RELATIONS OF RATIONALITY FOR SPECIAL VALUES OF RANKIN–SELBERG L-FUNCTIONS OF $\mathrm{GL}_n \times \mathrm{GL}_m$ OVER CM-FIELDS

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Abstract. In this paper we present a bridge between automorphic forms of general reductive groups and motives over number fields, hinting a translation of Deligne's conjecture for motivic L-functions into a precise automorphic context. The bulk of this article provides evidence for such an automorphic translation by considering the case of Rankin–Selberg L-functions $L(s, \Pi \times \Pi')$ of $\mathrm{GL}_n \times \mathrm{GL}_m$ over CM-fields F. Our main results are of two different kinds: Firstly, for arbitrary integers $1 \leq m \leq n$, and suitable pairs (Π, Π') of cohomological automorphic representations, we relate critical values of $L(s, \Pi \times \Pi')$ with a product of Whittaker periods attached to Π and Π', Blasius's CM-periods of Hecke-characters and certain non-zero values of standard L-functions. These relations lead, on the other hand, consequently to quite broad generalizations of fundamental rationality-results of Waldspurger, Harder–Raghuram and others.

Introduction

What triggers the problem? Let F be a number field and let $\mathscr{M}$ be a Grothendieck motive over F with coefficients in a number field $E(\mathscr{M}) \subset \mathbb{C}$.[1] For $s = k \in \mathbb{Z}$ a critical point of $L(s, \mathscr{M})$, Deligne has envisioned a fundamental, conjectural description of $L(k, \mathscr{M})$ in terms of periods $c^{\pm}(\mathscr{M}) \in \mathbb{C}^{\times}$ and integral powers $(2\pi i)^{d(k)}$, generalizing Euler's classical result on the nature of $\zeta(k)$, $k > 0$ even,

$$L(k, \mathscr{M}) \sim_{E(\mathscr{M})} (2\pi i)^{d(k)} c^{(-1)^k}(\mathscr{M}).$$

This relation should be read as the left hand side equals the right hand side up to multiplication by an element in $E(\mathscr{M})$.

Except for particular cases, Deligne's conjecture is open ever since. However, invoking automorphic techniques, significant progress has been made whenever $\mathscr{M}$ corresponds to an automorphic representation. Given the importance of these developments it is a little surprising that the only precise statements on the conjectural dictionary between motives and automorphic forms in the vast literature are Clozel's conjectures for $\mathrm{GL}_n(\mathbb{A}_F)$, cf. [Clo90], Conj. 4.5 and [Clo14] Conj. (2.1). In order to close this gap, we extend Clozel's conjecture for GL_n to arbitrary connected reductive groups G and explain how irreducible motives $\mathscr{M}$ are (or should be) linked to cohomological cuspidal automorphic representations π of $G(\mathbb{A}_F)$ in the appendix of this paper. For short, our conjecture says that whenever r denotes a rational representation of the L-group of G, we claim that there exists a certain automorphic twist π^{θ} such that there is an equality of partial L-functions

$$L^S(s, \mathscr{M}) = L^S(s, \pi^{\theta}, r),$$

where S is any finite set of places of F containing the archimedean and ramified ones. We refer to Conjecture M for all details (in particular for the definition of the twist π^{θ}). As already pointed out, the advantage of such a conjecture relies in the possibility of giving an interpretation of Deligne's

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[1] We leave out the question whether or not a (tannakian) category of Grothendieck motives really exists. Regardless of the status of this deep problem, we are inclined to adapt Dedekind's point of view on numbers, [Ded88], to motives: We do not focus on what they are, but rather on what they *should be*.

conjecture in a purely automorphic context. If $s = k \in \mathbb{Z}$ denotes any critical integer for $L(s, \pi^\theta_\infty, r)$, then there should be periods $p(\pi) \in \mathbb{C}^\times$ such that

$$L^S(k, \pi^\theta, r) \sim_{E(\pi_f)} (2\pi i)^{d(k)} p(\pi).$$

In this relation the field $E(\pi_f)$ denotes a finite extension of the field of rationality $\mathbb{Q}(\pi_f)$ of π_f, cf. [Wal85a]. Following the arguments of [Gro-Rag14a, §8.1], $\mathbb{Q}(\pi_f)$ (and hence $E(\pi_f)$) is a number field.

What do we prove?

The basic problem (in its context). In our paper we take up the challenge to establish a translation of Deligne's conjecture for the major example of $G = \mathrm{GL}_n \times \mathrm{GL}_m$ over an arbitrary CM-field F. More precisely, we are interested in proving relations of rationality for the critical values of Ranking-Selberg L-functions of type $L(s, \Pi \times \Pi')$, Π and Π' cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, such that the rationality-relations so obtained resemble Deligne's prediction. In particular, the contributions of the archimedean components Π_∞ and Π'_∞ shall be expressed by precise powers of $(2\pi i)$, matching the ones conjectured by Deligne.

As compared to the extensive literature for the case $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ – for F a CM-field we refer in particular to [Kur78, Kur79], [GHar83], [Hid94], [Lin15], [Rag16], [Gro-MHar16], [Gro18], [Gro-Lin17] – the rank m of our second GL-factor GL_m being in principle *any integer* $1 \leq m \leq n$ (for suitable pairs (Π, Π')) is arguably one of the most striking features of this article. In this regard, the results of this paper should not only be seen as an extension of the series of results mentioned above, but also of the approach taken in [Gre03], [Lin15] and most recently by the second named author in her thesis [Gun17].

Main results and applications. The main outcome of the present article are two results (Results A & B below) and two applications of the latter (Application I & II). However, the two aforementioned results are very technical and moreover quite involved in their assumptions and assertions, so they carry what one would call *den Mühlstein der Uneleganz* (the "millstone of inelegancy") in german; on the other hand, the two aforementioned applications are much lighter statements, providing wide generalizations of important results of Waldspruger (Application I) and Harder–Raghuram (Application II), as well as of other people, but certainly these applications are just corollaries of our technical Results A & B. We frankly admit that this makes it hard for the authors to really say which pair of the two is more important. We leave it to the kind reader to decide which of our achievements (s)he prefers to view as our main theorems and simply go over to presenting Results A & B and Applications I & II below now.

Results (as opposed to applications). From now on F denotes an arbitrary CM-field with maximal totally real subfield F^+. The quadratic Hecke character associated with F/F^+ admits a unitary extension to $\mathbb{A}_F^\times$ which is denoted η. Then by construction $\eta \,\|\cdot\|^{-1/2}$ is algebraic. For an integer $n \geq 2$ we let Π be an (irreducible) subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_n(F)\mathbb{R}_+\backslash \mathrm{GL}_n(\mathbb{A}_F))$. As in §I above, we shall assume that Π is cohomological, i.e., there exists a finite-dimensional, irreducible algebraic representation $\mathcal{E}_\mu$ of the real Lie group $G_{n,\infty} := \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{R})$ such that Π_∞ has non-trivial relative Lie algebra cohomology with respect to $\mathcal{E}_\mu$. Here, μ stands for the highest weight of $\mathcal{E}_\mu$ (depending on a choice of a Borel subgroup $B_n \subset G_n$). Choosing coordinates one may indeed identify it with $\mu = (\mu_v)_{v \in S_\infty}$ where $\mu_v = (\mu_{v,1}, ..., \mu_{v,n}) \in \mathbb{Z}^n$ and $\mu_{v,1} \geq ... \geq \mu_{v,n}$.

As in the very results Results A & B, we first let $1 \leq m < n$. We define Π' analogously as an (irreducible), conjugate self-dual subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_m(F)\mathbb{R}_+\backslash \mathrm{GL}_m(\mathbb{A}_F))$ but with the property that $\Pi'^{\mathsf{alg}} := \Pi' \otimes \eta^e$ is cohomological. Here, $e \in \{0, 1\}$ and $e = 0$ if and only if $n \not\equiv m \mod 2$.

The reason for introducing the twist Π'^{alg}, i.e., for assuming different conditions on cohomology for n and m is explained by the following construction: In order to be able to use the main result of [Gro18] (which is the starting-point of our proof), we choose any conjugate self-dual Hecke characters $\chi_1,...,\chi_{n-m-1}$ of $\mathbb{A}_F^\times$ such that the isobaric sum $\Sigma := \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ is cohomological with respect to algebraic coefficients $\mathcal{E}_{\mu'}$; such a choice can be made if and only if Π'^{alg}, rather than Π' itself, is cohomological. With these assumptions non-zero *Whittaker periods* $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$, respectively *CM-periods* $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$, have been defined in [Rag-Sha08], [Gro18], respectively [Bla86]. By their very construction their product is well-defined up to multiplication by non-zero elements in a certain number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi'^{\mathsf{alg}})E^{\mathsf{cm}}$, where E^{cm} is an abbreviation for a number field, which depends on the chosen characters χ_i and contains a Galois closure $F^{Gal} \subset \bar{\mathbb{Q}}$ of the extension $F/\mathbb{Q}$. See §1.7 and in particular 2.4 below for details.

We are now in the position to state

Result A. *Assume that* $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ *is non-trivial and let* $s_0 = \frac{1}{2} + k$ *be any critical point of* $L(s, \Pi \times \Sigma)$. *If* $k \neq 0$, *then*

$$L^S(\tfrac{1}{2}+k, \Pi\times\Pi') \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\eta\ \|\cdot\|^{-1/2})E^{\mathsf{cm}}} (2\pi i)^{[F^+:\mathbb{Q}]((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\ p(\Pi)\ p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi'\chi_j^{-1})}{L^S(\frac{1}{2}+k, \Pi\chi_j)}. \tag{0.1}$$

If $k = 0$, *i.e., if* $s_0 = \frac{1}{2}$ *denotes the central critical point, then the same relation holds under certain conditions of regularity on* Π_∞ *and* Σ_∞ *as well as a global non-vanishing hypothesis, see Thm. 3.1. Moreover, if* n *is even and* m *is odd, then all* L*-values* $L^S(\frac{1}{2}+k, \Pi\times\Pi')$, $L^S(1, \Pi'\chi_j^{-1})$ *and* $L^S(\frac{1}{2}+k, \Pi\chi_j)$ *in* (0.1) *are critical and the relation holds over the smaller number field* $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')\mathbb{Q}(\{\chi_1, ..., \chi_{n-m-1}\})F^{Gal}$.

We remark that if $k \neq 0$, then the denominators $L^S(\frac{1}{2}+k, \Pi\chi_j)$ in (0.1) are non-zero, which is part of the global non-vanishing hypothesis for the central case $k = 0$ mentioned in Result A. If $m = n-1$, then Result A becomes Thm. 5.2 from [Gro-Lin17] for cuspidals, which refined the main result of [Rag16] over CM-fields by giving an explicit power of $(2\pi i)$ instead of an abstract archimedean period. It is worth noting that this power is precisely what is predicted by Deligne's conjecture, see [Gro-Lin17], Rem. 5.8 for a detailed exposition.

On the other extreme, if $m = 1$, i.e., if we look at the twisted standard L-function of Π, then we retrieve at once Thm. 3.9, Cor. 5.7 and Thm. 6.11 of [Gro-MHar16]. For general m our Result A should hence be viewed as a theorem relating special values of $L(s, \Pi\times\Pi')$ with periods and quotients of special values of (other) standard L-functions. We kindly refer to Thm. 3.1 for a proof.

Next assume that $\Pi'' := (\Pi_1 \boxplus \Pi_2)\|\det\|^{1/2}$, where each Π_t, $t = 1, 2$, is a cuspidal automorphic representation of $\mathrm{GL}_{m_t}(\mathbb{A}_F)$ satisfying the conditions imposed on Π' above. In particular, this entails a choice of conjugate self-dual Hecke characters $\chi_1^{(t)}$, ..., $\chi_{n-m_t-1}^{(t)}$, $t = 1, 2$, such that the isobaric automorphic sums $\Sigma_t = \Pi_t \boxplus \chi_1^{(t)} \boxplus ... \boxplus \chi_{n-m_t-1}^{(t)}$ are both cohomological with respect to some irreducible algebraic coefficients $\mathcal{E}_{\mu_t}$ of $G_{n-1,\infty}$. We abbreviate

$$\Omega(\Pi_t, \{\chi_i^{(t)}\}) := \prod_{1\leq i<j\leq n-m_t-1} p(\chi_i^{(t)}\chi_j^{(t),-1}, \Psi_{\chi_i^{(t)}\chi_j^{(t),-1}}) \prod_{i=1}^{n-m_t-1} L^S(1, \Pi_t\ \chi_i^{(t),-1}).$$

Now let $n = m_1 + m_2$, so Π'' is also an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Obviously, Π'' is not square-integrable, hence in particular not cuspidal. Our second result, proved as Thm. 3.6 below, reads as follows:

Result B. *Assume that* $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu_t}, \mathbb{C}]$ *is non-trivial for* $t = 1, 2$ *and let* $s_0 = \frac{1}{2} + k$ *be any common critical point of* $L(s, \Pi \times \Sigma_1)$ *and* $L(s, \Pi \times \Sigma_2)$. *Then* k *is critical for* $L(s, \Pi \times \Pi'')$ *and if* $k \neq 0$, *then*

$$L^S(k, \Pi \times \Pi'') \sim (2\pi i)^{[F:\mathbb{Q}](n-1)((k-\frac{1}{2})n-1)+d\frac{1}{2}((m_1-1)(m_1-2)+(m_2-1)(m_2-2))}\ p(\Pi)^2\ p(\Pi_1^{\mathsf{alg}})\ p(\Pi_2^{\mathsf{alg}})$$

$$\Omega(\Pi_1, \{\chi_i^{(1)}\})\ \Omega(\Pi_2, \{\chi_j^{(2)}\}) \prod_{i=1}^{m_2-1} L^S(\tfrac{1}{2}+k, \Pi\ \chi_i^{(1)})^{-1} \prod_{j=1}^{m_1-1} L^S(\tfrac{1}{2}+k, \Pi\ \chi_j^{(2)})^{-1}, \tag{0.2}$$

this relation being over the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_1)\mathbb{Q}(\Sigma_2)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E_1^{\mathsf{cm}}E_2^{\mathsf{cm}}$. *If* $k = 0$, *then the same relation holds under certain conditions of regularity on* Π_∞ *and* $\Sigma_{t,\infty}$ *as well as a global non-vanishing hypothesis, see Thm. 3.6. If both* m_1 *and* m_2 *are odd, then* (0.2) *holds over the smaller number field* $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi_1)\mathbb{Q}(\Pi_2)\mathbb{Q}(\{\chi_1^{(1)}, ..., \chi_{n-m_1-1}^{(1)}\})\mathbb{Q}(\{\chi_1^{(2)}, ..., \chi_{n-m_2-1}^{(2)}\})F^{Gal}$.

Applications (as opposed to results). Results A & B have the following two implications, which generalize important results of Waldspurger (see Application I) and Harder–Raghuram (see Application II). Indeed, in [Wal85b] Waldspurger has established a rationality result for the quotient $L(\frac{1}{2}, \pi \otimes \alpha)/L(\frac{1}{2}, \pi \otimes \beta)$ of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi \otimes \alpha$ and $\pi \otimes \beta$ of GL_2 over any number field at their joint critical value $s_0 = \frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty = \beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L(\frac{1}{2}, \pi \otimes \beta)$ is supposed to be non-zero. Under these assumptions, Waldspurger's rationality-relation is of the form

$$\frac{L(\frac{1}{2}, \pi \otimes \alpha)}{L(\frac{1}{2}, \pi \otimes \beta)} \sim_{\mathbb{Q}(\pi)} \frac{p(\alpha)}{p(\beta)},$$

the two period-invariants $p(\alpha)$, $p(\beta)$ only depending on α respectively β and the archimedean component of the cuspidal representations π. See [Wal85b], p. 174.

In this paper we generalize Waldspurger's result to the case of quotients of standard L-functions of GL_n/F where $n \geq 2$ is arbitrary, $s_0 = \frac{1}{2} + k$ a more general special value while F is any CM-field. More precisely, we let α and β be any conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ such that $\alpha_\infty = \beta_\infty$ and such that, writing $\alpha_v(z) = z^{a_v}\bar{z}^{-a_v}$ at $v \in S_\infty$, the following two conditions are satisfied: $a_v \in \frac{n}{2} + \mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{v,n}$. These two simple conditions ensure that there is always a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sum $\Sigma = \alpha \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-2}$ is cohomological and such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ is non-trivial. We obtain

Application I. *Choose any conjugate self-dual Hecke characters* χ_1, ..., χ_{n-2}, *such that the isobaric automorphic sum* $\Sigma = \alpha \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-2}$ *is cohomological and such that* $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ *is non-trivial. Let* $s_0 = \frac{1}{2} + k$ *be any critical point of* $L(s, \Pi \times \Sigma)$. *If* n *is even, then all the* $s_0 = \frac{1}{2} + k$ *are indeed critical for* $L(s, \Pi \otimes \alpha)$ *and* $L(s, \Pi \otimes \beta)$ *and*

$$\frac{L^S(\frac{1}{2}+k, \Pi \otimes \alpha)}{L^S(\frac{1}{2}+k, \Pi \otimes \beta)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\alpha)\mathbb{Q}(\beta)\mathbb{Q}(\{\chi_1,...,\chi_{n-2}\})F^{Gal}} \prod_{i=1}^{n-2} \frac{p(\alpha, \Psi_{\alpha\chi_i^{-1}})}{p(\beta, \Psi_{\beta\chi_i^{-1}})}.$$

If $k = 0$, *then have to assume certain conditions of regularity on* Π_∞ *and* Σ_∞ *as well as a global non-vanishing hypothesis, see Thm. 4.1.*

We point out that Application I should furthermore be viewed as a generalization as well as a certain refinement of a consequence of the main result of [Gro-Rag14b], Thm. 7.1.2, and [Jan16], Thm. 8.2, established there for totally real fields F^+ and achieved here for general CM-fields F.

Our final application deals with a generalization[2] of the main result of [GHar-Rag17]. There, Harder–Raghuram achieved a fine relation of rationality between the quotients of consecutive critical values of Ranking-Selberg L-functions over totally real fields F^+ and so-called relative periods denoted $\Omega^{\epsilon'}(\pi_f)$: Let π and π' be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, resp. $\mathrm{GL}_m(\mathbb{A}_{F^+})$, and let S be any finite set of non-archimedean places, where π or π' are ramified. If n is even and m is odd [GHar-Rag17], Thm. 7.40, shows that

$$\frac{L^S(\frac{1}{2}+k,\pi\otimes\pi'^{\mathsf{v}})}{L^S(\frac{3}{2}+k,\pi\otimes\pi'^{\mathsf{v}})} \sim_{\mathbb{Q}(\pi)\mathbb{Q}(\pi')} \Omega^{\epsilon'}(\pi_f).$$

In [Gro-Lin17] this result has recently been given a generalization and refinement for cohomological cusp forms of $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_{n-1}(\mathbb{A}_F)$, again F denoting any CM-field.

Here we take up the CM-case for general even n and odd m. We obtain

Application II. *Suppose that $1\leq m\leq n$ are integers, $n\geq 2$. We assume that Π is obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$.*
Case $m<n$: *In this case we assume in addition that n is even and m is odd. Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, satisfying the conditions of Result A. Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two critical points of $L(s,\Pi\times\Sigma)$ different from $s_0=\frac{1}{2}$. Then $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are indeed critical for $L(s,\Pi\times\Pi')$ and the ratio of critical values satisfies*

$$\frac{L^S(\frac{1}{2}+k,\Pi\times\Pi')}{L^S(\frac{1}{2}+\ell,\Pi\times\Pi')} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{[F^+:\mathbb{Q}](k-\ell)nm}.$$

Case $m=n$: *We assume in addition that $n=m_1+m_2$, with m_1 and m_2 both odd. Let $\Pi''=(\Pi_1\boxplus\Pi_2)\|\det\|^{1/2}$ be an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, satisfying the conditions of Result B.*
Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two joint critical points of $L(s,\Pi\times\Sigma_1)$ and $L(s,\Pi\times\Sigma_2)$ different from $s_0=\frac{1}{2}$. Then k and ℓ are critical for $L(s,\Pi\times\Pi'')$ and the ratio of critical values satisfies

$$\frac{L^S(k,\Pi\times\Pi'')}{L^S(\ell,\Pi\times\Pi'')} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi'')F^{Gal}} (2\pi i)^{[F^+:\mathbb{Q}](k-\ell)n^2}.$$

See Cor. 4.4 for a detailed statement and a proof. Here we only remark that the appearance of base change is due to the fact that our proof uses the results of [Gue16], which in turn proved a conjecture of Lin [Lin15]. As this already indicates, our Application II is hence a decent generalization of a consequence of Thm. 10.8.1 from [Lin15]. In this regard, the inclusion of a statement for the case $m=n$ is its most innovative part.

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[2]In order to avoid any potential misunderstanding, let us point out that by the very nature of the approach taken in [GHar-Rag17] and our approach taken here, our Application II obviously does not recover Thm. 7.40 from [GHar-Rag17] but is complementary to it.

1. Notation

1.1. **Number fields.** Let F be a CM-field, $2d = [F : \mathbb{Q}]$ and let F^+ be its maximal totally real subfield. Abusing notation we identify the set of archimedean places S_∞ of F and F^+ (by implicitly fixing a CM-type of F). The normalized absolute value on the ring of adeles $\mathbb{A}_F$ is denoted $\|\cdot\|$. We extend the quadratic Hecke character $\varepsilon : (F^+)^\times\backslash\mathbb{A}^\times_{F^+} \to \mathbb{C}^\times$, associated to F/F^+ via class field theory, to a conjugate self-dual unitary Hecke character $\eta : F^\times\backslash\mathbb{A}^\times_F \to \mathbb{C}^\times$. At $v \in S_\infty$ we have $\eta_v(z) = z^t\bar{z}^{-t}$, for $z \in F_v$, where $t = t_v \in \frac{1}{2} + \mathbb{Z}$. We assume from now on that $t = 0$, i.e., $\eta_v(z) = z^{1/2}\bar{z}^{-1/2}$. We may define a non-unitary algebraic Hecke character $\phi : F^\times\backslash\mathbb{A}^\times_F \to \mathbb{C}^\times$, by $\phi := \eta\,\|\cdot\|^{-1/2}$.

1.2. **Algebraic groups and real Lie groups.** Let $n \geq 1$ be an integer. We will abbreviate $G_n := \mathrm{GL}_n/F$. Let $R_{F/\mathbb{Q}}$ be Weil's restriction of scalars. We write $G_{n,\infty} := R_{F/\mathbb{Q}}(G_n)(\mathbb{R})$ for the real Lie group being the archimedean factor of $G_n(\mathbb{A}_F)$ and let K_∞ be the product of the center $Z_{n,\infty}$ of $G_{n,\infty}$ and a maximal compact subgroup. By $\mathfrak{g}_{n,\infty}$ we denote the real Lie algebra of $G_{n,\infty}$.

1.3. **Cohomological automorphic representations.** Let $1 \leq m < n$ be any integers. Throughout the paper, we will let Π be a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$ and let Π' be a unitary cuspidal automorphic representation of $G_m(\mathbb{A}_F) = \mathrm{GL}_m(\mathbb{A}_F)$, in the sense of [Bor-Jac79, §4.6]. Generally, for convenience, we will not distinguish between a cuspidal automorphic representation, its smooth limit-Fréchet-space completion of moderate growth or its (non-smooth) Hilbert space completion in the L^2-spectrum. We will now specify our assumptions on their archimedean components Π_∞ and Π'_∞.

1.3.1. *The representation* Π_∞. Unless otherwise stated, throughout the paper we always assume that Π_∞ is *cohomological*, i.e., there is an irreducible finite-dimensional algebraic representation $\mathcal{E}_\mu$ of $G_{n,\infty}$, with respect to which Π_∞ has non-trivial $(\mathfrak{g}_{n,\infty}, K_{n,\infty})$-cohomology. More precisely, as Π_∞ is assumed to be unitary, $\mathcal{E}_\mu$ is conjugate self-dual and hence breaks as $\mathcal{E}_\mu = E_\mu \otimes E_{\mu^\vee}$, where $E_\mu = \otimes_{v\in S_\infty} E_{\mu_v}$ and we view each irreducible $\mathrm{GL}_n(F_v) = \mathrm{GL}_n(\mathbb{C})$-factor E_{μ_v} as being given by its highest weight μ_v. In terms of the standard choice of a maximal split torus in GL_n, positivity on the attached set of roots and standard coordinates, this highest weight is an n-tuple of integers $\mu_v = (\mu_{v,1}, ..., \mu_{v,n}) \in \mathbb{Z}^n$ with $\mu_{v,1} \geq ... \geq \mu_{v,n}$. Let

$$\rho_n = \left(\tfrac{n-1}{2}, \tfrac{n-3}{2}, \cdots, -\tfrac{n-3}{2}, -\tfrac{n-1}{2}\right),$$

be the half-sum of positive roots of GL_n with respect to the same conventions. Using [Gro18, Section 1.2.4], we see that the condition that Π_∞ is cohomological with respect to $\mathcal{E}_\mu$ is equivalent to the much more explicit condition that

$$\Pi_v \cong \mathrm{Ind}_{B_n(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})}[z_1^{\ell_{v,1}}\bar{z}_1^{-\ell_{v,1}} \otimes \cdots \otimes z_n^{\ell_{v,n}}\bar{z}_n^{-\ell_{v,n}}]$$

with

$$\ell_{v,i} = -\mu_{v,n-i+1} + \rho_{n,i} \tag{1.1}$$

at each $v \in S_\infty$. Here, B_n is the standard Borel subgroup of G_n (determined by our choice of positivity on the set of roots) and induction is normalized to maintain unitarity.

We recall that a set of n real numbers $\{l_{v,i}\}_{1\leq i\leq n}$ is called an *infinity type at* $v \in S_\infty$, if

$$l_{v,1} > l_{v,2} > ... > l_{v,n}$$

i.e., if its members form a strictly decreasing string. As it is obvious from (1.1), $\{\ell_{v,i}\}_{1\leq i\leq n}$ from above is such a set, called the infinity type of Π at $v \in S_\infty$. Recalling the well-known classification of irreducible unitary cohomological representations of $G_n(\mathbb{C})$ from [Enr79] (see also [Gro-Rag14a], §5.5, for a presentation tailor-made for our purposes here), the following lemma is obvious:

Lemma 1.4. *There is a bijection, defined by* (1.1)*, between the equivalence classes of irreducible unitary cohomological tempered representations of* $G_n(\mathbb{C})$ *and the infinity types* $\{l_{v,i}\}_{1\leq i\leq n}$*, for which* $l_{v,i}\in\frac{n+1}{2}+\mathbb{Z}$ *for all* $1\leq i\leq n$

As a last ingredient we remark that a highest weight μ as above is called *sufficiently regular*, if $\mu_{v,i}-\mu_{v,i+1}\geq 2$ for all $v\in S_\infty$ and $1\leq i\leq n-1$.

1.4.1. *The representation* Π'_∞. Similar to our assumptions on Π we will suppose that the twisted representation $\Pi'_\infty\|\det\|^{(n-m-1)/2}$ is cohomological, or, equivalently, that

$$\Pi'^{\mathsf{alg}}:=\begin{cases}\Pi' & \text{if } n-1\equiv m \mod 2,\\ \Pi'\otimes\eta & \text{otherwise.}\end{cases}\tag{1.2}$$

is cohomological. In terms of infinity types, this means that for each $v\in S_\infty$ and $1\leq i\leq m$, there are $a_{v,i}\in\frac{n}{2}+\mathbb{Z}$ with $a_{v,i}>a_{v,i+1}$, such that

$$\Pi'_v\cong\mathrm{Ind}_{B_m(\mathbb{C})}^{\mathrm{GL}_m(\mathbb{C})}[z_1^{a_{v,1}}\bar{z}_1^{-a_{v,1}}\otimes\cdots\otimes z_m^{a_{v,m}}\bar{z}_m^{-a_{v,m}}].$$

1.4.2. *An auxiliary representation* Σ *in piano-position.* We extend the infinity type $\{a_{v,i}\}_{1\leq i\leq m}$ of Π'_v, at each place $v\in S_\infty$ to an infinity type of length $n-1$, simply by choosing any $b_{j,v}\in\frac{n}{2}+\mathbb{Z}$ for $1\leq j\leq n-m-1$, such that

$$\{a_{v,i}\}_{1\leq i\leq m}\cap\{b_{j,v}\}_{1\leq j\leq n-m-1}=\emptyset.\tag{1.3}$$

Denote this new infinity type by $\{\ell'_{v,i}\}_{1\leq i\leq n-1}$. As by construction $\ell'_{v,i}\in\frac{n}{2}+\mathbb{Z}$ for all $1\leq i\leq n-1$, this is the infinity type of a unique cohomological irreducible unitary tempered representation of $G_{n-1}(\mathbb{C})$ by Lemma 1.4.

Turning back to global representations, let χ_1, ..., χ_{n-m-1} be unitary Hecke characters with $\chi_{j,v}(z)=z^{b_{j,v}}\bar{z}^{-b_{j,v}}$ for all $v\in S_\infty$, i.e., such that $\chi_j\|\cdot\|^{n/2}$ – or, specifying $m=1$ in (2.3) that χ_j^{alg} – is algebraic. By its very construction the isobaric automorphic sum

$$\Sigma:=\Pi'\boxplus\chi_1\boxplus...\boxplus\chi_{n-m-1}$$

has our infinity type $\{\ell'_{v,i}\}_{1\leq i\leq n-1}$ from above and is therefore cohomological. Let $\mathcal{E}_{\mu'}$ be the unique irreducible algebraic coefficients module of $G_{n-1,\infty}$ with respect to which Σ_∞ has non-trivial $(\mathfrak{g}_{n-1,\infty},K_{n-1,\infty})$-cohomology. By the same reasoning as above, $\mathcal{E}_{\mu'}=E_{\mu'}\otimes E^\vee_{\mu'}$ and writing $\mu'_v=(\mu'_{v,1},...,\mu'_{v,n-1})\in\mathbb{Z}^{n-1}$ at $v\in S_\infty$ one has $\mu'_{v,1}\geq...\geq\mu'_{v,n-1}$ and

$$\ell'_{v,i}=-\mu'_{v,n-i}+\rho_{n-1,i}\tag{1.4}$$

From now on we make the following standing assumption: Π_∞ and Σ_∞ are *piano*, i.e.,

$$\mu_{v,1}\geq-\mu'_{v,n-1}\geq\mu_{v,2}\geq-\mu'_{v,n-2}\geq...\geq-\mu'_{v,1}\geq\mu_{v,n}.\tag{1.5}$$

Equivalently, $\mathrm{Hom}_{G_{n-1,\infty}}(E_\mu\otimes E_{\mu'},\mathbb{C})$ is non-zero (and hence one-dimensional). According to out previous definition, we call μ' *sufficiently regular*, if $\mu'_{v,i}-\mu'_{v,i+1}\geq 2$ for all $v\in S_\infty$ and $1\leq i\leq n-2$.

1.5. **Critical points of L-functions.** For a moment let $N,M\geq 1$ be any integers and let π be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_F)\times\mathrm{GL}_M(\mathbb{A}_F)$ for which a completed standard L-function $L(s,\pi)=\prod_v L(s,\pi_v)$ is defined satisfying a global functional equation $L(s,\pi)=\varepsilon(s,\pi)\cdot L(1-s,\pi^\vee)$, cf. [Bor1, §IV]. The following definition is modelled after [Del79], Prop.-Def. 2.3 and our Conjecture M, cf. App. A.

Definition 1.6. A complex number $s_0\in\frac{N-M}{2}+\mathbb{Z}$ is called *critical* for $L(s,\pi)$ if both $L(s,\pi_\infty)$ and $L(1-s,\pi^\vee_\infty)$ are holomorphic at $s=s_0$. We write $\mathrm{Crit}(\pi)$ for the set of critical points of $L(s,\pi)$.

We proceed with the following simple observation

Observation 1.7. Recalling that $\Gamma(s)$ does not vanish, the set of holomorphic points of $L(s,\pi_\infty)$ coincides with the intersection of the sets of holomorphic points of the archimedean L-functions of the Langlands datum of π_∞, cf. [Kna94], §4.

As a consequence we obtain the following lemma, which relates the critical points of $L(s,\Pi\times\Sigma)$ to the critical points of the isobaric summands of Σ.

Lemma 1.6. *The following hold*

(i) *If* $n\not\equiv m \mod 2$, *then* $\mathrm{Crit}(\Pi\times\Sigma)\subseteq\mathrm{Crit}(\Pi\times\Pi')$.

(ii) *If n is even and m is odd, then* $\mathrm{Crit}(\Pi\times\Sigma)=\mathrm{Crit}(\Pi\times\Pi')\cap\bigcap_{j=1}^{n-m-1}\mathrm{Crit}(\Pi\chi_j)$.

(iii) *If m is odd, then* $s_0=1\in\mathrm{Crit}(\Pi'\chi_j^{-1})$ *for all* $1\leq j\leq n-m-1$. *In any case* $s_0=\frac{1}{2}\in\mathrm{Crit}(\Pi\times\Sigma)$.

Proof. After our Observation 1.7 only (iii) needs a short argument. Writing down $L(s,\Pi'_\infty\chi_{j,\infty}^{-1})$ and $L(1-s,\Pi'^{\vee}_\infty\chi_{j,\infty})$, cf. [Kna94], §4, we see that the behaviour of holomorphy of these two L-factors is the same as the one of

$$\prod_{v\in S_\infty}\prod_{i=1}^{m}\Gamma(s+|a_{i,v}-b_{v,j}|)\qquad\text{and}\qquad\prod_{v\in S_\infty}\prod_{i=1}^{m}\Gamma(1-s+|-a_{i,v}+b_{v,j}|).$$

By (1.3), $a_{i,v}-b_{v,j}\neq 0$ for all $v\in S_\infty$, $1\leq i\leq m$ and $1\leq j\leq n-m-1$. Whence, $|a_{i,v}-b_{v,j}|=|-a_{i,v}+b_{v,j}|\geq 1$, and so all the above Γ-factors are holomorphic at $s_0=1$. Hence, $s_0=1\in\mathrm{Crit}(\Pi'\chi_j^{-1})$, if m is odd. The last assertion finally follows from the piano-hypothesis (1.5) and [Gro18], §1.6.1.(4) or [Rag16], Thm. 2.21. □

1.7. **Whittaker periods, σ-twists and fields of rationality.** Let Σ be as above. Unitarity of all isobaric summands implies that

$$\Sigma\cong\mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_{n-1}(\mathbb{A}_F)}[\Pi'\otimes\chi_1\otimes\ldots\otimes\chi_{n-1-m}]$$

is fully induced from the standard parabolic subgroup $P\subseteq\mathrm{GL}_{n-1}$ with Levi component $L_P\cong\mathrm{GL}_m\times\prod_{j=1}^{n-m-1}\mathrm{GL}_1$. Hence, a *Whittaker period* $p(\Sigma)\in\mathbb{C}^\times$ has been constructed in [Gro18], §1.8. It recovers the original construction of Raghuram–Shahidi for cuspidal representations. See also [Gro-Lin17], Prop. 1.8 and Cor. 1.13 for details of this construction and an even more general approach. Hence, also $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$ and $p(\chi_j^{\mathsf{alg}})$ are all defined.

We remark that it is intrinsic to the construction of these Whittaker periods, that they are uniquely defined only up to multiplication by non-zero numbers in the respective *field of rationality*, i.e., if ν is any of the above representations, then $p(\nu)$ may be replace by $q\cdot p(\nu)$ for any $q\in\mathbb{Q}(\nu)^\times:=\mathbb{C}^{\mathfrak{S}(\nu)}$ where $\mathfrak{S}(\nu):=\{\sigma\in\mathrm{Aut}(\mathbb{C})|\nu_f\cong{}^\sigma\nu_f\}$ and ${}^\sigma\nu_f:=\nu_f\otimes_{\sigma^{-1}}\mathbb{C}$. For cohomological automorphic representations ν as above, the rationality fields $\mathbb{Q}(\nu)$ are number fields and ${}^\sigma\nu_f$ is the finite component of a uniquely determined, cohomological automorphic representation – denoted ${}^\sigma\nu$ – justifying the notation $\mathbb{Q}(\nu)$. We define $E(\nu)$ as in [Gro-Lin17] §1.4.2 to be the union of $\mathbb{Q}(\nu)$ and F^{Gal}, a fixed Galois closure of $F/\mathbb{Q}$ in $\bar{\mathbb{Q}}$.

As a last ingredient, for each critical point $s_0=\frac{1}{2}+k$ of $L(s,\Pi\times\Sigma)$, an *archimedean period* $p(k,\Pi_\infty,\Sigma_\infty)\in\mathbb{C}^\times$ has been defined in [Gro18] 1.6.1.(6) as the weighted sum of archimedean zeta-integrals. We do not repeat its precise definition here and rather refer to [Gro18], because $p(k,\Pi_\infty,\Sigma_\infty)$ will not show up in the final results of this paper, but plays the role of an auxiliary quantity on the way there.

2. Revisiting four results on period-relations

2.1. **Our starting point: The main result of** [Gro18]**.** After having done all necessary preliminary work in §1 we may now recall the following algebraicity result for the critical points of the L-function attached to a pair (Π, Σ):

Proposition 2.1 ([Gro18], Thm. 1.8)**.** *Let Π and Σ be cohomological automorphic representations as in §1.3. In particular, Π is a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F)$, $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ the isobaric sum of a unitary cuspidal automorphic representation Π' of $G_m(\mathbb{A}_F)$ and unitary Hecke characters χ_j, such that Π_∞ and Σ_∞ are piano. Then, for every critical point $s_0 = \frac{1}{2} + k$ of $L(s, \Pi \times \Sigma)$,*

$$L^S(\tfrac{1}{2} + k, \Pi \times \Sigma) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)} p(k, \Pi_\infty, \Sigma_\infty)\ p(\Pi)\ p(\Sigma) \tag{2.2}$$

This result is the starting point of our considerations. In the next three subsection we collect three additional results from the theory of special values – one of them achieved by Blasius in [Bla86], whereas the other two have only been quite recently established in joint work of the first-named author in [Gro-Lin17]. These three results shall then be used afterwards in order to rewrite the above statement (2.2) in a much more refined way, being the key-step in the proof of our first main result.

2.2. **Step I: The archimedean period as a power of $2\pi i$.** Under certain conditions, J. Lin and first-named author have computed the archimedean factor $p(k, \Pi_\infty, \Sigma_\infty)$ from (2.2) as a power of $(2\pi i)$, see [Gro-Lin17], Cor. 4.30.

In order to recall this result also for the archimedean period $p(0, \Pi_\infty, \Sigma_\infty)$ attached to the central critical point $s_0 = \frac{1}{2}$ of $L(s, \Pi \times \Sigma)$, cf. Lemma 1.6, consider two cyclic extensions L and L' of F, of degree n resp. $n-1$, which are still CM-fields. For an algebraic Hecke character χ of $\mathbb{A}_L^\times$ (resp. χ' of $\mathbb{A}_{L'}^\times$) let $\Pi(\chi)$ (resp. $\Pi(\chi')$) be the automorphic induction from χ to $G_n(\mathbb{A}_F)$ (resp. χ' to $G_{n-1}(\mathbb{A}_F)$), cf. [Art-Clo89], Chp. 3, Thm. 6.2 (as completed in [Hen12], Thm. 3 (see also [Clo17])). We denote

$$\Pi_\chi := \begin{cases} \Pi(\chi) & \text{if } n \text{ is odd,} \\ \Pi(\chi) \otimes \eta & \text{if } n \text{ is even.} \end{cases} \quad \text{and} \quad \Pi_{\chi'} := \begin{cases} \Pi(\chi') & \text{if } n \text{ is even,} \\ \Pi(\chi') \otimes \eta & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

It is argued in [Gro-Lin17], §4.5, that, given Π and Σ as in §1.3, one may always choose conjugate self-dual algebraic Hecke characters χ and χ' such that Π_χ and $\Pi_{\chi'}$ are cuspidal automorphic representations for which $\Pi_{\chi,\infty} \cong \Pi_\infty$ and $\Pi_{\chi',\infty} \cong \Sigma_\infty$. Whenever we use the symbols Π_χ and $\Pi_{\chi'}$ it is from now on silently assumed that such a choice has been made.

Our definition now allow us to state

Proposition 2.4 ([Gro-Lin17], Cor. 4.30)**.** *Let Π and Σ be cohomological automorphic representation as in §1.3 and let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma)$. Only if $k = 0$, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point of $L(s, \Pi \times \Sigma)$, we additionally assume that μ and μ' are both sufficiently regular and that there exists a choice of χ, χ' such that $L^S(\frac{1}{2}, {}^\sigma\Pi_\chi \times {}^\sigma\Pi_{\chi'}) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. With these assumptions*

$$p(k, \Pi_\infty, \Sigma_\infty) \sim_{E(\Pi)E(\Sigma)} (2\pi i)^{d(n-1)((k-\frac{1}{2})n+1))}. \tag{2.5}$$

2.3. **Step II: Breaking the period of Σ.** As the next ingredient for rewriting (2.2), we will decompose the Whittaker period $p(\Sigma)$ in terms of the isobaric summands of Σ. The following is a special case of another result of the first-named author's recent work with J. Lin:

Proposition 2.6. *Let $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ be a cohomological isobaric automorphic representation as in §1.3. Assume in addition that all summands Π' and χ_j are conjugate self-dual. Then*

$$p(\Sigma) \sim_{E(\Sigma)E(\phi)} p(\Pi'^{\mathsf{alg}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi'\chi_j^{-1}) \prod_{1\le i<j\le n-m-1} L^S(1, \chi_i\chi_j^{-1}). \tag{2.7}$$

Proof. By their definition, Whittaker periods of algebraic Hecke characters may be chosen to be in $\mathbb{Q}^\times$, so at the cost of readjusting $p(\chi_j^{\mathsf{alg}})$ appropriately, we obtain $\prod_{j=1}^{n-m-1} p(\chi_j^{\mathsf{alg}}) \in \mathbb{Q}^\times$ and hence the formula is shown in [Gro-Lin17], Cor. 2.12. □

2.4. **Step III: Relating $L^S(1, \chi_i\chi_j^{-1})$ to CM-periods.** The last necessary ingredient for rewriting (2.2) has been established by Blasius, cf. [Bla86]. He described the critical L-values $L^S(1, \chi_i\chi_j^{-1})$ showing up in the formula (2.7) in terms of *CM-periods* $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$.

2.4.1. *Review of CM-periods.* The reader familiar with Blasius's construction may skip this small subsection and proceed directly to Prop. 2.8 below.
As a first observation, for any pair (i, j) with $1 \le i < j \le n-m-1$, the Hecke character $\xi := \chi_i\chi_j^{-1}$ is critical in the sense of Deligne, [Del79]: That means that ξ is algebraic and has non-trivial archimedean components ξ_v for all $v \in S_\infty$. Clearly, the latter assertion follows from our definition of $\{b_{v,j}\}_{1\le j\le n-m-1}$ being an infinity type for all $v \in S_\infty$, i.e., a set of strictly decreasing real numbers. Hence, one may define another CM-type Ψ_ξ of F by the rule

$$\Psi_\xi := \{v \in S_\infty | b_{v,i} < b_{v,j}\} \cup \overline{\{v \in S_\infty | b_{v,i} > b_{v,j}\}}.$$

Let now Ψ_F be any CM-type of F. Attached to (ξ, Ψ_F) one may define a CM Shimura-datum as in [MHar93], Sect. 1.1 and a number field $E(\xi, \Psi_F)$, which contains $\mathbb{Q}(\xi)$ and the reflex field of the CM Shimura datum defined by Ψ_F. In particular, if $\Psi_F = \Psi_\xi$, one may associate a non zero complex number $p(\xi, \Psi_\xi)$ to this datum, as explained in the appendix of [MHar-Kud91]. This number $p(\xi, \Psi_\xi)$ is well-defined modulo $E(\xi, \Psi_\xi)^\times$ and called the CM-period attached to ξ. Resuming the notation $\chi_i\chi_j^{-1}$, let us abbreviate

$$E^{\mathsf{cm}} := E^{\mathsf{cm}}(\chi_1, ..., \chi_{n-m-1}) := \prod_{1\le i<j\le n-m-1} \prod_{\Psi_F} E(\chi_i\chi_j^{-1}, \Psi_F).$$

This field E^{cm} is a number field by construction, which contains the finite composition of number fields $\prod_{1\le i<j\le n-m-1} E(\chi_i\chi_j^{-1})$, as defined in §1.7, but may be bigger than that. If it is clear from the context, we suppress its dependency on the choice of the characters χ_j.

The following result is proved in [Bla86]. We also refer to [MHar93], Prop. 1.8.1 (and the attached erratum [MHar97], p. 82) or Thm 4.7 of [Gro-Lin17] for a slightly more tailor-made presentation.

Proposition 2.8. *Let $\Sigma = \Pi' \boxplus \chi_i \boxplus ... \boxplus \chi_{n-m-1}$ be a cohomological isobaric automorphic representation as in §1.3. Assume in addition that all characters χ_j are conjugate self-dual. Then*

$$\prod_{1\le i<j\le n-m-1} L^S(1, \chi_i\chi_j^{-1}) \sim_{E^{\mathsf{cm}}} (2\pi i)^{d\frac{1}{2}(n-m-1)(n-m-2)} \prod_{1\le i<j\le n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}). \tag{2.9}$$

3. Two key-theorems

3.1. **Special values for $\mathrm{GL}_n \times \mathrm{GL}_m, 1 \le m < n$.** We are now ready to prove our first main result. To this end, recall that $1 \le m < n$ has been any pair of integers and that Π and Π' have been cohomological unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively, the latter assumed to be conjugate self-dual in §2. Our first main result will relate special values of the partial Rankin-Selberg L-function $L(s, \Pi \times \Pi')$ (all of them indeed critical, if n and m are of different parity), to quantities only depending on Π, Π' and a suitable choice of auxiliary characters χ_j (as in §1.4.2).

Rendering this more precise, recall the Whittaker periods $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$ attached to Π and Π'^{alg} and the CM-periods $p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$ attached to a choice of auxiliary characters χ_j from §1.7 and §2.4. Recall that when n is even and m is odd, then $\Pi'^{\mathsf{alg}} = \Pi'$ and $\chi_j^{\mathsf{alg}} = \chi_j$ for all $1 \leq j \leq n-m-1$. In this case we abbreviate

$$E(\{\chi_1, ..., \chi_{n-m-1}\}) := F^{Gal} \cdot \mathbb{C}^{\mathfrak{S}(\{\chi_1,...,\chi_{n-m-1}\})},$$

where $\mathfrak{S}(\{\chi_1, ..., \chi_{n-m-1}\})$ denotes the group of all $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that

$$\{{}^\sigma\chi_1, ..., {}^\sigma\chi_{n-m-1}\} = \{\chi_1, ..., \chi_{n-m-1}\}.$$

Obviously, $E(\{\chi_1, ..., \chi_{n-m-1}\}) \subseteq \prod_{j=1}^{n-m-1} E(\chi_j)$ by definition, hence is a number field. Here is our first key-result:

Theorem 3.1. *We let F be any CM-field and let $1 \leq m < n$ be any integers. Let Π be a cohomological unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. Choose any conjugate self-dual Hecke characters χ_1, ..., χ_{n-m-1}, such that the isobaric automorphic sum $\Sigma = \Pi' \boxplus \chi_1 \boxplus ... \boxplus \chi_{n-m-1}$ is cohomological and assume that $(\Pi_\infty, \Sigma_\infty)$ satisfies the piano-hypothesis, (1.5). Let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma)$ be any critical point of $L(s, \Pi \times \Sigma)$.*
In the special case when $k = 0$ only, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point, we additionally assume that the coefficient modules of Π_∞ and Σ_∞ are both sufficiently regular, cf. §1.3.1 and §1.4.2, and that there exists a choice of Hecke characters χ, χ' such that $L^S(\frac{1}{2}, {}^\sigma\Pi_\chi \times {}^\sigma\Pi_{\chi'}) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, cf. §2.2, as well as that $L^S(\frac{1}{2}, \Pi\chi_j) \neq 0$ for all $1 \leq j \leq n-m-1$.
Then

$$L^S(\tfrac{1}{2} + k, \Pi \times \Pi') \sim_{E(\Pi)E(\Sigma)E(\phi)E^{\mathsf{cm}}} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1 \leq i < j \leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi'\chi_j^{-1})}{L^S(\frac{1}{2} + k, \Pi\chi_j)}. \tag{3.2}$$

If n is even and m is odd, then all L-values $L^S(\frac{1}{2}+k, \Pi \times \Pi')$, $L^S(1, \Pi'\chi_j^{-1})$ and $L^S(\frac{1}{2}+k, \Pi\chi_j)$ are critical and the same relation holds over the smaller number field $E(\Pi)E(\Pi')E(\{\chi_1, ..., \chi_{n-m-1}\})$.

Proof. Putting our Steps I – III, i.e., equations (2.2), (2.5), (2.7), and (2.9), together and observing the non-vanishing of $L^S(1, \Pi'\chi_j^{-1})$, cf. [Sha81], Thm. 5.1, we obtain

$$L^S(\tfrac{1}{2} + k, \Pi \times \Sigma) \sim_{E(\Pi)E(\Sigma)E(\phi)E^{\mathsf{cm}}} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1 \leq i < j \leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi'\chi_j^{-1}).$$

As $L^S(\frac{1}{2} + k, \Pi \times \Sigma) = L^S(\frac{1}{2} + k, \Pi \times \Pi') \cdot \prod_{j=1}^{n-m-1} L^S(\frac{1}{2} + k, \Pi\chi_j)$ this yields,

$$L^S(\tfrac{1}{2} + k, \Pi \times \Pi') \sim_{E(\Pi)E(\Sigma)E(\phi)E^{\mathsf{cm}}} (2\pi i)^{d((n-1)((k-\frac{1}{2})n-1))+\frac{1}{2}(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathsf{alg}})$$
$$\cdot \prod_{1 \leq i < j \leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi'\chi_j^{-1})}{L^S(\frac{1}{2} + k, \Pi\chi_j)}, \tag{3.3}$$

implying the first assertion. Hence, assume now that n is even and m is odd. Criticality of $L^S(\frac{1}{2}+k, \Pi\times\Pi')$, $L^S(1,\Pi'\chi_j^{-1})$ and $L^S(\frac{1}{2}+k,\Pi\chi_j)$ is implied by Lem. 1.6, so the theorem finally follows once we can show that this relation holds over $E(\{\Pi,\Pi',\chi_1,...,\chi_{n-m-1}\})$. To this end, observe that the right hand side of (3.2) only depends on $\{\Pi,\Pi',\chi_1,...,\chi_{n-m-1}\}$, not on the individual members: This follows from the fact that the characters χ_j are all conjugate self-dual by assumption, implying

$$L^S(1,\chi_i^{-1}\chi_j) = L^S(1,\overline{\chi_i\chi_j^{-1}}) = L^S(1,\chi_i\chi_j^{-1}),$$

whence we may replace $p(\chi_i\chi_j^{-1},\Psi_{\chi_i\chi_j^{-1}})$ by $p(\chi_i^{-1}\chi_j,\Psi_{\chi_i^{-1}\chi_j})$ without changing the relation. As a consequence, both sides of (3.2) are invariant under all $\sigma\in\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')\cap\mathfrak{S}(\{\chi_1,...,\chi_{n-m-1}\})$ and the theorem follows from [Gro-Lin17], Lem. 1.19. $\square$

Remark 3.4. In the proof of Thm. 3.1 we have used [Gro-Lin17], Lem. 1.19, called the "minimizing lemma". This is a simple, but very useful result, which allows us to reduce any given relation of rationality $a\sim b$ appearing in this paper to any subfield of $\mathbb{C}$ containing F^{Gal} and the fixed field of all $\sigma\in$Aut($\mathbb{C}$), leaving both sides of the relation invariant. Here, invariance is meant in the following way: We view a and b as the natural complex-valued functions Aut($\mathbb{C}$) $\to\mathbb{C}$, defined by them and we say that $\sigma\in$Aut($\mathbb{C}$) leaves a and b invariant, if $a(\sigma)=a(id)$ and $b(\sigma)=b(id)$.

3.2. **Special values for** $\mathrm{GL}_n\times\mathrm{GL}_n$**,** $n\geq 2$**.** Taking our algebraicity result for L-functions attached to representations of $\mathrm{GL}_n\times\mathrm{GL}_m$, $1\leq m<n$, as an anchor, we may also derive an algebraicity result for critical values of L-functions attached to a pair of representation (Π,Π') on $\mathrm{GL}_n\times\mathrm{GL}_n$, that is, when both factors are of the same rank n. Unlike the previous case $m<n$, Π' will now be a *non*-cuspidal automorphic representation.

In order to put ourselves *in medias res*, consider an automorphic, not square-integrable representation Π' on $\mathrm{GL}_n(\mathbb{A}_F)$ defined as,

$$\Pi' := (\Pi_1\boxplus\Pi_2)\|\det\|^{1/2}$$

where Π_1 and Π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{m_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{m_2}(\mathbb{A}_F)$ respectively, such that $m_1+m_2=n$. Furthermore let $\{\chi_i^{(1)}\}_{1\leq i\leq n-m_1-1}$ and $\{\chi_j^{(2)}\}_{1\leq j\leq n-m_2-1}$ be two (so far arbitrary) choices of conjugate self-dual algebraic Hecke characters $\chi_i^{(t)}$ of $\mathbb{A}_F^\times$ and define automorphic representations Σ_t, $t=1,2$ of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ by

$$\Sigma_t := \Pi_t\boxplus\chi_i^{(t)}\boxplus\cdots\boxplus\chi_{n-m_t-1}^{(t)}.$$

We suppose that both Σ_t satisfy the condition posed on our original representation Σ from §1.4.2, i.e., both Σ_t are cohomological representations, such that the pair $(\Pi_\infty,\Sigma_{t,\infty})$ satisfies the piano-condition (1.5). As in §1.4.2 this is a condition on the archimedean components of the two representations Π_t, $t=1,2$ only. We point out that the condition that Σ_t is cohomological implies that Π' is cohomological due to the presence of the twist $\|\det\|^{1/2}$. Moreover, following our Observation 1.7, it is easy to see that if $\frac{1}{2}+k\in\mathrm{Crit}(\Pi\times\Sigma_1)\cap\mathrm{Crit}(\Pi\times\Sigma_2)$ is critical for $L(s,\Pi\times\Sigma_1)$ and $L(s,\Pi\times\Sigma_2)$, then automatically $k\in\mathrm{Crit}(\Pi\times\Pi')$. Now for the notational brevity define for $t=1,2$

$$\Omega(\Pi_t,\{\chi_i^{(t)}\}) := \prod_{1\leq i<j\leq n-m_t-1} p(\chi_i^{(t)}\chi_j^{(t),-1},\Psi_{\chi_i^{(t)}\chi_j^{(t),-1}})\prod_{i=1}^{n-m_t-1}L^S(1,\Pi_t\ \chi_i^{(t),-1}) \tag{3.5}$$

and

$$E_t^{\mathsf{cm}} := E^{\mathsf{cm}}(\chi_1^{(t)},...,\chi_{n-m_t-1}^{(t)}).$$

It is clear that $\Omega(\Pi_t,\{\chi_i^{(t)}\})$ only depends on Π_t and the set $\{\chi_i^{(t)}\}_{1\leq i\leq n-m_t-1}$. Here is our second key-result.

Theorem 3.6. *We let F be any CM-field and let $1 \leq m_1, m_2 < n$ be any integers such that $m_1 + m_2 = n$. Let Π be a cohomological unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let $\Pi' = (\Pi_1 \boxplus \Pi_2)\|\det\|^{1/2}$ be the twisted isobaric sum of two conjugate self-dual cuspidal automorphic representations Π_1, Π_2 of $\mathrm{GL}_{m_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{m_1}(\mathbb{A}_F)$, respectively. Choose any conjugate self-dual Hecke characters $\chi_1^{(t)}, \ldots, \chi_{n-m_t-1}^{(t)}$, $t = 1, 2$, such that the isobaric automorphic sums $\Sigma_t = \Pi_t \boxplus \chi_1^{(t)} \boxplus \ldots \boxplus \chi_{n-m_t-1}^{(t)}$ are both cohomological and assume that they satisfy the piano-hypothesis, (1.5), with respect to Π_∞. Let $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma_1) \cap \mathrm{Crit}(\Pi \times \Sigma_2)$ be any critical point of $L(s, \Pi \times \Sigma_1)$ and $L(s, \Pi \times \Sigma_2)$.*
In the special case when $k = 0$ only we additionally assume that the coefficient modules of Π_∞ and $\Sigma_{t,\infty}$ are all sufficiently regular, cf. §1.3.1 and §1.4.2, and that there exists a choice of Hecke characters χ, $\chi'^{(t)}$ such that $L^S(\frac{1}{2}, {}^\sigma\Pi_\chi \times {}^\sigma\Pi_{\chi'^{(t)}}) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, cf. §2.2, as well as that $L^S(\frac{1}{2}, \Pi\ \chi_i^{(t)}) \neq 0$ for all $1 \leq i \leq n - m_t - 1$.

Then k is critical for $L(s, \Pi \times \Pi')$ and

$$\begin{aligned} L^S(k, \Pi \times \Pi') \sim (2\pi i)^{2d(n-1)((k-\frac{1}{2})n-1)+d\frac{1}{2}((m_1-1)(m_1-2)+(m_2-1)(m_2-2))}\ p(\Pi)^2\ p(\Pi_1^{\mathsf{alg}})\ p(\Pi_2^{\mathsf{alg}}) \\ \Omega(\Pi_1, \{\chi_i^{(1)}\})\ \Omega(\Pi_2, \{\chi_j^{(2)}\}) \prod_{1 \leq i \leq m_2-1} L^S(\tfrac{1}{2} + k, \Pi\ \chi_i^{(1)})^{-1} \prod_{1 \leq j \leq m_1-1} L^S(\tfrac{1}{2} + k, \Pi\ \chi_j^{(2)})^{-1}, \end{aligned}$$

this relation being over the number field $E(\Pi)E(\Sigma_1)E(\Sigma_2)E_1^{\mathsf{cm}}E_2^{\mathsf{cm}}E(\phi)$.

If both m_1 and m_2 are odd, then the same relation holds over the smaller number field $E(\Pi)E(\Pi_1)E(\Pi_2)E(\{\chi_1^{(1)}, \ldots, \chi_{n-m_1-1}^{(1)}\})E(\{\chi_1^{(2)}, \ldots, \chi_{n-m_2-1}^{(2)}\})$.

Proof. We write

$$L^S(k, \Pi \times \Pi') = L^S(\tfrac{1}{2} + k, \Pi \times (\Pi_1 \boxplus \Pi_2)) = L^S(\tfrac{1}{2} + k, \Pi \times \Pi_1) \cdot L^S(\tfrac{1}{2} + k, \Pi \times \Pi_2). \tag{3.7}$$

Each pair (Π, Π_t) defines a representation on $\mathrm{GL}_n \times \mathrm{GL}_{m_t}$ with $m_t < n$. Thus we can apply Thm. 3.1 to each L-factor appears on the right hand side of (3.7). Recalling the definition of E_t^{cm}, $\Omega(\Pi_t, \{\chi_i^{(t)}\})$ from (3.5) above and that $n = m_1 + m_2$ the result follows. □

Remark 3.8. One can obtain an analogous result if the representation Π' is the twisted isobaric sum of $r \geq 3$ conjugate self-conjugate cuspidal automorphic representations Π_t.

4. Main applications

4.1. We understand that the reader may be interested in statements and results that read "slimmer" or more reduced than our rather involved two main theorems Thm. 3.1 and Thm. 3.6. In particular, one my be interested in results which avoid any reference to the (at first sight) rather obscure, numerous period-invariants and even more so the various remaining L-values $L(1, \Pi'\chi_j^{-1})$, $L(\frac{1}{2} + k, \Pi\chi_j)$ showing up in our main formulas.

In this section we provide such results as applications of our main theorems, exemplifying the strength of period-relations such as the ones established in Thm. 3.1 and Thm. 3.6.

4.2. Quotients of twisted standard L-functions at a joint special value. Our first main application concerns the twisted standard L-function and is a broad generalization of the main result of [Wal85b].

Ibidem, Waldspurger has shown a rationality result for the quotient $L(\frac{1}{2}, \pi \otimes \alpha)/L(\frac{1}{2}, \pi \otimes \beta)$ of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi\otimes\alpha$ and $\pi\otimes\beta$ of GL_2 over any number field at their joint critical value $s_0=\frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty=\beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L(\frac{1}{2},\pi\otimes\beta)$ is supposed to be non-zero. Under these assumptions, Waldspurger established a relation of the form

$$\frac{L(\frac{1}{2},\pi\otimes\alpha)}{L(\frac{1}{2},\pi\otimes\beta)}\sim_{\mathbb{Q}(\pi)}\frac{p(\alpha)}{p(\beta)},$$

the two period-invariants $p(\alpha)$, $p(\beta)$ only depending on α respectively β and the archimedean component of the cuspidal representations π. See [Wal85b], p. 174.

Here we generalize Waldspurger's result to the case of quotients of standard L-functions of GL_n/F where $n\geq 2$ is arbitrary, $s_0=\frac{1}{2}+k$ a more general special value while F is any CM-field. We obtain

Theorem 4.1. *Let F be any CM-field and let $n\geq 2$ be an integer. We assume that Π is a cohomological unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let α and β be conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ having the same archimedean components $\alpha_v(z)=\beta_v(z)=z^{a_v}\bar{z}^{-a_v}$, $v\in S_\infty$. If $a_v\in\frac{n}{2}+\mathbb{Z}$ and $\mu_{v,1}\geq a_v\geq\mu_{n,v}$ for all $v\in S_\infty$, then there is a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sum $\Sigma=\alpha\boxplus\chi_1\boxplus\ldots\boxplus\chi_{n-2}$ is cohomological and such that $(\Pi_\infty,\Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis, (1.5). Fix any such choice and let $s_0=\frac{1}{2}+k\in\mathrm{Crit}(\Pi\times\Sigma_\alpha)$ be any critical point of $L(s,\Pi\times\Sigma_\alpha)$.*
In the special case when $k=0$ only, i.e., if $s_0=\frac{1}{2}$ denotes the central critical point, we additionally assume that the coefficient modules of Π_∞ and $\Sigma_{\alpha,\infty}$ are both sufficiently regular, cf. §1.3.1 and §1.4.2, and that there exists a choice of Hecke characters χ, χ' such that $L^S(\frac{1}{2},{}^\sigma\Pi_\chi\times{}^\sigma\Pi_{\chi'})\neq 0$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$, cf. §2.2, as well as that $L^S(\frac{1}{2},\Pi\otimes\beta)$ and $L^S(\frac{1}{2},\Pi\otimes\chi_j)\neq 0$ for all $1\leq j\leq n-2$. Then

$$\frac{L^S(\frac{1}{2}+k,\Pi\otimes\alpha)}{L^S(\frac{1}{2}+k,\Pi\otimes\beta)}\sim\prod_{i=1}^{n-2}\frac{p(\alpha,\Psi_{\alpha\chi_i^{-1}})}{p(\beta,\Psi_{\beta\chi_i^{-1}})}. \tag{4.2}$$

where the relation is over the number field

$$E(\Pi)E(\Sigma_\alpha)E(\Sigma_\beta)E(\phi)E^{\mathsf{cm}}(\alpha,\chi_1,...,\chi_{n-2})E^{\mathsf{cm}}(\beta,\chi_1,...,\chi_{n-2})E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\beta)\prod_i^{n-2}E^{\mathsf{cm}}(\chi_i^{-1}).$$

If n is even, then all the $s_0=\frac{1}{2}+k$ are indeed critical for $L(s,\Pi\otimes\alpha)$ and $L(s,\Pi\otimes\beta)$ and (4.2) holds over the smaller number field $E(\Pi)E(\alpha)E(\beta)E(\{\chi_1,...,\chi_{n-2}\})$.

Proof. The discussion in §1.4.2 together with our assumption that $a_v\in\frac{n}{2}+\mathbb{Z}$ and $\mu_{v,1}\geq a_v\geq\mu_{n,v}$ for all $v\in S_\infty$ implies immediately that there is a choice of conjugate self-dual Hecke characters χ_1, ..., χ_{n-2}, such that the isobaric automorphic sum $\Sigma=\alpha\boxplus\chi_1\boxplus\ldots\boxplus\chi_{n-2}$ is cohomological and such that $(\Pi_\infty,\Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis. Hence, putting $m=1$ and $\Pi'=\alpha$, the pair (Π,α) of cuspidal representations on $\mathrm{GL}_n(\mathbb{A}_F)\times\mathrm{GL}_1(\mathbb{A}_F)$ satisfies all the conditions of our Thm. 3.1 (with $\Sigma=\Sigma_\alpha$). As it is again immediate, our assumption $\alpha_\infty=\beta_\infty$ implies that the isobaric sum $\Sigma_\beta:=\beta\boxplus\chi_1\boxplus\ldots\boxplus\chi_{n-2}$ has the same archimedean component as Σ_α. Another short moment of thought convinces us that consequently Σ_α and Σ_β may be interchanged in the statement of Thm. 4.1 without changing any assertion: Otherwise put, the pair (Π,β) automatically satisfies all the conditions of our Thm. 3.1, letting $\Sigma = \Sigma_\beta$. Hence, simply by inserting we obtain

$$\frac{L^S(\frac{1}{2}+k,\Pi\otimes\alpha)}{L^S(\frac{1}{2}+k,\Pi\otimes\beta)} \sim_{E(\Pi)E(\Sigma_\alpha)E(\Sigma_\beta)E(\phi)E^{\mathsf{cm}}(\chi_1,\dots,\chi_{n-2})} \frac{p(\alpha^{\mathsf{alg}})}{p(\beta^{\mathsf{alg}})} \prod_{i=1}^{n-2} \frac{L^S(1,\alpha\chi_i^{-1})}{L^S(1,\beta\chi_i^{-1})}.$$

Observing that the Whittaker periods $p(\alpha^{\mathsf{alg}})$ and $p(\beta^{\mathsf{alg}})$ may both be chosen to be in $\mathbb{Q}^\times$, and applying Balsius's result, cf. Prop. 2.8, once more to $\prod_{i=1}^{n-2} L^S(1,\alpha\chi_i^{-1})$ and $\prod_{i=1}^{n-2} L^S(1,\beta\chi_i^{-1})$, we get

$$\frac{L^S(\frac{1}{2}+k,\Pi\otimes\alpha)}{L^S(\frac{1}{2}+k,\Pi\otimes\beta)} \sim_{E(\Pi)E(\Sigma_\alpha)E(\Sigma_\beta)E(\phi)E^{\mathsf{cm}}(\alpha\chi_1,\dots,\chi_{n-2})E^{\mathsf{cm}}(\beta,\chi_1,\dots,\chi_{n-2})} \prod_{i=1}^{n-2} \frac{p(\alpha\chi_i^{-1},\Psi_{\alpha\chi_i^{-1}})}{p(\beta\chi_i^{-1},\Psi_{\beta\chi_i^{-1}})}.$$

For each $1\leq i\leq n-2$ one has

$$p(\alpha\chi_i^{-1},\Psi_{\alpha\chi_i^{-1}}) \sim_{E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\chi_i)} p(\alpha,\Psi_{\alpha\chi_i^{-1}})\ p(\chi_i^{-1},\Psi_{\alpha\chi_i^{-1}})$$

and likewise for β taking the role of α, see [Gro-Lin17] Prop. 4.4. As $\alpha_\infty = \beta_\infty$ by assumption, $\Psi_{\alpha\chi_i^{-1}} = \Psi_{\beta\chi_i^{-1}}$ by definition, cf. §2.4. This implies the first assertion of the theorem. The second assertion follows now from Thm. 3.1 and [Gro-Lin17], Lem. 1.19. □

Remark 4.3 (Dependency on χ_i)**.** If n is even, then [Gro-Lin17], Lem. 1.19 actually implies more, namely that (4.2) holds over the even smaller field $E(\Pi)E(\alpha)E(\beta)E(\{\chi_{1,\infty},\dots,\chi_{n-2,\infty}\})$. Hence, relation (4.2) will be true over $\mathbb{Q}(\Pi)\mathbb{Q}(\alpha)\mathbb{Q}(\beta)F^{Gal}$ (and hence *independent* of the actual choice of the auxiliary characters χ_i) once the set of exponents $\{b_{i,v}\}_{v\in S_\infty}^{1\leq i\leq n-2}$ is stable under permutation and multiplication by -1.

Remark 4.4 (Further interpretations)**.** In [Gro-Rag14b] a rationality result for the critical values of the twisted standard L-function $L(s,\Pi\otimes\chi)$ using unspecified archimedean periods (later on made explicit in [Jan16]) has been established by Raghuram and the first named author. Here, Π denotes a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, admitting a Shalika model. This necessarily implies that n is even as in the refined second assertion of our Thm. 4.1 above. In this regard, Thm. 4.1 provides a generalization as well as a certain refinement of a consequence of the main result of [Gro-Rag14b] and [Jan16] over general CM-fields, instead of totally real fields F^+.

4.3. Quotients of a fixed Ranking-Selberg L-function at different critical values. Our second application concerns quotients of a given Rankin-Selberg L-function $L^S(s,\Pi\times\Pi')$ of general type $n\times m$, $1\leq m\leq n$, at different critical values $s=\frac{1}{2}+k$ and $s=\frac{1}{2}+\ell$.

Our result may be viewed as a generalization of

(i) the main result of Harder-Raghuram [GHar-Rag17], obtained there for Rankin-Selberg L-functions of general type $n\times m$, nm even, but over totally real fields F^+, as well as
(ii) one of the main results of [Gro-Lin17], cf. Thm. 5.5, obtained there for general CM-fields F, but for Rankin-Selberg L-functions of type $n\times(n-1)$ only.

It should be pointed out though, that if $m<n$, our result below is a mild generalization of a consequence of the main result of [Lin15]: There Lin has achieved a very general, fine rationality-result for Rankin-Selberg L-functions of type $n\times m$, stated in [Lin15], Thm. 10.8.1 under a list of additional local assumptions (and conjectures, but those which were later on proved in [Gue16] and [Gro-Lin17]). We hence do not claim a big amount of originality from our side, but rather include the following corollary of Thm.s 3.1 & 3.6 for sake of covering the new cases when $m=n$ and in order to give an example of the use of period-relations.

In order to explain our result, recall weak base change BC from an arbitrary rational unitary similitude group $GU(V)/\mathbb{Q}$ attached to a non-degenerate Hermitian space V of $\dim_{F^+} V = n$, as established in [Shi14]. Strictly speaking, the construction of BC in [Shi14] entails the claim that $F = \mathcal{K}F^+$ for some imaginary quadratic field $\mathcal{K}$, which we shall henceforth assume. This is no proper restriction for us, because the same assumption has been made in [Gue16], §5, which we shall use in the proof of our Cor. 4.4 below. Then, for every cohomological cuspidal automorphic representation π of $GU(V)(\mathbb{A}_\mathbb{Q})$ a base change $BC(\pi) = \chi_\pi \otimes \Pi$ has been constructed in [Shi14]: Here, χ_π is a Hecke character of $\mathbb{A}_\mathcal{K}^\times$, while Π is a conjugate self-dual isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. By results of Delorme-Enright, cf. [Enr79], Π_∞ is cohomological as well. See also [Lab11], §5.1 and [Clo91], §3.4.

Corollary 4.4. *Let $F = \mathcal{K}F^+$ be a CM-field and suppose that $1 \leq m \leq n$ are integers, $n \geq 2$. We let $\Pi = BC(\pi)|_{\mathrm{GL}_n(\mathbb{A}_F)}$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which we assume to be obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$.*
Case I: $m<n$ *In this case we assume in addition that n is even and m is odd. Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, satisfying the conditions of Thm. 3.1. Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two critical points of $L(s, \Pi \times \Sigma)$ different from $s_0 = \frac{1}{2}$. Then $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are indeed critical for $L(s, \Pi \times \Pi')$ and the ratio of critical values satisfies*

$$\frac{L^S(\frac{1}{2}+k, \Pi \times \Pi')}{L^S(\frac{1}{2}+\ell, \Pi \times \Pi')} \sim_{E(\Pi)E(\Pi')} (2\pi i)^{d(k-\ell)nm}.$$

Case II: $m=n$ *We assume in addition that $n = m_1 + m_2$, with m_1 and m_2 both odd. Let $\Pi' = (\Pi_1 \boxplus \Pi_2)\|\det\|^{1/2}$ be an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, satisfying the conditions of Thm. 3.6.*
Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two joint critical points of $L(s, \Pi \times \Sigma_1)$ and $L(s, \Pi \times \Sigma_2)$ different from $s_0 = \frac{1}{2}$. Then k and ℓ are critical for $L(s, \Pi \times \Pi')$ and the ratio of critical values satisfies

$$\frac{L^S(k, \Pi \times \Pi')}{L^S(\ell, \Pi \times \Pi')} \sim_{E(\Pi)E(\Pi')} (2\pi i)^{d(k-\ell)n^2}.$$

Proof. Case I: By Thm. 3.1, the quotient of critical values satisfies

$$\frac{L^S(\frac{1}{2}+k, \Pi \times \Pi')}{L^S(\frac{1}{2}+\ell, \Pi \times \Pi')} \sim_{E(\Pi)E(\Pi')E(\{\chi_1,\ldots,\chi_{n-m-1}\})} (2\pi i)^{d(k-\ell)(n-1)n} \cdot \prod_{j=1}^{n-m-1} \frac{L^S(\frac{1}{2}+\ell, \Pi\chi_j)}{L^S(\frac{1}{2}+k, \Pi\chi_j)}.$$

Observe that by Lem. 1.6 $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are both critical for all $L(s, \Pi\chi_j)$, $1 \leq j \leq n-m-1$. As $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are also assumed to be different from the central critical value, our additional assumption on Π being obtained by base change from π hence allows us to use Guerberoff's recent theorem, [Gue16], Thm. 1, on non-central critical values of standard L-functions. (The careful reader may want to use §4.2 in [Gro-Har-Lin18a] in combination with [KMSW14], Thm. 1.7.1, which confirms Guerberoff's Hypothesis 4.5.1 for our representation π.) Hence, simply by inserting in Guerberoff's formula we obtain

$$\prod_{j=1}^{n-m-1} \frac{L^S(\frac{1}{2}+\ell, \Pi\chi_j)}{L^S(\frac{1}{2}+k, \Pi\chi_j)} \sim_{E^{\mathsf{aux}}(\pi,\{\chi_j\})} (2\pi i)^{d(n-m-1)n(\ell-k)}.$$

Here, $E^{\mathsf{aux}}(\pi, \{\chi_j\})$ denotes any number field over which π_f and all characters $\chi_{j,f}$ are defined (such a field exists, e.g., by [Gro-Seb17], Thm. A.2.4). Collecting the powers of $(2\pi i)$ we obtain

$$\frac{L^S(\frac{1}{2}+k, \Pi \times \Pi')}{L^S(\frac{1}{2}+\ell, \Pi \times \Pi')} \sim_{E(\Pi)E(\Pi')E(\{\chi_1,\dots,\chi_{n-m-1}\})E^{\mathsf{aux}}(\pi,\{\chi_j\})} (2\pi i)^{d(k-\ell)nm}.$$

However, applying [Gro-Lin17], Lem. 1.19, we see that this relation actually holds over the much smaller number field $E(\Pi)E(\Pi')$, which shows the claim in Case I.
Case II: Using Thm. 3.6 instead of Thm. 3.1, one argues exactly as in Case I. □

Remark 4.5 (Further interpretations)**.** In [Jan17] Januszweski has achieved a conditional rationality-result for Rankin-Selberg L-functions of type $n \times (n-1)$ over totally real fields with precise powers of $(2\pi i)$ as archimedean contribution, recovering the result of Harder–Raghuram, [GHar-Rag17] in the case of $n \times (n-1)$ as a consequence (under the given hypotheses). Hence, Case I of our Cor. 4.4 may also be seen as an unconditional generalization of a consequence of the main result of [Jan17] for more general pairs $n \times m$ and over CM-fields $F = \mathcal{K}F^+$. Regarding Case II, Grenié has established a rationality-result for Rankin-Selberg L-functions of type $n \times n$ over totally imaginary number fields, but with an unspecified archimedean period. Case II of our Cor. 4.4 may therefore also be seen as a complementary result, as well as a certain refinement, of a consequence of the main theorem of [Gre03].

Appendix A. Motivic framework: Concluding thoughts

A.1. Automorphic motives for reductive groups: A conjecture. We would like to end with collecting a few general thoughts about automorphic forms and their connection to motives. Let G be a connected reductive linear algebraic group, defined over a number field $F \subset \mathbb{C}$ and let $\pi = \pi_\infty \otimes \pi_f$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$.

If π_∞ is *cohomological*, i.e., has non-trivial relative Lie algebra cohomology with respect to an irreducible, algebraic coefficient module, then π enjoys two remarkable properties: Firstly, the field of rationality $\mathbb{Q}(\pi_f)$ is a number field, as it is well-known (and follows, e.g., by the argument in [Gro-Rag14a, §8.1]). Secondly, by passing over to a suitable covering group $\tilde{G}/F$ (a z-extension in the sense of Kottwitz), one may always induce π to an automorphic representation of $\tilde{G}(\mathbb{A}_F)$, which allows a certain automorphic twist $\pi^\theta = \otimes_v \pi^\theta_v$ with *algebraic* associated Weil-group homomorphisms $r_{\pi^\theta_v} : W_{F_v} \to {}^L\tilde{G}(\mathbb{C})$ at the archimedean places. Algebraicity here refers to the obstruction that $r_{\pi^\theta_v}|_{\bar{F}_v^\times}(z) = z^{\alpha_v}\bar{z}^{\beta_v}$ for some rational characters α_v, $\beta_v \in X^*(\tilde{T})$ of a maximal torus $\tilde{T} \subseteq \tilde{G}$. In this regard, the twisting by θ simply denotes multiplication by any suitable automorphic character of $\tilde{G}(\mathbb{A}_F)$, such that the infinitesimal character of π's lift, say χ_λ, becomes algebraic, i.e., $\chi_{\pi^\theta} = \chi_{\lambda+\theta}$ with $\lambda + \theta \in X^*(\tilde{T})$. See [Buz-Gee14], Cor. 5.2.7.

Let $r : {}^L\tilde{G}(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ be a rational representation and let S be a finite set of places of F containing all archimedean places and all places where π^θ ramifies. According to Langlands functoriality, we shall expect the existence of an automorphic representation $r(\pi^\theta)$ of $\mathrm{GL}_N(\mathbb{A}_F)$, being determined at v outside $S \backslash S_\infty$ by $r \circ r_{\pi^\theta_v}$. Obviously, composition with a rational representation r preserves algebraicity of Weil-group homomorphisms at the archimedean places, [Buz-Gee14], Lem. 6.1.2. This leads us to conjecture the following:

Conjecture M (Automorphic motives for reductive groups)**.** *Assume that $r(\pi^\theta)$ is an irreducible cuspidal automorphic representation. Then there exists an absolutely irreducible motive $\mathscr{M} = \mathscr{M}(\pi, \theta, r)$ over F of rank N with coefficients in a subfield $E(\mathscr{M})$ of $\mathbb{C}$, which is a finite extension of $\mathbb{Q}(\pi_f)$, such that $L^S(s, \mathscr{M}) = L^S(s, \pi^\theta, r)$.*

Let us consider the major example $G = \mathrm{GL}_N$. We may take $\tilde{G} = G$, $\pi^\theta = \pi \otimes \|\det\|^{(N-1)/2}$ and r to be the tautological representation. Then our conjecture becomes Clozel's Conj. 4.5 in [Clo90].

For an even more paradigmatic example consider a CM-field E with maximal totally real subfield F and an E-split unitary group $G = U(V)$, attached to an n-dimensional, totally definite hermitian space V over F. Then we find ourselves in the framework of [Clo-Har-Tay08]. The space V being totally definite is equivalent to that $G(F_v)$ is compact at all archimedean places v, hence every irreducible automorphic representation π of $G(\mathbb{A}_F)$ comes under the purview of the above. However, if n is even, then we cannot simply let $\tilde{G} = G$, but have to pass to a bigger quotient $\tilde{G} = (G^{\mathsf{der}} \times Z(G) \times \mathbb{G}_m)/\ker(G^{\mathsf{der}} \times Z(G) \to G)$, see [Buz-Gee14]. Indeed, the L-group ${}^L\tilde{G}$ so described is nothing but the group denoted $\mathcal{G}_n$ in [Clo-Har-Tay08], see §1.1 *ibd.*, hence our conjectural motive $\mathscr{M}$ should be seen as attached to the Galois representations into $\mathcal{G}_n(\bar{\mathbb{Q}}_\ell) = {}^L\tilde{G}(\bar{\mathbb{Q}}_\ell)$ associated to π, constructed in [Clo-Har-Tay08].

A.2. As an important application of our Conjecture M we may *translate Deligne's conjecture* [Del79] on critical values of motivic L-functions into a conjecture for the automorphic L-functions $L^S(s, \pi^\theta, r)$ of a reductive group. Indeed, if $s = k \in \mathbb{Z}$ denotes any critical integer for $L(s, \pi^\theta_\infty, r)$, i.e., an integer where both $L(s, \pi^\theta_\infty, r)$ and $L(1-s, (\pi^\theta_\infty)^\mathsf{v}, r)$ are holomorphic, then there should be periods $p(\pi) \in \mathbb{C}^\times$, cf. [Kon-Zag01], and an integer $d(k)$ such that

$$L^S(k, \pi^\theta, r) \sim_{E(\pi_f)} (2\pi i)^{d(k)} p(\pi).$$

Here, $E(\pi_f)$ denotes a finite extension of the field of rationality $\mathbb{Q}(\pi_f)$.

For our L-function the twist by θ finally amounts to a (choice of a) shift in the variable s. This shift depends on θ *and* r and can be made explicit in any practical situation.

As a major explicit example for this twist, the reader may in fact take our definition of critical points for Ranking-Selberg L-functions, see Def. 1.6. Indeed, if $\pi = \pi_1 \otimes \pi_2$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_M(\mathbb{A}_F)$, then Def. 1.6 resembles nothing but what one should expect according to Conjecture M for the tensor product of standard representations $r = r_N \otimes r_M$: We should have

$$L^S(s, \mathscr{M}) = L^S(s, \pi^\theta) = L^S(s + \tfrac{N-M}{2}, \pi_1 \times \pi_2).$$

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