RELATIONS OF RATIONALITY FOR SPECIAL VALUES OF RANKIN–SELBERG L-FUNCTIONS OF $\mathrm{GL}_n \times \mathrm{GL}_m$ OVER CM-FIELDS

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Abstract. In this article we establish an "automorphic version" of Deligne's conjecture for motivic L-functions in the case of Rankin–Selberg L-functions $L(s, \Pi \times \Pi')$ of $\mathrm{GL}_n \times \mathrm{GL}_m$ over arbitrary CM-fields F. Our main results are of two different kinds: Firstly, for arbitrary integers $1 \leq m < n$, and suitable pairs (Π, Π') of cohomological automorphic representations, we relate critical values of $L(s, \Pi \times \Pi')$ with a product of Whittaker periods attached to Π and Π', Blasius's CM-periods of Hecke-characters and certain non-zero values of standard L-functions. Secondly, these relations lead to quite broad generalizations of fundamental rationality-results of Waldspurger, Harder–Raghuram and others.

Introduction

Motivated by conjectures of Deligne, Bellinson and Bloch-Kato, significant progress has been made in the study of special values of automorphic L-functions within the last decades. In this paper we continue this series of results on by treating the case of Rankin–Selberg L-functions over arbitrary CM-fields F. More precisely, let $1 \leq m < n$ be arbitrary integers and let Π and Π' be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively. If the infinity types of Π and Π' are compatible (in a sense to be made precise below) we will prove relations of rationality for a certain string of special values of the attached Rankin-Selberg L-function $L(s, \Pi \times \Pi')$, which turn out to fit Deligne's prediction: As a particular example, the contributions of the archimedean components Π_∞ and Π'_∞ to our rationality-relations will be expressed by explicit powers of $(2\pi i)$, matching the ones conjectured by Deligne.

As compared to the extensive literature for the case $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ – for F a CM-field we refer in particular to [Kur78, Kur79], [GHar83], [Hid94], [Lin15], [Rag16], [Gro-MHar16], [Gro18], [Gro-Lin19], [Jan19] – the rank m of our second GL-factor GL_m being in principle *any integer* $1 \leq m < n$ (for suitable pairs (Π, Π')) is arguably one of the most notable features of this article. In this regard, the results of this paper should not only be seen as an extension of the series of results mentioned above, but also of the approach taken in [Lin15] and by the second named author in her thesis [Sac19]. We also refer to the very recent [Rag19], where an application of our main result in the special case of the standard L-function (i.e., $m = 1$) has been proven by a different approach.

Main results and applications. The main outcome of the present article is Thm. 3.1 (reviewed as Main Result below) and its two applications (reviewed as Application I & II). However, the aforementioned main result reads somewhat technical and moreover turns out to be quite involved in its assumptions and assertions; on the other hand, the two aforementioned applications are much lighter statements, providing wide generalizations of important results of Waldspurger (Application I) and Harder–Raghuram (Application II), as well as of other people, but certainly

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these applications are just corollaries of our technical main result. We frankly admit that this makes it hard for the authors to really say which is more important. We leave it to the reader to decide.

The main result. From now on F denotes an arbitrary CM-field with maximal totally real subfield F^+. The quadratic Hecke character associated with F/F^+ admits a unitary extension to $\mathbb{A}_F^\times$ which is denoted η. Then by construction $\eta \parallel \cdot \parallel^{-1/2}$ is algebraic. See §1.1 for details. For an integer $n \geq 2$ we let Π be an (irreducible) subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_n(F)\mathbb{R}_+\backslash \mathrm{GL}_n(\mathbb{A}_F))$. As above, we shall assume that Π is cohomological, i.e., there exists a finite-dimensional, irreducible algebraic representation $\mathcal{E}_\mu$ of the real Lie group $G_{n,\infty} := \mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ such that Π_∞ has non-trivial relative Lie algebra cohomology with respect to $\mathcal{E}_\mu$. Here, μ stands for the highest weight of $\mathcal{E}_\mu$ (depending on a choice of a Borel subgroup $B_n \subset G_n$). Choosing coordinates one may indeed identify it with $\mu = (\mu_v)_{v \in S_\infty}$ where $\mu_v = (\mu_{v,1}, ..., \mu_{v,n}) \in \mathbb{Z}^n$ and $\mu_{v,1} \geq ... \geq \mu_{v,n}$.

Consider now another integer m, such that $1 \leq m < n$. We define Π' in analogy to Π above as an (irreducible) subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_m(F)\mathbb{R}_+\backslash \mathrm{GL}_m(\mathbb{A}_F))$ but which is now assumed to be conjugate self-dual with respect to the non-trivial Galois automorphisms of F/F^+ and with the property that $\Pi'^{\mathsf{alg}} := \Pi' \otimes \eta^e$ is cohomological. Here, $e \in \{0,1\}$ and $e = 0$ if and only if $n \not\equiv m \mod 2$.

The reason for introducing the twist $\Pi'^{\mathsf{alg}} = \Pi' \otimes \eta^e$, i.e., for assuming different conditions on cohomology for n and m is explained by the following construction: In order to be able to use the main result of [Gro18] (which is the starting-point of our proof), we choose any conjugate self-dual Hecke characters χ_1,...,χ_{n-m-1} of $\mathbb{A}_F^\times$ such that the isobaric sum $\Sigma := \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ is cohomological with respect to algebraic coefficients $\mathcal{E}_{\mu'}$: It turns out that such a choice can be made if and only if Π'^{alg}, rather than Π' itself, is cohomological. We will say that the infinity types of Π and Π' are compatible (or, more metaphorically, satisfy the "piano-condition", see (1.8) and below), if a choice of Σ can be made such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ is non-trivial.

With these assumptions non-zero *Whittaker periods* $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$, respectively *CM-periods* $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$, have been defined in [Rag-Sha08], [Gro18], respectively [Bla86]. By their very construction their product is well-defined up to multiplication by non-zero elements in a certain number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi'^{\mathsf{alg}})E^{\mathsf{cm}}$, where E^{cm} is an abbreviation for a number field, which depends on the chosen characters χ_i and contains a Galois closure $F^{Gal} \subset \bar{\mathbb{Q}}$ of the extension $F/\mathbb{Q}$. We refer to §1.6 and in particular 2.4 below for details.

We are now in the position to state our

Main Result. *Assume that* $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ *is non-trivial and let* $s_0 = \frac{1}{2} + k$ *be any critical point of* $L(s, \Pi \times \Sigma)$. *If* $k \neq 0$, *then*

$$L^S(\tfrac{1}{2} + k, \Pi \times \Pi') \sim (2\pi i)^{[F^+:\mathbb{Q}]((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\ p(\Pi)\ p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1 \leq i < j \leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi' \otimes \chi_j^{-1})}{L^S(\frac{1}{2} + k, \Pi \otimes \chi_j)}, \tag{0.1}$$

the relation "$\sim$" being over the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}$.

If $k = 0$, *i.e., if* $s_0 = \frac{1}{2}$ *denotes the central critical point, then the same relation holds under certain conditions of regularity on* Π_∞ *and* Σ_∞ *as well as a global non-vanishing hypothesis, see*

Thm. 3.1. Moreover, if n is even and m is odd, then all L-values $L^S(\frac{1}{2}+k, \Pi\times\Pi')$, $L^S(1, \Pi'\otimes\chi_j^{-1})$ and $L^S(\frac{1}{2}+k, \Pi\otimes\chi_j)$ in (0.1) are critical.

The reader should observe that this result is "best possible" in as such that the individual quantities on the right hand side are only well-defined up to multiplication by an element in the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}$. We also remark that if $k\neq 0$, then the denominators $L^S(\frac{1}{2}+k, \Pi\otimes\chi_j)$ in (0.1) are non-zero, which is in turn part of the global non-vanishing hypothesis for the central case $k=0$ mentioned in our Main Result.

If $m=n-1$, then our Main Result becomes Thm. 5.2 from [Gro-Lin19] for cuspidal automorphic representations, which refined the main result of [Rag16] over CM-fields by giving an explicit power of $(2\pi i)$ instead of an abstract archimedean period. It is worth noting that this power is precisely what is predicted by Deligne's conjecture on critical values of motivic L-functions, cf. [Del79], generalizing Euler's classical result on the nature of $\zeta(k)$ at even, positive integers. We refer to [Gro-Lin19], Rem. 5.8, for a more detailed exposition.

On the other extreme, if $m=1$, i.e., if we look at the twisted standard L-function of Π, then we retrieve at once Thm. 3.9, Cor. 5.7 and Thm. 6.11 of [Gro-MHar16], as well as a variant of the main result of [Rag19] over CM-fields. For general m our Main Result should hence be viewed as a theorem relating special values of $L(s, \Pi\times\Pi')$ with periods and quotients of special values of (other) standard L-functions. As already mentioned above, we refer to Thm. 3.1 for a proof.

Main applications. Our Main Result has the following two implications, which generalize important results of Waldspurger (see Application I) and Harder–Raghuram (see Application II). Indeed, in [Wal85b] Waldspurger has established a rationality result for the quotient $L(\frac{1}{2}, \pi\otimes\alpha)/L(\frac{1}{2}, \pi\otimes\beta)$ of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi\otimes\alpha$ and $\pi\otimes\beta$ of GL_2 over any number field at their joint critical value $s_0=\frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty=\beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L(\frac{1}{2}, \pi\otimes\beta)$ is supposed to be non-zero. Under these assumptions, Waldspurger's rationality-relation is of the form

$$\frac{L(\frac{1}{2}, \pi\otimes\alpha)}{L(\frac{1}{2}, \pi\otimes\beta)} \sim_{\mathbb{Q}(\pi)} \frac{p(\alpha)}{p(\beta)},$$

the two period-invariants $p(\alpha)$, $p(\beta)$ only depending on α respectively β and the archimedean component of the cuspidal representations π. See [Wal85b], p. 174.

In this paper we generalize Waldspurger's result to the case of quotients of standard L-functions of GL_n/F where $n\geq 2$ is arbitrary, $s_0=\frac{1}{2}+k$ a more general special value while F is any CM-field. More precisely, we let α and β be any conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ such that $\alpha_\infty=\beta_\infty$ and such that, writing $\alpha_v(z)=\beta_v(z)=z^{a_v}\bar{z}^{-a_v}$ at $v\in S_\infty$, the following two conditions are satisfied: $a_v\in\frac{n}{2}+\mathbb{Z}$ and $\mu_{v,1}\geq a_v\geq\mu_{v,n}$. This ensures that there is always a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sums $\Sigma_\alpha=\alpha\boxplus\chi_1\boxplus\ldots\boxplus\chi_{n-2}$ and $\Sigma_\beta=\beta\boxplus\chi_1\boxplus\ldots\boxplus\chi_{n-2}$ are cohomological with respect to an algebraic coefficient module $\mathcal{E}_{\mu'}$ and such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu\otimes\mathcal{E}_{\mu'},\mathbb{C}]$ is non-trivial. We obtain

Application I. *Choose any conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sums Σ_α and Σ_β are cohomological and such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu\otimes\mathcal{E}_{\mu'},\mathbb{C}]$ is non-trivial. Let $s_0=\frac{1}{2}+k$ be any critical point of $L(s,\Pi\times\Sigma_\alpha)$. If n is even, then all the $s_0=\frac{1}{2}+k$ are indeed critical for $L(s,\Pi\otimes\alpha)$ and $L(s,\Pi\otimes\beta)$ and*

$$\frac{L^S(\frac{1}{2}+k,\Pi\otimes\alpha)}{L^S(\frac{1}{2}+k,\Pi\otimes\beta)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}(\alpha;\beta;\chi_1;\ldots;\chi_{n-2})} \prod_{i=1}^{n-2}\frac{p(\alpha,\Psi_{\alpha\chi_i^{-1}})}{p(\beta,\Psi_{\beta\chi_i^{-1}})}.$$

If $k=0$, then we assume certain conditions of regularity on Π_∞ and Σ_∞ as well as a global non-vanishing hypothesis, see Thm. 4.1.

We point out that Application I should furthermore be viewed as a generalization as well as a certain refinement of a consequence of the main result of [Gro-Rag14b], Thm. 7.1.2, and [Jan16], Thm. 8.2, established there for totally real fields F^+ and achieved here for general CM-fields F.

Our final application deals with an extension of the main result of [GHar-Rag20]. There, Harder–Raghuram achieved a fine relation of rationality between the quotients of consecutive critical values of Rankin-Selberg L-functions over totally real fields F^+ and so-called relative periods denoted $\Omega^{\epsilon'}(\pi_f)$: Let π and π' be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, resp. $\mathrm{GL}_m(\mathbb{A}_{F^+})$, and let S be any finite set of non-archimedean places, where π or π' are ramified. Suppose that both $-\frac{n+m}{2}$ and $1-\frac{n+m}{2}$ are critical for $L(s,\pi\times\pi'^\vee)$ and that $L(1-\frac{n+m}{2},\pi\times\pi'^\vee)$ is non-zero. If n is even and m is odd [GHar-Rag20], Thm. 7.40, shows that

$$\frac{L^S(-\frac{n+m}{2},\pi\otimes\pi'^\vee)}{L^S(1-\frac{n+m}{2},\pi\otimes\pi'^\vee)} \sim_{\mathbb{Q}(\pi)\mathbb{Q}(\pi')} \Omega^{\epsilon'}(\pi_f).$$

In [Gro-Lin19] this result has recently been given a generalization and refinement for cohomological cusp forms of $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_{n-1}(\mathbb{A}_F)$, again F denoting any CM-field.

Here we take up the CM-case for general even n and odd m. We obtain

Application II. *Suppose that $1\le m<n$ are integers, m odd and n even. We assume that Π is obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$. Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, satisfying the conditions of our Main Result and let S be any finite set of places of F, containing the archimedean ones, such that Π and Π' are unramified outside S.*
Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two critical points of $L(s,\Pi\times\Sigma)$ with $k,\ell>\frac{n}{2}$, then $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are indeed critical for $L(s,\Pi\times\Pi')$ and the ratio of partial critical values satisfies

$$\frac{L^S(\frac{1}{2}+k,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{[F^+:\mathbb{Q}](k-\ell)nm}.$$

See Cor. 4.4 for all details and a proof. Here we only remark that the appearance of base change is due to the fact that our proof uses the results of [Gue16], which in turn proved a conjecture of Lin [Lin15]. As this already indicates, our Application II is hence a generalization of a consequence of Thm. 10.8.1 from [Lin15], however, obtained by somewhat different techniques.

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1. Notation

1.1. **Number fields and some particular Hecke characters.** We let F be a CM-field, i.e., a totally imaginary quadratic extension of a totally real field F^+. Consequently, the degree of F over the rational numbers $\mathbb{Q}$ is even and we let $2d = [F : \mathbb{Q}]$. Abusing notation we identify the set of archimedean places S_∞ of F and F^+: More precisely, we fix a so-called CM-type of F first, i.e., we fix a choice of pairs of complex embeddings $(\iota_v, \bar{\iota}_v)$, and then we identify the so represented places $v = (\iota_v, \bar{\iota}_v)$ of F, with the places of F^+ through the first component ι_v. We will also fix a Galois closure F^{Gal} of $F/\mathbb{Q}$ in $\bar{\mathbb{Q}}$.

We denote by $\|\cdot\|$ the normalized absolute value on the ring of adèles $\mathbb{A}_F$. Let

$$\varepsilon : (F^+)^\times \backslash \mathbb{A}_{F^+}^\times \to \mathbb{C}^\times$$

be the quadratic Hecke character associated to F/F^+ via class field theory. As it is well-known, (for what follows see, for instance [Bel-Che09, §6.9.2]) it is possible to extend ε to a conjugate self-dual unitary Hecke character

$$\eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times,$$

so that at $v \in S_\infty$ we have $\eta_v(z) = (z/|z|)^{2t}$ for $z \in F_v \cong \mathbb{C}$ and $t = t_v \in \frac{1}{2} + \mathbb{Z}$. As in [Bel-Che09, §6.9.2] we will abbreviate this by writing $\eta_v(z) = z^t \bar{z}^{-t}$, keeping in mind the possible sign ambiguities throughout. Furthermore, we may (and hence will) assume from now on that $t = \frac{1}{2}$, i.e., $\eta_v(z) = z^{1/2}\bar{z}^{-1/2}$, cf. [Bel-Che09, Lem. 6.9.2].

Finally, we may define a non-unitary, but algebraic Hecke character $\phi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$, by letting

$$\phi := \eta \, \|\cdot\|^{-1/2}.$$

1.2. **Algebraic groups and real Lie groups.** Let $n \geq 1$ be an integer. We will denote by $G_n := \mathrm{GL}_n/F$ the linear algebraic, general linear group over F. Let $R_{F/\mathbb{Q}}$ be Weil's restriction of scalars. We will abbreviate $G_{n,\infty} := R_{F/\mathbb{Q}}(G_n)(\mathbb{R})$: It is important to notice that the group

$$G_{n,\infty} = \prod_{v \in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v \in S_\infty} \mathrm{GL}_n(\mathbb{C})$$

– although in principle carrying a complex structure – is thought of as a *real* Lie group, namely as the archimedean factor of $G_n(\mathbb{A}_F)$. Furthermore, we let $K_{n,\infty}$ be the product of the center $Z_{n,\infty}$ of $G_{n,\infty}$ and a fixed maximal compact subgroup, i.e.,

$$K_\infty \cong \prod_{v \in S_\infty} \mathbb{C}^* U(n) \cong \prod_{v \in S_\infty} \mathbb{R}_+ U(n).$$

From this it is clear of how to embed $\mathbb{R}_+$ as a subgroup of $Z_{n,\infty}$ (and hence of $K_{n,\infty}$ and of $G_{n,\infty}$): An $x \in \mathbb{R}_+$ is sent onto the d-tuple of diagonal matrices $(\mathrm{diag}(x, x, ..., x))_{v \in S_\infty} \in G_{n,\infty}$. By $\mathfrak{g}_{n,\infty}$ we denote the real Lie algebra of $G_{n,\infty}$ and use the analogous notation for the Lie algebras of other Lie groups.

1.3. **Equivalence relations and Galois equivariance.**

Definition 1.1 (i)**.** Let $L \subseteq \mathbb{C}$ a subfield and let $x, y \in \mathbb{C}$. We write

$$x \sim_L y,$$

if there is an $\ell \in L$ such that $x = \ell y$ or $\ell x = y$.
(ii) Let $K, L \subset \mathbb{C}$ be subfields. Let $\underline{x} = \{x(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ and $\underline{y} = \{y(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ be two families of complex numbers. We write

$$\underline{x} \sim_L \underline{y}$$

and say that this relation *is equivariant under* $\mathrm{Aut}(\mathbb{C}/K)$, if either $y(\sigma)=0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$, or if $y(\sigma)\neq 0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$ and the following two conditions are verified:

(1) $x(\sigma)\sim_{\sigma(L)} y(\sigma)$ for all σ.

(2) $\sigma\left(\dfrac{x(\tau)}{y(\tau)}\right)=\dfrac{x(\sigma\tau)}{y(\sigma\tau)}$ for all $\sigma\in\mathrm{Aut}(\mathbb{C}/K)$ and all $\tau\in\mathrm{Aut}(\mathbb{C})$.

Obviously, one may replace the first condition by requiring it only for all σ running through representatives of $\mathrm{Aut}(\mathbb{C})/\mathrm{Aut}(\mathbb{C}/K)$. In particular, if $K=\mathbb{Q}$, one only needs to verify it for the identity $id\in\mathrm{Aut}(\mathbb{C})$.

The following lemma is well-known, see, e.g., [Gro-Lin19], Lem. 1.29:

Lemma 1.2. *Let $L\subset\mathbb{C}$ be a number field, containing F^{Gal}. Let $x=\{x(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ and $y=\{y(\sigma)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$ be as in Def. 1.1 and suppose that $y(\sigma)\neq 0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. If the complex numbers $x(\sigma)$ and $y(\sigma)$ depend only on the restriction of σ to L, then the second condition of Def. 1.1 implies the first.*

Proof. Fix $\sigma_0\in\mathrm{Aut}(\mathbb{C})$. For any $\sigma\in\mathrm{Aut}(\mathbb{C})$ fixing $\sigma_0(L)$, one has $\sigma\sigma_0\mid_L=\sigma_0\mid_L$. Hence $x(\sigma\sigma_0)=x(\sigma_0)$ and $y(\sigma\sigma_0)=y(\sigma_0)$ by our assumptions. Moreover, since $L\supset F^{Gal}$, we know $\sigma\in\mathrm{Aut}(\mathbb{C}/F)$. By the second condition, we have:

$$\sigma\left(\frac{x(\sigma_0)}{y(\sigma_0)}\right)=\frac{x(\sigma\sigma_0)}{y(\sigma\sigma_0)}=\frac{x(\sigma_0)}{y(\sigma_0)}.$$

Therefore, $\dfrac{x(\sigma_0)}{y(\sigma_0)}\in\sigma_0(L)$ for all σ_0 as claimed.

□

1.4. **Cohomological automorphic representations.** Let $1\leq m<n$ be any integers. Throughout the paper, we will let Π be a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F)=\mathrm{GL}_n(\mathbb{A}_F)$ and let Π' be a unitary cuspidal automorphic representation of $G_m(\mathbb{A}_F)=\mathrm{GL}_m(\mathbb{A}_F)$, in the sense of [Bor-Jac79, §4.6]. However, for convenience, we will not distinguish between a cuspidal automorphic representation, its smooth automorphic LF-space completion or its (non-smooth) Hilbert space completion in the L^2-spectrum, cf. [Gro20] or [Gro-Žun20] for a detailed account. In this regard, we will now specify our standing assumptions on their archimedean components Π_∞ and Π'_∞.

1.4.1. *The representation* Π_∞. Unless otherwise stated, throughout the paper we always assume that Π_∞ is *cohomological*, i.e., there exists an irreducible finite-dimensional algebraic representation $\mathcal{E}_\mu$ of $G_{n,\infty}$ on a complex vector space, with respect to which Π_∞ has non-trivial $(\mathfrak{g}_{n,\infty},K_{n,\infty})$-cohomology, cf. [Bor-Wal00]. As Π_∞ is assumed to be unitary, $\mathcal{E}_\mu$ must be conjugate self-dual and hence breaks as $\mathcal{E}_\mu=E_\mu\otimes E_{\mu^\vee}$, where $E_\mu=\otimes_{v\in S_\infty}E_{\mu_v}$ and we view again each irreducible $\mathrm{GL}_n(F_v)=\mathrm{GL}_n(\mathbb{C})$-factor E_{μ_v} as being given by its highest weight μ_v. In terms of the standard choice of a maximal split torus in GL_n, positivity on the attached set of roots and standard coordinates, this highest weight is an n-tuple of integers $\mu_v=(\mu_{v,1},...,\mu_{v,n})\in\mathbb{Z}^n$ with $\mu_{v,1}\geq...\geq\mu_{v,n}$. Let

$$\rho_n=\left(\tfrac{n-1}{2},\tfrac{n-3}{2},\cdots,-\tfrac{n-3}{2},-\tfrac{n-1}{2}\right),$$

be the half-sum of positive roots of GL_n with respect to the same conventions. As a consequence of classical results of Delorme–Enright, cf. [Enr79, Thm. 6.1 & 7.1], we see that the condition

that Π_∞ is cohomological with respect to $\mathcal{E}_\mu$ is equivalent to the much more explicit condition that

$$\Pi_v \cong \mathrm{Ind}_{B_n(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})}[z_1^{\ell_{v,1}}\bar{z}_1^{-\ell_{v,1}} \otimes \cdots \otimes z_n^{\ell_{v,n}}\bar{z}_n^{-\ell_{v,n}}]$$

with

$$\ell_{v,i} = -\mu_{v,n-i+1} + \rho_{n,i} \tag{1.3}$$

at each $v \in S_\infty$. Here, B_n is the standard Borel subgroup of G_n (determined by our choice of positivity on the set of roots) and induction is normalized to preserve unitarity.

We will call a set of n real numbers $\{l_{v,i}\}_{1\leq i\leq n}$ an *infinity type at* $v \in S_\infty$, if

$$l_{v,1} > l_{v,2} > ... > l_{v,n}$$

i.e., if its members form a strictly decreasing string. As it is obvious from (1.3), $\{\ell_{v,i}\}_{1\leq i\leq n}$ from above is such a set, called the infinity type of Π at $v \in S_\infty$. Recalling the well-known classification of irreducible unitary cohomological representations of $G_n(\mathbb{C})$ from [Enr79] (see also [Gro-Rag14a], §5.5, for a presentation tailor-made for our purposes here), the following lemma is obvious:

Lemma 1.4. *There is a bijection, defined by* (1.3)*, between the equivalence classes of irreducible unitary cohomological tempered representations of* $G_n(F_v) \cong G_n(\mathbb{C})$ *and the infinity types* $\{l_{v,i}\}_{1\leq i\leq n}$*, for which* $l_{v,i} \in \frac{n+1}{2} + \mathbb{Z}$ *for all* $1 \leq i \leq n$*.*

As a last ingredient we will call a highest weight μ as above *sufficiently regular*, if $\mu_{v,i} - \mu_{v,i+1} \geq 2$ for all $v \in S_\infty$ and $1 \leq i \leq n-1$.

1.4.2. *The representation* Π'_∞. Similar to our assumptions on Π we will suppose that the twisted representation $\Pi'_\infty\|\det\|^{(n-m-1)/2}$ is cohomological, or, equivalently, that

$$\Pi'^{\mathsf{alg}} := \begin{cases} \Pi' & \text{if } n-1 \equiv m \mod 2, \\ \Pi' \otimes \eta & \text{otherwise.} \end{cases} \tag{1.5}$$

is cohomological. In terms of infinity types, this means that for each $v \in S_\infty$ and $1 \leq i \leq m$, there are $a_{v,i} \in \frac{n}{2} + \mathbb{Z}$ with $a_{v,i} > a_{v,i+1}$, such that

$$\Pi'_v \cong \mathrm{Ind}_{B_m(\mathbb{C})}^{\mathrm{GL}_m(\mathbb{C})}[z_1^{a_{v,1}}\bar{z_1}^{-a_{v,1}} \otimes \cdots \otimes z_m^{a_{v,m}}\bar{z_m}^{-a_{v,m}}].$$

1.4.3. *An auxiliary representation* Σ *in piano-position.* We extend the infinity type $\{a_{v,i}\}_{1\leq i\leq m}$ of Π'_v, at each place $v \in S_\infty$ to an infinity type of length $n-1$, simply by choosing any distinct $b_{j,v} \in \frac{n}{2} + \mathbb{Z}$ for $1 \leq j \leq n-m-1$, such that

$$\{a_{v,i}\}_{1\leq i\leq m} \cap \{b_{j,v}\}_{1\leq j\leq n-m-1} = \emptyset. \tag{1.6}$$

Denote this new infinity type by $\{\ell'_{v,i}\}_{1\leq i\leq n-1}$. As by construction $\ell'_{v,i} \in \frac{n}{2} + \mathbb{Z}$ for all $1 \leq i \leq n-1$, this is the infinity type of a unique cohomological irreducible unitary tempered representation of $G_{n-1}(\mathbb{C})$ by Lemma 1.4.

Turning back to global representations, let χ_1, ..., χ_{n-m-1} be unitary Hecke characters with $\chi_{j,v}(z) = z^{b_{j,v}}\bar{z}^{-b_{j,v}}$ for all $v \in S_\infty$, i.e., such that $\chi_j\|\cdot\|^{n/2}$ – or, specifying $m = 1$ in (1.5) that χ_j^{alg} – is algebraic. We note that such characters exist, as it follows from [Wei56]. By its very construction the isobaric automorphic sum

$$\Sigma := \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$$

has our infinity type $\{\ell'_{v,i}\}_{1\leq i\leq n-1}$ from above and is therefore cohomological.

Let $\mathcal{E}_{\mu'}$ be the unique irreducible algebraic coefficients module of $G_{n-1,\infty}$ with respect to which Σ_∞ has non-trivial $(\mathfrak{g}_{n-1,\infty}, K_{n-1,\infty})$-cohomology. By the same reasoning as above, $\mathcal{E}_{\mu'} = E_{\mu'} \otimes E_{\mu'}^\vee$ and writing $\mu'_v = (\mu'_{v,1}, ..., \mu'_{v,n-1}) \in \mathbb{Z}^{n-1}$ at $v \in S_\infty$ one has $\mu'_{v,1} \geq ... \geq \mu'_{v,n-1}$ and

$$\ell'_{v,i} = -\mu'_{v,n-i} + \rho_{n-1,i}. \tag{1.7}$$

Henceforth we will assume that Π_∞ and Σ_∞ satisfy the *piano-condition*, by which we mean that

$$\mu_{v,1} \geq -\mu'_{v,n-1} \geq \mu_{v,2} \geq -\mu'_{v,n-2} \geq ... \geq -\mu'_{v,1} \geq \mu_{v,n}. \tag{1.8}$$

Equivalently, $\mathrm{Hom}_{G_{n-1,\infty}}(\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C})$ is non-zero (and hence one-dimensional), see [Goo-Wal09], Thm. 8.1.1.

According to out previous definition, we call μ' *sufficiently regular*, if $\mu'_{v,i} - \mu'_{v,i+1} \geq 2$ for all $v \in S_\infty$ and $1 \leq i \leq n-2$.

1.5. **Critical points of L-functions.** For a moment let $N, M \geq 1$ be any integers and let π be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_F) \times \mathrm{GL}_M(\mathbb{A}_F)$ for which a completed standard L-function $L(s,\pi) = \prod_v L(s,\pi_v)$ is defined satisfying a global functional equation $L(s,\pi) = \varepsilon(s,\pi) \cdot L(1-s,\pi^\vee)$, cf. [Bor79, §IV]. The following definition is modelled after [Del79], Prop.-Def. 2.3:

Definition 1.9. A complex number $s_0 \in \frac{N-M}{2} + \mathbb{Z}$ is called *critical* for $L(s,\pi)$ if both $L(s,\pi_\infty)$ and $L(1-s,\pi_\infty^\vee)$ are holomorphic at $s = s_0$. We write $\mathrm{Crit}(\pi)$ for the set of critical points of $L(s,\pi)$.

We proceed with the following simple observation

Observation 1.10. Recalling that $\Gamma(s)$ does not vanish, the set of holomorphic points of $L(s,\pi_\infty)$ coincides with the intersection of the sets of holomorphic points of the archimedean L-functions attached to the characters in the Langlands datum of π_∞, cf. [Kna94], §4.

As a consequence we obtain the following lemma, which relates the critical points of $L(s,\Pi\times\Sigma)$ to the critical points of the isobaric summands of Σ.

Lemma 1.11. *The following hold*

(i) *If $n \not\equiv m \mod 2$, then* $\mathrm{Crit}(\Pi\times\Sigma) \subseteq \mathrm{Crit}(\Pi\times\Pi')$.
(ii) *If n is even and m is odd, then* $\mathrm{Crit}(\Pi\times\Sigma) = \mathrm{Crit}(\Pi\times\Pi') \cap \bigcap_{j=1}^{n-m-1} \mathrm{Crit}(\Pi\otimes\chi_j)$.
(iii) *If m is odd, then $s_0 = 1 \in \mathrm{Crit}(\Pi'\otimes\chi_j^{-1})$ for all $1 \leq j \leq n-m-1$. In any case $s_0 = \frac{1}{2} \in \mathrm{Crit}(\Pi\times\Sigma)$.*

Proof. After our Observation 1.10 only (iii) needs a short argument. Writing down $L(s,\Pi'_\infty\chi_{j,\infty}^{-1})$ and $L(1-s,\Pi_\infty'^\vee\chi_{j,\infty})$, cf. [Kna94], §4, we see that the behaviour of holomorphy of these two L-factors is the same as the one of

$$\prod_{v\in S_\infty}\prod_{i=1}^m \Gamma(s+|a_{i,v}-b_{v,j}|) \qquad \text{and} \qquad \prod_{v\in S_\infty}\prod_{i=1}^m \Gamma(1-s+|-a_{i,v}+b_{v,j}|).$$

By (1.6), $a_{i,v} - b_{v,j} \neq 0$ for all $v \in S_\infty$, $1 \leq i \leq m$ and $1 \leq j \leq n-m-1$. Whence, $|a_{i,v}-b_{v,j}| = |-a_{i,v}+b_{v,j}| \geq 1$, and so all the above Γ-factors are holomorphic at $s_0 = 1$. Hence, $s_0 = 1 \in \mathrm{Crit}(\Pi'\otimes\chi_j^{-1})$, if m is odd. The last assertion finally follows from the piano-hypothesis (1.8) and [Gro18], §1.6.1.(4) or [Rag16], Thm. 2.21. □

1.6. **Whittaker periods, σ-twists and fields of rationality.** Let Σ be as above. Unitarity of all isobaric summands implies that

$$\Sigma \cong \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_{n-1}(\mathbb{A}_F)}[\Pi' \otimes \chi_1 \otimes ... \otimes \chi_{n-1-m}]$$

is fully induced from the standard parabolic F-subgroup $P \subseteq \mathrm{GL}_{n-1}$ with Levi component $L_P \cong \mathrm{GL}_m \times \prod_{j=1}^{n-m-1} \mathrm{GL}_1$. Hence, a *Whittaker period* $p(\Sigma) \in \mathbb{C}^\times$ has been constructed in [Gro-Lin19], Prop. 1.11 and Cor. 1.20. It recovers the original construction of Raghuram–Shahidi for cuspidal representations. Hence, also $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$ and $p(\chi_j^{\mathsf{alg}})$ are all defined. We recall that in [Gro-Lin19] the period $p(\chi)$ is normalized to $p(\chi) \sim_{\mathbb{Q}(\chi)} 1$ for all algebraic Hecke characters, which we also may and do assume here.

We remark that it is intrinsic to the construction of these Whittaker periods, that they are uniquely defined only up to multiplication by non-zero numbers in the respective *field of rationality*, i.e., if ν is any of the above representations, then $p(\nu)$ may be replaced by $q \cdot p(\nu)$ for any $q \in \mathbb{Q}(\nu)^\times := \mathbb{C}^{\mathfrak{S}(\nu)}$ where $\mathfrak{S}(\nu) := \{\sigma \in \mathrm{Aut}(\mathbb{C}) | \nu_f \cong {}^\sigma\nu_f\}$ and ${}^\sigma\nu_f := \nu_f \otimes_{\sigma^{-1}} \mathbb{C}$. For cohomological automorphic representations ν as above, the rationality fields $\mathbb{Q}(\nu)$ are number fields and ${}^\sigma\nu_f$ is the finite component of a uniquely determined, cohomological automorphic representation – denoted ${}^\sigma\nu$ – justifying the notation $\mathbb{Q}(\nu)$.

As a last ingredient, for each critical point $s_0 = \frac{1}{2} + k$ of $L(s, \Pi \times \Sigma)$, an *archimedean period* $p(k, \Pi_\infty, \Sigma_\infty) \in \mathbb{C}^\times$ has been defined in [Gro18] 1.6.1.(6) as the weighted sum of archimedean zeta-integrals. We do not repeat its precise definition here and rather refer to [Gro18], because $p(k, \Pi_\infty, \Sigma_\infty)$ will not show up in the final results of this paper, but plays the role of an auxiliary quantity on the way there. Here we only point out that the normalization $p(\chi) \sim_{\mathbb{Q}(\chi)} 1$ for all algebraic Hecke characters, together with two conditions of compatibility in the construction of $p(\Sigma)$ (cf. [Gro-Lin19], §1.5.3 for all details and further discussion) pin down $p(k, \Pi_\infty, \Sigma_\infty)$ uniquely.

2. Revisiting four results on period-relations

2.1. **Our starting point: The main result of** [Gro18]**.** We now recall the following algebraicity result for the critical points of the L-function attached to a pair (Π, Σ):

Proposition 2.1 ([Gro18], Thm. 1.8)**.** *Let Π and Σ be cohomological automorphic representations as in §1.4. In particular, Π is a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F)$, $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ the isobaric sum of a unitary cuspidal automorphic representation Π' of $G_m(\mathbb{A}_F)$ and unitary Hecke characters χ_j, such that Π_∞ and Σ_∞ satisfy the piano-condition. Then, for every critical point $s_0 = \frac{1}{2} + k$ of $L(s, \Pi \times \Sigma)$,*

$$L^S(\tfrac{1}{2} + k, \Pi \times \Sigma) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)} p(k, \Pi_\infty, \Sigma_\infty)\ p(\Pi)\ p(\Sigma), \tag{2.2}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C})$.

In this result, we interpreted the left and the right hand side in relation (2.2) as families $\underline{x} = \{x(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ and $\underline{y} = \{y(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ in the obvious way and we will continue to do so henceforth in analogous situations. In the next three subsection we collect three additional results from the theory of special values – one of them achieved by Blasius in [Bla86], whereas the other two have only been quite recently established in joint work of the first-named author in [Gro-Lin19]. These three results shall then be used afterwards in order to rewrite the above statement (2.2) in a much more refined way, being the key-step in the proof of our first main, new result.

2.2. **Step I: The archimedean period as a power of $2\pi i$.** Under certain conditions, J. Lin and first-named author have computed the archimedean period-factor $p(k, \Pi_\infty, \Sigma_\infty)$ from (2.2) as a power of $(2\pi i)$, see [Gro-Lin19], Cor. 4.30.

In order to recall this result also for the archimedean period $p(0, \Pi_\infty, \Sigma_\infty)$ attached to the central critical point $s_0 = \frac{1}{2}$ of $L(s, \Pi \times \Sigma)$, cf. Lemma 1.11, consider two cyclic extensions L and L' of F, of degree n resp. $n-1$, which are still CM-fields. For an algebraic Hecke character χ of $\mathbb{A}_L^\times$ (resp. χ' of $\mathbb{A}_{L'}^\times$), let $\Pi(\chi)$ (resp. $\Pi(\chi')$) be the automorphic induction from χ to $G_n(\mathbb{A}_F)$ (resp. χ' to $G_{n-1}(\mathbb{A}_F)$), cf. [Art-Clo89], Chp. 3, Thm. 6.2 (as completed in [Hen12], Thm. 3 (see also [Clo17])). We denote

$$\Pi_\chi := \begin{cases} \Pi(\chi) & \text{if } n \text{ is odd,} \\ \Pi(\chi) \otimes \eta & \text{if } n \text{ is even.} \end{cases} \qquad \text{and} \qquad \Pi_{\chi'} := \begin{cases} \Pi(\chi') & \text{if } n \text{ is even,} \\ \Pi(\chi') \otimes \eta & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

It is argued in [Gro-Lin19], §4.5, that, given Π and Σ as in §1.4, one may always choose conjugate self-dual algebraic Hecke characters χ and χ' such that Π_χ and $\Pi_{\chi'}$ are cuspidal automorphic representations for which $\Pi_{\chi,\infty} \cong \Pi_\infty$ and $\Pi_{\chi',\infty} \cong \Sigma_\infty$. Whenever we use the symbols Π_χ and $\Pi_{\chi'}$ it is from now on silently assumed that such a choice has been made. Our definition now allows us to state

Proposition 2.4 ([Gro-Lin19], Cor. 4.30)**.** *Let Π and Σ be cohomological automorphic representation as in §1.4 and let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma)$. Only if $k = 0$, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point of $L(s, \Pi \times \Sigma)$, we additionally assume that μ and μ' are both sufficiently regular and that there exists a choice of χ, χ' such that $L^S(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) \neq 0$. With these assumptions*

$$p(k, \Pi_\infty, \Sigma_\infty) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)F^{Gal}} (2\pi i)^{d(n-1)((k-\frac{1}{2})n+1)}, \tag{2.5}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Remark 2.6. The reader, who is interested in the nature of our non-vanishing hypothesis for the central critical value $L^S(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'})$ of the Rankin–Selberg L-function attached to a suitable choice of twisted automorphically induced representations Π_χ and $\Pi_{\chi'}$, may find the following remark illuminating: Let $K \subset LL'$ be a subfield and let $N_{\mathbb{A}_{LL'}/\mathbb{A}_K}$ be the adelic extension of the norm attached to the field extensions LL'/K. Then, it follows from the very construction of Π_χ and $\Pi_{\chi'}$, cf. [Art-Clo89], Chp. 3, that we have an equality of partial L-values

$$L^S(\tfrac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) = L^S(\tfrac{1}{2}, (\chi \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_L})(\chi' \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_{L'}})(\eta \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_F})).$$

Our non-vanishing assumption – made only in the case, when we want to consider the archimedean period $p(0, \Pi_\infty, \Sigma_\infty)$ at the central critical value – hence reduces to a non-vanishing assumption for a Hecke L-function on LL', i.e., may be reduced from considering Rankin-Selberg L-functions of type $n \times (n-1)$ over F to standard L-functions of GL_1/LL'. In turn, the non-vanishing of the latter L-functions is studied in many sources in the literature: The results of Rohlich [Roh89], Ginzburg–Jiang–Rallis [Gin-Jia-Ral04], Eischen [Eis17] and most recently Jiang–Zhang [Jia-Zha20] provide evidence that our non-vanishing assumption is indeed satisfied in all cases that we consider. For a formal argument and analysis of this latter assertion we refer to [Gro-Lin19], §4.5.1.

2.3. **Step II: Breaking the period of Σ.** As the next ingredient for rewriting (2.2), we will decompose the Whittaker period $p(\Sigma)$ in terms of the isobaric summands of Σ. The following is a special case of another result of the first-named author's recent work with J. Lin:

Proposition 2.7. *Let $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ be a cohomological isobaric automorphic representation as in §1.4. Assume in addition that all summands Π' and χ_j are conjugate self-dual. Then*

$$p(\Sigma) \sim_{\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)F^{Gal}} p(\Pi'^{\mathsf{alg}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi' \otimes \chi_j^{-1}) \prod_{1\leq i<j\leq n-m-1} L^S(1, \chi_i\chi_j^{-1}), \tag{2.8}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. We recall that by our conventions, cf. §1.6, $\prod_{j=1}^{n-m-1} p(\chi_j^{\mathsf{alg}}) \in \mathbb{Q}^\times$, whence the assertion follows from [Gro-Lin19], Cor. 2.16. □

2.4. **Step III: Relating $L^S(1, \chi_i\chi_j^{-1})$ to CM-periods.** The last necessary ingredient for rewriting (2.2) has been established by Blasius, cf. [Bla86]. He described the critical L-values $L^S(1, \chi_i\chi_j^{-1})$ showing up in the formula (2.8) in terms of *CM-periods* $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$.

2.4.1. *Review of CM-periods.* The reader familiar with Blasius's construction may skip this small subsection and proceed directly to Prop. 2.9 below.
As a first observation, for any pair (i, j) with $1 \leq i < j \leq n-m-1$, the Hecke character $\xi := \chi_i\chi_j^{-1}$ is critical in the sense of Deligne, [Del79]: That means that ξ is algebraic and has non-trivial archimedean components ξ_v for all $v \in S_\infty$. Clearly, the latter assertion follows from our definition of $\{b_{v,j}\}_{1\leq j\leq n-m-1}$ being an infinity type for all $v \in S_\infty$, i.e., a set of strictly decreasing real numbers. Hence, one may define another CM-type Ψ_ξ of F by the rule

$$\Psi_\xi := \{v \in S_\infty | b_{v,i} < b_{v,j}\} \cup \overline{\{v \in S_\infty | b_{v,i} > b_{v,j}\}}.$$

Let now Ψ_F be any CM-type of F. Attached to (ξ, Ψ_F) one may define a CM-Shimura-datum as in [MHar93], Sect. 1.1 and a number field $E(\xi, \Psi_F)$, which contains $\mathbb{Q}(\xi)$ and the reflex field of the CM-Shimura-datum defined by Ψ_F. In particular, if $\Psi_F = \Psi_\xi$, one may associate a non zero complex number $p(\xi, \Psi_\xi)$ to this datum, as explained in the appendix of [MHar-Kud91]. This number $p(\xi, \Psi_\xi)$ is well-defined modulo $E(\xi, \Psi_\xi)^\times$ and called the *CM-period attached to* ξ. Let us abbreviate $E^{\mathsf{cm}}(\xi) := \prod_{\Psi_F} E(\xi, \Psi_F)$ and, resuming the notation $\chi_i\chi_j^{-1}$,

$$E^{\mathsf{cm}}(\chi_1, ..., \chi_{n-m-1}) := \prod_{1\leq i<j\leq n-m-1} E^{\mathsf{cm}}(\chi_i\chi_j^{-1}).$$

This field is a number field by construction, which contains the finite compositum of the number fields $F^{Gal} \prod_{1\leq i<j\leq n-m-1} \mathbb{Q}(\chi_i\chi_j^{-1})$, as defined in §1.6, but may be bigger than that.

The following result is proved in [Bla86]. We also refer to [MHar93], Prop. 1.8.1 (and the attached erratum [MHar97], p. 82) or Thm 4.7 of [Gro-Lin19] for a slightly more tailor-made presentation.

Proposition 2.9. *Let $\chi_1, ..., \chi_{n-m-1}$ be conjugate self-dual Hecke characters, such that $\chi_i\chi_j^{-1}$ is algebraic and critical for all $1 \leq i < j \leq n-m-1$. Then*

$$\prod_{1\leq i<j\leq n-m-1} L^S(1, \chi_i\chi_j^{-1}) \sim_{E^{\mathsf{cm}}(\chi_1,...,\chi_{n-m-1})} (2\pi i)^{d\frac{1}{2}(n-m-1)(n-m-2)} \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}), \tag{2.10}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

3. Our main theorem

3.1. **Special values for $\mathrm{GL}_n \times \mathrm{GL}_m, 1 \leq m < n$.** We are now ready to prove our first new result. To this end, recall that $1 \leq m < n$ has been any pair of integers and that Π and Π' have been cohomological unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively, the latter assumed to be conjugate self-dual as in §2. Our first main result will relate special values of the partial Rankin-Selberg L-function $L(s, \Pi \times \Pi')$ (all of them indeed critical, if n and m are of different parity), to quantities only depending on Π, Π' and a suitable choice of auxiliary characters χ_j (as in §1.4.3).

Rendering this more precise, recall the Whittaker periods $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$ attached to Π and Π'^{alg} and the CM-periods $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$ attached to a choice of auxiliary characters χ_j from §1.6 and §2.4. Recall that when n is even and m is odd, then $\Pi'^{\mathsf{alg}} = \Pi'$ and $\chi_j^{\mathsf{alg}} = \chi_j$ for all $1 \leq j \leq n-m-1$. The following result may be seen as our main theorem:

Theorem 3.1. *We let F be any CM-field and let $1 \leq m < n$ be any integers. Let Π be a cohomological unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. Choose any conjugate self-dual Hecke characters χ_1, ..., χ_{n-m-1}, such that the isobaric automorphic sum $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ is cohomological and assume that $(\Pi_\infty, \Sigma_\infty)$ satisfies the piano-hypothesis,* (1.8). *Let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma)$ be any critical point of $L(s, \Pi \times \Sigma)$.*
In the special case when $k = 0$ only, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point, we additionally assume that the coefficient modules of Π_∞ and Σ_∞ are both sufficiently regular, cf. §1.4.1 and §1.4.3, and that there exists a choice of Hecke characters χ, χ' such that $L^S(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) \neq 0$, cf. §2.2, as well as that $L^S(\frac{1}{2}, \Pi \otimes \chi_j) \neq 0$ for all $1 \leq j \leq n-m-1$.
Then

$$L^S(\tfrac{1}{2}+k, \Pi \times \Pi') \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\chi_1,\ldots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathsf{alg}}) \cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi' \otimes \chi_j^{-1})}{L^S(\frac{1}{2}+k, \Pi \otimes \chi_j)}, \tag{3.2}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. If n is even and m is odd, then all L-values $L^S(\frac{1}{2}+k, \Pi \times \Pi')$, $L^S(1, \Pi' \otimes \chi_j^{-1})$ and $L^S(\frac{1}{2}+k, \Pi \otimes \chi_j)$ are critical.

Proof. Putting our Steps I – III, i.e., equations (2.2), (2.5), (2.8), and (2.10), together and observing the non-vanishing of $L^S(1, \Pi' \otimes \chi_j^{-1})$, cf. [Sha81], Thm. 5.1, we obtain

$$L^S(\tfrac{1}{2}+k, \Pi\times\Sigma) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\chi_1,\ldots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathsf{alg}}) \cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi' \otimes \chi_j^{-1}),$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. As $L^S(\frac{1}{2}+k,\Pi\times\Sigma) = L^S(\frac{1}{2}+k,\Pi\times\Pi')\cdot\prod_{j=1}^{n-m-1} L^S(\frac{1}{2}+k,\Pi\otimes\chi_j)$ this yields,

$$L^S(\tfrac{1}{2}+k,\Pi\times\Pi') \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\chi_1,\ldots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\ p(\Pi)\ p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1},\Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1,\Pi'\otimes\chi_j^{-1})}{L^S(\frac{1}{2}+k,\Pi\otimes\chi_j)}, \tag{3.3}$$

as a relation, which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$, implying the first assertion. Hence, assume now that n is even and m is odd. Criticality of $L^S(\frac{1}{2}+k,\Pi\times\Pi')$, $L^S(1,\Pi'\otimes\chi_j^{-1})$ and $L^S(\frac{1}{2}+k,\Pi\otimes\chi_j)$ is then implied by Lem. 1.11. This completes the proof. □

Remark 3.4. Our Thm. 3.1 has an obvious corollary / consequence for critical values $L^S(k,\Pi\times\Pi')$ of Rankin–Selberg L-functions of type $n\times n$ – i.e., when both factors are of the same rank n – but when Π' is a (non-cuspidal) isobaric sum. Indeed, let $\Pi' := (\Pi_1\boxplus\Pi_2)\|\det\|^{1/2}$, where Π_1 and Π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{m_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{m_2}(\mathbb{A}_F)$ respectively, such that $m_1+m_2=n$. If we assume that both summands Π_i may be completed using conjugate self-dual algebraic Hecke characters to cohomological isobaric sums Σ_i on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, which satisfy the piano-hypothesis with respect to Π_∞, then any $\frac{1}{2}+k\in\mathrm{Crit}(\Pi\times\Sigma_1)\cap\mathrm{Crit}(\Pi\times\Sigma_2)$ gives rise to a critical integer $k\in\mathrm{Crit}(\Pi\times\Pi')$. Hence, writing

$$L^S(k,\Pi\times\Pi') = L^S(\tfrac{1}{2}+k,\Pi\times(\Pi_1\boxplus\Pi_2)) = L^S(\tfrac{1}{2}+k,\Pi\times\Pi_1)\cdot L^S(\tfrac{1}{2}+k,\Pi\times\Pi_2)$$

we may apply Thm. 3.1 to both of the latter factors and derive a rationality result for the critical values $L^S(k,\Pi\times\Pi')$. We leave the obvious details to the reader. Finally, it is now also clear how one can obtain an analogous result if the representation Π' is the twisted isobaric sum of $r\geq 3$ conjugate self-conjugate cuspidal automorphic representations.

4. Main applications

In this section we provide a couple of applications of our main theorem, exemplifying the strength of period-relations such as the ones established in Thm. 3.1.

4.1. Quotients of twisted standard L-functions at a joint special value.

Our first main application concerns the twisted standard L-function and is a broad generalization of the main result of [Wal85b].

Ibidem, Waldspurger has shown a rationality result for the quotient $L(\frac{1}{2},\pi\otimes\alpha)/L(\frac{1}{2},\pi\otimes\beta)$ of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi\otimes\alpha$ and $\pi\otimes\beta$ of GL_2 over any number field at their joint critical value $s_0=\frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty=\beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L(\frac{1}{2},\pi\otimes\beta)$ is supposed to be non-zero. Under these assumptions, Waldspurger established a relation of the form

$$\frac{L(\frac{1}{2},\pi\otimes\alpha)}{L(\frac{1}{2},\pi\otimes\beta)} \sim_{\mathbb{Q}(\pi)} \frac{p(\alpha)}{p(\beta)},$$

the two period-invariants $p(\alpha)$, $p(\beta)$ only depending on α respectively β and the archimedean component of the cuspidal representations π. See [Wal85b], p. 174.

Here we generalize Waldspurger's result to the case of quotients of standard L-functions of

GL_n/F where $n \geq 2$ is arbitrary, $s_0 = \frac{1}{2} + k$ a more general special value while F is any CM-field. Our result reads as follows.

Theorem 4.1. *Let F be any CM-field and let $n \geq 2$ be an integer. We assume that Π is a cohomological unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let α and β be conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ having the same archimedean components $\alpha_v(z) = \beta_v(z) = z^{a_v}\bar{z}^{-a_v}$, $v \in S_\infty$. If $a_v \in \frac{n}{2}+\mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{n,v}$ for all $v \in S_\infty$, then there is a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sum $\Sigma = \alpha \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-2}$ is cohomological and such that $(\Pi_\infty, \Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis,* (1.8). *Fix any such choice and let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma_\alpha)$ be any critical point of $L(s, \Pi \times \Sigma_\alpha)$.*
In the special case when $k = 0$ only, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point, we additionally assume that the coefficient modules of Π_∞ and $\Sigma_{\alpha,\infty}$ are both sufficiently regular, cf. §1.4.1 and §1.4.3, and that there exists a choice of Hecke characters χ, χ' such that $L^S(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) \neq 0$, cf. §2.2, as well as that $L^S(\frac{1}{2}, \Pi \otimes \beta)$ and $L^S(\frac{1}{2}, \Pi \otimes \chi_j) \neq 0$ for all $1 \leq j \leq n-2$.
We have

$$\frac{L^S(\frac{1}{2}+k, \Pi \otimes \alpha)}{L^S(\frac{1}{2}+k, \Pi \otimes \beta)} \sim \prod_{i=1}^{n-2} \frac{p(\alpha, \Psi_{\alpha\chi_i^{-1}})}{p(\beta, \Psi_{\beta\chi_i^{-1}})}. \tag{4.2}$$

where the relation "$\sim$" is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$ and over the number field

$$\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\alpha, \beta, \chi_1, ..., \chi_{n-2})E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\beta) \prod_i^{n-2} E^{\mathsf{cm}}(\chi_i^{-1}).$$

If n is even, then all the $s_0 = \frac{1}{2} + k$ are indeed critical for $L(s, \Pi \otimes \alpha)$ and $L(s, \Pi \otimes \beta)$.

Proof. The discussion in §1.4.3 together with our assumption that $a_v \in \frac{n}{2}+\mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{n,v}$ for all $v \in S_\infty$ implies immediately that there is a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sum $\Sigma = \alpha \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-2}$ is cohomological and such that $(\Pi_\infty, \Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis. Hence, putting $m = 1$ and $\Pi' = \alpha$, the pair (Π, α) of cuspidal representations on $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_1(\mathbb{A}_F)$ satisfies all the conditions of our Thm. 3.1 (with $\Sigma = \Sigma_\alpha$). As it is again immediate, our assumption $\alpha_\infty = \beta_\infty$ implies that the isobaric sum $\Sigma_\beta := \beta \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-2}$ has the same archimedean component as Σ_α. Another short moment of thought convinces us that consequently Σ_α and Σ_β may be interchanged in the statement of Thm. 4.1 without changing any assertion: Otherwise put, the pair (Π, β) automatically satisfies all the conditions of our Thm. 3.1, letting $\Sigma = \Sigma_\beta$. Hence, simply by inserting we obtain

$$\frac{L^S(\frac{1}{2}+k, \Pi \otimes \alpha)}{L^S(\frac{1}{2}+k, \Pi \otimes \beta)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\chi_1,...,\chi_{n-2})} \frac{p(\alpha^{\mathsf{alg}})}{p(\beta^{\mathsf{alg}})} \prod_{i=1}^{n-2} \frac{L^S(1, \alpha\chi_i^{-1})}{L^S(1, \beta\chi_i^{-1})}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. As the Whittaker periods $p(\alpha^{\mathsf{alg}})$ and $p(\beta^{\mathsf{alg}})$ are both chosen to be 1, cf. §1.6, and applying Blasius's result, cf. Prop. 2.9, once more to $\prod_{i=1}^{n-2} L^S(1, \alpha\chi_i^{-1})$ and $\prod_{i=1}^{n-2} L^S(1, \beta\chi_i^{-1})$, we get

$$\frac{L^S(\frac{1}{2}+k, \Pi \otimes \alpha)}{L^S(\frac{1}{2}+k, \Pi \otimes \beta)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\alpha,\chi_1,...,\chi_{n-2})E^{\mathsf{cm}}(\beta,\chi_1,...,\chi_{n-2})} \prod_{i=1}^{n-2} \frac{p(\alpha\chi_i^{-1}, \Psi_{\alpha\chi_i^{-1}})}{p(\beta\chi_i^{-1}, \Psi_{\beta\chi_i^{-1}})}.$$

Obviously, we may replace $E^{\mathsf{cm}}(\alpha\chi_1, ..., \chi_{n-2})E^{\mathsf{cm}}(\beta, \chi_1, ..., \chi_{n-2})$ by $E^{\mathsf{cm}}(\alpha, \beta, \chi_1, ..., \chi_{n-2})$ in the latter relation without any harm. For each $1 \leq i \leq n-2$ one has

$$p(\alpha\chi_i^{-1}, \Psi_{\alpha\chi_i^{-1}}) \sim_{E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\chi_i^{-1})} p(\alpha, \Psi_{\alpha\chi_i^{-1}})\ p(\chi_i^{-1}, \Psi_{\alpha\chi_i^{-1}})$$

and likewise for β taking the role of α, see [Gro-Lin19] Prop. 4.4. As $\alpha_\infty = \beta_\infty$ by assumption, $\Psi_{\alpha\chi_i^{-1}} = \Psi_{\beta\chi_i^{-1}}$ by definition, cf. §2.4. This implies the first assertion of the theorem. The second assertion follows applying Lem. 1.11. □

Remark 4.3 (Further interpretations)**.** In [Gro-Rag14b] Raghuram and the first named author have achieved a rationality result for the critical values of the twisted standard L-function $L(s, \Pi \otimes \chi)$ using unspecified archimedean periods. Here, Π denotes a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, admitting a Shalika model and χ is a Hecke character of finite order. These assumptions necessarily imply that n is even as in the refined second assertion of our Thm. 4.1 above. In this regard, Thm. 4.1 provides a generalization as well as a certain refinement of a consequence of the main result of [Gro-Rag14b] over general CM-fields, instead of totally real fields F^+.

4.2. **Quotients of a fixed Rankin-Selberg L-function at different critical values.** Our second application concerns quotients of a given Rankin-Selberg L-function $L^S(s, \Pi \times \Pi')$ of general type $n \times m$, $1 \leq m \leq n$, at different critical values $s = \frac{1}{2} + k$ and $s = \frac{1}{2} + \ell$.

Our result may be viewed as a generalization of

(i) the main result of Harder-Raghuram [GHar-Rag20], obtained there for Rankin-Selberg L-functions of general type $n \times m$, nm even, but over totally real fields F^+, as well as
(ii) one of the main results of [Gro-Lin19], cf. Thm. 5.5, obtained there for general CM-fields F, but for Rankin-Selberg L-functions of type $n \times (n-1)$ only.

It should be pointed out though, that our result below is a rather mild generalization of a consequence of the main result of [Lin15]: There Lin has achieved a very general, fine rationality-result for Rankin-Selberg L-functions of type $n \times m$, stated in [Lin15], Thm. 10.8.1 under a list of additional local assumptions (and conjectures, but those were later proved in [Gue16] and [Gro-Lin19]). We hence do not claim much originality from our side, but rather include the following corollary of Thm. 3.1 for the sake of giving a new approach and an example of the use of our period-relations.

In order to explain our result, recall weak base change BC from an arbitrary rational unitary similitude group $GU(V)/\mathbb{Q}$ attached to a non-degenerate Hermitian space V of $\dim_{F^+} V = n$, as established in [Shi14]. Strictly speaking, the construction of BC in [Shi14] entails the claim that $F = \mathcal{K}F^+$ for some imaginary quadratic field $\mathcal{K}$, which we shall henceforth assume. The same assumption has been made in [Gue16], §5, which we shall use in the proof of our Cor. 4.4 below. Then, for every cohomological cuspidal automorphic representation π of $GU(V)(\mathbb{A}_\mathbb{Q})$ a base change $BC(\pi) = \chi_\pi \otimes \Pi$ has been constructed in [Shi14]: Here, χ_π is a Hecke character of $\mathbb{A}_\mathcal{K}^\times$, while Π is a conjugate self-dual isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. By results of Delorme-Enright, cf. [Enr79], Π_∞ is cohomological as well. See also [Lab11], §5.1 and [Clo91], §3.4.

Corollary 4.4. *Let $F = \mathcal{K}F^+$ be a CM-field and suppose that $1 \leq m < n$ are integers, n even and m odd. We let $\Pi = BC(\pi)|_{\mathrm{GL}_n(\mathbb{A}_F)}$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which we assume to be obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$.*
Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, satisfying the conditions of Thm. 3.1.
Let $\frac{1}{2} + k$ and $\frac{1}{2} + \ell$ be two critical points of $L(s, \Pi \times \Sigma)$ with $k, \ell > \frac{n}{2}$. Then $\frac{1}{2} + k$ and $\frac{1}{2} + \ell$ are

indeed critical for $L(s,\Pi\times\Pi')$ *and the ratio of critical values satisfies*

$$\frac{L^S(\frac{1}{2}+k,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')}\sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}}(2\pi i)^{d(k-\ell)nm},$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$.

Proof. By Thm. 3.1, the quotient of critical values satisfies

$$\frac{L^S(\frac{1}{2}+k,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')}\sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)E^{\mathsf{cm}}(\chi_1,\dots,\chi_{n-m-1})}(2\pi i)^{d(k-\ell)(n-1)n}\cdot\prod_{j=1}^{n-m-1}\frac{L^S(\frac{1}{2}+\ell,\Pi\otimes\chi_j)}{L^S(\frac{1}{2}+k,\Pi\otimes\chi_j)},\tag{4.5}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Moreover, obviously, both sides of this relation are invariant under all $\sigma\in\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')\cap\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$, where $\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$ denotes the group of all $\sigma\in\mathrm{Aut}(\mathbb{C})$ such that $\{{}^\sigma\chi_1,\dots,{}^\sigma\chi_{n-m-1}\}=\{\chi_1,\dots,\chi_{n-m-1}\}$. Therefore, by Lem. 1.2, relation (4.5) holds over every field L, which contains F^{Gal} and the subfield of $\mathbb{C}$, fixed by $\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')\cap\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$. In particular, (4.5) holds over the compositum of number fields $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')\mathbb{Q}(\{\chi_1,\dots,\chi_{n-m-1}\})F^{Gal}$.

Now observe that by Lem. 1.11 $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are both critical for all $L(s,\Pi\otimes\chi_j)$, $1\le j\le n-m-1$. As we also assumed that $k,\ell>\frac{n}{2}$, our additional assumption on Π being obtained by base change from π hence allows us to use Guerberoff's theorem, [Gue16], Thm. 4.5.1, on non-central critical values of standard L-functions. (The careful reader may want to use §4.2 in [Gro-Har-Lin18] in combination with [KMSW14], Thm. 1.7.1, which confirms Guerberoff's Hypothesis 4.5.1 for our representation π.) Hence, simply by inserting into Guerberoff's formula, we obtain

$$\prod_{j=1}^{n-m-1}\frac{L^S(\frac{1}{2}+\ell,\Pi\otimes\chi_j)}{L^S(\frac{1}{2}+k,\Pi\otimes\chi_j)}\sim_{\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})}(2\pi i)^{d(n-m-1)n(\ell-k)},$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$. Here, $\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})$ denotes any number field over which π_f and all characters $\chi_{j,f}$ are defined (such a field exists, e.g., by [Gro-Seb17], Thm. A.2.4). Collecting the powers of $(2\pi i)$ we hence obtain

$$\frac{L^S(\frac{1}{2}+k,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')}\sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')\mathbb{Q}(\{\chi_1,\dots,\chi_{n-m-1}\})\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})F^{Gal}}(2\pi i)^{d(k-\ell)nm}.\tag{4.6}$$

As we have seen, relation (4.6) is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{Gal})$ and both sides are invariant under every $\sigma\in\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')$. Hence, applying Lem. 1.2 once more, we see that this relation actually holds over any field containing F^{Gal} and the field of rationality of $\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')$. In particular, (4.6) holds over the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}$, which shows the claim. □

Remark 4.7 (Further interpretations)**.** In [Jan19] Januszewski recently achieved a conditional rationality-result for Rankin-Selberg L-functions of type $n\times(n-1)$ with precise powers of $(2\pi i)$ as archimedean contributions, recovering the result of Harder–Raghuram, [GHar-Rag20] in the case of $n\times(n-1)$ as a consequence (under the given hypotheses). Hence, our Cor. 4.4 may also be seen as an unconditional generalization of a consequence of the main result of [Jan19] for more general pairs $n\times m$ and over CM-fields $F=\mathcal{K}F^+$.

Most recently, Raghuram has presented a different approach to our corollary in the special case of $m=1$ (i.e., the standard L-function) through automorphic induction, see Thm. 1 in [Rag19]

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References

[Art-Clo89] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. Math. Studies, **120** (Princeton University Press, 1989)

[Bel-Che09] J. Bellaïche, G. Chenevier, Families of Galois representations and Selmer groups, Astérisque **324** (2009)

[Bla86] D. Blasius, On the critical values of Hecke L-series, *Ann. of Math.* **124** (1986) 23–63

[Bor79] A. Borel, *Automorphic L-functions*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979), pp. 27-61

[Bor-Jac79] A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979), pp. 189-202

[Bor-Wal00] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies **94**, (Princeton Univ. Press, New Jersey, 2000)

[Clo91] L. Clozel, Représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$, *Publ. IHES* **73** (1991) 97–145

[Clo17] L. Clozel, *Solvable descent for cuspidal automorphic representations of $GL(n)'$*, talk at IAS Princeton, dated 10th November (2017)
https://video.ias.edu/MotivesGaloisRepsandCohomology/2017/1110-LaurentClozel

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales, With an appendix by N. Koblitz and A. Ogus* in: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979), pp. 313–346

[Eis17] E. E. Eischen, *Nonvanishing theorems for twisted L-functions on unitary groups*, arXiv:1708.04026 (2017)

[Enr79] T. J. Enright, Relative Lie algebra cohomology and unitary representations of complex Lie groups, *Duke Math. J.* **46** (1979) 513–525

[Gin-Jia-Ral04] D. Ginzburg, D. Jiang, S. Rallis, On the nonvanishing of the central value of the Rankin-Selberg L-functions, JAMS **17** (2004) 679–722

[Goo-Wal09] R. Goodman, N. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, **255** (Springer, Dordrecht, 2009)

[Gro18] H. Grobner, Rationality for isobaric automorphic representations: The CM-case, *Monatshefte Math.* **187** (2018) 79 – 94

[Gro20] H. Grobner, *Smooth automorphic forms and smooth automorphic representations*, book in preparation (2020)

[Gro-MHar16] H. Grobner, M. Harris, Whittaker periods, motivic periods, and special values of tensor product L-functions, *J. Inst. Math. Jussieu* **15** (2016) 711–769

[Gro-Har-Lin18] H. Grobner, M. Harris, J. Lin, *Deligne's conjecture for automorphic motives over CM-fields, Part I: Factorization*, preprint (2018)

[Gro-Lin19] H. Grobner, J. Lin, *Special values of L-functions and the refined Gan-Gross-Prasad conjecture*, preprint (2019)

[Gro-Rag14a] H. Grobner, A. Raghuram, On some arithmetic properties of automorphic forms of GL_m over a division algebra, *Int. J. Number Theory* **10** (2014) 963–1013

[Gro-Rag14b] H. Grobner, A. Raghuram, On the arithmetic of Shalika models and the critical values of L-functions for GL_{2n}, with an appendix by Wee Teck Gan, *Amer. J. Math.* **136** (2014) 675–728

[Gro-Seb17] H. Grobner, R. Sebastian, Period relations for cusp forms of GSp_4 and some more general aspects of rationality, teXed notes, University of Vienna, (Vienna, 2017), available online at http://homepage.univie.ac.at/harald.grobner/papers/OccultPeriods.pdf

[Gro-Žun20] H. Grobner, S. Žunar, *On the notion of smooth automorphic forms and smooth automorphic representations*, preprint (2020)

[Gue16] L. Guerberoff, Period relations for automorphic forms on unitary groups and critical values of L-functions, *Documenta Math.* **21** (2016) 1397–1458

[GHar83] G. Harder, *General aspects in the theory of modular symbols*, in: *Séminaire de Théorie des Nombres, Paris 1981-82*, ed M.-J. Bertin, Progress in Mathematics **38** (Boston, Basel, Stuttgart 1983), pp. 73-88

[GHar-Rag20] G. Harder, A. Raghuram, *Eisenstein cohomology for $GL(N)$ and ratios of critical values of Rankin-Selberg L-functions*, to appear in *Ann. Math. Studies* **203** (2020)

[MHar93] M. Harris, L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms, *JAMS* **6** (1993) 637–719

[MHar97] M. Harris, L-functions and periods of polarized regular motives, *J. reine angew. Math.* **483** (1997) 75–161

[MHar-Kud91] M. Harris, S. S. Kudla, The central critical value of a triple product L-functions, *Ann. Math.* **133** (1991) 605–672

[Hen12] G. Henniart, Induction automorphe globale pour les corps des nombres, *Bull. Soc. math. France* **140** (2012) 1–17

[Hid94] H. Hida, On the critical values of L-functions of GL(2) and GL(2) $\times$ GL(2), *Duke Math. J.* **74** (1994) 431–529

[Jan19] F. Januszewski, On Period Relations for Automorphic L-functions I, *Trans. AMS* **371** (2018) 6547–6580

[Jan16] F. Januszewski, *On Period Relations for Automorphic L-functions II*, preprint (2016)

[Jia-Zha20] D. Jiang, L. Zhang, On the Non-vanishing of the Central Value of Certain L-functions: Unitary Groups, *J. Eur. Math. Soc.* **22** (2020) 1759–1783

[KMSW14] T. Kaletha, A. Minguez, S. W. Shin, P.-J. White, *Endoscopic classification of representations: inner forms of unitary groups*, preprint (2014)

[Kna94] A. W. Knapp, *Local Langlands correspondence: the Archimedean case*, in: *Motives*, Proc. Sympos. Pure Math., Vol. 55, Part 2, AMS, Providence, R.I., (1994), pp. 393–410

[Kur78] P. F. Kurchanov, Cohomology of discrete groups and Dirichlet series connected with Jacquet-Langlands cusp forms, *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978) 588–601

[Kur79] P. F. Kurchanov, Dirichlet series of Jacquet-Langlands cusp forms over fields of CM-type, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979) 67–86

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, in: *On the Stabilization of the Trace Formula*, Vol. I, eds. L. Clozel, M. Harris, J.-P. Labesse, B.-C. Ngô, International Press, Boston, MA, 2011, pp. 429–470

[Lin15] J. Lin, *Special values of automorphic L-functions for $GL_n \times GL_{n'}$ over CM fields, factorization and functoriality of arithmetic automorphic periods*, thèse de docotorat, (2015)

[Rag16] A. Raghuram, Critical values of Rankin-Selberg L-functions for $GL(n) \times GL(n-1)$ and the symmetric cube L-functions for $GL(2)$, *Forum Math.* **28** (2016) 457–489

[Rag19] A. Raghuram, Special values of L-functions for GL_N over a CM field, to appear in *IMRN* (2020)

[Rag-Sha08] A. Raghuram, F. Shahidi, On certain period relations for cusp forms on GL_n, *Int. Math. Res. Not.* Vol. (2008), doi:10.1093/imrn/rnn077.

[Roh89] D. E. Rohrlich, Nonvanishing of L-functions for GL(2), *Invent. Math.* **97** (1989) 381–403

[Sac19] G. Sachdeva, Critical values of L-functions for $\mathrm{GL}_3 \times \mathrm{GL}_1$ and symmetric square L-functions for GL(2) over a CM field, *Journal of Number Theory* **211** (2019) 43–74

[Sha81] F. Shahidi, On certain L-functions, *Amer. J. Math.* **103** (1981) 297–355

[Shi14] S. W. Shin, On the cohomological base change for unitary similitude groups, appendix to: W. Goldring, Galois representations associated to holomorphic limits of discrete series I: Unitary Groups, Compos. Math. **150** (2014) 191–228

[Wal85b] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de syémtrie, *Comp. Math.* **54** (1985) 173–242

[Wei56] A. Weil, *On a certain type of characters of the idèle-class group of an algebraic number field.*, in: Proceeding of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, Science Council of Japan, Tokyo (1956), pp. 1–7