PERIOD RELATIONS FOR CUSP FORMS OF GSp_4 AND SOME MORE GENERAL ASPECTS OF RATIONALITY

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Abstract. Let F be a totally real number field and let π be a cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$, which contributes irreducibly to coherent cohomology. If π has a Bessel model, we may attach a period $p(\pi)$ to this datum. In the present paper, we establish a relation of these Bessel-periods $p(\pi)$ and all of its twists $p(\pi \otimes \xi)$ under arbitrary algebraic Hecke characters ξ. In the appendices we show that $(\mathfrak{g}, K)$-cohomological cusp forms of $\mathrm{GSp}_4(\mathbb{A}_F)$ all qualify to be of the above type – providing a large source of examples. Moreover, in these appendices we also provide proofs of some well-known results on the rationality of semisimple modules of Hecke algebras for further reference. (This is an extended, slightly deviating version of our paper *Period relations for cusp forms of* GSp_4, containing an additional appendix A on rational structures of semisimple Hecke modules.)

Contents

1. Introduction

1.1. Generalizing Euler's classical theorem on the values $\zeta(2k)$ of the Euler-Riemann ζ-function at positive even integers $s = 2k$, Deligne has stated a far-reaching conjecture about the behaviour of motivic L-functions $L(s, M)$ at their critical points $s = k \in \mathbb{Z}$. Deligne's conjectured formula expresses the critical L-values in question up to multiplication by elements in a number-field $E(M)$, depending on the motive M, in terms of certain geometric period-invariants $c^{\pm}(M)$, as well as certain explicit integral powers $(2\pi i)^{d(k)}$, [Del79, Conj. 2.8]:

$$L(k, M) \in (2\pi i)^{d(k)} c^{(-1)^k}(M) \cdot E(M).$$

2010 *Mathematics Subject Classification.* Primary: 11F67; Secondary: 11F41, 11F70, 11F75, 22E55.
H.G. is supported by the Austrian Science Fund (FWF), START-prize Y966-N35 and stand-alone-research project P 25974-N25. R.S. was partly supported by a DST INSPIRE grant.

It is important to notice that, stated this way, Deligne's conjecture entails the following relation of the periods attached to the motive M and its twist $M(k)$,

$$c^{\pm}(M(k)) = (2\pi i)^{d(k)} c^{\pm(-1)^k}(M).$$

proved by Deligne in [Del79, equation (5.1.8)].

Apart from particular cases Deligne's conjecture is still wide open. However, even when such a precise relation between periods $c^{\pm}(M)$ and critical values of $L(s, M)$ is unknown, it is still an important task to investigate *period-relations* of the above type: Inspired by Deligne's result, relating the periods of M and $M(k)$, Blasius [Bla97] and Panchishkin [Pan94] have formulated precise expectations of how Deligne's periods transfer under twisting by Artin motives, and – using the conjectured dictionary between motives and automorphic representations – Harris has established period-relations for motives coming from cuspidal automorphic representations from unitary groups [MHar97].

Matching the spirit of the latter approach, it is the automorphic side, where most of the recent results on period-relations have been achieved. Lacking the well-shaped rigidity of the motivic world, automorphic periods allow more freedom in the choice of their definition[1]: As a general principle, automorphic periods are defined by a comparison of two rational structures: One on a space of cohomology and one of a certain model-space of the given automorphic representation π. While the rational structure on the first space is of geometric origin, the rational structure on the latter space is defined by reference to the uniqueness of the chosen model. Finally, in order to actually compare the two rational structures one has to make a choice of an embedding of the model space at hand into the given cohomological realization of π: One possible technique to make this choice of an embedding is by fixing a cohomological vector at infinity.

Following this principle, in [Rag-Sha08] Raghuram-Shahidi have used the Whittaker model of a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, F any number field, the space of $(\mathfrak{g}, K)$-cohomology in lowest degree and a choice of a cohomological vector at infinity, $\mathbf{w}_\infty$, in order to define Whittaker-periods $p(\pi) = p(\pi, \mathbf{w}_\infty)$ attached to this datum. Inspired by Blasius' results mentioned above, they then derive a theorem on period-relations between $p(\pi)$ and $p(\pi \otimes \chi)$ for an algebraic Hecke character χ: As predicted, up to multiplication by an element in the rationality field $\mathbb{Q}(\mathcal{W}(\pi), \chi)$ of the Whittaker model $\mathcal{W}(\pi)$ and χ, $p(\pi)$ and $p(\pi \otimes \chi)$ differ only by a certain power of the Gauß sum $\mathscr{G}(\chi_f)$.

In [Gro-Rag14], Raghuram and the first mentioned author of the present article achieved period-relations for Shalika-periods $\omega(\pi)$ and $\omega(\pi \otimes \chi)$. These periods are defined by reference to the Shalika model of a cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, F a totally real number field, the space of $(\mathfrak{g}, K)$-cohomology in highest degree and a chosen cohomological vector at infinity, denoted $[\pi_\infty]^\epsilon$. Again, $\omega(\pi)$ and $\omega(\pi \otimes \chi)$ only differ by a

[1] This freedom happens at a price of uncertainty: The question whether or not *automorphic periods* are periods (in the sense of Kontsevich-Zagier [Kon-Zag01]) seems almost as hard to decide as showing Deligne's conjecture in the respective case.

certain power of the Gauß sum $\mathscr{G}(\chi_f)$ up to multiplication by an element in the rationality field $\mathbb{Q}(\mathcal{S}(\pi), \chi)$ of the Shalika model $\mathcal{S}(\pi)$ and χ.

1.2. The present paper continues this series of results, but focuses on a completely new aspect of the theory: In this article, we describe the relation of what we call Bessel-periods: These are periods $p(\pi)$ for cuspidal automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_F)$, F any totally real number field, which contribute irreducibly to coherent cohomology and allow a Bessel model. We refer to Def. 3.3.11 for their precise definition and only mention here that they also depend on the choice of a $(\mathfrak{p}, K)$-cohomological vector $\phi_{\pi,\infty}$.

For an arbitrary algebraic Hecke character ξ, let $\pi \otimes \xi = \xi(\mu(.)) \cdot \pi$ be the cuspidal automorphic representation obtained by multiplying the functions in π by the composite of ξ with the symplectic similitude character μ. Our main result on the relation of our Bessel-periods $p(\pi) = p(\pi, \phi_{\pi,\infty})$ and $p(\pi \otimes \xi) = p(\pi \otimes \xi, \phi_{\pi\otimes\xi,\infty})$ reads as follows

Theorem. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ which contributes irreducibly to coherent cohomology and allows a Bessel model $\mathcal{B}(\pi)$, see §3.1 and §3.2. Let ξ be any algebraic Hecke character of $\mathbb{A}_F^*$. Then there are periods $p(\pi) = p(\pi, \phi_{\pi,\infty})$ and $p(\pi \otimes \xi) = p(\pi \otimes \xi, \phi_{\pi\otimes\xi,\infty})$, which satisfy the following relation*

$$p(\pi \otimes \xi) \;\sim\; p(\pi) \cdot \mathscr{G}(\xi_f),$$

where "$\sim$" means up to multiplication by a non-zero element in a finite extension of the rationality-field $\mathbb{Q}(\mathcal{B}(\pi), \xi)$ of $\mathcal{B}(\pi)$ and ξ and $\mathscr{G}(\xi_f)$ denotes the Gauß-sum of ξ_f, cf. §2.1.3.

We would like to point out that for cuspidal representations π admitting a Langlands-transfer to $\mathrm{GL}_4(\mathbb{A}_F)$, our theorem is compatible with the results obtained in [Gro-Rag14], see Rem. 4.3.4, and – modulo the conjectural translation of cuspidal algebraic representations into motives – expected to be compatible with the conjectures of Blasius and Panchishkin, mentioned above. We refer to our Prop. 3.3.12 and Thm. 4.3.3 for further details concerning the construction of our periods $p(\pi)$ and $p(\pi \otimes \xi)$ as well as for the various dependencies in their relation.

In our two Appendices we enlarge the focus of the present paper: While Appendix B is mainly written to provide a large class of candidate-representations π for our main theorem above, it also establishes a general reference for the relation of $(\mathfrak{g}, K)$- and $(\mathfrak{p}, K)$-cohomology for all connected reductive groups G over $\mathbb{Q}$. In Appendix A we give detailed proofs of some results on existence of rational structures on admissible $G(\mathbb{A}_f)$-modules, where G denotes an arbitrary reductive group over a general number field F. These results seem to be well known to experts, but the authors were not able to find any references for the same, and so this appendix has been included for the benefit of the reader (and the authors) as a hopefully valuable source of reference for future articles in the field.

We hope that our main theorem on period-relations will allow the conceptual treatment of questions of rationality of the special values of L-functions attached to π as above. Exploring the precise relation between $p(\pi)$ and such L-values is work in progress and we hope to be able to report on this subject in the forthcoming part II of this paper.

Acknowledgements: Both authors would like to thank A. Raghuram for suggesting to work on this project. Thanks are also due to him for useful discussions. We are also grateful to the anonymous referee for her/his careful reading and numerous, very helpful suggestions, which improved the presentation of the present article.

2. Basic notation and conventions

2.1. Number fields and algebraic characters.

2.1.1. Unless otherwise stated, F will be a totally real number field of degree $d = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}$. For any place v we write F_v for the topological completion of F at v. Let S_∞ be the set of archimedean places of F. If $v \notin S_\infty$, we let $\mathcal{O}_v$ be the local ring of integers of F_v with unique maximal ideal $\wp_v$. Moreover, $\mathbb{A}$ denotes the ring of adèles of F and $\mathbb{A}_f$ its finite part. We use the local and global normalized absolute values and denote each of them by $|\cdot|$. Further, $\mathfrak{D}_F$ stands for the absolute different of F, that is, $\mathfrak{D}_F^{-1} = \{x \in F : Tr_{F/\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z}\}$.

2.1.2. We fix a non-trivial, additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^*$ following Tate's thesis, see [Gro-Rag14] §2.7. In particular, $\psi = \otimes_v \psi_v$ takes values in the subgroup $\boldsymbol{\mu}_\infty$ of $\mathbb{C}^*$ consisting of all roots of unity and if we factor the different $\mathfrak{D}_F = \prod_\wp \wp^{r_\wp}$, with the product running over all prime ideals $\wp$ of $\mathcal{O}$, then $\wp_v^{-r_\wp}$ is the conductor of the character ψ_v at $v \notin S_\infty$.

2.1.3. Let $\chi = \chi_\infty \otimes \chi_f$ be any algebraic Hecke character of $\mathbb{A}$. We define the Gauß sum of its finite part χ_f, following Weil [Wei67, VII, Sect. 7]: Let $\mathfrak{c}_\chi$ stand for the conductor ideal of χ_f and let $y = (y_v)_{v \notin S_\infty} \in \mathbb{A}_f^\times$ be chosen such that $\mathrm{ord}_v(y_v) = -\mathrm{ord}_v(\mathfrak{c}_\chi) - \mathrm{ord}_v(\mathfrak{D}_F)$. The Gauß sum of χ_f is now defined as $\mathscr{G}(\chi_f, \psi_f, y) = \prod_{v \notin S_\infty} \mathscr{G}(\chi_v, \psi_v, y_v)$, where the local Gauß sum $\mathscr{G}(\chi_v, \psi_v, y_v)$ is defined as

$$\mathscr{G}(\chi_v, \psi_v, y_v) = \int_{\mathcal{O}_v^\times} \chi_v(u_v)^{-1} \psi_v(y_v u_v)\, du_v.$$

For almost all v, we have $\mathscr{G}(\chi_v, \psi_v, y_v) = 1$, and for all v we have $\mathscr{G}(\chi_v, \psi_v, y_v) \neq 0$. (See, for example, Godement [God70, Eq. 1.22].) Note that, unlike in [Wei67], we do not normalize the Gauß sum to make it have absolute value one. Suppressing the dependence on ψ and y, we denote $\mathscr{G}(\chi_f, \psi_f, y)$ simply by $\mathscr{G}(\chi_f)$.

2.2. The symplectic similitude group and its variants.

2.2.1. *Algebraic groups and varieties.* Let I_n be the $n \times n$-identity matrix and let

$$J_4 := \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Unless otherwise stated, in this paper, we let

$$G := \mathrm{GSp}_4 := \left\{ g \in \mathrm{GL}_4 | {}^t g J_4 g = \mu(g) J_4 \right\},$$

the F-split symplectic similitude group of degree 4. The kernel of the character $\mu : \mathrm{GSp}_4 \to \mathrm{GL}_1$ is the F-split symplectic group $G^{\mathsf{ss}} = \mathrm{Sp}_4$. We let $\theta : G \to G$ be the Cartan involution on G being defined by $\theta(g) := {}^t g^{-1} = \mu(g)^{-1} J_4 g J_4^{-1}$.

Let $R_{\mathbb{K}/\mathbb{F}}$ stand for Weil's restriction of scalars from $\mathbb{K}$ to $\mathbb{F}$. We consider the homomorphism $h : R_{\mathbb{C}/\mathbb{R}}(GL_1) \to R_{F/\mathbb{Q}}(G) \times_{\mathbb{Q}} \mathbb{R}$ which maps $x + iy \in \mathbb{C}^*$ to d identical copies of the matrix

$$\begin{pmatrix} xI_2 & yI_2 \\ -yI_2 & xI_2 \end{pmatrix}.$$

The $R_{F/\mathbb{Q}}(G)(\mathbb{R})$-conjugacy class X of h is diffeomorphic to the d disjoint unions of the Siegel upper and lower half space of genus 2. The pair $(R_{F/\mathbb{Q}}(G), X)$ is (hence) a Shimura datum in the sense of [MHar85], 1.1. See also Sect. B.1.

2.2.2. *Real Lie groups.* We will abbreviate $G_\infty := \prod_{v \in S_\infty} G(F_v)$ (respectively, $G^{\mathsf{ss}}_\infty := \prod_{v \in S_\infty} G^{\mathsf{ss}}(F_v)$). Lie algebras of real Lie groups are denoted by the same letter, but in lower case gothics.

The connected component of the identity of the group of fixed points of θ in G_∞ is isomorphic to d copies of $\mathrm{U}(2)$, the compact unitary group, and defines a maximal compact subgroup K^{ss}_∞ of G^{ss}_∞. We let K_∞ be the product of K^{ss}_∞ (being identified with $\mathrm{U}(2)^d$) and the center $Z_{G,\infty}$ of G_∞ (being identified with $(\mathbb{R}^* I_4)^d$). The group K_∞ is isomorphic to the centralizer of a fixed point $h \in X$. With these identifications, we have $\mathrm{U}(2) \cap \mathbb{R}^* = \{\pm I_4\}$, so $K_\infty = Z^\circ_{G,\infty} \times K^{\mathsf{ss}}_\infty \cong (\mathbb{R}_{>0} \times \mathrm{U}(2))^d$

Much confusion is avoided, if the reader bears in mind that this group K_∞ does not contain a maximal compact subgroup of G_∞ (which has $|\pi_0(G_\infty)| = 2^d$ connected components), but rather the connected component of the identity of such a group. As a consequence, the $(\mathfrak{g}_\infty, K_\infty)$-module of K_∞-finite vectors in the archimedean component of a given automorphic representation is in general not irreducible (but decomposes as the direct sum of at most 2^d irreducible $(\mathfrak{g}_\infty, K_\infty)$-modules).

2.3. $(\mathfrak{p}_h, K_\infty)$-cohomology and coherent cohomology.

2.3.1. *Relative Lie algebra cohomology.* The Lie algebra $\mathfrak{k}_\infty$ of K_∞ operates by the adjoint action on $\mathfrak{g}_{\infty,\mathbb{C}} := \mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$ and we obtain a $\mathfrak{k}_\infty$-invariant decomposition

$$\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\infty,\mathbb{C}} \oplus \mathfrak{p}_-. \tag{2.3.1}$$

Here, $\mathfrak{p}_-$ (resp. $\mathfrak{p}_+$) is the holomorphic (resp. anti-holomorphic) tangent space of X at h. We let

$$\mathfrak{p}_h := \mathfrak{k}_{\infty,\mathbb{C}} \oplus \mathfrak{p}_+.$$

This is a parabolic subalgebra of $\mathfrak{g}_{\infty,\mathbb{C}}$ with Levi subalgebra $\mathfrak{k}_{\infty,\mathbb{C}}$ and nilpotent, even abelian, radical $\mathfrak{p}_+$. Observe that $\mathfrak{p}_h$ lies somewhat "skew" to the real structure of $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \oplus i\mathfrak{g}_\infty$ as $\mathfrak{p}_h \cap \mathfrak{g}_\infty = \mathfrak{k}_\infty$.

For us, a $\mathfrak{g}_\infty$-module V (on a complex locally convex vector space), which is also a representation of K_∞, is called a $(\mathfrak{g}_\infty, K_\infty)$-module, if it is a $(\mathfrak{g}^{\mathsf{ss}}_\infty, K^{\mathsf{ss}}_\infty)$-module in the sense of Borel-Wallach [Bor-Wal00], §0.2, by restriction.

The $(\mathfrak{p}_h, K_\infty)$-cohomology of a $(\mathfrak{g}_\infty, K_\infty)$-module V is the cohomology of the complex

$$C^q(\mathfrak{p}_h, K_\infty, V) := \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}_h/\mathfrak{k}_{\infty,\mathbb{C}}), V) \cong \mathrm{Hom}_{K_\infty}(\Lambda^q \mathfrak{p}_+, V),$$

with the usual derivatives, cf. [MHar90] (4.1.3) and [Bor-Wal00] §I.1.1. Following [MHar90] §4.1.1, we say that a $(\mathfrak{g}_\infty, K_\infty)$-module V is $(\mathfrak{p}_h, K_\infty)$-*cohomological*, if there is an irreducible finite-dimensional K_∞-module V_τ such that $H^q(\mathfrak{p}_h, K_\infty, V \otimes V_\tau) \neq 0$ for some degree q. We will furthermore assume that the representation V_τ is *algebraic*, that is, the $\mathbb{C}$-linear extension of V_τ to a $\mathfrak{k}_{\infty,\mathbb{C}}$-module extends to a representation of the algebraic group $K_{\infty,\mathbb{C}}$ (defined as the Levi subgroup of the unique parabolic complex algebraic subgroup $\mathscr{P}_h(\mathbb{C})$ of $G(\mathbb{C})$ with Lie algebra $\mathfrak{p}_h$).

2.3.2. *Coherent sheaf cohomology.* In this subsection we briefly summarize the discussion in [MHar90, §1,2], which the reader is referred to for more details. Define

$$M(G,X)_\mathbb{C} := G(F)\backslash(X \times G(\mathbb{A}_f)).$$

If $K \subset G(\mathbb{A}_f)$ is a neat, open compact subgroup, then

$${}_KM := M(G,X)_\mathbb{C}/K = G(F)\backslash(X \times G(\mathbb{A}_f))/K$$

is a smooth quasi-projective variety. For suitable compactifying data, denoted Σ, there is a smooth toroidal compactification of ${}_KM$, denoted ${}_KM_\Sigma$, with boundary a snc divisor, denoted Z_Σ. All these varieties are defined over the reflex field $E(G,X)$.

Let V_τ be an algebraic irreducible finite-dimensional K_∞-module, one obtains an automorphic vector bundle ${}_K[\mathscr{V}_\tau]$ on the quasi-projective variety ${}_KM$. This vector bundle has a canonical extension to ${}_KM_\Sigma$, denoted ${}_K[\mathscr{V}_\tau]^{can}_\Sigma$, and a subcanonical extension, defined as ${}_K[\mathscr{V}_\tau]^{sub}_\Sigma := {}_K[\mathscr{V}_\tau]^{can}_\Sigma(-Z_\Sigma)$. Since V_τ was chosen algebraic, these bundles are defined over a finite extension $E = E(\tau)$ of the reflex field $E(G,X)$, which can be computed explicitly, [Del71] Prop. 3.8.

It is proved in [MHar90] that the cohomology $H^*({}_KM_\Sigma, {}_K[\mathscr{V}_\tau]^{can}_\Sigma)$ and $H^*({}_KM_\Sigma, {}_K[\mathscr{V}_\tau]^{sub}_\Sigma)$ are independent of the toroidal compactification. Thus, we may drop the 'Σ' from the notation and define

$$H^q([\mathscr{V}_\tau]^{\mathrm{can}}) := \varinjlim_K H^*({}_KM, {}_K[\mathscr{V}_\tau]^{can}) \qquad \text{and} \qquad H^q([\mathscr{V}_\tau]^{\mathrm{sub}}) := \varinjlim_K H^*({}_KM, {}_K[\mathscr{V}_\tau]^{sub}).$$

These are admissible $G(\mathbb{A}_f)$-modules, defined over the field of definition $E = E(\tau)$ of $[\mathscr{V}_\tau]$ (in the sense of Waldspurger [Wal85], I.1; and our Appendix A.1), see [MHar90] Prop. 2.8. This also applies to the image $\bar{H}^q([\mathscr{V}_\tau])$ of the natural map $H^q([\mathscr{V}_\tau]^{\mathrm{sub}}) \to H^q([\mathscr{V}_\tau]^{\mathrm{can}})$.

2.3.3. *Automorphic coherent cohomology.* Given an irreducible, algebraic representation V_τ of K_∞ as above, let χ_τ be the restriction to $Z^\circ_{G_\infty}$ of its central character $Z_{K_\infty} \to \mathbb{C}^*$. Observe that $Z^\circ_{G_\infty}$ appears as a direct factor in G_∞, so by abuse of notation, χ_τ also defines a character of the group G_∞ (and even $G(\mathbb{A})$). We let $\mathscr{A}_{(2)}(G,\chi_\tau)$ be the space of all automorphic forms $f : G(\mathbb{A}) \to \mathbb{C}$, which are square-integrable modulo χ_τ, i.e.,

$$\int_{Z^\circ_{G_\infty} G(F)\backslash G(\mathbb{A})} |\chi_\tau(g) f(g)|^2 dg < \infty.$$

As $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-module, it decomposes as a countable direct sum,

$$\mathscr{A}_{(2)}(G,\chi_\tau) = \bigoplus_\pi \pi^{m_{(2)}(\pi)} \tag{2.3.2}$$

over all (equivalence classes of) irreducible $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-modules π, each appearing with finite multiplicity $0 \leq m_{(2)}(\pi) < \infty$. Let $\mathscr{A}_0(G,\chi_\tau)$ be the subspace of cuspidal automorphic forms in $\mathscr{A}_{(2)}(G,\chi_\tau)$. It inherits from (2.3.2) a direct sum decomposition $\mathscr{A}_0(G,\chi_\tau)=\bigoplus_\pi \pi^{m_0(\pi)}$, where clearly $0\le m_0(\pi)\le m_{(2)}(\pi)$.

There is the following commutative diagram of admissible $G(\mathbb{A}_f)$-modules, cf. [MHar90, Prop. 3.6, Thm. 5.3]

$$\begin{tikzcd}
H^q(\mathfrak{p}_h,K_\infty,\mathscr{A}_0(G,\chi_\tau)\otimes V_\tau) \arrow[rr, hook] \arrow[d, hook, "\alpha"] & & H^q(\mathfrak{p}_h,K_\infty,\mathscr{A}_{(2)}(G,\chi_\tau)\otimes V_\tau) \arrow[d, "\beta"'] \\
H^q([\mathscr{V}_\tau]^{\mathrm{sub}}) \arrow[r, "\gamma"] & \bar{H}^q([\mathscr{V}_\tau]) \arrow[r, hook, "\iota"] & H^q([\mathscr{V}_\tau]^{\mathrm{can}})
\end{tikzcd}$$

with $\ker(\gamma\circ\alpha)=\{0\}$ and $\mathrm{Im}(\beta)\supseteq\mathrm{Im}(\iota)$. In particular, $\bar{H}^q([\mathscr{V}_\tau])$ inherits from (2.3.2) the structure of a semisimple $G(\mathbb{A}_f)$-module: For each of its isotypic components $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)$, the representation π_f appears as the finite part of an irreducible automorphic subrepresentation of $\mathscr{A}_{(2)}(G,\chi_\tau)$.

3. Bessel models and Bessel periods for GSp$_4$

3.1. **The "Bessel subgroup"** R**.** Let $P=M\cdot U$ be the Siegel parabolic subgroup of G. Explicitly, its unipotent radical is the abelian group

$$U=\left\{\begin{pmatrix} I_2 & X\\ 0 & I_2\end{pmatrix}\Big| X\in M_2 \text{ and } {}^tX=X\right\}$$

and its Levi factor

$$M=\left\{\begin{pmatrix} g & 0\\ 0 & x\,{}^tg^{-1}\end{pmatrix}\Big| g\in \mathrm{GL}_2 \text{ and } x\in\mathrm{GL}_1\right\}\cong \mathrm{GL}_2\times\mathrm{GL}_1.$$

A symmetric matrix $\beta\in M_2(F)$ will be called non-degenerate if $\det(\beta)\neq 0$. For such a β there is a linear form ℓ_β on U given by

$$\ell_\beta\begin{pmatrix} I_2 & u\\ 0 & I_2\end{pmatrix}=\mathrm{Tr}(\beta u).$$

The group M acts on U by conjugation and so on the space of linear forms. Let $D_\beta\subset M$ denote the connected component of the identity of the stabilizer of ℓ_β under the conjugation action. It has the following explicit description: Let

$$\beta=\begin{pmatrix} a & \frac{b}{2}\\ \frac{b}{2} & c\end{pmatrix}\qquad\text{and}\qquad \delta=\begin{pmatrix} \frac{b}{2} & c\\ -a & -\frac{b}{2}\end{pmatrix}$$

and let $d:=b^2-4ac\neq 0$. One checks easily that $\delta^2=\frac{d}{4}I_2$. Denote by $F(\delta):=F\oplus F\delta$ the semisimple quadratic subalgebra of $M_2(F)$ over F, obtained from F by adjoining δ. Then there is an isomorphism of algebraic groups $R_{F(\delta)/F}(\mathrm{GL}_1)\xrightarrow{\sim} D_\beta\subset M$, given by

$$(x+y\delta)\mapsto\begin{pmatrix} x+y\delta & 0\\ 0 & x-y\,{}^t\delta\end{pmatrix}.$$

In order to simplify notation, from now on we suppress the dependence of D_β on β. Finally we get an algebraic subgroup $R:=DU\subset P\subset G$ over F. Observe that $D\cap U=\{e\}$ and so to give a character of R it suffices to give characters of D and U. For more details the reader may consult [PS97, Section 2] and [Fur93, Section 1.1].

3.2. **Definition of Bessel Models.** Let ψ be the non-trivial additive character from §2.1. Fix a non-degenerate symmetric matrix $\beta \in M_2(F)$ and consider the character ψ_β of $U(F)\backslash U(\mathbb{A})$ defined by $\psi_\beta(u) := \psi(\mathrm{Tr}(\beta u))$. Let ν be an algebraic character of $D(F)\backslash D(\mathbb{A})$, i.e, ν_f takes values in $\bar{\mathbb{Q}}^*$. Combining these two gives rise to an algebraic character of $R(\mathbb{A})$ defined by $\alpha_{\nu,\beta}(du) := \nu(d) \cdot \psi_\beta(u)$.

Let (π, V) be a cuspidal automorphic representation of $G(\mathbb{A})$. For convenience we will not distinguish between a cuspidal automorphic representation, its smooth Fréchet-space completion of moderate growth and its (non-smooth) Hilbert space completion associated with the L^2-spectrum. Assume that there is a pair (ν, β) as above and a cusp form $\varphi \in V$ such that

$$B_\varphi(g) := \int_{Z(\mathbb{A})R(F)\backslash R(\mathbb{A})} \varphi(rg)\alpha_{\nu,\beta}(r)^{-1} dr \neq 0, \tag{3.2.1}$$

for some invariant Haar measure dr on $R(\mathbb{A})$. Obviously, in order for the integrand to be well-defined mod $Z(\mathbb{A})$, this entails the assertion that $\nu(z) = \omega_\pi(z)$ for all $z \in Z(\mathbb{A})$, where ω_π denotes the central character of π. By the irreducibility of π, B_φ being non-zero for some $\varphi \in V$ is equivalent to the assumption that the map

$$V \to \mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})} \alpha_{\nu,\beta}, \qquad \varphi \mapsto B_\varphi$$

is a $G(\mathbb{A})$-equivariant inclusion. We shall denote the image of V by $\mathcal{B}^\nu_\beta(\pi)$ and call it a (ν, β)-*Bessel model* of the representation π. With (π, V) as above, given a place v of F and the irreducible admissible representation (π_v, V_v) of $G(F_v)$, suppose there is a pair (ν_v, β) and a nonzero $G(F_v)$-equivariant map

$$B : V_v \to \mathrm{Ind}_{R(F_v)}^{G(F_v)} \alpha_{\nu_v,\beta},$$

then we say that π_v has a (ν_v, β)-Bessel model $\mathcal{B}^{\nu_v}_\beta(\pi_v)$. Clearly, if a global Bessel model exists, then one also gets local Bessel models by restriction, in particular $\mathcal{B}^{\nu_f}_\beta(\pi_f)$ is well-defined in this case. The space $\mathrm{Hom}_{R(F_v)}(\pi_v, \alpha_{\nu_v,\beta})$ is at most one dimensional, see [PS97, Theorem 3.1] and [Pra-Tak11], Thm. 1. It follows that if (π, V) is as above and has a global Bessel model, then the space $\mathrm{Hom}_{R(\mathbb{A}_f)}(\pi_f, \alpha_{\nu_f,\beta})$ is exactly one dimensional, i.e., $\mathcal{B}^{\nu_f}_\beta(\pi_f)$ is unique.

3.3. **Definition of the Cohomological Bessel-periods.** In this and the next section we work with an irreducible cuspidal automorphic representation (π, V) of $G(\mathbb{A})$ which satisfies the following assumptions:

(1) (π, V) has a (ν, β)-Bessel model. (I.e., the map B defined in equation (3.2.1) is nonzero.)
(2) (π_∞, V_∞) is $(\mathfrak{p}_h, K_\infty)$-cohomological in degree q with respect to the coefficient module V_τ (recall that V_τ is an irreducible finite-dimensional K_∞-module), §2.3.
(3) The isotypical component $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)$ is an irreducible $G(\mathbb{A}_f)$ representation.

Remark 3.3.1. (1) Condition (3) is clearly the strongest assumption on (π, V) as it somehow imitates a Multuplicity One and a Strong Multiplicity One result for parts of the square-integrable automorphc spectrum of $G(\mathbb{A})$. It is hence a legitimate question whether or not such cusp forms exist. Invoking our Thm. B.2.1 from our appendix below, it looks very plausible however that Ikeda and Yamana just recently constructed a whole family of cuspidal automorphic representations of $G(\mathbb{A})$, which satisfy (1) - (3). We refer to their Thm. 1.2 in [Ike-Yam15].

(2) Note that condition (3) forces the $(\mathfrak{p}_h, K_\infty)$-cohomology $H^q(\mathfrak{p}_h, K_\infty, V_\infty \otimes V_\tau)$ to be one dimensional and also forces that there is a canonical isomorphism $H^q(\mathfrak{p}_h, K_\infty, V \otimes V_\tau) \xrightarrow{\sim} \bar{H}^q([\mathscr{V}_\tau])(\pi_f)$, see §2.3.3.

We will make use of the following intermediate

Proposition 3.3.2. *Let G be a group and let (ρ, V) be a complex representation of G. Suppose this representation is defined over a field $L \subset \mathbb{C}$, that is, there is a L-subspace $V^0 \subset V$ which is invariant under G and the natural map $V^0 \otimes_L \mathbb{C} \to V$ is an isomorphism of complex G-representations. Let H be a subgroup of G with an L-valued character $\chi : H \to L^* \subseteq \mathbb{C}^*$ such that $\dim_{\mathbb{C}} \mathrm{Hom}_H(\rho, \chi) = 1$. Let $l \neq 0$ be an element of $\mathrm{Hom}_H(\rho, \chi)$, then there is a scalar $p(l) \in \mathbb{C}^*$ such that $(l/p(l))(V^0) \subset L$.*

Proof. Let $\{\lambda_i\}_{i\in I}$ be a basis for $\mathbb{C}$ over L. The restriction of l to V^0 is a L-linear and H-equivariant map $V^0 \to \bigoplus_{i\in I} L \cdot \lambda_i$, where the H action on the right is via its action on $L \cdot \lambda_i$ by χ. Let $l_i : V^0 \to L$ denote the coefficient of the i^{th} basis vector, so that $l(v) = \bigoplus_{i\in I} l_i(v) \cdot \lambda_i$. Since $l \neq 0$ there is an $i \in I$ such that the projection $l_i : V^0 \to L$ is nonzero. It is clear that l_i is H-equivariant, L-linear and after tensoring with $\mathbb{C}$ gives a nonzero element $l_i \otimes 1 \in \mathrm{Hom}_H(\rho, \chi)$. Since $\mathrm{Hom}_H(\rho, \chi)$ is one dimensional we see that $l_i \otimes 1 = a\, l$, for some scalar $a \in \mathbb{C}^*$. It is also clear that $l_i \otimes 1(V^0) \subset L$, and so the proposition is proved by taking $p(l) := 1/a$. □

3.3.1. *The first intertwining ℓ_π.* There is a finite extension E' of the field of definition $E := E(\tau)$ of $[\mathscr{V}_\tau]$, such that the $G(\mathbb{A}_f)$-representation $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)$ is defined over E' and is compatible with the E' structure on $\bar{H}^q([\mathscr{V}_\tau])$, see Thm. A.2.4. Denote this E'-structure by $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'}$. By the one–dimensionality of the space of Bessel functionals, as remarked earlier, applying proposition 3.3.2 with $\rho = \bar{H}^q([\mathscr{V}_\tau])(\pi_f)$, $H = R(\mathbb{A}_f)$, $\chi = \alpha_{\nu_f,\beta}$ and $L = \bar{\mathbb{Q}}$, we get that there is a nonzero $\ell_\pi \in \mathrm{Hom}_{G(\mathbb{A}_f)}(\bar{H}^q([\mathscr{V}_\tau])(\pi_f), \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta})$ such that $\ell_\pi(\bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'}) \subset C^\infty(G(\mathbb{A}_f), \bar{\mathbb{Q}})$. Let $\mathbb{Q}(\nu_f)$ denote the subfield of $\mathbb{C}$ generated by the image of ν_f. Next we modify ℓ_π so that the image $\ell_\pi(\bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'})$ is contained in $C^\infty(G(\mathbb{A}_f), E'\mathbb{Q}(\nu_f))$.

To this end, consider the following composite of maps

$$\begin{array}{ccccccccc} \mathrm{Aut}(\mathbb{C}) & \to & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_\infty)/\mathbb{Q}) & \to & \hat{\mathbb{Z}}^* \cong \Pi_p \mathbb{Z}_p^* & \subset & \Pi_p \Pi_{v|p} \mathcal{O}_v^* \\ \sigma & \mapsto & \sigma|_{\bar{\mathbb{Q}}} & \mapsto & \sigma|_{\mathbb{Q}(\boldsymbol{\mu}_\infty)} & \mapsto & t_\sigma & \mapsto & \Pi_p \Pi_{v|p} t_v \end{array}$$

Denote by T_σ the matrix $\mathrm{diag}(t_\sigma^{-1}, t_\sigma^{-1}, 1, 1) \in G(\mathbb{A}_f)$. For $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\nu_f))$ define

$$\tilde{\sigma} : \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta} \to \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta}$$

by

$$\tilde{\sigma}(b)(g) := \sigma(b(T_\sigma g)).$$

One easily checks that this is a σ-linear isomorphism of $G(\mathbb{A}_f)$-representations (using the fact that $\sigma(\psi(t_\sigma^{-1}s)) = \psi(s)$ and $\sigma \circ \nu_f = \nu_f$). Now suppose that $\sigma \in \mathrm{Aut}(\mathbb{C}/E'\mathbb{Q}(\nu_f))$. Consider the arrows in the (not necessarily commutative) diagram

$$\begin{tikzcd} \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \mathbb{C} \arrow[r, "\ell_\pi"] \arrow[d, "(1\otimes\sigma)"'] & \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta} \arrow[d, "\tilde{\sigma}"] \\ \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \mathbb{C} \arrow[r, "\ell_\pi"] & \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta} \end{tikzcd}$$

The map $\tilde{\sigma}^{-1} \circ \ell_\pi \circ (1 \otimes \sigma)$ is a nonzero map of $G(\mathbb{A}_f)$-representations from $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)$ to $\mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \alpha_{\nu_f,\beta}$. As the space of such maps is one dimensional it follows that $\ell_\pi \circ (1 \otimes \sigma) = \sigma(a) \cdot (\tilde{\sigma} \circ \ell_\pi)$ for some scalar $a = a_\sigma \in \mathbb{C}^*$, depending on σ. Let $0 \neq v^0 \in \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'}$ and let $g \in G(\mathbb{A}_f)$ be such that $\ell_\pi(v^0)(g) \neq 0$ (such a g exists by the irreducibility of $\bar{H}^q([\mathscr{V}_\tau])(\pi_f)$). Then evaluating the equality at v^0 and g we get

$$\ell_\pi(v^0)(g) = \sigma(a) \cdot \tilde{\sigma}(\ell_\pi(v^0))(g) = \sigma(a) \cdot \sigma(\ell_\pi(v^0)(T_\sigma g)).$$

As $\ell_\pi(v^0)$ maps $G(\mathbb{A}_f)$ into $\bar{\mathbb{Q}}$, this shows that $\sigma(a)$, and hence also a, is in $\bar{\mathbb{Q}}^*$.

Definition 3.3.3. Recall that $\mathcal{B}_\beta^{\nu_f}(\pi_f) = \ell_\pi(\bar{H}^q([\mathscr{V}_\tau])(\pi_f))$ by uniqueness of Bessel-models and let $\mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}} := \ell_\pi(\bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{\bar{\mathbb{Q}}})$.

As $a \in \bar{\mathbb{Q}}^*$, we get that $\tilde{\sigma}$ takes $\mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}}$ to itself. For $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}/E'\mathbb{Q}(\nu_f))$, we infer from the previous diagram the following (not necessarily commutative) square

$$\begin{tikzcd} \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \bar{\mathbb{Q}} \arrow[r, "\ell_\pi"] \arrow[d, "(1\otimes\sigma)"'] & \mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}} \arrow[d, "\tilde{\sigma}"] \\ \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \bar{\mathbb{Q}} \arrow[r, "\ell_\pi"] & \mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}} \end{tikzcd} \tag{3.3.4}$$

As already observed above, there is a scalar $a_\sigma \in \bar{\mathbb{Q}}^*$ such that

$$\ell_\pi \circ (1 \otimes \sigma) = a_\sigma \, \tilde{\sigma} \circ \ell_\pi.$$

One checks easily that the a_σ satisfy the cocycle condition. Since $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/E'\mathbb{Q}(\nu_f)), \bar{\mathbb{Q}}^*) = 0$, there is hence a $b \in \bar{\mathbb{Q}}^*$ such that for all $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}/E'\mathbb{Q}(\nu_f))$

$$b\,\ell_\pi \circ (1 \otimes \sigma) = \tilde{\sigma} \circ b\,\ell_\pi. \tag{3.3.5}$$

In other words, $b \in \bar{\mathbb{Q}}^*$ is independent of σ (whereas the a_σ were). Note that b may be modified by any element of $(E'\mathbb{Q}(\nu_f))^*$. The following diagram is hence commutative for all $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}/E'\mathbb{Q}(\nu_f))$.

$$\begin{tikzcd} \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \bar{\mathbb{Q}} \arrow[r, "b\ell_\pi"] \arrow[d, "(1\otimes\sigma)"'] & \mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}} \arrow[d, "\tilde{\sigma}"] \\ \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \bar{\mathbb{Q}} \arrow[r, "b\ell_\pi"] & \mathcal{B}_\beta^{\nu_f}(\pi_f)_{\bar{\mathbb{Q}}} \end{tikzcd} \tag{3.3.6}$$

which finally defines the desired, uniform adjustment of ℓ_π announced above.

Remark 3.3.7. The above defines a rational structure on the Bessel model over the number field $E'\mathbb{Q}(\nu_f)$. The idea underlying our definition of the automorphism $\tilde{\sigma}$ using the matrix T_σ can be found in [GHar83, pp. 79–80] and [Mah05, p. 594]. It has been pursued in [Rag-Sha08] for Whittaker models of cuspidal automorphic representations of GL_n over general number fields and in [Gro-Rag14] for Shalika models of cuspidal automorphic representations of GL_{2n} over totally real fields.

3.3.2. *The second intertwining $B^{\phi_{\pi,\infty}}$.* Next we define another map in the space

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\bar{H}^q([\mathscr{V}_\tau])(\pi_f), \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\alpha_{\nu_f,\beta}).$$

To this end, recall that the vector space $H^q(\mathfrak{p}_h, K_\infty, V_\infty \otimes V_\tau)$ is naturally isomorphic to $\mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V_\infty)$, cf. [MHar90] Prop. 4.5. Because of our assumptions on (π, V), there are the following canonical isomorphisms

$$\begin{aligned}(3.3.8)\qquad \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V) &\xrightarrow{\sim} H^q(\mathfrak{p}_h, K_\infty, V \otimes V_\tau)\\ &\xrightarrow{\sim} \bar{H}^q([\mathscr{V}_\tau])(\pi_f).\end{aligned}$$

Definition 3.3.9. Denote the inverse of the composite of all the above arrows by Ψ_π.

The finite dimensional K_∞-representation $\wedge^q\mathfrak{p}_+\otimes V_\tau^*$ breaks up as a direct sum of irreducible representations. Since $\mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V_\infty)$ is one dimensional, it follows that there is an irreducible representation W of K_∞, such that W occurs in $\wedge^q\mathfrak{p}_+ \otimes V_\tau^*$ with multiplicity one and $\mathrm{Hom}_{K_\infty}(W, V_\infty) = \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V_\infty)$. Fix a lowest weight vector $\phi_{\pi,\infty} \in W \subset \wedge^q\mathfrak{p}_+ \otimes V_\tau^*$ (this choice is unique up to $\mathbb{C}^*$). Define a map $B^{\phi_{\pi,\infty}}$

$$(3.3.10)\qquad \bar{H}^q([\mathscr{V}_\tau])(\pi_f) \xrightarrow{\Psi_\pi} \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V) \longrightarrow \mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\alpha_{\nu_f,\beta}$$

given by

$$x \mapsto B^{\phi_{\pi,\infty}}(x) := B_{\Psi_\pi(x)(\phi_{\pi,\infty})},$$

where B is as in equation (3.2.1). A priori $B_{\Psi_\pi(x)(\phi_{\pi,\infty})}$ is an element of $\mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})}\alpha_{\nu,\beta}$, but clearly every element in this space gives rise to an element in $\mathrm{Ind}_{R(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\alpha_{\nu_f,\beta}$, by restricting a function on $G(\mathbb{A})$ to $G(\mathbb{A}_f)$.

Definition 3.3.11 (Bessel-periods)**.** The two linear maps $b\ell_\pi$ and $B^{\phi_{\pi,\infty}}$ differ by a non-zero complex number. Define $p(\pi, \phi_{\pi,\infty}) \in \mathbb{C}^*$ to be this scalar, that is, $b\ell_\pi = p(\pi, \phi_{\pi,\infty})B^{\phi_{\pi,\infty}}$. This period is a non-zero complex number, uniquely defined up to multiplication by elements of $(E'\mathbb{Q}(\nu_f))^*$.

Proposition 3.3.12. *The period $p(\pi, \phi_{\pi,\infty})$ has the property that it makes the following diagram commute for every $\sigma \in \mathrm{Aut}(\mathbb{C}/E'\mathbb{Q}(\nu_f))$.*

$$(3.3.13)\qquad \begin{array}{ccc} \mathcal{B}_\beta^{\nu_f}(\pi_f) & \xrightarrow{(B^{\phi_{\pi,\infty}})^{-1}/p(\pi,\phi_{\pi,\infty})} & \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \mathbb{C} \\ \Big\downarrow{\scriptstyle\tilde{\sigma}} & & \Big\downarrow{\scriptstyle(1\otimes\sigma)} \\ \mathcal{B}_\beta^{\nu_f}(\pi_f) & \xrightarrow{(B^{\phi_{\pi,\infty}})^{-1}/p(\pi,\phi_{\pi,\infty})} & \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'} \otimes_{E'} \mathbb{C} \end{array}$$

Proof. This is clear from equation (3.3.6) (note that $(B^{\phi_{\pi,\infty}})^{-1}/p(\pi,\phi_{\pi,\infty}) = (b\ell_\pi)^{-1}$). □

4. Period relations for cusp forms of GSp_4

4.1. **Twisting by algebraic Hecke characters.** Let (π, V) be a cuspidal automorphic representation of $G(\mathbb{A})$ and let ξ be an algebraic Hecke character $\xi : \mathrm{GL}_1(\mathbb{A}) \to \mathbb{C}^*$. Then $\xi = |\cdot|^m \cdot \xi^\circ$, where $m \in \mathbb{Z}$ and ξ° is a character of finite order. For $\varphi \in V$, we define a new cuspidal automorphic form $\varphi_\xi : G(\mathbb{A}) \to \mathbb{C}$ by $\varphi_\xi(g) := \varphi(g) \cdot \xi(\mu(g))$ and we denote by (π_ξ, V_ξ) the corresponding cuspidal automorphic representation of $G(\mathbb{A})$. Next, let ν be an algebraic character of $D(\mathbb{A})$ as in Sect. 3.2. We will consider the new algebraic character ν_ξ of $D(\mathbb{A})$ defined by $\nu_\xi(d) := \nu(d) \cdot \xi(\mu(d))$.

Lemma 4.1.1. *If (π, V) has a (ν, β)-Bessel model, then (π_ξ, V_ξ) has a (ν_ξ, β)-Bessel model. If (π, V) is $(\mathfrak{p}_h, K_\infty)$-cohomological with respect to V_τ, then (π_ξ, V_ξ) is $(\mathfrak{p}_h, K_\infty)$-cohomological with respect to $V_{\tau(m)} := V_\tau \otimes \mu^{-m}$.*

Proof. A direct calculation shows that $B_{\varphi_\xi}(g) = B_\varphi(g) \cdot \xi(\mu(g))$ from which the first claim follows. The second claim is obvious. □

Corollary 4.1.2. *If (π, V) satisfies the assumptions (1) - (3) of Sect. 3.3 with respect to (ν, β), the degree q and the coefficient module V_τ, then (π_ξ, V_ξ) satisfies the assumptions (1) - (3) of Sect. 3.3 with respect to (ν_ξ, β), the same degree q and the coefficient module $V_{\tau(m)}$.*

Proof. This is clear by Lemma 4.1.1 and the semisimplicity of the $G(\mathbb{A}_f)$-modules $\bar{H}^q([\mathscr{V}_\tau])$ and $\bar{H}^q([\mathscr{V}_{\tau(m)}])$. □

Recall the (fixed) lowest weight vector $\phi_{\pi,\infty} \in \wedge^q\mathfrak{p}_+ \otimes V_\tau^*$ from Sect. 3.3 above. The identity map defines an isomorphism of vector spaces $\mathbf{1}_\xi : \wedge^q\mathfrak{p}_+ \otimes V_\tau^* \xrightarrow{\sim} \wedge^q\mathfrak{p}_+ \otimes V_{\tau(m)}^*$ and we denote by $\phi_{\pi_\xi,\infty} = \mathbf{1}_\xi(\phi_{\pi,\infty})$ the image of $\phi_{\pi,\infty}$ in $\wedge^q\mathfrak{p}_+ \otimes V_{\tau(m)}^*$. Clearly, the assignment $\varphi \mapsto \varphi_\xi$ defines an isomorphism of vector spaces $V \xrightarrow{\sim} V_\xi$, whence we finally obtain a linear bijection $\mathscr{H}_\xi : \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V) \xrightarrow{\sim} \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_{\tau(m)}^*, V_\xi)$.

Similarly, $B_\varphi \mapsto B_{\varphi_\xi}$ defines a linear isomorphism of the finite part of the corresponding Bessel models $\mathscr{B}_\xi : \mathcal{B}_\beta^{\nu_f}(\pi_f) \xrightarrow{\sim} \mathcal{B}_\beta^{\nu_{\xi,f}}(\pi_{\xi,f})$ by restriction. Putting all of these maps into one diagram and observing that $B_{\varphi_\xi}(g) = B_\varphi(g) \cdot \xi(\mu(g))$, we finally obtain a commutative square of linear bijections

$$\begin{tikzcd}
\mathcal{B}_\beta^{\nu_f}(\pi_f) \arrow[d, "\mathscr{B}_\xi"] & \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_\tau^*, V) \arrow[l] \arrow[d, "\mathscr{H}_\xi"] \\
\mathcal{B}_\beta^{\nu_{\xi f}}(\pi_{\xi,f}) & \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}_+ \otimes V_{\tau(m)}^*, V_\xi) \arrow[l]
\end{tikzcd}$$

where the horizontal arrows are given by the Bessel-map $h \mapsto B_{h(\phi_{\pi,\infty})}$ (resp. $h' \mapsto B_{h'(\phi_{\pi_\xi,\infty})}$).

Definition 4.1.3. Define $\mathscr{C}_\xi := \Psi_{\pi_\xi}^{-1} \circ \mathscr{H}_\xi \circ \Psi_\pi$.

The following proposition is a direct consequence of our discussion above:

Proposition 4.1.4. *There is a commutative square of linear isomorphisms,*

$$\begin{tikzcd}
\mathcal{B}_\beta^{\nu_f}(\pi_f) \arrow[r, "(B^{\phi_\pi,\infty})^{-1}"] \arrow[d, "\mathscr{B}_\xi"] & \bar{H}^q([\mathscr{V}_\tau])(\pi_f) \arrow[d, "\mathscr{C}_\xi"] \\
\mathcal{B}_\beta^{\nu_{\xi_f}}(\pi_{\xi,f}) \arrow[r, "(B^{\phi_{\pi_\xi},\infty})^{-1}"] & \bar{H}^q([\mathscr{V}_{\tau(m)}])(\pi_{\xi,f})
\end{tikzcd}$$

where $B^{\phi_\pi,\infty}$ and $B^{\phi_{\pi_\xi},\infty}$ are the unnormalized period-maps from 3.3.10.

4.2. **Rationality of $\mathscr{C}_\xi$.** A cohomology class $v \in H^q([\mathscr{V}_\tau]^{\mathrm{sub}})$ is represented by a smooth map $\varphi : G(\mathbb{A}) \to V_\tau^q$ which satisfies various conditions. The element $\mathscr{C}_\xi(v)$ is then represented by the smooth map $\varphi_\xi : G(\mathbb{A}) \to V^q_{\tau(m)}$, where $\varphi_\xi(g) := \varphi(g)\xi(\mu(g))$. In this subsection we will show that if v is L-rational, L any subfield of $\mathbb{C}$ containing $E'\mathbb{Q}(\xi_f)$, then so is $\mathscr{C}_\xi(v)$. Although looking very suggestively, we invite the reader to convince him/herself that this is a priori far from being clear from the purely transcendental description using smooth forms.

Recall the space X from §2.2 and let D be a connected component of X. Let $G_\infty^+ \subset G_\infty$ denote the stabilizer of D and for any subgroup $S \subset G_\infty$ define $S^+ = S \cap G^+$, in particular, we may take $S = G(F)$. The group G_∞^{ss} acts transitively on D. Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup. Let $\{\gamma\}$ be a set of coset representatives for $G(F)^+\backslash G(\mathbb{A}_f)/K$. Recall that (see equations (1.2.1) in [MHar90])

$$_KM = M(G,X)_{\mathbb{C}}/K = \coprod_{\{\gamma\}} M_{\Gamma(\gamma)}$$

where $\Gamma(\gamma) = G(F)^+ \cap \gamma K \gamma^{-1}$ and $M_{\Gamma(\gamma)} := \Gamma(\gamma)\backslash D$. For $\gamma \in G(\mathbb{A}_f)$ as above, let $\Gamma(\gamma)\gamma K \subset G(\mathbb{A}_f)$ denote the subset consisting of elements of the type $g\gamma k$ with $g \in \Gamma(\gamma)$ and $k \in K$. Note that $\Gamma(\gamma)$ acts on $D \times \Gamma(\gamma)\gamma K$ diagonally. With this action one checks easily that

$$\Gamma(\gamma)\backslash D = \Gamma(\gamma)\backslash\big(G_\infty^{\mathrm{ss}}/K_\infty^{\mathrm{ss}} \times \Gamma(\gamma)\gamma K\big)/K. \tag{4.2.1}$$

Choose a collection of rational boundary components Σ appropriately so that $_KM_\Sigma$ is a smooth projective variety and $_KM_\Sigma \setminus {}_KM$ is a divisor with normal crossings, all defined over a field L as in [MHar90, 1.2.3.3]. Then one has (see equation (1.2.4.1) in [MHar90])

$$_KM_\Sigma = \coprod_{\{\gamma\}} M_{\Gamma(\gamma),\Sigma(\gamma)}.$$

So, [MHar90, Prop. 2.4] yields

$$H^q([\mathscr{V}_\tau]^{\mathrm{sub}})^K_L = H^q({}_KM_\Sigma, [\mathscr{V}_\tau]^{\mathrm{sub}}_\Sigma)_L = \left(\prod_{\{\gamma\}} H^q(M_{\Gamma(\gamma),\Sigma(\gamma)}, [\mathscr{V}_\tau]_{\Sigma(\gamma)})\right)_L. \tag{4.2.2}$$

Proposition 4.2.3. *Let L be any subfield of $\mathbb{C}$ containing $E'\mathbb{Q}(\xi_f)$ and $v \in \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_L$. Then $\mathscr{C}_\xi(v) \in \bar{H}^q([\mathscr{V}_{\tau(m)}])(\pi_{\xi,f})_L$.*

Proof. Let $v \in \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_L$. Then there is an open compact subgroup $K \subset G(\mathbb{A}_f)$ such that v is invariant under K and, making K even smaller, such that $\xi_f(\mu(K)) = 1$ is trivial. According to the decomposition (4.2.2), we may write $v = (v_\gamma) \in (\prod_{\{\gamma\}} H^q(M_{\Gamma(\gamma),\Sigma(\gamma)}, [\mathscr{V}_\tau]^{\mathrm{sub}}_{\Sigma(\gamma)}))_L$. The class $v \in H^q([\mathscr{V}_\tau]^{\mathrm{sub}})^K_L$ is represented by a smooth map

$$\varphi : G(\mathbb{A}) \to V^q_\tau.$$

From equation (4.2.1) it follows that the restriction of φ to the subset $G^{\mathrm{ss}}_\infty \times \Gamma(\gamma)\gamma K$ determines the smooth section of the smooth vector bundle on $M_{\Gamma(\gamma),\Sigma(\gamma)}$ associated to the representation V^q_τ of K_∞, which represents the class $v_\gamma \in H^q(M_{\Gamma(\gamma),\Sigma(\gamma)}, [\mathscr{V}_\tau]^{\mathrm{sub}}_{\Sigma(\gamma)})$.

Consider the new function

$$\varphi_\xi : G(\mathbb{A}) \to V^q_{\tau(m)}$$

defined by $\varphi_\xi(g) := \varphi(g)\xi(\mu(g))$. It represents the class $\mathscr{C}_\xi(v)$. On the subset $G^{\mathrm{ss}}_\infty \times \Gamma(\gamma)\gamma K$ we have

$$\varphi_\xi(g) = \varphi(g)\xi(\mu(g)) = \varphi(g)\xi_f(\mu(\gamma)),$$

since $\mu = 1$ on G^{ss}_∞ and $\xi_f(\mu(K)) = \xi_f(\mu(\Gamma(\gamma))) = 1$.

Twisting by $\xi(\mu(\cdot))$ on differential forms defines a complex-linear map

$$\begin{array}{ccc} H^q([\mathscr{V}_\tau]^{\mathrm{sub}})^K & \longrightarrow & H^q([\mathscr{V}_{\tau(m)}]^{\mathrm{sub}})^K \\ \| & & \| \\ \prod_{\{\gamma\}} H^q(M_{\Gamma(\gamma),\Sigma(\gamma)}, [\mathscr{V}_\tau]^{\mathrm{sub}}_{\Sigma(\gamma)}) & \longrightarrow & \prod_{\{\gamma\}} H^q(M_{\Gamma(\gamma),\Sigma(\gamma)}, [\mathscr{V}_{\tau(m)}]^{\mathrm{sub}}_{\Sigma(\gamma)}) \end{array}$$

It is easy to see that the bundles $[\mathscr{V}_\tau]$ and $[\mathscr{V}_{\tau(m)}]$ on the space $M_{\Gamma(\gamma)}$ are identical. For example, in the notation of [MHar90, §2.1], both these are already identical on $G(\mathbb{C})/\mathscr{P}_h(\mathbb{C})$ as this is identical with $G^{\mathrm{ss}}(\mathbb{C})/G^{\mathrm{ss}}(\mathbb{C}) \cap \mathscr{P}_h(\mathbb{C})$, and both the representations τ and $\tau(m)$ agree on $G^{\mathrm{ss}}(\mathbb{C}) \cap \mathscr{P}_h(\mathbb{C})$. Consequently, the subcanonical extensions $[\mathscr{V}_\tau]^{\mathrm{sub}}_{\Sigma(\gamma)}$ and $[\mathscr{V}_{\tau(m)}]^{\mathrm{sub}}_{\Sigma(\gamma)}$ to the space $M_{\Gamma(\gamma),\Sigma(\gamma)}$ are identical. The computation in the preceding paragraph shows that the horizontal arrow takes the vector v_γ to $\xi_f(\mu(\gamma))v_\gamma$. Thus, since $\mathbb{Q}(\xi_f) = \mathbb{Q}(\mathrm{Im}(\xi_f)) \subseteq L$, the effect of this map on cohomology is rational over the field L. □

Remark 4.2.4. We would like to point out that the argument of the above proposition can be adopted to give an alternative proof of [Rag-Sha08, Prop. 4.5], (which is based on a rather involved argument) which is a key ingredient of their main theorem [Rag-Sha08, Thm. 4.1], as well as of [Gro-Rag14, Thm. 5.2.1].

4.3. **Period relations.** Before we prove the main result of this paper, we need one last ingredient. In accordance with the above notation, let $L := E'\mathbb{Q}(\nu_f, \xi_f)$, $\sigma \in \mathrm{Aut}(\mathbb{C}/L)$ and consider the following (not necessarily commutative) square

$$\begin{array}{ccc} \mathcal{B}^{\nu_f}_\beta(\pi_f) & \xrightarrow{\mathscr{B}_\xi} & \mathcal{B}^{\nu_{\xi,f}}_\beta(\pi_{\xi,f}) \\ \downarrow{\scriptstyle\tilde{\sigma}} & & \downarrow{\scriptstyle\tilde{\sigma}} \\ \mathcal{B}^{\nu_f}_\beta(\pi_f) & \xrightarrow{\mathscr{B}_\xi} & \mathcal{B}^{\nu_{\xi,f}}_\beta(\pi_{\xi,f}). \end{array} \tag{4.3.1}$$

By a direct calculation one easily checks that

$$\tilde{\sigma} \circ \mathscr{B}_\xi = \xi_f(t_\sigma^{-1}) \cdot (\mathscr{B}_\xi \circ \tilde{\sigma}),$$

(where we used that σ fixes ξ_f). Moreover, our definition of the Gauß-sum $\mathscr{G}(\xi_f)$ of ξ_f, §2.1.3, involving our concretely chosen additive character ψ, §2.1.2, implies by a simple calculation (see [Rag-Sha07, Lem. 2.3.4], or [Gro-Har16, Proof of Thm. 3.9, p. 24] that $\xi_f(t_\sigma^{-1}) = \frac{\mathscr{G}(\xi_f)}{\sigma(\mathscr{G}(\xi_f))}$. This yields the following result about the algebraic behaviour of $\mathscr{B}_\xi$:

Proposition 4.3.2. *For all algebraic Hecke characters ξ of $\mathrm{GL}_1(\mathbb{A})$ and for all $\sigma \in \mathrm{Aut}(\mathbb{C}/L)$,*

$$\tilde{\sigma} \circ \mathscr{B}_\xi = \frac{\mathscr{G}(\xi_f)}{\sigma(\mathscr{G}(\xi_f))} \cdot (\mathscr{B}_\xi \circ \tilde{\sigma}).$$

We are now ready to prove

Theorem 4.3.3 (Period relations for Bessel periods)**.** *Let (π, V) be a cuspidal automorphic representation of $G(\mathbb{A})$ which satisfies the assumptions (1)-(3) of Sect. 3.3. Let ξ be any algebraic Hecke character of $\mathrm{GL}_1(\mathbb{A})$. Then the periods $p(\pi, \phi_{\pi,\infty})$ and $p(\pi_\xi, \phi_{\pi_\xi,\infty})$ satisfy the following relation*

$$p(\pi_\xi, \phi_{\pi_\xi,\infty}) \ \sim_{(E'\mathbb{Q}(\nu_f,\xi_f))^*} \ p(\pi, \phi_{\pi,\infty}) \cdot \mathscr{G}(\xi_f),$$

where "$\sim_{(E'\mathbb{Q}(\nu_f,\xi_f))^*}$" means up to multiplication by a non-zero number in $E'\mathbb{Q}(\nu_f, \xi_f)$.

Proof. Start from a non-zero vector $v^0 \in \bar{H}^q([\mathscr{V}_\tau])(\pi_f)_{E'}$. From Prop. 3.3.12 it follows that $B^{\phi_{\pi,\infty}}(p(\pi, \phi_{\pi,\infty})v^0) \in \mathcal{B}^{\nu_f}_{\psi_f}(\pi_f)$ is invariant under $\tilde{\sigma}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E'\mathbb{Q}(\nu_f))$. We get from Prop. 4.3.2 that $\mathscr{G}(\xi_f)\mathscr{B}_\xi(B^{\phi_{\pi,\infty}}(p(\pi, \phi_{\pi,\infty})v^0)) \in \mathcal{B}^{\nu_{\xi,f}}_\beta(\pi_{\xi,f})$ is invariant under $\tilde{\sigma}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E'\mathbb{Q}(\nu_f, \xi_f))$, so by the bijectivity and linearity of $B^{\phi_{\pi,\infty}}$, $\mathscr{B}_\xi$ and $B^{\phi_{\pi_\xi,\infty}}$ and Prop. 3.3.12 again,

$$0 \neq \frac{\mathscr{G}(\xi_f)p(\pi, \phi_{\pi,\infty})}{p(\pi_\xi, \phi_{\pi_\xi,\infty})}(B^{\phi_{\pi_\xi,\infty}})^{-1}(\mathscr{B}_\xi(B^{\phi_{\pi,\infty}}(v^0))) \in \bar{H}^q([\mathscr{V}_{\tau'}])(\pi_{\xi,f})_{E'\mathbb{Q}(\nu_f,\xi_f)}.$$

To complete the proof of the theorem it suffices to show that

$$(B^{\phi_{\pi_\xi,\infty}})^{-1}(\mathscr{B}_\xi(B^{\phi_{\pi,\infty}}(v^0))) \in \bar{H}^q([\mathscr{V}_{\tau'}])(\pi_{\xi,f})_{E'\mathbb{Q}(\nu_f,\xi_f)}.$$

But this is clear since by definition

$$(B^{\phi_{\pi_\xi,\infty}})^{-1}(\mathscr{B}_\xi(B^{\phi_{\pi,\infty}}(v^0))) = \mathscr{C}_\xi(v^0)$$

and from Proposition 4.2.3 it follows that $\mathscr{C}_\xi(v^0)$ is in $\bar{H}^q([\mathscr{V}_{\tau'}])(\pi_{\xi,f})_{E'\mathbb{Q}(\nu_f,\xi_f)}$. □

Remark 4.3.4 (Compatibility with lifting to $\mathrm{GL}_4(\mathbb{A}_F)$)**.** Assume that π from Thm. 4.3.3 admits a Langlands functorial lifting to a cuspidal automorphic representation $\mathrm{Lift}(\pi) =: \Pi$ on $\mathrm{GL}_4(\mathbb{A}_F)$ through the tautological representation of the attached dual groups

$${}^L\mathrm{GSp}_4^\circ = \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow {}^L\mathrm{GL}_4^\circ = \mathrm{GL}_4(\mathbb{C}).$$

(For generic representations π, for instance, such a lift – known to Jacquet, Piatetski–Shapiro and Shalika for quite some time – together with a criterion of cuspidality of Π has finally been established by Asgari–Shahidi in [Asg-Sha06].) Inspecting the Satake parameters of local representations π_v at unramified places $v \notin S_\infty$, cf. e.g., [Asg-Sha06], (1) & (2), one easily finds the global identity

$$\mathrm{Lift}(\pi_\xi) = \Pi \otimes \xi(\det)$$

for all algebraic Hecke characters ξ. We remark that since $\pi_\xi = \pi \otimes \xi(\mu)$ by definition, and $\mu^2 = \det$, this incorporates the following necessary relation of central characters

$$\omega^2_{\pi_\xi} = (\omega_\pi \cdot \xi(\mu))^2 = \omega_\Pi \cdot \xi(\det) = \omega_{\Pi\otimes\xi(\det)} = \omega_{\mathrm{Lift}(\pi_\xi)} \tag{4.3.5}$$

as demanded by [Asg-Sha06], Thm. 2.4.
In view of Prop. 3.1.4 of [Gro-Rag14], $\mathrm{Lift}(\pi) = \Pi$ has a (ω_π, ψ)-Shalika model and more generally all its twists $\mathrm{Lift}(\pi_\xi) = \Pi \otimes \xi(\det)$ have a (ω_{π_ξ}, ψ)-Shalika model. Due to (4.3.5) this is consistent with Lem. 5.1.1, *ibidem.*
As a consequence of this fundamental match, for those π, which are even $(\mathfrak{g}_\infty, K_\infty)$-cohomological (a condition which implies that π is $(\mathfrak{p}_h, K_\infty)$-cohomological, see Thm. B.2.1 below, and that its lift $\mathrm{Lift}(\pi) = \Pi$ is $(\mathfrak{gl}_4(\mathbb{R})^d, (\mathbb{R}_+ SO(4))^d)$-cohomological), the results of [Gro-Rag14] are compatible with the situation at hand.

Appendix A. General aspects in the theory of rational structures of admissible representations of global groups

A.1. **An action of** $\mathrm{Aut}(\mathbb{C})$**, rationality fields and rational strutures.** For the sake of a reference, we will now prove three general rationality-results for representations of reductive groups over general non-archimedean fields or over the finite adeles. These results will apply in particular to components of $(\mathfrak{g}, K)$- or $(\mathfrak{p}_h, K_\infty)$-cohomological cuspidal automorphic representations; and more generally to the components of those square-integrable automorphic representations, which appear either in inner cohomology or its analogue in coherent cohomology. The reader not familiar with these concepts is referred to Harder [GHar16] III, §4.2, Schwermer [Sch83] §1.6–1.8 and Harris [MHar90] Prop. 3.6 and Thm. 5.3.

We understand this section, which we believe to be interesting in its own right, as a source of reference, and so we try to provide details for all (intermediate) results. The contents of Appendix A are closely related to some of the results mentioned in [Shi-Tem14], see particularly §2.2 therein: There the authors prove various general theorems on the finiteness of the extension $\mathbb{Q}(\pi) \supseteq \mathbb{Q}$, π being an automorphic representation of an arbitrary reductive group, which is an interesting topic, but which will not be considered here. Instead, we will investigate the question when one may define a *rational structure* on representations π_f as above, which on the other hand is not the focus of [Shi-Tem14] (see explicitly their Rem. 2.3) but is dealt there implicitly in the proof of Prop. 2.15 (ii). This latter result is a special case of our Thm A.2.3, though the idea of the proof of [Shi-Tem14, Prop. 2.15 (ii)] overlaps with our arguments. We would like to thank one of the referees for pointing us to [Shi-Tem14], which we had not been aware of while writing this paper.

From now on we let G be an arbitrary reductive algebraic group over a number field F. Let (π, V) be any representation of $G(F_v)$, v non-archimedean, or of $G(\mathbb{A}_f)$ on a complex vector space V. For any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ we define the σ-twisted representation (π^σ, V^σ) as follows. The underlying abelian group of the representation space is V^σ is V, however, we change the complex structure of V^σ by defining for $\lambda \in \mathbb{C}$ and $\mu \in V^\sigma$ $\lambda \cdot \mu := \sigma^{-1}(\lambda) \cdot \mu$ (on the right we mean the complex structure on V). We remark that the identity map $V \to V^\sigma$ is not complex linear but σ-linear, that is, $\lambda \cdot \mu \mapsto \sigma(\lambda) \cdot \mu$, to emphasize this fact we will denote this map by Φ^σ. It is checked easily that the action of G on V^σ commutes with the action of $\mathbb{C}$ on V^σ, so we get a representation (π^σ, V^σ) of G on a complex vector space. This definition is in accordance with Waldspurger, [Wal85], I.1 (we remark that these definitions make sense for any group, as mentioned in [Wal85], however, we restrict our attention to the case of interest to us).

If (δ, V) is any of the above representations, we define the subgroup $S(\delta) \subset \mathrm{Aut}(\mathbb{C})$ to consist of the elements σ, for which $(\delta, V) \cong (\delta^\sigma, V^\sigma)$ as representations. Define the rationality field $\mathbb{Q}(\delta)$ to be $\mathbb{C}^{S(\delta)}$, the fixed field under $S(\delta)$.

Let $E \subset \mathbb{C}$ be any subfield. Recall that (δ, V) is said to have a rational structure over E, or, equivalently, to be defined over E, if there is an E-subspace, $V_E \subset V$ stable under the given group action, such that the natural map

$$\Psi : V_E \otimes_E \mathbb{C} \to V$$

is an isomorphism of complex representations of G. We remark that the action of G on $V_E \otimes_E \mathbb{C}$ is through its action on V_E.

Suppose (δ, V) has a rational structure over E, then for any $E \subset E' \subset \mathbb{C}$ we denote by $V_{E'}$ the E'-subspace of V generated by V_E. It is clear that (δ, V) also has a rational structure over E', that is, the natural map $V_{E'} \otimes_{E'} \mathbb{C} \to V$ is an isomorphism of complex representations of G.

In general, it is not the case that (δ, V) has a rational structure over $\mathbb{Q}(\delta)$. In the next subsection we prove three general results on existence of rational structures. Even though these results seem to be known to experts, we could not find references for these in the literature. The following theorem from field theory will be used in proving these results.

Theorem A.1.1. *Let K be an algebraically closed field of characteristic 0. Then for any subfield $k \subset K$, we have $K^{\mathrm{Aut(K/k)}} = k$.*

A.2. **Three general rationality-results.**

Result 1 - A useful lemma. We will start off with the following useful lemma, observed by [Shi-Tem14] (see Rem. 2.3), but stated there without proof.

Lemma A.2.1. *Let (π, V) be a representation of $G(F_v)$, v non-archimedean, or of $G(\mathbb{A}_f)$ on a complex vector space V. Assume that (π, V) is defined over a subfield E of $\mathbb{C}$, then $\mathbb{Q}(\pi) \subseteq E$.*

Proof. We show that if $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, then there is an isomorphism of complex representations $V \cong V^\sigma$. For $v \in V_E$, consider the arrows

$$V \overset{\Psi^{-1}}{\to} V_E \otimes_E \mathbb{C} \overset{1\otimes\sigma^{-1}}{\longrightarrow} V_E \otimes_E \mathbb{C} \overset{\Psi}{\longrightarrow} V \overset{\Phi^\sigma}{\longrightarrow} V^\sigma$$
$$\lambda \cdot v \mapsto v \otimes \lambda \mapsto v \otimes \sigma^{-1}(\lambda) \mapsto \sigma^{-1}(\lambda) \cdot v \mapsto \lambda \cdot v$$

It is clear that the G-action is preserved by each of these arrows. Thus, $\sigma \in S(\pi)$ and so $\mathrm{Aut}(\mathbb{C}/E) \subseteq S(\pi)$. As a consequence we get $\mathbb{Q}(\pi) \subset \mathbb{C}^{\mathrm{Aut}(\mathbb{C}/E)}$. The lemma follows now from Thm. A.1.1. □

Result 2. Our second general rationality result (after the lemma) will be of more importance. It applies, e.g., to all isotypic summands of the $G(\mathbb{A}_f)$-module defined by inner cohomology and – if G defines a Shimura datum – also to the analogue of inner cohomology in coherent cohomology.

Proposition A.2.2. *Let X be a complex representation of $G(\mathbb{A}_f)$ which is smooth and admissible. Assume that X has a E-structure and that X is semisimple. Let V be an isotypical component of X. Then there is a finite extension E' (depending on V) of E such that that V has a rational structure over E'.*

Proof. Let $K \subset G(\mathbb{A}_f)$ be a compact open such that $V^K \neq 0$. As X^K is finite dimensional, there are finitely many isotypical components (π_i, V_i) such that

$$X^K = \bigoplus_{i=1}^n V_i^K.$$

The isotypical component V is one of the V_i and we will show that there is a finite extension of E such that all the (π_i, V_i) have a rational structure over this finite extension.

Since we have an isomorphism of $G(\mathbb{A}_f)$-modules $X_E \otimes_E \mathbb{C} \xrightarrow{\sim} X$, taking K-invariants yields an isomorphism of Hecke-, i.e. $\mathcal{H}_K$-modules ($\mathcal{H}_K := C_c^\infty(G(\mathbb{A}_f)//K, \mathbb{C})$)

$$X_E^K \otimes_E \mathbb{C} \xrightarrow{\sim} X^K = \bigoplus_{i=1}^n V_i^K. \tag{A.2.3}$$

Let U_E denote the $G(\mathbb{A}_f)$-subrepresentation of X_E generated by X_E^K, that is, U_E is the E-subspace generated by the subset $\langle G(\mathbb{A}_f) \cdot x \,|\, x \in X_E^K \rangle$.

Each V_i being an isotypical component, breaks up as $V_i = \bigoplus_{j=1}^{r_i} W_{ij}$, where the W_{ij} are mutually isomorphic and irreducible $G(\mathbb{A}_f)$ representations. The index set is finite since $V_i^K = \bigoplus_j W_{ij}^K$ and V_i^K is finite dimensional. Then obviously

$$W_{ij}^K \subseteq V_i^K \hookrightarrow X_E^K \otimes_E \mathbb{C} \subseteq U_E \otimes_E \mathbb{C},$$

where the inclusion is given by (A.2.3), i.e., by forming the tensor product. Choose a nonzero element $w_{ij} \in W_{ij}^K$. Since W_{ij} is irreducible, it is the $\mathbb{C}$-span of the subset $\langle G(\mathbb{A}_f) \cdot w_{ij} \rangle$. Thus, to show that $W_{ij} \hookrightarrow U_E \otimes_E \mathbb{C}$, it suffices to show that $\langle G(\mathbb{A}_f) \cdot w_{ij} \rangle \hookrightarrow U_E \otimes_E \mathbb{C}$. Since

$$w_{ij} = \sum_{r=1}^{l} x_r \otimes \lambda_r \qquad x_r \in U_E, \lambda_r \in \mathbb{C}$$

the previous assertion follows as $g \cdot x_r \in U_E$ for $g \in G(\mathbb{A}_f)$. Thus, for every ij we have that the tensor-product-map (A.2.3) yields an inclusion of $G(\mathbb{A}_f)$-modules $W_{ij} \hookrightarrow U_E \otimes_E \mathbb{C}$, so

$$\bigoplus_{i=1}^{n} V_i = \bigoplus_{ij} W_{ij} \hookrightarrow U_E \otimes_E \mathbb{C}.$$

On the other hand, as $\mathcal{H}_K$-modules

$$U_E^K \otimes_E \mathbb{C} \subseteq X_E^K \otimes_E \mathbb{C} \xrightarrow{\sim} X^K = \bigoplus_{i=1}^{n} V_i^K,$$

whence forming the tensor product (A.2.3) also defines a natural inclusion of $G(\mathbb{A}_f)$-modules

$$U_E \otimes_E \mathbb{C} \hookrightarrow \bigoplus_{i=1}^{n} V_i.$$

Together with the above, this shows that the natural tensor-map gives rise to an isomorphism of $G(\mathbb{A}_f)$-modules $U_E \otimes_E \mathbb{C} \xrightarrow{\sim} \bigoplus_{i=1}^{n} V_i$ as desired.

For $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ consider the map

$$1 \otimes_E \sigma : X_E \otimes_E \mathbb{C} \to X_E \otimes_E \mathbb{C}.$$

It is clear that this map leaves the subspace $U_E \otimes_E \mathbb{C}$ invariant, and since an isotypical component of $U_E \otimes_E \mathbb{C}$ will map to an isotypical component, it follows that each $1 \otimes_E \sigma$ maps V_i to some V_j. This defines a group homomorphism $\mathrm{Aut}(\mathbb{C}/E) \to S_n$ (the symmetric group on n letters). Let H denote the kernel of this homomorphism and let $E' := \mathbb{C}^H$. If $\sigma \in \mathrm{Aut}(\mathbb{C}/E')$, then $(1 \otimes_E \sigma)V_i \cong V_i$ as complex $G(\mathbb{A}_f)$-modules, which forces that $1 \otimes_E \sigma$ maps V_i to itself.

Let $X_{E'}$ be the E' span of X_E in X, then the natural map $X_{E'} \otimes_{E'} \mathbb{C} \to X$ is an isomorphism. Define $V_{i,E'} := V_i \cap X_{E'}$, we claim that $V_{i,E'}$ is a E' rational structure for V_i. Consider the projection map from $X_{E'} \otimes_{E'} \mathbb{C}$ to the isotypical component V_i. Because of the discussion in the preceding paragraph we see that this map is $\mathrm{Aut}(\mathbb{C}/E')$ equivariant. Consequently the image of $X_{E'}$ is contained in $V_{iE'}$. Since the image of $X_{E'}$ generates V_i as a complex vector space, we get that the natural map $V_{iE'} \otimes_{E'} \mathbb{C} \to V_i$ is surjective. The injectivity of this map is clear since the map $X_{E'} \otimes_{E'} \mathbb{C} \to X$ is injective.

Finally we show that E' is a finite extension of E. First we claim that $\mathrm{Aut}(\mathbb{C}/E)$ leaves E' invariant. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and let $e' \in E'$. Since H is a normal subgroup, we have that $\sigma^{-1}h\sigma \in H$ for all $h \in H$, and so $\sigma^{-1}h\sigma(e') = e'$, which shows $h(\sigma(e')) = \sigma(e')$ for all $h \in H$, that is, $\sigma(e') \in \mathbb{C}^H = E'$. Thus, there is a group homomorphism

$$\begin{gathered} \mathrm{Aut}(\mathbb{C}/E) \to \mathrm{Aut}(E') \\ \sigma \mapsto \sigma|_{E'} \end{gathered}$$

Note that H is in the kernel of this homomorphism and so the image, which we call D, is a finite group. We claim $E'^D = E$, since if $\alpha \in E'$ and $\sigma|_{E'}(\alpha) = \alpha$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, this means that $\alpha \in \mathbb{C}^{\mathrm{Aut}(\mathbb{C}/E)} = E$ by Theorem A.1.1. $\square$

Result 3. We complete this section by our third rationality result. As already announced in the beginning of this section, it applies to the $G(\mathbb{A}_f)$-components of $(\mathfrak{g}, K)$- or $(\mathfrak{p}_h, K_\infty)$-cohomological cuspidal automorphic representations. Even more generally it may be applied to all those $G(\mathbb{A}_f)$-parts of square-integrable automorphic representations, which appear either in inner cohomology or its analogue in coherent cohomology.

Theorem A.2.4. *Let X be a complex representation of $G(\mathbb{A}_f)$ which is smooth and admissible. Assume that X is semisimple and that it has an E-structure which we denote by X_E. Given an irreducible representation (π, V) of $G(\mathbb{A}_f)$ which occurs in X, there is a finite extension E' of E and a $G(\mathbb{A}_f)$-subspace $V_{E'} \subset X_{E'}$ such that the natural map $V_{E'} \otimes_{E'} \mathbb{C} \to X$ induces an isomorphism $V_{E'} \otimes_{E'} \mathbb{C} \cong V$ as complex $G(\mathbb{A}_f)$-representations. In particular, (π, V) is defined over E'.*

Proof. We immediately reduce, using Proposition A.2.2, to the case where X is isotypical. Since X is semisimple and also admissible, it follows that $X \cong V^{\oplus l}$. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup such that $V^K \neq 0$. The Hecke algebra of E-valued functions $\mathcal{H}_{K,E} := C_c^\infty(G(\mathbb{A}_f) /\!\!/ K, E)$ acts on X_E^K.

Choose a E-basis $\{w_1, w_2, \ldots, w_n\}$ for X_E^K, obviously, this is also a $\mathbb{C}$ basis for $X_E^K \otimes_E \mathbb{C}$. Let $A : \mathcal{H}_{K,E} \to M_n(E)$ denote the map which associates to an element of $\mathcal{H}_{K,E}$ its matrix in the basis $\{w_1, w_2, \ldots, w_n\}$. Now choose a basis $\{w'_1, w'_2, \ldots, w'_r\}$ for V^K and extend this to a basis $\{w'_1, w'_2, \ldots, w'_n\}$ for $X_E^K \otimes_E \mathbb{C}$. Let $\mathcal{C}$ denote the matrix whose ith column is w'_i written in terms of the w_j's. Consider the map $\tilde{A} : \mathcal{H}_{K,E} \to M_n(\mathbb{C})$ which associates to an element of $\mathcal{H}_{K,E}$ its matrix in the basis $\{w'_1, w'_2, \ldots, w'_n\}$. For $f \in \mathcal{H}_{K,E}$ there is the obvious relationship $\tilde{A}(f) = \mathcal{C}^{-1} A(f) \mathcal{C}$. Since $\mathcal{H}_{K,E}$ leaves V^K invariant, the image of $\tilde{A}$ lands inside the maximal parabolic whose upper left block is an $r \times r$ block, call this parabolic P.

Consider the subvariety Y of $GL_n(E)$ defined by the condition $C^{-1}A(f)C \in P$ for every $f \in \mathcal{H}_{K,E}$. This subvariety is not empty since it has a complex point, thus, it also has a point over $\bar{E}$. Choose any such point, and call it C_0. In other words, we have found a subspace $U_0 \subsetneq X_{\bar{E}}^K$ (generated by the first r columns of C_0) which is invariant under the action of $\mathcal{H}_{K,E}$.

$$U_0 \otimes_{\bar{E}} \mathbb{C} \subset X_{\bar{E}}^K \otimes_{\bar{E}} \mathbb{C} \cong \underbrace{V^K \oplus V^K \cdots \oplus V^K}_{\text{l times}}$$

All the maps above are $\mathcal{H}_{K,\mathbb{C}}$-equivariant. Thus, for one of the projections, $U_0 \otimes_{\bar{E}} \mathbb{C} \to V^K$ is a nonzero map of $\mathcal{H}_{K,\mathbb{C}}$-modules. As V^K is a simple $\mathcal{H}_{K,\mathbb{C}}$-module ([Bum98, Prop. 4.2.3]), we get that this is a surjection, so an isomorphism as both have the same dimension as complex vector spaces. As U_0 is a finite dimensional subspace of $X_E^K \otimes_E \bar{E}$, it follows that it is defined over a finite extension E' of E. Let $U_{E'} \subset X_{E'}^K$ be such that $U_0 = U_{E'} \otimes_{E'} \bar{E}$. Thus,

$$U_{E'} \otimes_{E'} \mathbb{C} \to V^K$$

is an isomorphism of $\mathcal{H}_{K,\mathbb{C}}$-modules. Take $V_{E'}$ to be the E'-span in $X_{E'}$ of $\langle G(\mathbb{A}_f) \cdot U_{E'} \rangle$. It follows, for example, as in the proof of "(ii)⇒(iii)" in [Bum98, Proposition 4.2.3], that $V_{E'}^K = U_{E'}$.

The composite of the maps

$$V_{E'} \otimes_{E'} \mathbb{C} \to X \to V$$

is surjective since it is nonzero. The representation $V_{E'} \otimes_{E'} \mathbb{C}$ is isotypical since it is a subrepresentation of $X_{E'} \otimes \mathbb{C}$. Thus, $V_{E'} \otimes_{E'} \mathbb{C} \cong V^{\oplus i}$, taking K invariants on both sides and using $(V_{E'} \otimes_{E'} \mathbb{C})^K = V_{E'}^K \otimes_{E'} \mathbb{C} = U_{E'} \otimes_{E'} \mathbb{C}$, comparing dimensions we get that $i = 1$. This shows that the map $V_{E'} \otimes_{E'} \mathbb{C} \to V$ above is an isomorphism. $\square$

Appendix B. General aspects in the automorphic theory of the cohomology of Shimura varieties

B.1. **Two relative Lie algebra cohomology theories related to Shimura varieties.** In this section, we let (G, X) be the datum defining a Shimura variety in the sense made precise in Harris, [MHar85] 1.1 For the sake of completeness, we recall that this means that G is a connected reductive linear algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : R_{\mathbb{C}/\mathbb{R}}(GL_1) \to G \times_{\mathbb{Q}} \mathbb{R}$, such that

(1) The Hodge structure on the Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ of G given by $Ad \circ h$ is of type $(0, 0) + (1, -1) + (-1, 1)$.
(2) The automorphism $Ad(h(i))$ induces a Cartan involution on $G^{\mathsf{ss}}(\mathbb{R})$, G^{ss} being the derived group of G. The $\mathbb{R}$-group $G^{\mathsf{ss}} \times_{\mathbb{Q}} \mathbb{R}$ has no anisotropic factors over $\mathbb{Q}$.
(3) The weight map $h \circ w : GL_1 \times_{\mathbb{Q}} \mathbb{R} \to G \times_{\mathbb{Q}} \mathbb{R}$, where $w : GL_1 \times_{\mathbb{Q}} \mathbb{R} \to R_{\mathbb{C}/\mathbb{R}}(GL_1)$ is the canonical co-norm map, is defined over $\mathbb{Q}$
(4) For a maximal $\mathbb{Q}$-split torus $Z' \subset Z_G$, the quotient $Z_G(\mathbb{R})/Z'(\mathbb{R})$ is compact

With these assumptions, X is the finite disjoint union of Hermitian symmetric spaces of the form $G^{\mathsf{ss}}(\mathbb{R})^\circ/K^{\mathsf{ss}}$, where K^{ss} is a maximal connected compact subgroup of $G^{\mathsf{ss}}(\mathbb{R})$. We let K be the centralizer of a fixed point $h \in X$ in $G(\mathbb{R})$. It contains the product of K^{ss} and $Z_G(\mathbb{R})$. Let $\mathfrak{k}$ be the Lie algebra of K. It operates by the adjoint action on $\mathfrak{g}_{\mathbb{C}}$ and we obtain a $\mathfrak{k}$-invariant decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-. \tag{B.1.1}$$

Here, $\mathfrak{p}_-$ (resp. $\mathfrak{p}_+$) is the holomorphic (resp. anti-holomorphic) tangent space of X at h. We let

$$\mathfrak{p}_h := \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+.$$

This is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with Levi subalgebra $\mathfrak{k}_{\mathbb{C}}$ and nilpotent, even abelian, radical $\mathfrak{p}_+$. Observe that $\mathfrak{p}_h$ lies somewhat "skew" to the real structure of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ as $\mathfrak{p}_h \cap \mathfrak{g} = \mathfrak{k}$.

For us, a $\mathfrak{g}$-module V, which is also a representation of K, is called a $(\mathfrak{g}, K)$-module, if it is a $(\mathfrak{g}^{\mathsf{ss}}, K^{\mathsf{ss}})$-module in the sense of Borel-Wallach [Bor-Wal00], §0.2, by restriction. Mainly to set notation and for the sake of precision, we will now rapidly recall the definition of two relative Lie algebra cohomology theories.

The relative Lie algebra cohomology $H^q(\mathfrak{g}, \mathfrak{k}, V)$ of V was defined in [Bor-Wal00] I, 1.2. In the same reference, in I, 5.1, also the $(\mathfrak{g}, K)$-cohomology of V was defined. It is the cohomology $H^q(\mathfrak{g}, K, V)$ of the complex

$$C^q(\mathfrak{g}, K, V) := \mathrm{Hom}_K(\Lambda^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}), V) \cong \mathrm{Hom}_K(\Lambda^q(\mathfrak{p}_+ \oplus \mathfrak{p}_-), V)$$

$$\begin{aligned} df(X_0,...,X_q) \;&:=\; \sum_{i=0}^{q}(-1)^i X_i\cdot f(X_0,...,\hat{X}_i,...,X_q) \\ &+\; \sum_{i<j}(-1)^{i+j} f([X_i,X_j],X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_q). \end{aligned}$$

If K is connected, then $H^q(\mathfrak{g},\mathfrak{k},V)=H^q(\mathfrak{g},K,V)$. We recall that an irreducible representation V_λ of the real Lie group $G(\mathbb{R})$ on a finite-dimensional complex vector space is called *algebraic*, if the (extended) action of the complex Lie group $G(\mathbb{C})$ on V_λ is a representation of the linear algebraic group $G\times_{\mathbb{Q}}\mathbb{C}$ over $\mathbb{C}$. Finally, we say that a $(\mathfrak{g},K)$-module V is $(\mathfrak{g},K)$-*cohomological*, if there is an irreducible finite-dimensional algebraic $G(\mathbb{R})$-module V_λ such that $H^q(\mathfrak{g},K,V\otimes V_\lambda)\neq 0$ for some degree q.

The $(\mathfrak{p}_h,K)$-cohomology of a $(\mathfrak{g},K)$-module V is the cohomology of the complex

$$C^q(\mathfrak{p}_h,K,V):=\mathrm{Hom}_K(\Lambda^q(\mathfrak{p}_h/\mathfrak{k}_{\mathbb{C}}),V)\cong \mathrm{Hom}_K(\Lambda^q\mathfrak{p}_+,V),$$

with df defined as above. Following [MHar90], we say that a $(\mathfrak{g},K)$-module V is $(\mathfrak{p}_h,K)$-*cohomological*, if there is an irreducible finite-dimensional K-module V_τ such that the space $H^q(\mathfrak{p}_h,K,V\otimes V_\tau)\neq 0$ for some degree q.

B.2. **A general result on the relation of $(\mathfrak{g},K)$-cohomology and $(\mathfrak{p}_h,K)$-cohomology.** We denote by $\mathcal{C}_G$ (resp. $\mathcal{C}_K$) the collection of all equivalence classes of irreducible algebraic representations of $G(\mathbb{R})$ (resp. irreducible finite-dimensional representations of K). We assume to have fixed a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$ and a set of positive roots $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$. Because of (B.1.1), $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, too, and we assume that $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ is a choice of positive roots for $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$, extending the given choice $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ for $\mathfrak{k}_{\mathbb{C}}$. We assume that $\mathfrak{p}_h$ is a standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$, i.e., all roots in $\mathfrak{p}_+$ are positive (which also explains the notation). Clearly, $\overline{\mathfrak{p}}_h:=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}_-$ is the parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, which is opposite to $\mathfrak{p}_h$. If $V_\lambda\in\mathcal{C}_G$, then V_λ is determined by its highest weight λ with respect to $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$. Similarly, if $V_\tau\in\mathcal{C}_K$, then V_τ is determined by its highest weight τ with respect to $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$. Let $\rho:=\frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}\alpha$ (resp. $\rho_c:=\frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}\alpha$) be the half-sum of roots in $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ (resp. $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$) and let W (respectively $W_{\mathfrak{k}}$) be the Weyl group of $\mathfrak{g}_{\mathbb{C}}$ (respectively $\mathfrak{k}_{\mathbb{C}}$) with respect to $\mathfrak{t}_{\mathbb{C}}$. Then, the infinitesimal characters χ_{V_λ} (resp. χ_{V_τ}) of V_λ (resp. V_τ) are determined by $\lambda+\rho$ (resp. $\tau+\rho_c$) up to the action of W (resp. $W_{\mathfrak{k}}$). We will use the notation $\chi_{V_\lambda}=\chi_{\lambda+\rho}$ and $\chi_{V_\tau}=\chi_{\tau+\rho_c}$. Recall that there is the obvious surjection

$$\xi:\mathrm{Hom}(Z(\mathfrak{k}_{\mathbb{C}}),\mathbb{C})\twoheadrightarrow \mathrm{Hom}(Z(\mathfrak{g}_{\mathbb{C}}),\mathbb{C}),$$

cf. [MHar90], p. 31, mapping χ_Λ onto $\xi(\chi_\Lambda)=\chi_{\Lambda+\rho_n}$. Here, $Z(\mathfrak{k}_{\mathbb{C}})$ (resp. $Z(\mathfrak{g}_{\mathbb{C}})$) denotes the centre of the universal enveloping algebra of $\mathfrak{k}_{\mathbb{C}}$ (resp. $\mathfrak{g}_{\mathbb{C}}$) and $\rho_n=\rho-\rho_c$ is the half-sum of non-compact roots in $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ (i.e., the roots appearing in $\mathfrak{p}_+$).

Theorem B.2.1. *Let V be an irreducible unitary $(\mathfrak{g},K)$-module and let $V_\lambda\in\mathcal{C}_G$. If V is $(\mathfrak{g},K)$-cohomological with respect to V_λ in degree q, then V is $(\mathfrak{p}_h,K)$-cohomological in some degree $a\leq q$.*

Proof. Let V and V_λ be as in the statement of the proposition and assume that V is $(\mathfrak{g}, K)$-cohomological with respect to V_λ. Hence,

$$\mathrm{Hom}_K(\Lambda^q(\mathfrak{p}_+ \oplus \mathfrak{p}_-), V \otimes V_\lambda) \neq 0$$

for some degree q. As

$$\mathrm{Hom}_K(\Lambda^q(\mathfrak{p}_+ \oplus \mathfrak{p}_-), V \otimes V_\lambda) \cong \bigoplus_{a+b=q} \mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+ \otimes \Lambda^b \mathfrak{p}_-, V \otimes V_\lambda),$$

there are $0 \leq a, b \leq q$ such that $a + b = q$ and

$$\mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+ \otimes \Lambda^b \mathfrak{p}_-, V \otimes V_\lambda) \cong \mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+, V \otimes (V_\lambda \otimes \Lambda^b \mathfrak{p}_-^*)) \neq 0.$$

Observe that there is an isomorphism of K-representations $V_\lambda \otimes \Lambda^b \mathfrak{p}_-^* \cong H^b(\mathfrak{p}_-, V_\lambda)$. We may hence use Kostant's description of the K-module $H^b(\mathfrak{p}_-, V_\lambda)$: To this end, we identify $\mathfrak{p}_-$ as the nilpotent radical of the parabolic subalgebra $\overline{\mathfrak{p}}_h \subset \mathfrak{g}_\mathbb{C}$ opposite to $\mathfrak{p}_h$. Let $W^{\mathfrak{p}_h}$ be the set of Weyl group elements w for which $w^{-1}(\alpha) \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ for all $\alpha \in \Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Going over to the opposite ordering of roots, we obtain the set of Kostant representatives for $\overline{\mathfrak{p}}_h$ as $W^{\overline{\mathfrak{p}}_h} = w_G W^{\mathfrak{p}_h} w_G$. Here, w_G denotes the longest element of W. Therefore, [Kos61], Thm. 5.14, implies that, there is an isomorphism of K-modules

$$H^b(\mathfrak{p}_-, V_\lambda) \cong \bigoplus_{\substack{w' \in W^{\overline{\mathfrak{p}}_h} \\ \ell(w')=b}} V_{w_c(w'(w_G(\lambda)+\overline{\rho})-\overline{\rho})}.$$

where w_c is the longest element in the Weyl group $W_\mathfrak{k}$ of K, $\overline{\rho}$ the half-sum of positive roots with respect to the opposite ordering in $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ (whence equal to $\overline{\rho} = w_G(\rho) = -\rho$) and V_τ denotes the irreducible K-representation of highest weight τ with respect to $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. A simple calculation using [Bor-Wal00, V, 1.4] hence implies that as K-modules

$$H^b(\mathfrak{p}_-, V_\lambda) \cong \bigoplus_{\substack{w \in W^{\overline{\mathfrak{p}}_h} \\ \ell(w)=\dim_\mathbb{R}(\mathfrak{p}_-)-b}} V_{w(\lambda+\rho)+w_c(\rho)}. \tag{B.2.2}$$

Abbreviate $b' := \dim_\mathbb{R}(\mathfrak{p}_-) - b$, $\tau_w := w(\lambda + \rho) + w_c(\rho)$ and let V_{τ_w} be the corresponding irreducible K-representation appearing in (B.2.2). Hence, by what we have seen above, we have proved that there are $0 \leq a, b \leq q$ such that $a + b = q$ and

$$0 \neq \mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+, V \otimes (V_\lambda \otimes \Lambda^b \mathfrak{p}_-^*)) \cong \bigoplus_{\substack{w \in W^{\overline{\mathfrak{p}}_h} \\ \ell(w)=b'}} \mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+, V \otimes V_{\tau_w}).$$

Hence, there is a $w \in W^{\overline{\mathfrak{p}}_h}$ of length $\ell(w) = b'$, such that

$$\mathrm{Hom}_K(\Lambda^a \mathfrak{p}_+, V \otimes V_{\tau_w}) \neq 0.$$

Fix such a Kostant representative $w \in W^{\overline{\mathfrak{p}}_h}$. The infinitesimal character of the contragredient $V_{\tau_w}^\vee$ is given by

$$\chi_{V_{\tau_w}^\vee} = \chi_{-w_c(\tau_w)+\rho_c} = \chi_{-w_c w(\lambda+\rho)-\rho+\rho_c} = \chi_{-w_c w(\lambda+\rho)-\rho_n},$$

and hence maps by the surjection ξ onto the infinitesimal character of the contragredient of the algebraic $G(\mathbb{R})$-representation V_λ:

$$\xi(\chi_{V^\vee_{\tau_w}}) = \chi_{(-w_c w(\lambda+\rho)-\rho_n)+\rho_n} = \chi_{-w_c w(\lambda+\rho)} = \chi_{-\lambda-\rho} = \chi_{V^\vee_\lambda}. \tag{B.2.3}$$

Let $C_{\mathfrak{g}}$ be the Casimir operator in $Z(\mathfrak{g}_{\mathbb{C}})$. Then, as by assumption V is $(\mathfrak{g}, K)$-cohomological with respect to V_λ, the infinitesimal character χ_V of V and the infinitesimal character $\chi_{V^\vee_\lambda}$ of $V^\vee_\lambda$ agree on $C_{\mathfrak{g}}$ as a consequence of [Bor-Wal00, I, Thm. 5.3.(ii)]. In particular, (B.2.3) implies that

$$\chi_V(C_{\mathfrak{g}}) = \xi(\chi_{V^\vee_{\tau_w}})(C_{\mathfrak{g}}).$$

The analogue of Kuga's formula for $(\mathfrak{p}_h, K)$-cohomology, established as Thm. 4.1 in [Oka-Oze67] (see also [MHar90, Prop. 4.4.3]), hence implies verbatim as in the proof of [Bor-Wal00, II, Prop. 3.1.(b)] that

$$\mathrm{Hom}_K(\Lambda^a\mathfrak{p}_+, V \otimes V_{\tau_w}) = H^a(\mathfrak{p}_h, K, V \otimes V_{\tau_w}).$$

So, by our above discussion, we finally obtain

$$H^a(\mathfrak{p}_h, K, V \otimes V_{\tau_w}) \neq 0.$$

Thus, V is $(\mathfrak{p}_h, K)$-cohomological in degree $a \leq q$ with respect to some irreducible K-summand V_{τ_w} of $V_\lambda \otimes \Lambda^b\mathfrak{p}^*_-$. Q.E.D. □

We conclude this section by the following

Corollary B.2.4. *Let V be an irreducible unitary $(\mathfrak{g}, K)$-module and let $V_\lambda \in \mathcal{C}_G$. The degrees q, where V is $(\mathfrak{g}, K)$-cohomological with respect to V_λ, are all of the same parity.*

Proof. Using the Künneth rule, [Bor-Wal00] I.1.3, and [Vog-Zuc84], Thm. 5.5, it is enough to show this for the trivial $(\mathfrak{g}, K)$-module $V = \mathbf{1}$ and the trivial coefficient system $V_\lambda = \mathbb{C}$ of $G(\mathbb{R})$. It is clear that $V = \mathbf{1}$ has non-trivial $(\mathfrak{g}, K)$-cohomology with respect to $V_\lambda = \mathbb{C}$ in degree $q = 0$. Hence, we have to show that all degrees q, where $H^q(\mathfrak{g}, K, \mathbf{1} \otimes \mathbb{C}) \neq 0$, are even. Let q be any such degree. Then, we have seen in the proof of Prop. B.2.1, that there are a, b such that $q = a + b$ and

$$\mathrm{Hom}_K(\Lambda^a\mathfrak{p}_+, \Lambda^b\mathfrak{p}^*_-) \neq 0.$$

In other words, the K-representations $\Lambda^a\mathfrak{p}_+$ and $\Lambda^b\mathfrak{p}^*_- \cong \Lambda^b\mathfrak{p}_+$ share an irreducible K-type. For any r, we have

$$\Lambda^r\mathfrak{p}_+ \cong \bigoplus_{i_1,\ldots,i_r} \bigotimes_{j=1}^{r} \mathfrak{g}_{\alpha_{i_j}},$$

where $\mathfrak{g}_{\alpha_{i_j}}$ is the one-dimensional root eigenspace of $\mathfrak{g}_{\mathbb{C}}$ of the non-compact, positive root α_{i_j}. Hence, for $\Lambda^a\mathfrak{p}_+$ and $\Lambda^b\mathfrak{p}_+$ to have an irreducible K-type in common, we have to have $a = b$, whence, $q = 2a$ is even as predicted. □

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