assigns rational numbers to compact open subsets. We view the above expression as a formal Laurent series $A^\psi(W_1, W_2, \Phi) \in \mathbb{C}((X))$ in $X = q^{-s}$ whose m-th coefficient $c_m^\psi(W_1, W_2, \Phi)$ is

$$\int_{U(F)\backslash \mathrm{GL}_n(F) : |\det g| = q^{-m}} W_1(g) W_2(g) \Phi(e_n g) \; dg.$$

The last integral reduces to a finite sum, and vanishes for $m \ll 0$, because of the support of Whittaker functions. It is therefore clear (by a simple change of variable) that

$$\sigma(c_m^\psi(W_1, W_2, \Phi)) = c_m^\psi({}^\sigma W_1, {}^\sigma W_2, \sigma\Phi)$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. Thus,

$$A^\psi({}^\sigma W_1, {}^\sigma W_2, \sigma\Phi) = (A^\psi(W_1, W_2, \Phi))^\sigma$$

where σ acts on $\mathbb{C}((X))$ in the obvious way. The linear span of

$$A^\psi(W_1, W_2, \Phi), \; W_1 \in \mathcal{W}^\psi(\pi_1), \; W_2 \in \mathcal{W}^{\psi^{-1}}(\pi_2), \; \Phi \in \mathcal{S}(F^n)$$

is a fractional ideal $\mathfrak{I}^\psi(\pi_1, \pi_2)$ of $\mathbb{C}[X, X^{-1}]$. Thus, by the above,

$$\mathfrak{I}^\psi(\pi_1, \pi_2)^\sigma = \mathfrak{I}^\psi({}^\sigma\pi_1, {}^\sigma\pi_2)$$

[1] Essentially the same argument is given in the middle of the proof of Theorem 2 of [Gre03]. We have restated it separately for convenience.

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. Hence, if we write $L(s, \pi_1 \times \pi_2) = (P_{\pi_1,\pi_2}(q^{-s}))^{-1}$ where $(P_{\pi_1,\pi_2}(X))^{-1}$ is the generator of $\mathfrak{I}^{\psi}(\pi_1, \pi_2)$ such that $P_{\pi_1,\pi_2} \in \mathbb{C}[X]$ and $P_{\pi_1,\pi_2}(0) = 1$ then it follows that $P_{\sigma\pi_1,\sigma\pi_2} = P^{\sigma}_{\pi_1,\pi_2}$.

Of course, this argument applies equally well to other L-factors defined by the Rankin-Selberg method.

5. A cohomological interpretation of $\mathrm{Res}_{s=1} L(s, \Pi \times \Pi^{\vee})$

5.1. **A pairing.** For any compact open subgroup K_f of $G(\mathbb{A}_{E,f})$ we use the de Rham isomorphism to define a canonical map of vector spaces

$$H^d_c(\mathcal{S}_{E,K_f}, \underline{\mathbb{C}}) \xrightarrow{\int_{\mathcal{S}_{E,K_f}}} \mathbb{C}$$

where $\underline{\mathbb{C}}$ denotes the constant sheaf. Thus, if $p+q = d$ then we get a canonical non-degenerate pairing

$$H^p_c(\mathcal{S}_{E,K_f}, \mathcal{E}_{\mu}) \times H^q(\mathcal{S}_{E,K_f}, \mathcal{E}^{\vee}_{\mu}) \to \mathbb{C} \tag{5.1}$$

which is defined by taking the cap product to $H^d_c(\mathcal{S}_E, \mathcal{E}_{\mu} \otimes \mathcal{E}^{\vee}_{\mu})$, mapping it to $H^d_c(\mathcal{S}_E, \underline{\mathbb{C}})$ using the canonical map $\mathcal{E}_{\mu} \otimes \mathcal{E}^{\vee}_{\mu} \to \underline{\mathbb{C}}$ and finally applying $\int_{\mathcal{S}_{E,K_f}}$. Note however that, as defined, the maps $\int_{\mathcal{S}_{E,K_f}}$ do not fit together compatibly to a map $H^d_c(\mathcal{S}_E, \underline{\mathbb{C}}) \to \mathbb{C}$, since we have to take into account the degrees of the covering maps $\mathcal{S}_{E,K_f} \to \mathcal{S}_{E,K'_f}$, $K_f \subset K'_f$. To rectify the situation, we fix once and for all a $\mathbb{Q}$-valued Haar measure γ on $G(\mathbb{A}_{E,f})$ (which is unique up to multiplication by $\mathbb{Q}^*$). The normalized integrals $\int'_{\mathcal{S}_{E,K_f}} : H^d_c(\mathcal{S}_{E,K_f}, \underline{\mathbb{C}}) \to \mathbb{C}$ given by $\int'_{\mathcal{S}_{E,K_f}} = \mathrm{vol}_{\gamma}(K_f) \int_{\mathcal{S}_{E,K_f}}$ are compatible with the pull-back with respect to the covering maps $\mathcal{S}_{E,K_f} \to \mathcal{S}_{E,K'_f}$, $K_f \subset K'_f$. Thus, we get a map

$$H^d_c(\mathcal{S}_E, \underline{\mathbb{C}}) \xrightarrow{\int'_{\mathcal{S}_E}} \mathbb{C}.$$

We denote the resulting pairing

$$\mathbb{P}^p : H^p_c(\mathcal{S}_E, \mathcal{E}_{\mu}) \times H^q(\mathcal{S}_E, \mathcal{E}^{\vee}_{\mu}) \to \mathbb{C}$$

It depends implicitly on the choice of γ, but this ambiguity is only up to an element of $\mathbb{Q}^*$.[2] At any rate, the maps $\int_{\mathcal{S}_{E,K_f}}$ (and consequently, $\int'_{\mathcal{S}_{E,K_f}}$ and $\int'_{\mathcal{S}_E}$) are $\mathrm{Aut}(\mathbb{C})$-equivariant with respect to the standard rational structure of $H^d_c(\mathcal{S}_{E,K_f}, \underline{\mathbb{C}})$ and $H^d_c(\mathcal{S}_E, \underline{\mathbb{C}})$. The same is therefore true for the pairing (5.1) and $\mathbb{P}^p$ (with respect to the rational structure of $\mathcal{E}_{\mu}$).

As noted in [Clo90, p. 124], the pairing $\mathbb{P}^p$ restricts to a non-degenerate pairing

$$H^p_{\mathrm{cusp}}(\mathcal{S}_E, \mathcal{E}_{\mu}) \times H^q_{\mathrm{cusp}}(\mathcal{S}_E, \mathcal{E}^{\vee}_{\mu}) \to \mathbb{C}$$

and therefore to a non-degenerate pairing

$$H^p_{\Pi_f}(\mathcal{S}_E, \mathcal{E}_{\mu}) \times H^q_{\Pi^{\vee}_f}(\mathcal{S}_E, \mathcal{E}^{\vee}_{\mu}) \to \mathbb{C}$$

[2] This point is implicit in [Clo90, p. 124].

where $H^p_{\Pi_f}(\mathcal{S}_E, \mathcal{E}_\mu)$ is the Π_f-isotypic part of $H^p_{\mathrm{cusp}}(\mathcal{S}_E, \mathcal{E}_\mu)$ and similarly for $H^q_{\Pi_f^\vee}(\mathcal{S}_E, \mathcal{E}^\vee_\mu)$. Composing with Δ^p and Δ^q we finally get a non-degenerate pairing

$$\mathbb{K}^{\mathrm{Pet},p} : H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E^\vee_\mu) \to \mathbb{C}.$$

This pairing coincides with the volume of $\mathcal{S}_E$ with respect to the complex-valued measure $\mathbb{K}^{\mathrm{coh},p}_{(\Pi_\infty, \Pi^\vee_\infty, (\cdot,\cdot)_{\mathrm{Pet}})} \otimes \gamma$ of $G(\mathbb{A}_E)^1/K_\infty \cong G_\infty/A_G K_\infty \times G(\mathbb{A}_{E,f})$. Here we identify $(\wedge^d \mathfrak{p})^*$ with the space of invariant measures on $G_\infty/A_G K_\infty$.

5.2. **Measures over E.** At this point it will be convenient to introduce some Haar measures on various groups. If w is non-archimedean we take the Haar measure on E_w which gives volume one to the integers $\mathcal{O}_w$. On $\mathbb{C}$ we take twice the Lebesgue measure. Having fixed measures on E_w for all w we can define (unnormalized) Tamagawa measures on local groups by providing a gauge form (up to a sign). On the groups G, $\mathcal{P}$ and U we take the gauge form $\wedge dx_{i,j}/(\det x)^k$ where (i,j) range over the coordinates of the non-constant entries in the group and k is n, $n-1$ and 0 respectively. Note that if w is non-archimedean then the volume of $G(\mathcal{O}_w)$ is $\Delta^{-1}_{G,w}$ where $\Delta_{G,w} = \prod_{j=1}^n L(j, \mathbf{1}_{E^*_w})$. On $G(\mathbb{A}_E)$ and $G(\mathbb{A}_{E,f})$ we will take the measure

$$\prod_w \Delta_{G,w}\ dg_w$$

where w ranges over all places (resp., all finite places). On A_G we take the Haar measure whose push-forward (to $\mathbb{R}_+$) under $|\det|_{\mathbb{A}^*_E}$ is dx/x where dx is the Lebesgue measure. The isomorphism $G(\mathbb{A}_E) \cong A_G \times G(\mathbb{A}_E)^1$ gives a measure on $G(\mathbb{A}_E)^1$. Then

$$\mathrm{vol}(G(E)\backslash G(\mathbb{A}_E)^1) = |D_E|^{n^2/2} \,\mathrm{Res}_{s=1} \prod_{j=1}^n \zeta^*_E(s+j-1)$$

where D_E is the discriminant of E and $\zeta^*_E(s)$ is the completed Dedekind zeta function of E.

Let $\xi \in (\wedge^d \tilde{\mathfrak{p}})^*$ correspond to the invariant measure on $A_G\backslash G_\infty/K_\infty$ obtained by the push-forward of the Haar measure on $A_G\backslash G_\infty$ chosen above. Let $\Lambda_0 \in \wedge^d \tilde{\mathfrak{p}}$ be the element such that $\xi(\Lambda_0) = 1$.

5.3. **The Whittaker realization of the Petersson inner product.** Let S be a finite set of places of E and let $E_S = \prod_{v\in S} E_v$. Given an irreducible generic essentially unitarizable representation π_S of $G(E_S)$ with Whittaker model $\mathcal{W}^{\psi_S}(\pi_S)$ define

$$[W, W^\vee]_S := \frac{\Delta_{G,S}}{L(1, \pi_S \times \pi_S^\vee)} \cdot \int_{U(E_S)\backslash \mathcal{P}(E_S)} W(g) W^\vee(g)\ dg, \quad W \in \mathcal{W}^\psi(\pi_S),\ W^\vee \in \mathcal{W}^{\psi^{-1}}(\pi_S^\vee)$$

where $\Delta_{G,S} = \prod_{w\in S} \Delta_{G,w}$. It is well known that this integral converges and defines a $G(E_S)$-invariant pairing on $\mathcal{W}^\psi(\pi_S) \times \mathcal{W}^{\psi^{-1}}(\pi_S^\vee)$. If S consists of the archimedean places of E, we simply write $[W, W^\vee]_\infty$.

We note that if S consists of non-archimedean places only, then $[W, W^\vee]_S$ is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e.,

$$[{}^\sigma W, {}^\sigma W^\vee]_S = \sigma([W, W^\vee]_S).$$

Indeed, by uniqueness, it suffices to check this relation when the restriction of W to $\mathcal{P}(E_S)$ is compactly supported modulo $U(E_S)$, in which case the integral reduces to a finite sum and the assertion follows from §4.3 and the fact that the measures chosen on $U(E_S)$ and $\mathcal{P}(E_S)$ assign rational values to compact open subgroups.

With our choice of measures, given cuspidal automorphic forms φ, $\varphi^\vee$ in the space of Π and $\Pi^\vee$, respectively, we abbreviate $W_\varphi^\psi = W^\psi(\varphi)$, $W_{\varphi^\vee}^{\psi^{-1}} = W^{\psi^{-1}}(\varphi^\vee)$ and obtain

$$(\varphi, \varphi^\vee)_{\mathrm{Pet}} = |D_E|^{n(n+1)/4} \operatorname{Res}_{s=1} L(s, \Pi \otimes \Pi^\vee)[W_\varphi^\psi, W_{\varphi^\vee}^{\psi^{-1}}]_S \tag{5.2}$$

(see [LM15, p. 477] which is of course based on [JS81]) provided that S is a finite set of places of E containing all the archimedean ones as well as all the non-archimedean places for which either φ or $\varphi^\vee$ is not $G(\mathcal{O}_w)$-invariant or the conductor of ψ_w is different from $\mathcal{O}_w$. (Note that $[W_\varphi^\psi, W_{\varphi^\vee}^{\psi^{-1}}]_S$ is unchanged by enlarging S because of the extra factor $\Delta_{G,S}$ in the numerator.)

We will also write $[W, W^\vee]_f = [W, W^\vee]_S$ for any $W \in \mathcal{W}^{\psi_f}(\Pi_f)$ and $W^\vee \in \mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$ where S is any sufficiently large set of non-archimedean places of E (depending on W and $W^\vee$).

5.4. **A relation between the Whittaker CRSs and** $\operatorname{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$**.**

Theorem 5.3. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic representation E_μ. For all degrees p, the number $\left(|D_E|^{n(n+1)/4} \operatorname{Res}_{s=1} L(s, \Pi \times \Pi^\vee)\right)^{-1}$ spans the one-dimensional $\mathbb{Q}(\Pi_f)$-vector space*

$$\Lambda \circ \mathbb{K}^{\mathrm{coh},p}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee), [\cdot,\cdot]_\infty)}(\mathbb{S}_\Pi^{\psi,p}, \mathbb{S}_{\Pi^\vee}^{\psi^{-1},d-p}) \subset \mathbb{C}$$

where $\Lambda : (\wedge^d \tilde{\mathfrak{p}})^* \to \mathbb{C}$ is the evaluation at the element $\Lambda_0 \in \wedge^d \tilde{\mathfrak{p}}$ defined in §5.2.

Proof. We have two pairings $\mathbb{K}^{\mathrm{local},p}_{\Pi_\infty}$ and $\mathbb{K}^{\mathrm{global},p}_{\Pi}$ on

$$H^p(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty^\vee) \otimes E_\mu^\vee)$$

(with $p + q = d$): Firstly, the local pairing $\mathbb{K}^{\mathrm{local},p}_{\Pi_\infty}$ is defined to be

$$\mathbb{K}^{\mathrm{local},p}_{\Pi_\infty} := \Lambda \circ \mathbb{K}^{\mathrm{coh},p}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee), [\cdot,\cdot]_\infty)}.$$

Secondly, in order to define $\mathbb{K}^{\mathrm{global},p}_{\Pi}$ we use the isomorphism (4.2): Namely, the pairing $\mathbb{K}^{\mathrm{global},p}_{\Pi}$ is the one which is compatible under (4.2) with the pairing $\mathbb{K}^{\mathrm{Pet},p}$ (defined in §5.1) on $H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E_\mu^\vee)$ and the pairing $[\cdot,\cdot]_f$ on $\mathcal{W}^{\psi_f}(\Pi_f) \times \mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$. By (5.2) and our convention of measures we have

$$\mathbb{K}^{\mathrm{global},p}_{\Pi} = |D_E|^{n(n+1)/4} \operatorname{Res}_{s=1} L(s, \Pi \times \Pi^\vee) \cdot \mathbb{K}^{\mathrm{local},p}_{\Pi_\infty}.$$

On the other hand, $\mathbb{K}^{\mathrm{global},p}_{\Pi}(\mathbb{S}_\Pi^{\psi,p}, \mathbb{S}_{\Pi^\vee}^{\psi^{-1},d-p}) = \mathbb{Q}(\Pi_f)$. This follows from the definition of $\mathbb{S}_\Pi^{\psi,p}$ and the fact that (1) $[\cdot,\cdot]_f$ is $\mathbb{Q}(\Pi_f)$-rational with respect to the $\mathbb{Q}(\Pi_f)$-structures on $\mathcal{W}^{\psi_f}(\Pi_f)$

and $\mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$ and (2) $\mathbb{K}^{\mathrm{Pet},p}$ is $\mathbb{Q}(\Pi_f)$-rational with respect to the $\mathbb{Q}(\Pi_f)$-structures on $H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$ and $H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E_\mu^\vee)$ (see §5.1). The theorem follows. □

6. A cohomological interpretation of $\mathrm{Res}_{s=1} L(s, \Pi, \mathrm{As}^{(-1)^{n-1}})$

6.1. **Locally symmetric spaces over F.** We write $G'_\infty = R_{F/\mathbb{Q}}G(\mathbb{R})$, where $R_{F/\mathbb{Q}}$ denotes restriction of scalars from F to $\mathbb{Q}$, and denote by K'_∞ the connected component of the identity of the intersection $K_\infty \cap G'_\infty$. It is isomorphic to $\mathrm{SO}(n)^{[F:\mathbb{Q}]}$. We write A'_G for the group of positive reals embedded diagonally in the center of G'_∞. (It will be convenient to distinguish between the isomorphic groups A_G and A'_G.) As before, we have $G(\mathbb{A}_F) \cong G(\mathbb{A}_F)^1 \times A'_G$ where $G(\mathbb{A}_F)^1 := \{g \in G(\mathbb{A}_F) : |\det g|_{\mathbb{A}_F^*} = 1\}$. We write $\mathfrak{g}'_\infty = \mathrm{Lie}(G'_\infty)$, $\mathfrak{k}'_\infty = \mathrm{Lie}(K'_\infty)$, $\mathfrak{a}'_G = \mathrm{Lie}(A'_G)$, $\mathfrak{p}' = \mathfrak{g}'_\infty/\mathfrak{k}'_\infty$ and $\tilde{\mathfrak{p}}' = \mathfrak{p}'/\mathfrak{a}'_G$.

Let

$$\mathcal{S}_F := G(F)\backslash G(\mathbb{A}_F)^1/K'_\infty$$

be the "locally symmetric space" attached to $G(F)$. The closed (non-injective) map $\mathcal{S}_F \to \mathcal{S}_E$ gives rise to a map $H_c^q(\mathcal{S}_E, \mathcal{E}_\mu) \to H_c^q(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F})$ of $G(\mathbb{A}_{F,f})$-modules.

Finally, let ϵ' be character on $G(\mathbb{A}_F)$ given by $\varepsilon \circ \det$ if n is even and 1 if n is odd, where ε is the quadratic Hecke character associated to the extension E/F via class field theory.

As before, let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E)$ which is cohomological and conjugate self-dual. Then Π is $(G(\mathbb{A}_F), \epsilon')$-distinguished in the sense that

$$\int_{G(F)\backslash G(\mathbb{A}_F)^1} \varphi(h)\epsilon'(h)\, dh$$

is non-zero for some φ in Π. (Equivalently, (by [FZ95]) $L(s, \Pi, \mathrm{As}^{(-1)^{n-1}})$ has a pole at $s = 1$.) Indeed, otherwise $L(s, \Pi, \mathrm{As}^{(-1)^n})$ would have a pole, and hence in particular Π_∞ would be (G'_∞, χ)-distinguished where $\chi = \varepsilon \circ \det$ if n is odd and $\chi = 1$ if n is even. However, it is easy to see that this is incompatible with the description of tempered distinguished representations [Pan01]. See [HL04] and [Mok14] for the relation between distinction and base change from a unitary group.

6.2. **An archimedean period on cohomology.** Suppose that ρ is a tempered irreducible $(\mathfrak{g}_\infty, K_\infty)$-module which is (G'_∞, ϵ')-distinguished, i.e., there exists a non-zero (G'_∞, ϵ')-equivariant functional ℓ on ρ. (Such a functional is unique up to a constant. This follows from [AG09] and the automatic continuity in this context [BD92, vdBD88].) Observe that

$$t = \frac{n(n+1)}{2}[F : \mathbb{Q}] - 1 = \dim_{\mathbb{R}} \mathcal{S}_F = \dim_{\mathbb{R}} \tilde{\mathfrak{p}}',$$

where t is the highest degree for which $H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ can be non-zero for a generic irreducible (essentially unitary) representation ρ. Moreover, $H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ is one-dimensional.

Let V_λ be a highest weight representation of $\mathrm{GL}_n(\mathbb{R})$ with parameter $\lambda = (\lambda_1, \ldots, \lambda_n)$ and let $\lambda^\vee = (-\lambda_n, \ldots, -\lambda_1)$. Let $(\cdot,\cdot)_\lambda$ be the standard pairing on $V_\lambda \times V_{\lambda^\vee}$. Since by assumption E_μ is conjugate self-dual, we can define a G'_∞-invariant form $\ell_\mu : E_\mu \to \mathbb{C}$ by taking the tensor product of the pairings above over all archimedean places of E.

We define a functional

$$\mathbb{L}^{\mathrm{coh},t}_{(\rho,\ell)} : H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) \to (\wedge^t \tilde{\mathfrak{p}}')^*$$

as follows. Suppose that $\tilde{\omega} \in \mathrm{Hom}_{K_\infty}(\wedge^t \tilde{\mathfrak{p}}, \rho \otimes E_\mu)$ represents $\omega \in H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$. We compose $\tilde{\omega}$ with the embedding $\wedge^t \tilde{\mathfrak{p}}' \hookrightarrow \wedge^t \tilde{\mathfrak{p}}$ and with $\ell \otimes \ell_\mu$ to get an element of $\mathbb{L}^{\mathrm{coh},t}_{(\rho,\ell)}(\omega) \in (\wedge^t \tilde{\mathfrak{p}}')^*$. We will make the following assumption:

Hypothesis 6.1. $\mathbb{L}^{\mathrm{coh},t}_{(\rho,\ell)}$ is non-zero.

Hopefully, this will be proved in the near future using the method of Binyong Sun (cf. [Sun13, Sun11]).

Next, we fix a $\mathbb{Q}$-valued Haar measure γ' on $G(\mathbb{A}_{F,f})$: As in §5.1 we use γ' to define the normalized integrals

$$H^t_c(\mathcal{S}_F, \underline{\mathbb{C}}) \xrightarrow{\int'_{\mathcal{S}_F,\epsilon'}} \mathbb{C},$$

except that now we take the cup product with the class $[\epsilon'] \in H^0(\mathcal{S}_F, \underline{\mathbb{C}})$ represented by ϵ' before integrating. By composing $\int'_{\mathcal{S}_F,\epsilon'}$ with the map $H^t_c(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F}) \to H^t_c(\mathcal{S}_F, \underline{\mathbb{C}})$ induced from ℓ_μ and the map $H^t_c(\mathcal{S}_E, \mathcal{E}_\mu) \to H^t_c(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F})$, we get a period map $H^t_c(\mathcal{S}_E, \mathcal{E}_\mu) \to \mathbb{C}$. As before, this map is $\mathrm{Aut}(\mathbb{C})$-equivariant. Composing with Δ^t we finally obtain a linear form

$$\mathbb{L}^{\mathrm{per},t} : H^t(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \to \mathbb{C}.$$

It coincides with the volume of $\mathcal{S}_F$ with respect to the complex-valued measure $\mathbb{L}^{\mathrm{coh},t}_{(\Pi_\infty,\ell_{\mathrm{aut}})} \otimes \gamma'$ of $G(\mathbb{A}_F)^1/K'_\infty \cong G'_\infty/A'_G K'_\infty \times G(\mathbb{A}_{F,f})$ where

$$\ell_{\mathrm{aut}}(\varphi) = \int_{G(F)\backslash G(\mathbb{A}_F)^1} \varphi(h)\epsilon'(h)\, dh$$

and we identify an element of $(\wedge^t \tilde{\mathfrak{p}}')^*$ with an invariant measures on $G'_\infty/A'_G K'_\infty$. In particular, $\mathbb{L}^{\mathrm{per},t}$ is non-zero if we assume Hypothesis 6.1.

6.3. **Measures over F.** We now fix some measures. If v is a non-archimedean place of F, we take the Haar measure on F_v which gives volume one to the integers $\mathcal{O}_v$. On $\mathbb{R}$ we take the Lebesgue measure. This gives rise to Tamagawa measures on the local groups $G(F_v)$, $\mathcal{P}(F_v)$ and $U(F_v)$ by taking the standard gauge form as in §5.2. Thus, if v is non-archimedean then the volume of $G(\mathcal{O}_v)$ is $\Delta^{-1}_{G,v}$ where $\Delta_{G',v} = \prod_{j=1}^n L(j, \mathbf{1}_{F_v^*})$. On $G(\mathbb{A}_F)$ and $G(\mathbb{A}_{F,f})$ we will take the measure

$$\prod_v \Delta_{G',v}\, dg_v$$

where v ranges over all places (resp., all finite places). The measure on A'_G will be determined by the isomorphism $|\det|_{\mathbb{A}_F^*} : A'_G \to \mathbb{R}_+$ and the measure dx/x on $\mathbb{R}_+$ where dx is the Lebesgue measure. The isomorphism $G(\mathbb{A}_F) \cong A'_G \times G(\mathbb{A}_F)^1$ gives rise to a measure on $G(\mathbb{A}_F)^1$. Then

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A}_F)^1) = |D_F|^{n^2/2} \,\mathrm{Res}_{s=1} \prod_{j=1}^n \zeta^*_F(s+j-1)$$

where D_F is the discriminant of F and $\zeta_F^*(s)$ is the completed Dedekind zeta function of F.

Let $\xi' \in (\wedge^t \tilde{\mathfrak{p}}')^*$ correspond to the invariant measure on $G'_\infty/A'_G K'_\infty$ obtained by the push-forward of the Haar measure on G'_∞/A'_G chosen above. Let $\Lambda'_0 \in \wedge^t \tilde{\mathfrak{p}}'$ be the element such that $\xi'(\Lambda'_0) = 1$.

6.4. **The Whittaker realization of ℓ_{aut}.** Recall that ψ was a fixed character of $E\backslash\mathbb{A}_E$. We assume from now on that the restriction of ψ to $F\backslash\mathbb{A}_F$ is trivial.

Given a finite set of places S of F and an irreducible generic unitarizable $(G(F_S), \epsilon')$-distinguished representation π_S of $G(E_S)$ with Whittaker model $\mathcal{W}^{\psi_S}(\pi_S)$ define

$$\ell_S(W) := \frac{\Delta_{G',S}}{L(1, \pi_S, \mathrm{As}^{(-1)^{n-1}})} \cdot \int_{U(F_S)\backslash\mathcal{P}(F_S)} W(h)\epsilon'(h)\, dh, \quad W \in \mathcal{W}^{\psi}(\pi_S)$$

where $\Delta_{G',S} = \prod_{v\in S} \Delta_{G',v}$. The integral converges and defines a $(G(F_S), \epsilon')$-equivariant form on $\mathcal{W}^{\psi_S}(\pi_S)$ ([Off11, Kem12]).

If v is a non-archimedean place of F, then by the same argument as in §4.3 we have

$$L(s, {}^\sigma\Pi_v, \mathrm{As}^{(-1)^{n-1}}) = L(s, \Pi_v, \mathrm{As}^{(-1)^{n-1}})^\sigma$$

(where on the right-hand side, σ acts on $\mathbb{C}(q^{-s})$ in the obvious way). We recall that the local Asai L-function is defined as the gcd of the Rankin–Selberg integrals. This is consistent with the local Langlands correspondence – see [Mat11]. In particular,

$$L(1, {}^\sigma\Pi_v, \mathrm{As}^{(-1)^{n-1}}) = \sigma(L(1, \Pi_v, \mathrm{As}^{(-1)^{n-1}})). \tag{6.2}$$

Thus, if S consists only of non-archimedean places, then $\ell_S(W)$ is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e.,

$$\ell_S({}^\sigma W) = \sigma(\ell_S(W)).$$

Indeed, by uniqueness ([Fli91]), it suffices to check this relation when the restriction of W to $\mathcal{P}(E_S)$ is compactly supported modulo $U(E_S)$, in which case the integral reduces to a finite sum and the assertion follows from (6.2) and the rationality of the measure on $U(F_S)\backslash\mathcal{P}(F_S)$.

With our choice of measures, for any cuspidal automorphic form φ in the space of Π and $W_\varphi^\psi = W^\psi(\varphi)$, we have

$$\ell_{\mathrm{aut}}(\varphi) = |D_F|^{n(n+1)/4} \operatorname{Res}_{s=1} L(s, \Pi, \mathrm{As}^{(-1)^{n-1}}) \cdot \ell_S(W_\varphi^\psi) \tag{6.3}$$

provided that S is a sufficiently large finite set of places of F. (Cf. Gelbart–Jacquet–Rogawski, [GJR01], pp. 184–185 or Zhang [Zha14], Sect. 3.2.) Note that $\ell_S(W_\varphi^\psi)$ is unchanged by enlarging S because of the extra factor $\Delta_{G',S}$ in the numerator.

As before we write $\ell_f(W) = \ell_S(W)$ for $W \in \mathcal{W}^{\psi_f}(\Pi_f)$ where S is any sufficiently large set of places.

6.5. **A relation between the top Whittaker CRS and $\operatorname{Res}_{s=1} L(1, \Pi, \mathrm{As}^{(-1)^{n-1}})$.** We may now prove our first main theorem on the Asai L-function.

Theorem 6.4. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic*

representation E_μ. Assume Hypothesis 6.1. Then, $\left(|D_F|^{n(n+1)/4}\operatorname{Res}_{s=1}L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})\right)^{-1}$ *spans the one-dimensional $\mathbb{Q}(\Pi_f)$-vector space*

$$\Lambda'\circ\mathbb{L}^{\mathrm{coh},t}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}(\mathbb{S}^{\psi,t}_{\Pi})\subset\mathbb{C}$$

where $\Lambda':(\wedge^t\tilde{\mathfrak{p}}')^*\to\mathbb{C}$ is the evaluation at the element $\Lambda'_0\in\wedge^t\tilde{\mathfrak{p}}'$ defined above.

Proof. Arguing as in the proof of Thm. 5.3, the result follows using relation (6.3), which compares the global period-map $\mathbb{L}^{\mathrm{per},t}$ with the local period-map $\Lambda'\circ\mathbb{L}^{\mathrm{coh},t}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}$, and the fact that $\ell_f({}^\sigma W)=\sigma(\ell_f(W))$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. □

7. A relation between the bottom Whittaker CRS and $L(1,\Pi,\mathrm{As}^{(-1)^n})$

7.1. We will now put the contents of §5 and §6 together, in order to obtain a rationality result for $L(1,\Pi,\mathrm{As}^{(-1)^n})$. To that end, recall the pairing $\mathbb{K}^{\mathrm{coh},i}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee),[\cdot,\cdot]_\infty)}$ from §2.4. We use it to identify $\Lambda'\circ\mathbb{L}^{\mathrm{coh},t}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}$ as an element of

$$H^b(\mathfrak{m}_G,K_\infty,\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee)\otimes E_\mu^\vee)\otimes\wedge^d\tilde{\mathfrak{p}},$$

where $b=\frac{n(n-1)}{2}[F:\mathbb{Q}]$. (Recall that $b+t=d$.) We write this element as $\mathbb{M}^{\mathrm{coh},b}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}\otimes\Lambda_0$, where

$$\mathbb{M}^{\mathrm{coh},b}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}\in H^b(\mathfrak{m}_G,K_\infty,\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee)\otimes E_\mu^\vee).$$

We may now prove our second main theorem on the Asai L-function.

Theorem 7.1. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E)=\mathrm{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic representation E_μ. Assume Hypothesis 6.1. Then,*

$$\left(|D_E/D_F|^{n(n+1)/4}L(1,\Pi,\mathrm{As}^{(-1)^n})\right)^{-1}\cdot\mathbb{M}^{\mathrm{coh},b}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty),\ell_\infty)}$$

spans the one-dimensional $\mathbb{Q}(\Pi_f)$-vector subspace $\mathbb{S}^{\psi^{-1},b}_{\Pi^\vee}$ of $H^b(\mathfrak{m}_G,K_\infty,\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee)\otimes E_\mu^\vee)$.

Proof. The theorem follows readily from Thm. 5.3 and Thm. 6.4 and the relation $L(s,\Pi\times\Pi^\tau)=L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})\cdot L(s,\Pi,\mathrm{As}^{(-1)^n})$. □

Remark 7.2. Theorem 7.1 generalizes a result of the first two named authors, see [GH16] Thm. 6.22.

References

[AG09] Avraham Aizenbud and Dmitry Gourevitch, *Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem*, Duke Math. J. **149** (2009), no. 3, 509–567, With an appendix by the authors and Eitan Sayag. MR 2553879 (2011c:22026)

[BR14] Baskar Balasubramanyam and A. Raghuram, *Special values of adjoint L-functions and congruences for automorphic forms on GL(n) over a number field*, 2014, arXiv:1408.3896.

[BD92] Jean-Luc Brylinski and Patrick Delorme, *Vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein*, Invent. Math. **109** (1992), no. 3, 619–664. MR MR1176208 (93m:22016)

[BW00] Armand Borel and Nolan Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR MR1721403 (2000j:22015)

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. MR 1044819 (91k:11042)

[Fli88] Yuval Z. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France **116** (1988), no. 3, 295–313. MR 984899 (89m:11049)

[Fli91] ———, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172. MR 1111204 (92i:22019)

[FZ95] Yuval Z. Flicker and Dmitrii Zinoviev, *On poles of twisted tensor L-functions*, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), no. 6, 114–116. MR 1344660 (96f:11075)

[GJR01] Stephen Gelbart, Hervé Jacquet, and Jonathan Rogawski, *Generic representations for the unitary group in three variables*, Israel J. Math. **126** (2001), 173–237. MR MR1882037 (2003k:11081)

[Gha99] Eknath Ghate, *Critical values of the twisted tensor L-function in the imaginary quadratic case*, Duke Math. J. **96** (1999), no. 3, 595–638. MR 1671216 (2000a:11076)

[Gre03] Loïc Grenié, *Critical values of automorphic L-functions for* $\mathrm{GL}(r) \times \mathrm{GL}(r)$, Manuscripta Math. **110** (2003), no. 3, 283–311. MR 1969002 (2004b:11065)

[GH16] Harald Grobner and Michael Harris, *Whittaker periods, motivic periods, and special values of tensor product L-functions*, J. Inst. Math. Jussieu **15** (2016) 711–769

[GR14] Harald Grobner and A. Raghuram, *On some arithmetic properties of automorphic forms of* GL_m *over a division algebra*, Int. J. Number Theory **10** (2014), no. 4, 963–1013. MR 3208871

[HL04] Michael Harris and Jean-Pierre Labesse, *Conditional base change for unitary groups*, Asian J. Math. **8** (2004), no. 4, 653–683. MR 2127943 (2006g:11098)

[JS81] Hervé Jacquet and Joseph A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815. MR 623137 (82m:10050b)

[Kem12] Alexander Kemarsky, *Distinguished representations of* $GL(n, \mathbb{C})$, 2012, arXiv:1212.6436.

[LM15] Erez Lapid and Zhengyu Mao, *A conjecture on Whittaker–Fourier coefficients of cusp forms*, J. Number Theory **146** (2015), 448–505. MR 3267120

[Mah05] Joachim Mahnkopf, *Cohomology of arithmetic groups, parabolic subgroups and the special values of L-functions on* GL_n, J. Inst. Math. Jussieu **4** (2005), no. 4, 553–637. MR 2171731 (2006m:11076)

[Mat11] Nadir Matringe, *Distinguished generic representations of* $\mathrm{GL}(n)$ *over p-adic fields*, Int. Math. Res. Not. IMRN (2011), no. 1, 74–95. MR 2755483 (2012f:22032)

[Mok14] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Memoirs of the AMS, **235**, no. 1108, 2014

[Off11] Omer Offen, *On local root numbers and distinction*, J. Reine Angew. Math. **652** (2011), 165–205. MR 2787356

[Pan01] Marie-Noëlle Panichi, *Charactérisations du spectre tempéré de* $GL_n(\mathbb{C})/GL_n(\mathbb{R})$, Ph.D. thesis, Université Paris 7, 2001, www.institut.math.jussieu.fr/theses/2001/panichi/.

[RS08] A. Raghuram and Freydoon Shahidi, *On certain period relations for cusp forms on* GL_n, Int. Math. Res. Not. IMRN (2008), Art. ID rnn 077, 23. MR 2439563 (2010e:11046)

[Roh96] Jürgen Rohlfs, *Projective limits of locally symmetric spaces and cohomology*, J. Reine Angew. Math. **479** (1996), 149–182. MR MR1414393 (97g:20049)

[Sun11] Binyong Sun, *Cohomologically induced distinguished representations and a non-vanishing hypothesis for algebraicity of critical L-values*, 2011, arXiv:1111.2636.

[Sun13] ———, *The nonvanishing hypothesis at infinity for Rankin-Selberg convolutions*, 2013, arXiv:1307.5357.

[Urb95] Éric Urban, *Formes automorphes cuspidales pour* GL_2 *sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. **99** (1995), no. 3, 283–324. MR 1361742 (96m:11039)

[vdBD88] Erik van den Ban and Patrick Delorme, *Quelques propriétés des représentations sphériques pour les espaces symétriques réductifs*, J. Funct. Anal. **80** (1988), no. 2, 284–307. MR 961900 (89j:22025)

[Wal85] J.-L. Waldspurger, *Quelques propriétés arithmétiques de certaines formes automorphes sur* GL(2), Compositio Math. **54** (1985), no. 2, 121–171. MR 783510 (87g:11061a)

[Zha14] Wei Zhang, *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. **27** (2014), no. 2, 541–612. MR 3164988

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