ON THE COHOMOLOGY OF $SL_n(\mathbb{Z})$ BEYOND THE STABLE RANGE

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Abstract. This paper investigates the cohomology of $SL_n(\mathbb{Z})$, $n \geq 2$, "right outside" what one calls the "stable range". More precisely, a qualitative non-vanishing result for the cohomology $H^q(SL_n(\mathbb{Z}))$ in degrees $q = n-1$ and $q = n$ is shown. This relies on a description of $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ (for all $n \geq 2$ and all degrees q) in terms of automorphic forms, which turns out to be very simple if $n \leq 11$. In the last section a question of F. Brown on $SL_6(\mathbb{Z})$, respectively a question of A. Ash on $SL_8(\mathbb{Z})$, is answered.

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Introduction

In order to describe the context and the results of this paper, let $G/\mathbb{Q}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and fix a choice of a maximal compact subgroup K of the real Lie group $G(\mathbb{R})$, i.e., of the group of $\mathbb{R}$-points of G. We denote by $X = G(\mathbb{R})/K$ the associated symmetric space. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$.

Half a century ago, cf. [Bor74], A. Borel showed that the cohomology $H^q(\Gamma, \mathbb{C})$ of Γ is – below a certain degree $q(G)$ – entirely spanned by classes, which are represented by $G(\mathbb{R})$-invariant differential forms on X. Although Borel's bound is not sharp in general, his result implies that below degree $q(G)$, the cohomology $H^q(\Gamma, \mathbb{C})$ falls into what one calls ever since the "stable range", i.e., the maximal range of degrees of cohomology, in which $H^q(\Gamma, \mathbb{C})$ does not change, even if the rank of

2010 *Mathematics Subject Classification.* 11F66, 11F70, 11F72, 22E55.

Key words and phrases. $SL_n(\mathbb{Z})$, group cohomology, automorphic representation, Franke's filtration.

Both authors are supported by the Croatian Science Foundation research project HRZZ-IP-2022-10-4615. The first named author is also supported by the Croatian Science Foundation project HRZZ-IP-2019-04-4216.

The second named author is supported by the Stand-alone research project P32333 and the Principal Investigator project PAT4628923, both financed by the Austrian Science Fund (FWF).

G in its Cartan-type classification is allowed to grow to infinity (and Γ varies among the arithmetic subgroups of G).

If Γ is a congruence subgroup, then the above can be rephrased in the more modern language of adèles $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ (over $\mathbb{Q}$) and automorphic forms: It can be expressed by saying that in a certain maximal range of degrees $0 \leq q \leq st(G)$, all classes in the cohomology $H^q(\Gamma, \mathbb{C})$ are obtained from $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$, i.e., from the $(\mathfrak{g}, K)$-cohomology of the global trivial automorphic representation $\mathbf{1}_{G(\mathbb{A})}$ of $G(\mathbb{A})$, realized as a square-integrable automorphic representation on the space of constant functions $G(\mathbb{A}) \to \mathbb{C}$. In other words, given the Lie group $G(\mathbb{R})$, it is enough to study the Poincaré-polynomial of $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{R})})$, which is usually well-understood in terms of differential geometry, in order to understand $H^q(\Gamma, \mathbb{C})$ for all congruence subgroups Γ of $G(\mathbb{Q})$ and degrees $q \leq st(G)$.

In this paper we explore what happens in the case of the special linear group $G = SL_n$, if one passes right beyond the "stable range" $st(SL_n) = n - 2$. More precisely, we show qualitative results for the growth of the $\mathbb{Z}$-rank of the free part of the $\mathbb{Z}$-module $H^q(SL_n(\mathbb{Z}))$ in degrees $q = n-1$ and $q = n$. Our main result in this direction reads as follows. Let $a(q)$ be the number of ways to write a positive integer q as the sum of different integers of the form $4\ell + 1$, $\ell \geq 1$. Then we obtain

Theorem A. *Let $n \geq 4$. Then,*

$$\dim_{\mathbb{C}} H^n(SL_n(\mathbb{Z}), \mathbb{C}) \geq \begin{cases} a(n) - 1 & \text{if } n \text{ is even} \\ a(n) & \text{if } n \text{ is odd} \end{cases}$$

In particular, the free part of the $\mathbb{Z}$-module $H^n(SL_n(\mathbb{Z}))$ is non-zero, in the following cases:

- *for odd n, if either $n \geq 25$, or $n \in \{5, 9, 13, 17, 21\}$;*
- *for even n, if either $n \geq 50$, or $n \in \{22, 26, 30, 34, 38, 42, 46\}$.*

Moreover,

$$\dim_{\mathbb{C}} H^{n-1}(SL_n(\mathbb{Z}), \mathbb{C}) \geq \begin{cases} a(n-1) + 1 & \text{if } n \text{ is even} \\ a(n-1) & \text{if } n \text{ is odd} \end{cases}$$

In particular, the free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$ is non-zero, whenever $H^{n-1}(\mathfrak{g}, K, \mathbb{C})$ is non-zero. In addition to this non-vanishing result, the free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$ is non-zero, if $n = 4, 8, 12$.

We refer to a combination of our Thm. 4.6, Cor. 5.3, Lem. 3.2 and Thm. 5.4 for this result.

The reader is invited to compare our result with Thm. 1.1 of the recent preprint [Bro23], which also implies a growth-condition on the dimension of the cohomology of $SL_n(\mathbb{Z})$. Though our methods here are totally different, it is interesting to notice that for odd n the dimension of the space of n-forms of "non-compact type" (as they are used and called in [Bro23], Thm. 1.1) is the same as our constant $a(n)$. It should also be noted that there are several complementary (and sometimes partly overlapping) results in the recent literature: We would like to mention [Ash24, KMP21, PSS20, Chu-Put17, CFP14] as a chronologically decreasing selection of interesting recent sources.

Another feature of this paper – and, in fact, the basis of the proof of Thm. A above – is a description of $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ in terms of automorphic forms, which turns out to be very simple, if

$n \leq 11$. Our main result in this direction builds on the deep work of Franke [Fra98], Franke–Schwermer [Fra-Schw98] and a fundamental result of Chenevier-Lannes [Che-Lan19] and says, that

Theorem B. *For all $n \geq 2$ and all degrees q of cohomology, there is an isomorphism of modules over the Hecke algebra of $SL_n(\mathbb{Z})$ (or, equivalently, of the maximal open compact subgroup $K_f = SL_n(\hat{\mathbb{Z}})$ in $SL_n(\mathbb{A}_f)$, where $\hat{\mathbb{Z}}$ denotes the Prüfer ring, i.e., the profinite completion of $\mathbb{Z}$)*

$$H^q(SL_n(\mathbb{Z}),\mathbb{C}) \cong \bigoplus_{\{P\}} \bigoplus_{\substack{\varphi(\pi):\\ \chi_{\tilde{\pi}_\infty}=\rho_M \\ \pi_f \text{ is of level 1}}} H^q(\mathfrak{g},K,\mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f}.$$

Here, the first sum ranges over all associate classes of standard parabolic $\mathbb{Q}$-subgroups P of SL_n and the second sum ranges over all associate classes $\varphi(\pi)$ of cuspidal automorphic representations $\pi = \tilde{\pi} \cdot e^{\langle \lambda_\pi, H_P(\cdot)\rangle}$ of the Levi subgroup of P, satisfying the listed conditions.

If $n \leq 11$, then the following much simpler description holds:

$$H^q(SL_n(\mathbb{Z}),\mathbb{C}) \cong H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)}(G))^{K_f},$$

where $\varphi(\chi)$ is the cuspidal support represented by the Hecke character

$$\chi = e^{\langle \rho_B, H_B(\cdot)\rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the adèlic points of the maximal torus T of SL_n.

See Thm. 2.2 for all details and Sect. 2.2.2 for unexplained notation.

It should be noted, however, that this intriguingly simple automorphic structure of $H^q(SL_n(\mathbb{Z}),\mathbb{C})$ shall not be expected to persist for all $n \geq 12$: Indeed, as it was indicated to us by Chenevier (and correcting a blunder in an earlier version of this manuscript), one may easily construct a level 1 cusp form of $GL_2(\mathbb{A}) \times \prod_{i=1}^{10} GL_1(\mathbb{A})$ out of the Ramanujan Delta-function and 10 suitable Hecke characters, such that the right infinitesimal character is matched.

We remark that for $n \geq 27$ one should even expect a non-trivial contribution of cuspidal cohomology to $H^q(SL_n(\mathbb{Z}),\mathbb{C})$: Indeed, though it is known due to the work of Miller [Mil02] and Salamanca-Riba [Sal-Rib98], that there are no irreducible unitary cuspidal automorphic representations of $GL_n(\mathbb{A})$, with $1 < n \leq 26$, which have the same infinitesimal character as the trivial representation of $GL_n(\mathbb{R})$ and which are also unramified at all non-archimedean places, computations of Chenevier and Taïbi suggest that there are lots of such level 1 cusp forms of $GL_n(\mathbb{A})$ with n "not much bigger than" 27, cf. [Che23]; while it was only recently discovered by Boxer, Calegari and Gee – using the newly established proof of functoriality of all symmetric powers of level 1 cusp forms of $GL_2(\mathbb{A})$ by Newton–Thorne [New-Tho21] – that there is indeed an irreducible unitary cuspidal automorphic representation of $GL_n(\mathbb{A})$, with $n = 79, 105, 106$, which has the same infinitesimal character as the trivial representation of $GL_n(\mathbb{R})$ and which is also unramified at all non-archimedean places. See [BCG23].

In the last section of this paper we answer a question of F. Brown on the cohomology of $SL_6(\mathbb{Z})$, respectively a question of A. Ash on the cohomology of $SL_8(\mathbb{Z})$.

More precisely, Brown's recent paper [Bro23] establishes a powerful technique to obtain non-trivial cohomology classes for $SL_n(\mathbb{Z})$ (in particular, it reproves [Bor74]). Still, his method is not suited to explain the non-vanishing of $H^8(SL_6(\mathbb{Z}))$, as shown by Elbaz-Vincent–Gangl–Soulé, [EVGS13]. Here, we give a structural explanation of the non-vanishing of $H^8(SL_6(\mathbb{Z}))$ and determine, which automorphic forms of $SL_6(\mathbb{A})$ represent the non-trivial classes in $H^8(SL_6(\mathbb{Z}),\mathbb{C})$. See §6.1 for details.

Similarly, as communicated to the second named author by Brown, A. Ash, has asked for a description of the cohomology of $SL_8(\mathbb{Z})$. Among others, degree $q = 15$ was of particular interest. Here we show that $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$ is two-dimensional, and we describe, which automorphic forms of $SL_8(\mathbb{A})$ represent the non-trivial classes in $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$. We refer to §6.2 for this result.

Degree $q = 8$ (for $SL_6(\mathbb{Z})$) and $q = 15$ (for $SL_8(\mathbb{Z})$) have in common that they are the value of $q = m^2 - 1$, if we write $n = 2m$. It should be noted that m^2 is the bottom degree in which a tempered cuspidal automorphic representation of $SL_n(\mathbb{A})$ may have cohomology, i.e., degree $q = m^2 - 1$ is just the degree "right outside" the tempered cuspidal range. Hence, finally, as a natural generalization of these considerations, it seems tempting to state the following

Open Problem. *Determine for which $m \geq 5$, the cohomology $H^{m^2-1}(SL_{2m}(\mathbb{Z}),\mathbb{C})$ is non-zero.*

Our results imply that $H^{m^2-1}(SL_{2m}(\mathbb{Z}),\mathbb{C})$ is non-zero for $m = 1, 2, 3, 4$. However, in higher rank, the problem becomes more and more complicated. See §6.3 for more details.

Acknowledgments:
We are grateful to Francis Brown, Gaëtan Chenevier and Christian Kaiser for several extremely valuable hints and discussions, which followed various, very different approaches to the topic. We are also truly indebted to Günter Harder, who (re-)initiated our interest in the cohomology of $SL_n(\mathbb{Z})$. Finally, H.G. wants to thank the Hotel "Das Schmidt" in Mörbisch, where major parts of this paper have been established in intensive work, whereas in a very pleasant atmosphere.

1. Preliminaries and notation

1.1. **Groups.** The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have their usual meaning. The ring of adèles of $\mathbb{Q}$ will be denoted by $\mathbb{A}$, its subring of non-archimedean elements by $\mathbb{A}_f$.

For $n \geq 1$, let GL_n be the general linear group defined over $\mathbb{Q}$. If H is any $\mathbb{Q}$-subgroup of GL_n, then $S(H)$ will denote its elements of determinant equal to 1. In particular, we will write $G := S(GL_n) = SL_n$ for the ($\mathbb{Q}$-split) special linear group defined over $\mathbb{Q}$.

1.2. **Parabolic data.** We fix once and for all the Borel subgroup B of G, consisting of upper-triangular matrices in G. Let $B = TU$ be the Levi decomposition of B, where T is a maximal split torus in B, and U the unipotent radical. Then,

$$T(R) = \left\{ \operatorname{diag}(t_1, \dots, t_n) \,:\, t_i \in R^\times, \prod_i t_i = 1 \right\}$$

for any abelian $\mathbb{Q}$-algebra R. More generally, let $P \supseteq B$ be a standard parabolic $\mathbb{Q}$-subgroup of G, cf. [Bor-Wal00], 0.3.4. They are parameterized by the tuples $(n_1, \dots, n_k)$, $k \geq 1$, $n_i \in \mathbb{N}$, according to the block-sizes of the corresponding Levi subgroup $L \cong S(GL_{n_1} \times \cdots \times GL_{n_k}) \subset P$. Its group of real points $L(\mathbb{R})$ admits a unique maximal semisimple direct factor, denoted M. Its Lie algebra is naturally complemented by the real Lie algebra $\mathfrak{a}_P$ of the split component A_P of L. Its (complexified) dual is as usual denoted by $\check{\mathfrak{a}}_P$ (resp. by $\check{\mathfrak{a}}_{P,\mathbb{C}}$). We write $S(\check{\mathfrak{a}}_{P,\mathbb{C}})$ for the attached symmetric tensor algebra. In general, if H is a real Lie group, we will use $\mathfrak{h}$ to denote its Lie algebra and $\mathfrak{h}_\mathbb{C}$ for its complexification.

1.3. **Compact subgroups.** We assume to have fixed a maximal compact subgroup K of $G(\mathbb{R})$ and K_f of $G(\mathbb{A}_f)$ in good position with respect to B and T. More explicitly, $K = SO(n)$, the compact special orthogonal group of $n \times n$-matrices and $K_f = SL_n(\hat{\mathbb{Z}}) = \prod_p SL_n(\mathbb{Z}_p)$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the Prüfer ring, i.e., the profinite completion of $\mathbb{Z}$.

1.4. **Certain characters.** We denote by $\mathrm{sgn} : \mathbb{R}^* \to \{\pm 1\}$ the sign-character of the multiplicative group $\mathbb{R}^*$ of non-zero real numbers. If $\lambda \in \check{\mathfrak{a}}_P$, then $\mathbb{C}_\lambda$ denotes the one-dimensional module of $L(\mathbb{R})$ of highest weight λ, i.e., if $L(\mathbb{R}) \cong S(GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_k}(\mathbb{R}))$ and $\lambda = (\lambda_1, \ldots \ldots, \lambda_k)$, then $\mathbb{C}_\lambda = \det_{n_1}^{\lambda_1} \otimes \cdots \otimes \det_{n_k}^{\lambda_k}$, where $\det_{n_i}$ denotes the determinant on $GL_{n_i}(\mathbb{R})$. Going adelic, if $\lambda \in \check{\mathfrak{a}}_P$, then $e^{\langle \lambda, H_P(\cdot) \rangle}$ denotes the one-dimensional representation of $L(\mathbb{A})$ constructed from λ and the Harish-Chandra height function $H_P(\cdot)$, cf. [Fra98], p. 185. If H is any subgroup of $G(\mathbb{A})$, then $\mathbf{1}_H$ denotes the trivial representation of H.

2. A sufficient condition for the non-vanishing of $H^q(SL_n(\mathbb{Z}))$

2.1. **Recap: The cohomology of $SL_n(\mathbb{Z})$ via automorphic forms.** For the sake of later reference, we shall shortly recall some facts about the cohomology of $SL_n(\mathbb{Z})$ and its interconnection to the cohomology of the space of automorphic forms of $SL_n(\mathbb{A})$.

In order to do so, we need to take a "transcendental" point of view, i.e., work with coefficient modules over $\mathbb{C}$. Just in this section, let us abbreviate $\Gamma = SL_n(\mathbb{Z})$ and let us also view $\mathbb{Z}$ and $\mathbb{C}$ as trivial modules under Γ. It is well-known that the group homology $H_*(\Gamma) := H_*(\Gamma, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. Indeed, this follows easily from the fact that Γ has a subgroup of finite index, which is torsion free, whence Γ itself is an arithmetic group of finite type, cf. [Ser79], §1.3. The universal coefficient theorem for group homology hence shows that as $\mathbb{C}$-vector spaces

$$H_q(\Gamma) \otimes_\mathbb{Z} \mathbb{C} \cong H_q(\Gamma, \mathbb{C}).$$

Using duality between singular homology and cohomology, we get again an isomorphism of group cohomology as $\mathbb{C}$-vector spaces

$$H^q(\Gamma) \otimes_\mathbb{Z} \mathbb{C} \cong H^q(\Gamma, \mathbb{C}).$$

It follows that the free part of the $\mathbb{Z}$-module $H^q(SL_n(\mathbb{Z}))$ must be non-zero, if $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ is.

Let now $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$, cf. [Bor-Jac79, Gro23], on which the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$ acts trivially. Then, it is well-known, that $H^q(\Gamma, \mathbb{C})$ allows a description as the space of K_f-invariant vectors in the $(\mathfrak{g}, K)$-cohomology of $\mathcal{A}(G)$, cf. [Bor-Wal00], Thm. VII.2.2 in combination with (Strong Approximation for G, cf. [Pla-Rap94], Thm. 7.12) and [Fra98], Thm. 18:

$$H^q(SL_n(\mathbb{Z}), \mathbb{C}) \cong H^q(\mathfrak{g}, K, \mathcal{A}(G))^{K_f}. \tag{2.1}$$

Therefore, each cohomology class in $H^q(SL_n(\mathbb{Z}),\mathbb{C})$ may be represented by everywhere unramified, i.e., K_f-right invariant, automorphic forms in $\mathcal{A}(G)$.

2.2. Automorphic background à la Franke and a first consequence for the cohomology of $SL_n(\mathbb{Z})$.

2.2.1. *Parabolic supports.* Let $\{P\}$ be the associate class of the parabolic $\mathbb{Q}$-subgroup $P = L_P N_P$ of $G = SL_n$: It consists by definition of all parabolic $\mathbb{Q}$-subgroups $Q = L_Q N_Q$ of G, for which L_Q and L_P are conjugate by an element in $SL_n(\mathbb{Q})$. We denote by $\mathcal{A}_{\{P\}}(G)$ the space of all $f \in \mathcal{A}(G)$, which are negligible along every parabolic $\mathbb{Q}$-subgroup $Q \notin \{P\}$: This means that for all $g \in SL_n(\mathbb{A})$, the function $L_Q(\mathbb{A}) \to \mathbb{C}$, which is given by $\ell \mapsto f_Q(\ell g)$, where f_Q denotes the constant term of f along Q, is orthogonal (with respect to the Petersson inner product) to the space of all cuspidal automorphic forms on $L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})$. Having set up these notations, there is the following decomposition of $\mathcal{A}(G)$ as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module, cf. [BLS96] Thm. 2.4:

$$\mathcal{A}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\{P\}}(G).$$

2.2.2. *Cuspidal supports.* We recall now, cf. [Fra-Schw98], 1.2, and [Gro23], §15.2, the notion of an *associate class* $\varphi(\pi)$ of cuspidal automorphic representations of the Levi subgroups of the elements in the class $\{P\}$. Therefore, let $\{P\}$ be represented by $P = LN$. Then, an associate class $\varphi(\pi)$ may be parameterized by $\pi = \tilde{\pi} \cdot e^{\langle \lambda_\pi, H_P(\cdot)\rangle}$, where

(1) $\tilde{\pi}$ is a unitary cuspidal automorphic representation of $L(\mathbb{A})$, whose central character vanishes on the identity component $A_P(\mathbb{R})^\circ$ of $A_P(\mathbb{R})$,
(2) $\lambda_\pi \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, which is compatible with the infinitesimal character $\chi_{\tilde{\pi}_\infty}$ of $\tilde{\pi}_\infty$ (cf. [Fra-Schw98], 1.2, or [Gro23], §15.2, in particular (15.13)).

We let $\mathcal{W}_{P,\tilde{\pi}}$ be the space of all smooth, K-finite functions

$$f : L(\mathbb{Q})N(\mathbb{A})A_P(\mathbb{R})^\circ\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $\ell \mapsto f(\ell g)$ on $L(\mathbb{A})$ is contained in the $\tilde{\pi}$-isotypic component of the cuspidal spectrum $L^2_{cusp}(L(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$. For a function $f \in \mathcal{W}_{P,\tilde{\pi}}$, $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ and $g \in G(\mathbb{A})$ an *Eisenstein series* is formally defined as

$$E_P(f,\lambda)(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) e^{\langle \lambda + \rho_P, H_P(\gamma g)\rangle}.$$

It is known to converge absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times \check{\mathfrak{a}}_{P,\mathbb{C}}$, if the real part of λ is sufficiently positive. In that case, $E_P(f,\lambda)$ is an automorphic form and the map $\lambda \mapsto E_P(f,\lambda)(g)$ can be analytically continued to a meromorphic function on all of $\check{\mathfrak{a}}_{P,\mathbb{C}}$, cf. [Mœ-Wal95], II.1.5, IV.1.8, IV.1.9, [Lan76], §7, or, most concretely, the main result of [Ber-Lap23]. Given $\varphi(\pi)$, represented by a cuspidal representation π of the above form, a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-submodule

$$\mathcal{A}_{\{P\},\varphi(\pi)}(G)$$

of $\mathcal{A}_{\{P\}}(G)$ was defined in [Fra-Schw98], 1.3 as follows: It is the span of all possible partial derivatives of holomorphic values or residues of all Eisenstein series attached to $\tilde{\pi}$, evaluated at the point $\lambda = \lambda_\pi$. This definition is independent of the choice of the representatives P and π, due to the functional equations satisfied by the Eisenstein series considered. For details, we refer the reader to

[Fra-Schw98] 1.2–1.4 as the original source, or to [Gro23], §15.2–15.3. We obtain, see, [Fra-Schw98], Thm. 1.4, or [Gro23], Thm. 15.21,

Theorem 2.1. *There is an isomorphism of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules*

$$\mathcal{A}_{\{P\}}(G) \cong \bigoplus_{\varphi(\pi)} \mathcal{A}_{\{P\},\varphi(\pi)}(G).$$

Using Thm. 2.1, the next result refines the above description of $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ in terms of automorphic forms and reveals that the cohomology of $SL_n(\mathbb{Z})$ is in fact quite simply structured, if $n \leq 11$. Namely, we will show that in the latter case it is strictly supported by the trivial character of the Borel subgroup $B = TU$ of G. For this recall that an irreducible cuspidal automorphic representation π of $L(\mathbb{A})$ is called *of level 1*, if its non-archimedean component π_f, as a representation of $L(\mathbb{A}_f)$, satisfies $\pi_f^{K_f \cap L(\mathbb{A}_f)} \neq \{0\}$, i.e., if π is unramified at all non-archimedean places.

Theorem 2.2. *For all $n \geq 2$ and all degrees q of cohomology, there is an isomorphism of modules of the Hecke algebra of $SL_n(\mathbb{Z})$ (or, equivalently of the maximal open compact subgroup $K_f = SL_n(\hat{\mathbb{Z}})$)*

$$H^q(SL_n(\mathbb{Z}), \mathbb{C}) \cong \bigoplus_{\{P\}} \bigoplus_{\substack{\varphi(\pi): \\ \chi_{\tilde{\pi}_\infty} = \rho_M \\ \pi_f \text{ is of level } 1}} H^q(\mathfrak{g}, K, \mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f},$$

where ρ_M denotes the restriction of ρ to the semisimple part of $L(\mathbb{R})$, i.e., the intrinsic modulus character of M.

If $n \leq 11$, then the following much simpler description holds:

$$H^q(SL_n(\mathbb{Z}), \mathbb{C}) \cong H^q(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)}(G))^{K_f}, \tag{2.2}$$

where $\varphi(\chi)$ is the cuspidal support represented by the Hecke character

$$\chi = e^{\langle \rho_B, H_B(\cdot) \rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$.

Proof. From our above explanations, we get

$$\begin{aligned} H^q(SL_n(\mathbb{Z}), \mathbb{C}) &\cong H^q(\mathfrak{g}, K, \mathcal{A}(G))^{K_f} \\ &\cong \bigoplus_{\{P\}} \bigoplus_{\varphi(\pi)} H^q(\mathfrak{g}, K, \mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f}. \end{aligned}$$

For any representative $\pi = \tilde{\pi} \cdot e^{\langle \lambda_\pi, H_P(\cdot) \rangle}$ of an associate class $\varphi(\pi)$, the natural $(\mathfrak{g}, K, G(\mathbb{A}_f))$-homomorphism,

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes S(\check{\mathfrak{a}}_{P,\mathbb{C}})) \longrightarrow \mathcal{A}_{\{P\},\varphi(\pi)}(G)$$

given by summation of locally regularized Eisenstein series around λ_π is surjective, cf. [Fra-Schw98], 3.3.(4). Hence, in order to obtain a non-zero space

$$H^q(\mathfrak{g}, K, \mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f},$$

it is necessary that $\tilde{\pi}_\infty$ has the same infinitesimal character like the trivial representation of the semisimple part M of $L(\mathbb{R})$, i.e., $\chi_{\tilde{\pi}_\infty} = \rho_M$, cf. [Bor-Wal00], Thm. I.5.3.(ii) and [Kna86], Prop.

8.22. Moreover, invoking Frobenious reciprocity for non-archimedean parabolic induction, it is clear that π_f must be unramified at every place, i.e., of level 1. This shows the first assertion.

Let now be $n \leq 11$. Then, by [Che-Lan19], Thm. F on p. 13 (see also [Che-Tai20], Thm. 3 and §2.4.6), there is no level 1 irreducible unitary cuspidal automorphic representation of $L(\mathbb{A})$, whose infinitesimal character matches the one of $\mathbf{1}_M$, if $L = S(GL_{n_1} \times ... \times GL_{n_r})$ contains a general linear group of rank $n_i > 1$. It therefore follows that $\varphi(\pi)$ must be represented by an irreducible cuspidal automorphic representation π with $P = B$ and $\tilde{\pi} = \mathbf{1}_{T(\mathbb{A})}$. Moreover, for $\varphi(\pi)$ to contribute non-trivially to $H^*(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\pi)}(G))$, we must have $\lambda_\pi = -w(\rho_B)$, where w is an element of the Weyl group W of SL_n (i.e., a Kostant representative for the Borel subgroup B, cf. [Bor-Wal00], III.1.4), such that $-w(\rho_B)$ is in the closed positive Weyl chamber of $\check{\mathfrak{t}}_{\mathbb{C}} = \check{\mathfrak{a}}_{B,\mathbb{C}}$, cf. [Fra-Schw98], 5.5. together with p. 772 *ibidem*. The latter condition, however, is only satisfied by the longest element w_G of W, which gives $\lambda_\pi = -w_G(\rho_B) = \rho_B$. This shows the claim. □

Remark 2.3. As indicated in the introduction, this simple description of $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ as in (2.2) will generally fail, if $n \geq 12$, because of the existence of an irreducible unitary cuspidal automorphic representation $\tilde{\tau}$ of $GL_2(\mathbb{A})$ of level 1 and of infinitesimal character $\chi_{\tilde{\tau}_\infty} = (\frac{11}{2}, -\frac{11}{2})$ (namely the one constructed out of a non-zero cuspidal modular form of weight 12 and full level, i.e., out of a non-zero element in $S_{12}(SL_2(\mathbb{Z}))$, e.g., the Ramanujan Delta-function). Indeed, suitably extended by 10 Hecke characters, one obtains an irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A}) \times \prod_{i=1}^{10} GL_1(\mathbb{A})$ of level 1 and of infinitesimal character $\chi_{\pi_\infty} = (\frac{11}{2}, \frac{9}{2}, \frac{7}{2}, ..., -\frac{9}{2}, -\frac{11}{2}) = \rho_{GL_{12}}$.

3. An examination of Franke's filtration and consequences for automorphic cohomology

3.1. **Franke's filtration of the cuspidal support of the trivial automorphic representation.** We recall that in [Fra98], §6, a certain, technically involved, finite-step filtration was defined for the spaces $\mathcal{A}_{\{P\}}(G)$, which can be refined to apply to the individual summands $\mathcal{A}_{\{P\},\varphi(\pi)}(G)$, cf. [Grb12], §3, [Gro13], §3.1, [Grb-Gro13], §3, or [Grb23], Chap. 4. The reader, who prefers to read a presentation of this subject, which is taylored to the (special) linear group, is invited to consult [Grb-Gro22], §2, for all relevant details. Our next result makes this filtration explicit for the datum $(\{B\}, \varphi(\chi))$, $\chi = e^{\langle \rho_B, H_B(\cdot)\rangle}$.

Theorem 3.1. *Let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character*

$$\chi = e^{\langle \rho_B, H_B(\cdot)\rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$. Then, Franke's filtration of the space $\mathcal{A}_{\{B\},\varphi(\chi)}$ of automorphic forms with cuspidal support in the associate class $\varphi(\chi)$ can be defined as the filtration

$$\mathcal{A}_{\{B\},\varphi(\chi)} = \mathcal{A}^0_{\{B\},\varphi(\chi)} \supsetneq \mathcal{A}^1_{\{B\},\varphi(\chi)} \supsetneq \cdots \supsetneq \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \supsetneq \{0\}$$

of length n, where the quotients of the filtration for $i = 0, 1, \ldots, n-1$ are isomorphic to

$$\begin{aligned}\mathcal{A}^{i}_{\{B\},\varphi(\chi)}/\mathcal{A}^{i+1}_{\{B\},\varphi(\chi)} &\cong \bigoplus_{\substack{\underline{n}=(n_1,\ldots,n_r)\\ \text{with } r=n-i}} \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}) \\ &\cong \bigoplus_{\substack{\underline{n}=(n_1,\ldots,n_r)\\ \text{with } r=n-i}} \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(\bigotimes_{j=1}^{r} |\det_{n_j}|^{\frac{n_{j+1}+\cdots+n_r-(n_1+\cdots+n_{j-1})}{2}}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\end{aligned}$$

as $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules, where the direct sum is over the set of all ordered partitions $\underline{n} = (n_1, \ldots, n_r)$ of n into positive integers with $r = n-i$, i.e., over all parabolic subgroups of rank i. In particular,

$$\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \cong \mathbf{1}_{G(\mathbb{A})},$$

where $\mathbf{1}_{G(\mathbb{A})}$ is the trivial representation of $G(\mathbb{A})$, realized as the residual automorphic representation on the space of constant functions on $G(\mathbb{A})$.

Proof. It follows from Theorem 4.1 in [Grb-Gro22], that Franke's filtration of the space of automorphic forms with cuspidal support in $\varphi(\chi)$ can be arranged in such a way that the contributions to the quotients of the filtration are determined by the rank of the parabolic subgroup on which the degenerate Eisenstein series are supported, i.e., by the rank of the parabolic subgroup from which the contribution is parabolically induced. The result then follows from the decomposition of the sequence of exponents of the cuspidal support into segments.

The exponents in the induced representation from the parabolic subgroup $P_{\underline{n}}$ may be easily obtained by a direct calculation, or, can be found e.g., in (1.10) of [Gro-Lin21]. □

3.2. Cohomology of the trivial representation of $SL_n(\mathbb{R})$.

Lemma 3.2. *Let $n \geq 1$. The Poincaré polynomial of the cohomology $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ of the trivial reprepsentation $\mathbf{1}_{SL_n(\mathbb{R})} = \mathbb{C}$ of $SL_n(\mathbb{R})$ is given by*

$$P_n(t) = \begin{cases} \prod_{i=1}^{k-1}(1+t^{4i+1}) \cdot (1+t^n) & \text{if } n = 2k \\ \prod_{i=1}^{k}(1+t^{4i+1}) & \text{if } n = 2k+1 \end{cases}$$

Consequently, $\dim_{\mathbb{C}} H^0(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) = 1$, whereas for $q \geq 1$ the complex dimension of the cohomology $H^q(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ is given as follows: Let $a(q)$ be the number of ways to write a positive integer q as the sum of different integers of the form $4\ell+1$, $\ell \geq 1$. Then,

$$\dim_{\mathbb{C}} H^q(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) = \begin{cases} a(q)+1 & \text{if } n = 2k \\ a(q) & \text{if } n = 2k+1 \end{cases}$$

Proof. As $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) \cong H^*_{\mathrm{dR}}(SU(n)/SO(n), \mathbb{C})$, the Poincaré polynomial of the cohomology space $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ can be read off [GHV76], Table 1, p. 493. The claim on the complex dimension of $\dim_{\mathbb{C}} H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ hence follows immediately. □

3.3. **The Kostant representatives.** Let W be the Weyl group of G with respect to the fixed maximal split torus T. Then W is isomorphic to the symmetric group of permutations $\mathfrak{S}_n$ of n letters. The action of $w \in W \cong \mathfrak{S}_n$ on the character of the torus given by the sequence of exponents

$(s_1, \dots, s_n)$ is by permutation of these exponents. Recall that $(s_1, \dots, s_n) \in \mathbb{C}^n$ corresponds to the character given by the assignment

$$(t_1, \dots, t_n) \mapsto |t_1|^{s_1} \dots |t_n|^{s_n},$$

where $(t_1, \dots, t_n) \in T(\mathbb{A})$. The Weyl group is generated by the reflections w_i, $i = 1, \dots, n-1$ with respect to the simple roots of G. The length $\ell(w)$ of an element $w \in W$ is the number of simple reflections in any reduced decomposition of w into a product of simple reflections.

Given a standard parabolic subgroup P of G, we denote by W_L the Weyl group of its Levi factor L with respect to $L \cap T$. Observe that if $P = P_{(n_1,\dots,n_k)}$ corresponds to the ordered partition $(n_1, \dots, n_k)$ of n into positive integers, then

$$W_L \cong \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}$$

where $\mathfrak{S}_{n_j}$ is the symmetric group of permutations of n_j letters.

The Kostant representatives for P [Bor-Wal00], III.1.4, are defined as the unique representatives of minimal length in the cosets $W_L \backslash W$. Let W^P denote the set of Kostant representatives for P. These representatives and their length play a crucial role in the calculation of cohomology of the quotients of Franke's filtration. Since the quotients of Franke's filtration described in Theorem 3.1 contain an induced representation from each standard parabolic subgroup, our next task is to determine the Kostant representatives producing the correct exponents for these induced representations. This is the subject of the following proposition.

Proposition 3.3. *Let $P_{\underline{n}}$ be the standard parabolic subgroup of G corresponding to the ordered partition $\underline{n} = (n_1, \dots, n_k)$ of n into positive integers, and let $L_{\underline{n}}$ be its Levi factor. Let $w_{\underline{n}}$ be the Kostant representative in $W^{P_{\underline{n}}}$ such that*

$$-w_{\underline{n}}(\rho)\Big|_{\check{\mathfrak{a}}_{P_{\underline{n}}}}$$

equals the exponents that appear in the induced representation from $P_{\underline{n}}$ of Theorem 3.1. Then the length of $w_{\underline{n}}$ is given by

$$\ell(w_{\underline{n}}) = \sum_{1 \le i < j \le k} n_i n_j. \tag{3.1}$$

In particular, $w_{\underline{n}}$ is the longest element in $W^{P_{\underline{n}}}$.

Proof. Since the exponents in ρ are all different, there is a unique representative $w_{\underline{n}}$ in $W^{P_{\underline{n}}}$ producing the required exponents for each $\underline{n}$. It is the Weyl group element which acts as the longest permutation of blocks of sizes $n_k, \dots, n_1$. More precisely, the block of last n_1 exponents should be sent to the beginning of the sequence, the next to the last n_2 exponents should be sent to the second block of n_2 exponents, and so on, without changing the order inside the blocks. The first step of moving the block of last n_1 exponents can be made in $n_1(n-n_1)$ simple reflections, obtained as interchange of position of all n_1 exponents in the last block with all $n - n_1$ exponents outside the last block. The second step of moving the next to the last block of n_2 exponents to become the second block can be made in $n_2(n-n_1-n_2)$ steps. And so on, we obtain

$$\begin{aligned}\ell(w_{\underline{n}}) &= n_1(n-n_1)+n_2(n-n_1-n_2)+\cdots+n_{k-1}(n-n_1-n_2-\cdots-n_{k-1})\\ &= n_1(n_2+\cdots+n_k)+n_2(n_3+\cdots+n_k)+\cdots+n_{k-1}n_k\\ &= \sum_{1\le i<j\le k} n_i n_j.\end{aligned}$$

From the fact that $w_{\underline{n}}$ is the longest permutation of blocks of the parabolic, it is clear that it is the longest element in $W^{P_{\underline{n}}}$. □

3.4. Automorphic cohomology in low degrees.

Proposition 3.4. *Let $n\ge 4$ and let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character*

$$\chi = |\cdot|^{\frac{n-1}{2}}\otimes|\cdot|^{\frac{n-3}{2}}\otimes|\cdot|^{\frac{n-5}{2}}\otimes\cdots\otimes|\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$. Then, there is an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)})\cong H^q(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$$

for all degrees $0\le q\le n$.

Proof. Let $n\ge 4$ as in the statement of the proposition and let $k\ge 3$. The short exact sequence of $(\mathfrak{g},K,G(\mathbb{A}_f))$-modules

$$\{0\}\to\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}\to\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}\to\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}\to\{0\}$$

gives rise to a long exact sequence of $G(\mathbb{A}_f)$-modules

$$\begin{aligned}\cdots\to &H^q(\mathfrak{g},K,\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)})\to H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)})\to H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)})\to\\ \to &H^{q+1}(\mathfrak{g},K,\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)})\to\dots\end{aligned}\tag{3.2}$$

It is hence enough to show that

$$H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)})=\{0\}\tag{3.3}$$

for all $k\ge 3$ and all $q\le n$.
Recalling Frobenius reciprocity (as it was used in the proof of [Bor-Wal00], Thm. III.3.3 or in Eq. (5) on p. 257 in [Fra98]) and the fact that for each parabolic subgroup $P_{\underline{n}}$ of G,

$$H^*(\mathfrak{a}_{P_{\underline{n}}},S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}))=H^0(\mathfrak{a}_{P_{\underline{n}}},S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}))\cong\mathbb{C},$$

see, [Fra98], p. 256, the $G(\mathbb{A}_f)$-module $H^q\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle\rho_{P_{\underline{n}}},H_{P_{\underline{n}}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right)$ is isomorphic to

$$H^{q-\ell(w_{\underline{n}})}(\mathfrak{m}_{\underline{n}},K\cap M_{\underline{n}},e^{\langle 2\rho_{P_{\underline{n}}},H_{P_{\underline{n}}}(\cdot)_\infty\rangle}\otimes\mathbb{C}_{-2\rho_{P_{\underline{n}}}})\otimes\mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{\underline{n}}},H_{P_{\underline{n}}}(\cdot)_f\rangle}\right),$$

where $\ell(w_{\underline{n}})=\sum_{1\le i<j\le k}n_in_j$ is the length of the uniquely determined Kostant representative $w_{\underline{n}}\in W^{P_{\underline{n}}}$, given by Prop. 3.3. Hence, our Thm. 3.1 implies that it is enough to prove that for all $k\ge 3$

$$\min_{\underline{n}=(n_1,\dots,n_k)}\sum_{1\le i<j\le k}n_in_j\ge n+1,\tag{3.4}$$

in order to show (3.3) for all $k \geq 3$ and all $q \leq n$. To this end, we rewrite

$$\begin{aligned}\sum_{1\leq i<j\leq k} n_i n_j &= \sum_{1\leq i<j\leq k-1} n_i n_j + \sum_{i=1}^{k-1} n_i (n - n_1 - n_2 - \cdots - n_{k-1}) \\ &= -\sum_{i=1}^{k-1} n_i^2 + n\cdot\sum_{i=1}^{k-1} n_i - \sum_{1\leq i<j\leq k-1} n_i n_j,\end{aligned}$$

revealing $\ell(w_{\underline{n}})$ as a quadratic polynomial in the variables n_i, $1 \leq i \leq k-1$. Since the coefficient of n_i^2 is always negative, the minimum over all ordered partitions $\underline{n} = (n_1, \ldots, n_k)$ is attained at the boundary of the domain of possible values, which, in the present case, is (all boundary values lead to the same outcome) at $n_1 = n_2 = \cdots = n_{k-1} = 1$. Hence, by inserting, we get that for all $k \geq 3$,

$$\min_{\underline{n}=(n_1,\ldots,n_k)} \sum_{1\leq i<j\leq k} n_i n_j = n(k-1) - \frac{k(k-1)}{2}.$$

Checking, when this expression satisfies (3.4), hence leads by a simple calculation to checking when

$$n \geq \frac{k^2-k+2}{2(k-2)}. \tag{3.5}$$

Viewing the right-hand side of (3.5) as a function $\phi(k)$ of a real variable k, it a matter of basic calculus to show that the only local extreme in the domain $3 \leq k \leq n$ is the local minimum at $k = 4$. Hence, the maximum of $\phi(k)$ is attained at the boundary of the domain $3 \leq k \leq n$, and it remains to check that

$$\begin{aligned} n &\geq \phi(3) = 4, \\ n &\geq \phi(n) = \frac{n^2-n+2}{2(n-2)}, \end{aligned}$$

holds for $n \geq 4$. The former inequality is obvious, and the latter follows by writing it as a quadratic inequality in n. Thus, the result follows. $\square$

4. Non-vanishing of $H^n(SL_n(\mathbb{Z}))$

4.1. **Quotient-cohomology in degree** $q = n-1$**.** Let $n \geq 4$ and let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$. We start with an analysis of the $G(\mathbb{A}_f)$-module $H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$. Recall from Thm. 3.1 that $\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \cong \mathbf{1}_{G(\mathbb{A})}$. We will first prove

Lemma 4.1. *There is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$\begin{aligned} H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong \\ H^{n-1}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(1,n-1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right) \\ \oplus H^{n-1}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right). \end{aligned}$$

Proof. Thm. 3.1 implies that

$$H^q(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})\cong\bigoplus_{\underline{n}=(n_1,n_2)}H^q\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle\rho_{P_{\underline{n}}},H_{P_{\underline{n}}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right).$$

Literally the same strategy, which we have seen in the proof of Prop. 3.4, shows that the lowest degree of cohomology, in which the cohomology space $H^q\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle\rho_{P_{\underline{n}}},H_{P_{\underline{n}}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right)$ may be non-zero is bounded from below by $\ell(w_{\underline{n}})=\sum_{1\leq i<j\leq 2}n_in_j=n_1\cdot n_2$. Since $n_1+n_2=n$, it is a very simple exercise to prove that $n_1\cdot n_2>n-1$ if $2\leq n_1\leq n-2$. Therefore the result follows. □

We continue be refining Lem. 4.1, distinguishing the parity of n.

4.2. **The case of odd n.** In this subsection we will suppose that $n\geq 5$ is odd. We get

Proposition 4.2. *Under the assumptions of Sect. 4.1*

$$H^{n-1}(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})=\{0\}$$

Proof. We first consider $H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)$. This cohomology space is obviously isomorphic to

$$\begin{aligned}H^{n-1}&\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{R})}_{P_{(n-1,1)}(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\&\otimes\mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(n-1,1)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right),\end{aligned}$$

so, this space being zero is equivalent to the vanishing of

$$H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{R})}_{P_{(n-1,1)}(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right).$$

Recall that by our Prop. 3.3, $\ell(w_{(n-1,1)})=n-1$. Hence, invoking [Bor-Wal00], Thm. III.3.3 and [Fra98], p. 256,

$$\begin{aligned}&H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{R})}_{P_{(n-1,1)}(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\&\cong H^0(\mathfrak{m}_{(n-1,1)},K\cap M_{(n-1,1)},e^{\langle 2\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\otimes\mathbb{C}_{-2\rho_{P_{(n-1,1)}}})\\&\cong H^0(\mathfrak{m}_{(n-1,1)},K\cap M_{(n-1,1)},\mathrm{sgn}(\det_{n-1})\otimes\mathrm{sgn}^{n-1}).\end{aligned}$$

Using [Bor-Wal00], I.1.3.(2) and I.5.1.(4), together with the fact that $n-1$ is even, the latter space is isomorphic to

$$\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathbb{C})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}$$

We may represent the only non-trivial member of $S(O(n-1)\times O(1))/SO(n-1)\times SO(1)$ by the pair $(\mathrm{diag}(id_{n-2},-1);-1)$, which obviously acts trivially by its adjoint action on $H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathbb{C})$ and by multiplication by -1 on $H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))$. Hence,

$$\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathbb{C})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}=\{0\},$$

and so also

$$H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)=\{0\}.$$

The case of $H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle\rho_{P_{(1,n-1)}},H_{P_{(1,n-1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right)$ is completely analogous. Hence, invoking Lem. 4.1, the result follows. □

Corollary 4.3. *Let $n\geq 5$ be odd. Then, the $G(\mathbb{A}_f)$-module $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})})$ embeds into $H^n(\mathfrak{g},K,\mathcal{A}(G))$.*

Proof. Since $H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)})$ is a direct $G(\mathbb{A}_f)$-summand of $H^q(\mathfrak{g},K,\mathcal{A}(G))$, the result follows from Prop. 3.4, Lem. 4.2 and (3.2). □

4.3. **The case of even n.** Let now $n\geq 4$ be even. Then,

Proposition 4.4. *Under the assumptions of Sect. 4.1*

$$\begin{aligned}H^{n-1}(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) &\cong \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(1,n-1)}},H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right)\\ &\oplus \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right)\end{aligned}$$

Proof. Again, we first consider $H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)$, which is isomorphic to

$$\begin{aligned}&H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\ &\quad\otimes \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right).\end{aligned}$$

As in the case of odd n above, we invoke [Bor-Wal00], Thm. III.3.3 and [Fra98], p. 256, together with [Bor-Wal00], I.1.3.(2) and I.5.1.(4), and obtain

$$H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)$$

$$\cong \left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn}^{n-1})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)},$$

which, since now n is assumed to be even, simplifies to

$$\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}.$$

In the present case, we may represent the only non-trivial member of $S(O(n-1)\times O(1))/SO(n-1)\times SO(1)$ by the pair $(-id_{n-1};-1)$, which obviously acts by multiplication by -1 on both, $H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))$ and on $H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn})$, hence, trivially on the tensor product $H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn})$. Therefore,

$$\begin{aligned}&\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}\\ &\quad\cong H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn})\\ &\quad\cong H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathbb{C})\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathbb{C}).\end{aligned}$$

Since the latter two cohomology spaces are both one-dimensional by Lem. 3.2, we finally get that

$$H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\cong\mathbb{C}$$

and so

$$\begin{aligned}&H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\ &\quad\cong \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right)\end{aligned}$$

as $G(\mathbb{A}_f)$-module. The same argument shows that

$$H^{n-1}\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right)$$
$$\cong \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right),$$

whence the results follows from Lem. 4.1.

□

Corollary 4.5. *Let $n \geq 4$ be even. Then, the image of the $G(\mathbb{A}_f)$-module $H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ in $H^n(\mathfrak{g}, K, \mathcal{A}(G))$ has dimension greater or equal to $\dim_{\mathbb{C}} H^n(\mathfrak{g}, K, \mathbb{C}) - 2$.*

Proof. Since $\mathcal{A}_{\{B\},\varphi(\chi)}$ is a direct $(\mathfrak{g}, K, G(\mathbb{A}_f))$-summand of $\mathcal{A}(G)$, it is enough to show this for $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$. We consider the respective part of the long exact sequence in cohomology (3.2), which by Prop. 4.4 and Thm. 3.1 reads as

$$\begin{array}{lll} \cdots \to & \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right) & \oplus \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right) \\ \to & H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) & \to H^n(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to \ldots \end{array}$$

Since the trivial representation $\mathbf{1}_{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$ appears precisely once as a quotient of the induced representation $\mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right)$, respectively of the induced representation $\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right)$, the connecting homomorphism above has at most two-dimensional image in $H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \cong H^n(\mathfrak{g}, K, \mathbb{C}) \otimes \mathbf{1}_{G(\mathbb{A}_f)}$. Hence, the image of the $G(\mathbb{A}_f)$-module $H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ has dimension greater or equal to $\dim_{\mathbb{C}} H^n(\mathfrak{g}, K, \mathbb{C}) - 2$ in $H^n(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$. However, by Prop. 3.4 the latter is nothing else than $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$, whence the corollary follows.

□

4.4. **Non-vanishing of $H^n(SL_n(\mathbb{Z}))$.** We recall the number $a(q)$ from Lem. 3.2, which denoted the number of ways to write a positive integer q as the sum of different integers of the form $4\ell + 1$, $\ell \geq 1$. We are now ready to prove our first main result:

Theorem 4.6. *Let $n \geq 4$. Then,*

$$\dim_{\mathbb{C}} H^n(SL_n(\mathbb{Z}), \mathbb{C}) \geq \begin{cases} a(n) - 1 & \text{if } n \text{ is even} \\ a(n) & \text{if } n \text{ is odd} \end{cases}$$

In particular, the free part of the $\mathbb{Z}$-module $H^n(SL_n(\mathbb{Z}))$ is non-zero, in the following cases:

- *for odd n, if either $n \geq 25$, or $n \in \{5, 9, 13, 17, 21\}$;*
- *for even n, if either $n \geq 50$, or $n \in \{22, 26, 30, 34, 38, 42, 46\}$.*

Proof. After all of our preparatory work, this is now a direct consequence of (2.1), our corollaries Cor. 4.5 and Cor. 4.3 and Lem. 3.2. □

5. Non-vanishing of $H^m(SL_n(\mathbb{Z}))$, $m \leq n - 1$

5.1. **A well-known result revisited.** We start with the following easy observation that $H^m(SL_n(\mathbb{Z}))$ is non-zero for $m \leq n - 1$, whenever $H^m(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ is non-zero, which is a simple consequence of the following lemma and its well-known corollary:

Lemma 5.1. *Let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$. Then,*

$$H^q(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) = \{0\}$$

for all $0 \leq q \leq n-2$.

Proof. The arguments presented in the proof of Prop. 3.4 show that $\ell(w_{\underline{n}})$, $\underline{n} = (n_1, n_2)$, is a lower bound for the degrees of cohomology q, in which $H^q(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$ may be non-zero. However, $\ell(w_{\underline{n}})$ is bounded from below by $n-1$, as we have seen in the proof of Lem. 4.1. □

As a direct consequence, we get the following, quite well-known lemma (see, for instance [Fra08], (7.2), p. 59, which indirectly contains this result):

Corollary 5.2. *The $G(\mathbb{A}_f)$-module $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ embeds into $H^q(\mathfrak{g}, K, \mathcal{A}(G))$ for all $0 \leq q \leq n-1$.*

Proof. Since $H^q(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ is a direct $G(\mathbb{A}_f)$-summand of $H^q(\mathfrak{g}, K, \mathcal{A}(G))$, this follows from Prop. 3.4, Lem. 5.1 and (3.2). □

As a corollary of the corollary we hence get

Corollary 5.3. *The free part of the $\mathbb{Z}$-module $H^m(SL_n(\mathbb{Z}))$ is non-zero for $m \leq n-1$, whenever $H^m(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ is non-zero.*

Proof. This is a direct consequence of (2.1) and Cor. 5.2. □

This is the well-known non-vanishing result for $H^m(SL_n(\mathbb{Z}))$, which we mentioned above. It is important to notice, however, that one can do slightly better than this, when $m = n-1$, exploiting the results of our Sect. 4, which shall be the subject of the next subsection.

5.2. **Additional non-vanishing of $H^{n-1}(SL_n(\mathbb{Z}))$ for small n.** We now consider the cases $n = 4, 8, 12$. It follows directly from Lem. 3.2 that $H^{n-1}(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) = \{0\}$, whence Cor. 5.2 (and its attached Cor. 5.3) does not help, in order to see that still $H^{n-1}(SL_n(\mathbb{Z})) \neq \{0\}$. We will now show

Theorem 5.4. *The free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$ is non-zero, if $n = 4, 8, 12$.*

Proof. Let $n = 4, 8, 12$ as in the statement of the theorem. Recalling (2.1), it suffices to prove that $H^{n-1}(\mathfrak{g}, K, \mathcal{A}(G))$ contains a copy of the trivial representation $\mathbf{1}_{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$. Let $\varphi(\chi)$ be again the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$. Since $\mathcal{A}_{\{B\},\varphi(\chi)}$ is a direct $(\mathfrak{g}, K, G(\mathbb{A}_f))$-summand of $\mathcal{A}(G)$, it is enough to show that $H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ contains a copy of the trivial representation $\mathbf{1}_{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$. Recalling our Prop. 3.4, it hence suffices to prove this for $H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$. To this end, we consider once more the long exact sequence in cohomology, which comes from the short exact sequence of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules

$$\{0\} \to \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \to \{0\}.$$

More precisely, we look at the part

$$\cdots \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to$$

$$\to H^n(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \dots,$$

which, according to Thm. 3.1 and Prop. 4.4 becomes

$$\cdots \to H^{n-1}(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to$$

$$\to \bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \to H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to \dots$$

and hence by Lem. 3.2

$$\{0\} \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to \bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \to \mathbf{1}_{G(\mathbb{A}_f)}. \tag{5.1}$$

Since $\bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right)$ captures precisely two copies of $\mathbf{1}_{G(\mathbb{A}_f)}$ as quotients, the kernel of the last morphism of $G(\mathbb{A}_f)$-modules in (5.1) must contain one copy of $\mathbf{1}_{G(\mathbb{A}_f)}$. Hence, by exactness of (5.1), $H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$ contains a copy of the trivial representation $\mathbf{1}_{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$. □

Remark 5.5. The non-vanishing of $H^3(SL_4(\mathbb{Z}))$ was also shown by completely different techniques in [Lee-Szc78]. In fact, their paper completely computes the cohomology of $SL_4(\mathbb{Z})$ in all degrees. See [Lee-Szc78], Thm. 2.

6. Applications for classes in the case of $SL_6(\mathbb{Z})$ and $SL_8(\mathbb{Z})$

6.1. **The mysterious class in $H^8(SL_6(\mathbb{Z}))$ and a question of F. Brown.** In [EVGS13], Elbaz-Vincent, Gangl and Soulé have calculated the cohomology of $SL_n(\mathbb{Z})$ for $n = 5, 6, 7$. In particular, they found a non-trivial cohomology class of $SL_6(\mathbb{Z})$ in degree $q = 8$, cf. [EVGS13], Thm. 7.3.(ii), for whose existence, however, there seems to be no proper conceptual explanation: We refer to Brown's recent preprint [Bro23], in particular to its Thm. 1.1 and Table 1, for a discussion of this phenomenon.

We present here a structural reason, arising from the point of view of automorphic forms, for the existence of this non-trivial class, i.e., we will explain which automorphic forms represent the one-dimensional space $H^8(SL_6(\mathbb{Z}), \mathbb{C})$.

To this end, we first apply our Thm. 3.1 to the case $i = n - 2$, i.e., to the second last non-trivial step in Franke's filtration of $\mathcal{A}_{\{B\},\varphi(\chi)}$, $\chi = e^{\langle \rho_B, H_B(\cdot)\rangle}$. Its cohomology is then computed as

$$H^q(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong \bigoplus_{\underline{n}=(n_1,n_2)} H^q\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right), \tag{6.1}$$

which, invoking [Bor-Wal00], Thm. III.3.3 and [Fra98], p. 256, together with [Bor-Wal00], I.1.3.(2) and I.5.1.(4), is isomorphic as $G(\mathbb{A}_f)$-module to

$$\bigoplus_{\underline{n}=(n_1,n_2)} \bigoplus_{r+s=q-n_1n_2} \Big(H^r(\mathfrak{sl}_{n_1}(\mathbb{R}), SO(n_1), \mathrm{sgn}^{n_2}) \otimes H^s(\mathfrak{sl}_{n_2}(\mathbb{R}), SO(n_2), \mathrm{sgn}^{n_1}) \Big)^{S(O(n_1)\times O(n_2))/SO(n_1)\times SO(n_2)} \otimes \mathrm{Ind}_{P_{(n_1,n_2)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{(n_1,n_2)}}, H_{P_{(n_1,n_2)}}(\cdot)_f \rangle} \right). \tag{6.2}$$

Put now $n = 6$ in (6.2). Then, $H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)})$ has five direct summands as due to (6.1), indexed by the partitions $(1,5)$, $(5,1)$, $(2,4)$, $(4,2)$, $(3,3)$. By equation (3.1), the partition $(3,3)$ only contributes to cohomology in degree $q \geq 3\cdot 3 = 9$. Similarly, by (3.1) together with Lem. 3.2, the partitions $(1,5)$ and $(5,1)$ may only contribute to degrees $q = 5, 10, 14$. While for the same reason, the partitions $(2,4)$, $(4,2)$ may only contribute to degrees $q = 8, 10, 13, 15$. It therefore follows that

$$\begin{aligned} &H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \cong \\ &\quad \Big(H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C}) \otimes H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C}) \Big)^{S(O(2)\times O(4))/SO(2)\times SO(4)} \otimes \mathrm{Ind}_{P_{(2,4)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f \rangle} \right) \\ &\bigoplus \Big(H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C}) \Big)^{S(O(4)\times O(2))/SO(4)\times SO(2)} \otimes \mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f \rangle} \right). \end{aligned}$$

The only non-trivial element of $S(O(2) \times O(4))/SO(2) \times SO(4)$ (resp. $S(O(4) \times O(2))/SO(4) \times SO(2)$) operates trivially on the one-dimensional spaces $H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C}) \otimes H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C})$ (resp. $H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C})$), hence

$$\begin{aligned} H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \cong{} & \mathrm{Ind}_{P_{(2,4)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f \rangle} \right) \\ & \oplus \mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f \rangle} \right). \end{aligned}$$

If we plug this (and the knowledge on $H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) = H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathbf{1}_{SL_6(\mathbb{A})})$, which is given by Lem. 3.2) into the long exact sequence in cohomology, which comes from the short exact sequence of $(\mathfrak{sl}_6(\mathbb{R}), SO(6), SL_6(\mathbb{A}_f))$-modules

$$\{0\} \to \mathcal{A}^5_{\{B\},\varphi(\chi)} \to \mathcal{A}^4_{\{B\},\varphi(\chi)} \to \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)} \to \{0\}.$$

i.e., into the exact sequence of $SL_6(\mathbb{A}_f)$-modules

$$\cdots \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) \to$$

$$\to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \to H^9(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) \to \ldots,$$

we obtain an exact sequence of $SL_6(\mathbb{A}_f)$-modules

$$\{0\} \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) \to$$

$$\to \mathrm{Ind}^{SL_6(\mathbb{A}_f)}_{P_{(2,4)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(2,4)}},H_{P_{(2,4)}}(\cdot)_f\rangle}\right) \oplus \mathrm{Ind}^{SL_6(\mathbb{A}_f)}_{P_{(4,2)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(4,2)}},H_{P_{(4,2)}}(\cdot)_f\rangle}\right) \to \mathbf{1}_{SL_6(\mathbb{A}_f)} \to \dots$$

Recalling that both $\mathrm{Ind}^{SL_6(\mathbb{A}_f)}_{P_{(2,4)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(2,4)}},H_{P_{(2,4)}}(\cdot)_f\rangle}\right)$ and $\mathrm{Ind}^{SL_6(\mathbb{A}_f)}_{P_{(4,2)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(4,2)}},H_{P_{(4,2)}}(\cdot)_f\rangle}\right)$ contain $\mathbf{1}_{SL_6(\mathbb{A}_f)}$ with multiplicity one, it follows that $H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)})$ contains (at least) one copy of $\mathbf{1}_{SL_6(\mathbb{A}_f)}$.

In order to determine automorphic forms that represent a non-trivial class in $H^8(SL_6(\mathbb{Z}),\mathbb{C})$, it hence suffices to show by Thm. 2.2 that

$$H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) = H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}_{\{B\},\varphi(\chi)}).$$

But this is clear, once we realize that all the other quotients $\mathcal{A}^{6-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{6-k+1}_{\{B\},\varphi(\chi)}$ $k \geq 3$, will only have non-trivial $(\mathfrak{sl}_6(\mathbb{R}), SO(6))$-cohomology in degrees $q \geq 9$ by inserting into (3.1). Therefore, in summary, as Hecke-modules

$$H^8(SL_6(\mathbb{Z}),\mathbb{C}) \cong H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)})^{K_f},$$

where recall that $K_f = SL_6(\hat{\mathbb{Z}})$ is the fixed maximal compact subgroup in $SL_6(\mathbb{A}_f)$. Therefore, a non-trivial class in $H^8(SL_6(\mathbb{Z}),\mathbb{C})$ is necessarily represented by the main values of some appropriate partial derivatives of degenerate Eisenstein series attached to the associate class of the everywhere unramified automorphic characters $e^{\langle\rho_{P_{(2,4)}},H_{P_{(2,4)}}(\cdot)\rangle}$ and $e^{\langle\rho_{P_{(4,2)}},H_{P_{(4,2)}}(\cdot)\rangle}$ of the Levi factor of $P_{(2,4)}$ and $P_{(4,2)}$, respectively.

6.2. **Two non-trivial classes in $H^{15}(SL_8(\mathbb{Z}))$ and a question of A. Ash.** As communicated to the second named author by Brown, A. Ash, has asked for a description of the cohomology of $SL_8(\mathbb{Z})$. Among others, degree $q = 15$ was of particular interest. Here we show that $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$ is two-dimensional, and we describe, which automorphic forms of $SL_8(\mathbb{A})$ represent the non-trivial classes in $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$.

We put $n = 8$ in (6.2). By the analogous arguments as presented in §6.1 above, i.e., by recalling Lem. 3.2 and using the long exact sequence in cohomology, that stems from Franke's filtration, we obtain an isomorphism of $SL_8(\mathbb{A}_f)$-modules

$$H^{15}(\mathfrak{sl}_8(\mathbb{R}), SO(8), \mathcal{A}^6_{\{B\},\varphi(\chi)}) \cong \mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(3,5)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(3,5)}},H_{P_{(3,5)}}(\cdot)_f\rangle}\right) \oplus \mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(5,3)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(5,3)}},H_{P_{(5,3)}}(\cdot)_f\rangle}\right).$$

Once more we use Lem. 3.2 and (3.1) and deduce that

$$H^{15}(\mathfrak{sl}_8(\mathbb{R}), SO(8), \mathcal{A}^6_{\{B\},\varphi(\chi)}) \cong H^{15}(\mathfrak{sl}_8(\mathbb{R}), SO(8), \mathcal{A}_{\{B\},\varphi(\chi)}).$$

Hence, invoking Thm. 2.2 and the fact that the induced representation $\mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(3,5)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(3,5)}},H_{P_{(3,5)}}(\cdot)_f\rangle}\right)$ as well as $\mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(5,3)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(5,3)}},H_{P_{(5,3)}}(\cdot)_f\rangle}\right)$ contain $\mathbf{1}_{SL_8(\mathbb{A}_f)}$ with multiplicity one as a quotient, it follows that

$$H^{15}(SL_8(\mathbb{Z}),\mathbb{C}) \cong H^{15}(\mathfrak{sl}_8(\mathbb{R}), SO(8), \mathcal{A}^6_{\{B\},\varphi(\chi)})^{K_f} \cong \mathbb{C}^2$$

as modules under the Hecke algebra attached to $K_f = SL_8(\hat{\mathbb{Z}})$. The cohomology classes in this case are represented by the main values of some appropriate partial derivatives of degenerate Eisenstein series attached to the associate classes of the everywhere unramified automorphic characters $e^{\langle \rho_{P_{(3,5)}}, H_{P_{(3,5)}}(\cdot)\rangle}$ and $e^{\langle \rho_{P_{(5,3)}}, H_{P_{(5,3)}}(\cdot)\rangle}$ of the Levi factors of $P_{(3,5)}$ and $P_{(5,3)}$, respectively.

6.3. **A final remark on possible generalizations.** Let now $n = 2m$ be an arbitrary positive even number. Then, it is well-known, cf. [Bor-Wal00], Prop. I.5.3, that the lowest degree in which a tempered cuspidal automorphic representation of $SL_{2m}(\mathbb{A})$ may have non-zero cohomology is given by $q = m^2$. If we combine our considerations of §6.1 and §6.2 with Thm. 2.2 above, it therefore seems tempting to ask – complementing our results on what happens right outside the stable range – for a description of the cohomology of $SL_{2m}(\mathbb{Z})$ "right below" the tempered cuspidal range, i.e., to consider the following

Open Problem. *Determine for which $m \geq 1$, the cohomology $H^{m^2-1}(SL_{2m}(\mathbb{Z}), \mathbb{C})$ is non-zero.*

By what we obtained above, $H^{m^2-1}(SL_{2m}(\mathbb{Z}), \mathbb{C})$ is non-zero for $m = 1, 2, 3, 4$. However, in higher rank, the problem gets more and more complicated. The possible contributions to cohomology in degree $m^2 - 1$ of the quotients of Franke's filtration associated to parabolic subgroups of lower rank cannot be excluded by a simple argument based on the length of the Kostant representative. In the cases of $m = 1, 2, 3$ there were no such contributions, and in the case of $m = 4$, the only possible contributions arise from the associate class of the parabolic subgroup $P_{(1,1,6)}$, but it cannot contribute to degree $q = m^2 - 1 = 15$ by the Poincaré polynomial, cf. Lem. 3.2. As m grows, the rank of parabolic subgroups associated to the quotients of Franke's filtration that may contribute to the cohomology in the considered degree can be bounded, but the bound is slightly larger than $m/2$, which gives quite a lot of possibilities, and Lem. 3.2 cannot exclude all of them. Therefore, although the problem is a natural generalization of our results, it seems that it is still out of reach.

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