AUTOMORPHIC FORMS, COHOMOLOGY AND CAP REPRESENTATIONS. THE CASE GL_2 OVER A DEFINITE QUATERNION ALGEBRA

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Abstract. In this paper we fully describe the cuspidal and the Eisenstein cohomology of the group $G = GL_2$ over a definite quaternion algebra $D/\mathbb{Q}$. Functoriality is used to show the existence of residual and cuspidal automorphic forms, having cohomology in degree 1. The latter ones turn out to be CAP-representations, though G satisfies Strong Multiplicity One.

Introduction

This paper, in a sense, follows up on a work of G. Harder, [19]. He considered the Eisenstein cohomology of arithmetic subgroups of GL_2 defined over a number field $\mathbb{K}$. Here we want to describe the cohomology of arithmetic congruence subgroups Γ of $G = GL_2'$, the group of invertible 2×2-matrices with entries in a definite quaternion algebra $D/\mathbb{Q}$. Furthermore, we do not restrict ourselves to determining the Eisenstein cohomology of such groups Γ, but also describe their cuspidal cohomology.

The Eisenstein cohomology of $G = GL_2'$ with respect to a finite-dimensional, irreducible complex-rational representation E of $G(\mathbb{R})$, denoted $H^*_{Eis}(G, E)$, is spanned by classes represented by (a bunch of) Eisenstein series, residues or derivatives of such. As suggested by the last sentence, in order to determine the very nature of a representative of an Eisenstein class, one needs to decide whether a given Eisenstein series is or is not holomorphic at a certain point. As shown by Langlands, this means to describe the residual spectrum of G.

Here the first problem arises. For a *quasi-split* group the Langlands-Shahidi method is available, which expresses the poles of Eisenstein series by poles of a certain intertwining operator $M(s, \widetilde{\pi})$. This operator links two representations induced from a cuspidal automorphic representation $\widetilde{\pi}$ of a proper Levi subgroup. If $\widetilde{\pi}$ is *generic*, then the Langlands-Shahidi method provides a normalization of $M(s, \widetilde{\pi})$ defined using L-functions, which theoretically allows one to read off the poles of $M(s, \widetilde{\pi})$. Clearly, $G = GL_2'$ is not quasi-split and - furthermore - there are also many cuspidal automorphic representations of $L = D^* \times D^*$ (sitting inside G as the Levi subgroup of the unique proper standard parabolic $\mathbb{Q}$-subgroup P), which have non-generic local components. Still, using [16], Prop. 3.1, we are able to reduce the task of normalizing the global intertwining operator

$$M(s, \widetilde{\pi}) : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\sigma \otimes \tau \otimes e^{sH_P(\cdot)}] \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\tau \otimes \sigma \otimes e^{-sH_P(\cdot)}]$$

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to normalizing its local components at split places. As before, $\widetilde{\pi} = \sigma \otimes \tau$ is a cuspidal automorphic representation of $L = D^* \times D^*$. To overcome the second problem, namely to actually normalize it at non-generic split places, we use the recent work of N. Grbac, [13]. With this result at hand, we are able to classify the cuspidal automorphic representations $\widetilde{\pi}$ of L, which give rise to residual Eisenstein series and show where their poles are located, see Proposition 3.4.
This gives the first main theorem of this paper, see Theorem 3.2, which comprehensively describes to Eisenstein cohomology of GL'_2 with respect to arbitrary coefficients E. In particular, necessary and sufficient conditions for the existence of non-trivial residual Eisenstein cohomology classes are established and it is explained how they are distributed (resp. separated from holomorphic Eisenstein classes) in (resp. by) the degrees of cohomology.

Degree one seems to be of most interest, since Eisenstein series contributing to it must form a non-trivial residual automorphic representation of G. After having shown their existence by having proved the non-vanishing of $H^1_{Eis}(G, E)$ in Theorem 3.2, we explicitly construct such residual representations in Theorem 3.3. Here we use functoriality, more precisely, the global Jacquet-Langlands Correspondence from GL'_2 to GL_4, as it was developed by I. Badulescu and D. Renard, [2].

As a next step we shortly reconsider Eisenstein cohomology in degree $q = 2$ and prove a theorem (cf., Theorem 3.4) that serves as the starting point for investigating rationality results of critical values of L-functions. It also answers in our particular case of $G = GL'_2$ a question raised by G. Harder.

Finally, in section 5, we describe the cuspidal cohomology of G, denoted $H^q_{cusp}(G, E)$. Its elements, i.e., cuspidal cohomology classes, are represented by cuspidal automorphic forms. As it follows from the classification of the cohomological unitary dual of $G(\mathbb{R})$, see Proposition 3.5, cuspidal cohomology classes gather in pairs $(q, 5-q)$, $q = 1, 2$, symmetrically around the middle degree of cohomology. They are quite different in their nature: The classes in degrees $(2, 3)$ are represented by a representation with tempered archimedean component, while the classes in degree $(1, 4)$ will give rise to the existence of cuspidal automorphic representations with *non*-tempered archimedean component.

Again we use functoriality, *expressis verbis*, the global Jacquet-Langlands Correspondence, to show the existence of cuspidal automorphic representations giving rise to cohomology in degrees $q = 1$ and 4, see Theorem 4.1. Those serve as examples for cohomological CAP-representations, although G satisfies Strong Multiplicity One. Further, it proves the non-vanishing of $H^q_{cusp}(G, E)$, $q = 1, 4$.
In the last part of this paper, we obtain the non-vanishing of cuspidal cohomology in degrees $q = 2$ and 3. It follows from the twisted version of Arthur's Trace Formula, respectively applying the results of [7] or [3].

On a final remark, let us point out that we pursued the analysis given in this paper in our recent work [15]. The reader may also be interested in our joint preprint with Raghuram [17].

1. Basic group data

1.1. **The groups.** Let D be a quaternion algebra over the field of rational numbers $\mathbb{Q}$ and $S(D)$ be the set of places over which D does not split. By the Brauer-Hasse-Noether Theorem, $S(D)$ determines D up to isomorphism and has always finite and even cardinality (cf. [27] Thm. 1.12). We assume throughout this paper that the archimedean place ∞ is in $S(D)$, so $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$, the usual Hamilton quaternions and therefore D is not split over $\mathbb{Q}$. If $x \in D$, we set $\nu(x) := x\overline{x}$, where $x \mapsto \overline{x}$ is the conjugation within D. The determinant $\det'$ of an $n \times n$-matrix $X \in M_n(D)$, $n \geq 1$, is then the generalization of ν to matrices: $\det'(X) := \det(\varphi(X \otimes 1))$, for some isomorphism $\varphi : M_n(D) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} M_{2n}(\overline{\mathbb{Q}})$. It is independent of φ (see, e.g., [27] 1.4.1) and a rational polynomial in the coordinates of the entries of X. So the group

$$G(\mathbb{Q}) = \{X \in M_n(D) | \det'(X) \neq 0\}$$

defines an algebraic group over $\mathbb{Q}$, which we denote GL'_n. It is reductive and has a natural simple subgroup SL'_n, the group of matrices of determinant 1. We will consider $G = GL'_2$ and $G_s = SL'_2$.

Fix a maximal $\mathbb{Q}$-split tours $T \subset G$. It is of the form $T \cong GL_1 \times GL_1$ embedded diagonally into G. As the centre Z of G consists of diagonal scalar matrices, i.e., $Z \cong GL_1$ embedded diagonally into G,

$$A := T \cap SL'_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$$

is a maximal $\mathbb{Q}$-split torus of SL'_2 and $T = Z \cdot A$. Observe that T is also maximally split over $\mathbb{R}$. Therefore we may identify the set of $\mathbb{Q}$-roots $\Delta_{\mathbb{Q}}$ and the set of $\mathbb{R}$-roots $\Delta_{\mathbb{R}}$ of G. They are given as $\Delta_{\mathbb{Q}} = \Delta_{\mathbb{R}} = \{\pm(\alpha - \beta)\}$, where α (resp. β) extracts the first (resp. the second) diagonal entry of T. On the level of the simple group SL'_2, the two roots degenerate to $\pm 2\alpha$. We chose positivity in the usual way as $\Delta^+_{\mathbb{Q}} = \Delta^+_{\mathbb{R}} = \{\alpha - \beta\}$. It follows that $rk_{\mathbb{Q}}(SL'_2) = 1$. Furthermore, the $\mathbb{Q}$-Weyl group $W_{\mathbb{Q}}$ of G_s (and G) consists precisely of 1 and one non-trivial quadratic element w which acts by exchanging α and β: $W_{\mathbb{Q}} = \{1, w\}$.

Observe that T and A are not maximally split over $\mathbb{C}$. In fact, $GL'_2(\mathbb{C}) \cong GL_4(\mathbb{C})$ and so we need to multiply T (resp. A) by a two-dimensional torus $B \cong GL_1 \times GL_1$, in order to get a maximally $\mathbb{C}$-split torus of GL'_2 (resp. SL'_2). Let $H := A \cdot B \hookrightarrow SL'_2$. Then $H(\mathbb{C})$ can be viewed as

$$H(\mathbb{C}) = \left\{ \begin{pmatrix} (a, b_1) & 0 \\ 0 & (a^{-1}, b_2) \end{pmatrix} \right\},$$

where the pair $(a, b_i) \in \mathbb{C}^* \times \mathbb{C}^*$, $i = 1, 2$, represents the first two coordinates of the quaternion division algebra D. The group $H(\mathbb{C})$ is a Cartan subgroup of $G_s(\mathbb{C}) \cong SL_4(\mathbb{C})$. If we set for $h \in \mathfrak{h}(\mathbb{C})$ =Lie($H(\mathbb{C})$),

$$\varepsilon_i(h) = \begin{cases} b_1 + a & i = 1 \\ a - b_1 & i = 2 \\ b_2 - a & i = 3 \\ -b_2 - a & i = 4 \end{cases}$$

then the set of absolute (i.e. $\mathbb{C}$-)roots of $G_s = SL'_2$ is generated by the following set of simple roots $\Delta^\circ = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2 - \varepsilon_3, \alpha_3 := \varepsilon_3 - \varepsilon_4\}$, where we observe that the choice of positivity given by Δ° is compatible with the one on $\Delta_{\mathbb{Q}}$. Accordingly, $rk_{\mathbb{C}}(SL'_2) = 3$ and the fundamental weights of G_s are $\omega_1 = \frac{3}{4}\alpha_1 +$

$\frac{1}{2}\alpha_2+\frac{1}{4}\alpha_3$, $\omega_2=\frac{1}{2}\alpha_1+\alpha_2+\frac{1}{2}\alpha_3$ and $\omega_3=\frac{1}{4}\alpha_1+\frac{1}{2}\alpha_2+\frac{3}{4}\alpha_3$. As usual their sum is denoted by ρ. For the simple reflection corresponding to the root α_i we write w_i and the Weyl group generated by these reflections is called $W=W(G,H)=W(G_s,H)$.

1.2. **Real Groups and Symmetric Spaces.** The real Lie groups $G(\mathbb{R})$ and $G_s(\mathbb{R})$ are isomorphic to $GL_2(\mathbb{H})$ and $SL_2(\mathbb{H})$, respectively. If we fix a Cartan involution $\theta(x):={}^t\overline{x}^{-1}$ on G, then the corresponding maximal compact subgroups K of $G(\mathbb{R})$ and K_s of $G_s(\mathbb{R})$ are isomorphic to $K\cong K_s\cong Sp(2)$. As $G(\mathbb{R})=Z(\mathbb{R})^\circ\times G_s(\mathbb{R})$, the corresponding symmetric spaces $X=Z(\mathbb{R})^\circ\backslash G(\mathbb{R})/K\cong\mathbb{R}^*_{>0}\backslash GL_2(\mathbb{H})/Sp(2)$ and $X_s=G_s(\mathbb{R})/K_s=SL_2(\mathbb{H})/Sp(2)$ are diffeomorphic, whence we will not distinguish between them. Their common real dimension is $\dim_{\mathbb{R}}X=5$.

1.3. Throughout this paper $E=E_\lambda$ denotes an irreducible, finite-dimensional representation of $G_s(\mathbb{R})$ on a complex vector space determined by its highest weight $\lambda=\sum_{i=1}^3 c_i\alpha_i$. We can and will view E also as a representation of $G(\mathbb{R})$ by extending it trivially on $Z(\mathbb{R})^\circ$. Furthermore, we will always assume that E is the complexification of an algebraic representation of $G/\mathbb{Q}$.

1.4. The Parabolic Subgroup. Since $G_s=SL_2'$ is a $\mathbb{Q}$-rank one group, we only have one single proper standard parabolic $\mathbb{Q}$-subgroup P of G (resp. $P_s=P\cap G_s$ of G_s). It is given by

$$P=\left\{\begin{pmatrix} a & n\\ 0 & b\end{pmatrix}|a,b\in GL_1',n\in D\right\}$$

with Levi- (resp. Langlands-) decomposition $P=LN$ (resp. $P=MTN$), where

$$L=GL_1'\times GL_1'$$
$$M=M_s=SL_1'\times SL_1'$$

and

$$N=N_s\cong D\text{ (as additive group).}$$

The set of absolute simple roots of $M=M_s$ with respect to $B=H\cap M$ is $\Delta_M^\circ=\{\alpha_1,\alpha_3\}$. For later use we already define and determine

$$\begin{aligned} W^P:=W^{P_s} &:= \{w\in W|w^{-1}(\alpha_i)>0\quad i=1,3\}\\ &= \{id,w_2,w_2w_1,w_2w_3,w_2w_1w_3,w_2w_1w_3w_2\}\end{aligned}$$

This set is called the set of Kostant representatives, cf. [9] III 1.4.

2. Generalities on the automorphic cohomology of G

2.1. A space of automorphic forms. Having fixed some basic facts and notation concerning G and G_s, we will now delve into cohomology, the object to be studied in this paper. The main reference for this general section is [11]. For the special case of a group of rank one, the reader may also consult the first sections of the author's paper [16], where the results of [11] are summarized for these groups.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ of the real Lie algebra $\mathfrak{g}=\mathfrak{gl}_2(\mathbb{H})$ of $G(\mathbb{R})$. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{J}\lhd\mathcal{Z}(\mathfrak{g})$ an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$. We denote by $\mathcal{A}(G)=\mathcal{A}(G,\mathcal{J})$ the space of those adelic automorphic forms on $Z(\mathbb{R})^\circ G(\mathbb{Q})\backslash G(\mathbb{A})$ (in the sense of [6], 4) which are annihilated by some power of $\mathcal{J}$. As we will be only interested in automorphic forms which have non-trivial $(\mathfrak{g}_s,K)$-cohomology with respect to E, we take $\mathcal{J}\lhd\mathcal{Z}(\mathfrak{g})$ to be the ideal annihilating the dual representation $\check{E}$, cf. [11] Rem. 3.4.

Having fixed our choice for $\mathcal{J}$, the space $\mathcal{A}(G)$ has a certain decomposition along the two $G(\mathbb{Q})$-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G (represented by P and G), which we shall now describe. First of all we recall that the group $G(\mathbb{A}_f)$ of finite adelic points acts on the space $C(G(\mathbb{A}_f))$ of continuous functions $f : G(\mathbb{A}_f) \to \mathbb{C}$ by right translation. Topologised via the inductive limit

$$C^\infty(G(\mathbb{A}_f)) = \varinjlim C(G(\mathbb{A}_f))^{K_f},$$

K_f running over all open, compact subgroups of $G(\mathbb{A}_f)$, the space $C^\infty(G(\mathbb{A}_f))$ of smooth functions $f : G(\mathbb{A}_f) \to \mathbb{C}$ is a complete Hausdorff, locally convex vector space with semi-norms $|.|_\alpha$ say, but generally not Fréchet. As a consequence,

$$C^\infty(G(\mathbb{R}), C^\infty(G(\mathbb{A}_f))) = C^\infty(G(\mathbb{A})) \quad \text{and} \quad C^\infty(Z(\mathbb{R})^\circ G(\mathbb{Q})\backslash G(\mathbb{A}))$$

are Hausdorff, locally convex vector spaces carrying smooth $G(\mathbb{R})$- and $G(\mathbb{A}_f)$-actions via right translation. We recall that a left $Z(\mathbb{R})^\circ G(\mathbb{Q})$-invariant smooth function $f \in C^\infty(G(\mathbb{A})) = C^\infty(G(\mathbb{R}), C^\infty(G(\mathbb{A}_f)))$ is of *uniform moderate growth*, if f satisfies

$$\forall D \in \mathcal{U}(\mathfrak{g}), \forall |.|_\alpha \quad \exists N = N(f, |.|_\alpha), C = C(f, D, |.|_\alpha) \in \mathbb{R}_{\geq 0} : \quad |Df(g)|_\alpha < C\|g\|^N$$

for all $g \in G(\mathbb{R})$. Here $\|g\| = \sqrt{\det'(g)^{-2} + 2 \cdot tr(g \cdot {}^t\overline{g})}$ is the usual norm on $G(\mathbb{R})$, cf. [6] 1.2. Now, let V_G be the space of smooth functions $f \in C^\infty(Z(\mathbb{R})^\circ G(\mathbb{Q})\backslash G(\mathbb{A}))$ which are of uniform moderate growth. It can be decomposed as

$$V_G = V_G(G) \oplus V_G(P),$$

where for $Q \in \{P, G\}$, $V_G(Q)$ denotes the space of $f \in V_G$ which are negligible along every parabolic $\mathbb{Q}$-subgroup P' of G not conjugate to Q. (This latter condition means that the constant term of f along P' is orthogonal to the space of cusp forms of the Levi subgroup of P', cf. [11] 1.1.) A proof of this result in a much more general context was already given by R. P. Langlands in a letter to A. Borel, but may also be found in [7], Thm. 2.4. Declaring

$$\mathcal{A}_{cusp}(G) := V_G(G) \cap \mathcal{A}(G) \quad \text{and} \quad \mathcal{A}_{Eis}(G) := V_G(P) \cap \mathcal{A}(G),$$

we therefore get the desired decomposition of $\mathcal{A}(G)$ as $(\mathfrak{g}_s, K, G(\mathbb{A}_f))$-module:

$$\mathcal{A}(G) = \mathcal{A}_{cusp}(G) \oplus \mathcal{A}_{Eis}(G).$$

We remark that $\mathcal{A}_{cusp}(G)$ consists precisely of all cuspidal automorphic forms in $\mathcal{A}(G)$ (which explains the subscript), so finding the above decomposition amounted essentially in finding an appropriate complement - namely $\mathcal{A}_{Eis}(G)$ - to the space of cuspidal automorphic forms in $\mathcal{A}(G)$. We will explain the choice of the subscript of $\mathcal{A}_{Eis}(G)$ in the next section.

2.2. Eisenstein series. The space $\mathcal{A}_{Eis}(G) = V_G(P) \cap \mathcal{A}(G)$ has a further description using Eisenstein series attached to cuspidal automorphic representations of $L(\mathbb{A})$. Therefore we need some technical assumptions:

Let Q be a proper parabolic $\mathbb{Q}$-subgroup of G with Levi- (resp. Langlands-) decomposition $Q = L_Q N_Q = M_Q T_Q N_Q$. In analogy to our section 1.4 we decompose the central torus T_Q of L_Q as $T_Q = A_Q Z$ with $A_Q = G_s \cap T_Q$. The torus A_Q acts on N_Q by the adjoint representation and hence defines a set of weights denoted by $\Delta(Q, A_Q)$. Summing up the halfs of these weights gives a character ρ_Q.

Now, let φ_Q be a finite set of irreducible representations of $L_Q(\mathbb{A})$, given by the following data:

(1) $\chi : A_Q(\mathbb{R})^\circ \to \mathbb{C}^*$ is a continuous character. Since $T_Q(\mathbb{R})^\circ = A_Q(\mathbb{R})^\circ \times Z(\mathbb{R})^\circ$, we can and will view the derivative $d\chi$ also as a character of $\mathfrak{t}_Q$.

(2) $\widetilde{\pi}$ is an irreducible, unitary $L_Q(\mathbb{A})$-subrepresentation of $L^2_{cusp}(L_Q(\mathbb{Q})T_Q(\mathbb{R})^\circ\backslash L_Q(\mathbb{A}))$, the cuspidal spectrum of $L_Q(\mathbb{A})$. We assume furthermore, that the infinitesimal character of $\widetilde{\pi}$ matches the one of the dual of an irreducible $M(\mathbb{R})$-subrepresentation of the Lie algebra cohomology $H^*(\mathfrak{n}_Q, E)$.

In short, this mean that φ_Q is a finite set of cuspidal automorphic representations $\pi := \chi\widetilde{\pi} := e^{\langle d\chi, H_Q(.)\rangle}\widetilde{\pi}$ of $L_Q(\mathbb{A})$, with χ and $\widetilde{\pi}$ as above. Here, $H_Q : G(\mathbb{A}) \to (\mathfrak{t}_Q)^*_\mathbb{C}$ is the usual Harish-Chandra height function.

Finally, three further "compatibility conditions" have to be satisfied between these sets φ_Q, skipped here and listed in [11], 1.2. The family of all collections $\varphi = \{\varphi_Q\}$ of such finite sets is denoted Ψ.

Let $\mathrm{I}_{Q,\widetilde{\pi}} := \mathrm{Ind}^{G(\mathbb{A})}_{Q(\mathbb{A})}[\widetilde{\pi}]$ (normalized induction). For a K-finite function $f \in \mathrm{I}_{Q,\widetilde{\pi}}$, $\Lambda \in (\mathfrak{a}_Q)^*_\mathbb{C}$ and $g \in G(\mathbb{A})$ an Eisenstein series is (at least) formally defined as

$$E(f,\Lambda)(g) = \sum_{\gamma\in Q(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g)e^{\langle\Lambda, H_Q(\gamma g)\rangle}.$$

If we set $\mathfrak{a}_Q^+ := \{\Lambda \in (\mathfrak{a}_Q)^*_\mathbb{C} | Re(\Lambda) \in \rho_Q + C\}$, where C equals the open, positive Weyl-chamber with respect to $\Delta(Q, A_Q)$, the series converges absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times \mathfrak{a}_Q^+$. It is known that $E(f,\Lambda)$ is an automorphic form there and that the map $\Lambda \mapsto E(f,\Lambda)(.)$ can be analytically continued to a meromorphic function on all of $(\mathfrak{a}_Q)^*_\mathbb{C}$, cf. [26] p. 140 or [24] §7. It has only a finite number of at most simple poles for $\Lambda \in (\mathfrak{a}_Q)^*_\mathbb{C} \hookrightarrow (\mathfrak{t}_Q)^*_\mathbb{C}$ with $Re(\Lambda) \geq 0$, see [26] Prop. IV.1.11. If there is a singularity at such a $\Lambda = \Lambda_0 \in (\mathfrak{a}_Q)^*_\mathbb{C} \cong \mathbb{C}$, the residue $\mathrm{Res}_{\Lambda_0}E(f,\Lambda)(g)$ is a smooth function and square-integrable modulo $Z(\mathbb{R})^\circ$, cf. [26] I.4.11. It defines hence an element of $L^2_{res}(G(\mathbb{Q})Z(\mathbb{R})^\circ\backslash G(\mathbb{A}))$, the residual spectrum of $G(\mathbb{A})$.

Now we are able to give the desired description of $\mathcal{A}_{Eis}(G)$ by Eisenstein series: For $\pi = \chi\widetilde{\pi} \in \varphi_P$ let $\mathcal{A}_\varphi(G)$ be the space of functions, spanned by all possible residues and derivatives of Eisenstein series defined via all K-finite $f \in \mathrm{I}_{P,\widetilde{\pi}}$ at the value $d\chi$ with $Re(d\chi) \geq 0$. It is a $(\mathfrak{g}_s, K, G(\mathbb{A}_f))$-module. Thanks to the functional equations (see [26] IV.1.10) satisfied by the Eisenstein series considered, this is well defined, i.e., independent of the choice of a representative for the class of P (whence we took P itself) and the choice of a representation $\pi \in \varphi_P$. Finally, we get

Theorem 2.1 ([11], Thm. 1.4; [26] III, Thm. 2.6)**.** *There is a direct sum decomposition as $(\mathfrak{g}_s, K, G(\mathbb{A}_f))$-module*

$$\mathcal{A}_{Eis}(G) = \bigoplus_{\varphi\in\Psi} \mathcal{A}_\varphi(G)$$

2.3. Definition of automorphic cohomology. Recall from section 2.1 that $\mathcal{A}(G)$ is a $(\mathfrak{g}_s, K, G(\mathbb{A}_f))$-module. We can define:

Definition. The $G(\mathbb{A}_f)$-module

$$H^q(G,E) := H^q(\mathfrak{g}_s, K, \mathcal{A}(G) \otimes E)$$

is called the *automorphic cohomology* of G with respect to E. Its natural complementary $G(\mathbb{A}_f)$-submodules

$$H^q_{cusp}(G,E) := H^q(\mathfrak{g}_s, K, \mathcal{A}_{cusp}(G)\otimes E) \quad \& \quad H^q_{Eis}(G,E) := H^q(\mathfrak{g}_s, K, \mathcal{A}_{Eis}(G)\otimes E)$$

are called *cuspidal cohomology* and *Eisenstein cohomology*, respectively.

We recall that the space $\mathcal{A}_{cusp}(G)$ consists of smooth and K-finite functions in $L^2_{cusp}(G(\mathbb{Q})Z(\mathbb{R})^\circ\backslash G(\mathbb{A}))$, which itself, acted upon by $G(\mathbb{A})$ via right translation, decomposes into a direct Hilbert sum of irreducible admissible $G(\mathbb{A})$-representations π, each of which occurring with finite multiplicity $m(\pi)$. By Thm. 18.1.(b) of [2] we have (Strong) Multiplicity One for automorphic representations of $G(\mathbb{A})$ appearing in the discrete spectrum, whence we get by [9], XIII, a direct sum decomposition

$$H^q_{cusp}(G,E) = \bigoplus_{\pi} H^q(\mathfrak{g}_s, K, \pi \otimes E), \tag{1}$$

the sum ranging over all (equivalence classes of) cuspidal automorphic representations π of $G(\mathbb{A})$.

Using Theorem 2.1 we can also decompose Eisenstein cohomology.

$$H^q_{Eis}(G,E) = \bigoplus_{\varphi\in\Psi} H^q(\mathfrak{g}_s, K, \mathcal{A}_\varphi(G) \otimes E). \tag{2}$$

Remark 2.1. The reason underlying these two decomposition is essentially the same. In fact, one can think of (1) as a degenerate version of (2), by rereading section 2.2 and replacing in mind each proper parabolic $\mathbb{Q}$-subgroup by G itself. Chronologically, however, the approach was as presented here.

3. Eisenstein cohomology

By (1) and (2) it is clear, how one can (at least in principle) describe the cuspidal cohomology and the Eisenstein cohomology of G. One has to determine the individual $G(\mathbb{A}_f)$-submodules $H^q(\mathfrak{g}_s, K, \pi\otimes E)$ and $H^q(\mathfrak{g}_s, K, \mathcal{A}_\varphi(G)\otimes E)$ (notation as in section 2.3). In this section we want to review a method how to construct Eisenstein cohomology, using the notion of "(π,w)-types" and carry it out concretely. The next subsection is devoted to the definition of the latter.

3.1. Classes of type (π, w)**.** Take $\pi = \chi\widetilde{\pi} \in \varphi_P$ and consider the symmetric tensor algebra

$$S_\chi(\mathfrak{a}^*) = \bigoplus_{n\geq 0} \bigodot^n \mathfrak{a}^*_{\mathbb{C}},$$

$\bigodot^n \mathfrak{a}^*_{\mathbb{C}}$ being the symmetric tensor product of n copies of $\mathfrak{a}^*_{\mathbb{C}}$, as module under $\mathfrak{a}$: Since $S_\chi(\mathfrak{a}^*)$ can be viewed as the space of polynomials on $\mathfrak{a}_{\mathbb{C}}$, we let $\xi\in\mathfrak{a}$ act via translation followed by multiplication with $\langle\xi, d\chi\rangle$. This explains the subscript "χ". We extend this action trivially on $\mathfrak{l}$ and $\mathfrak{n}$ to get an action of the Lie algebra $\mathfrak{p}$ on the space $S_\chi(\mathfrak{a}^*)$. We may also define a $P(\mathbb{A}_f)$-module structure via the rule

$$q\cdot X = e^{\langle d\chi, H_P(q)\rangle} X,$$

for $q\in P(\mathbb{A}_f)$ and $X\in S_\chi(\mathfrak{a}^*)$. There is a linear isomorphism

$$\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\widetilde{\pi}\otimes S_\chi(\mathfrak{a}^*)] \xrightarrow{\sim} \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*),$$

so in particular one can view the right hand side as a $G(\mathbb{A})$-module by transport of structure. Doing this, it is shown in [10], pp. 256-257, that

$$\begin{aligned} &H^q(\mathfrak{g}_s, K, \mathrm{I}_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*)\otimes E) \cong \\ &\bigoplus_{\substack{w\in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}} = d\chi}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}\left[H^{q-l(w)}(\mathfrak{m}, K_M, \widetilde{\pi}_\infty\otimes F_w)\otimes\mathbb{C}_{d\chi}\otimes\widetilde{\pi}_f\right]. \end{aligned} \tag{3}$$

Some notation needs to be explained: Above, $K_M = K \cap M(\mathbb{R})$, F_w is the finite dimensional representation of $M(\mathbb{C})$ with highest weight $\mu_w := w(\lambda+\rho) - \rho|_{\mathfrak{b}_\mathbb{C}}$ and $\mathbb{C}_{d\chi}$ the one-dimensional, complex $P(\mathbb{A}_f)$-module on which $q \in P(\mathbb{A}_f)$ acts by multiplication by $e^{\langle d\chi, H_P(q)\rangle}$. A non-trivial class in a summand of the right hand side is called a cohomology class *of type* (π, w), $\pi \in \varphi_P$, $w \in W^P$. (This notion was first introduced in [34].)
Further, as $L(\mathbb{R}) \cong M(\mathbb{R}) \times T(\mathbb{R})^\circ$, $\widetilde{\pi}_\infty$ can be regarded as an irreducible, unitary representation of $M(\mathbb{R})$. Therefore, a (π, w) type consists of an irreducible representation $\pi = \chi\widetilde{\pi}$ whose unitary part $\widetilde{\pi} = \widetilde{\pi}_\infty \hat{\otimes} \widetilde{\pi}_f$ (completed Hilbert tensor product) has at the infinite place an irreducible, unitary representation $\widetilde{\pi}_\infty$ of the semisimple group $M(\mathbb{R})$ with non-trivial $(\mathfrak{m}, K_M)$-cohomology with respect to F_w.

3.2. The possible archimedean components of π. Recall that $\pi_\infty = \chi\widetilde{\pi}_\infty$, so classifying all possible archimedean components π_∞ of $\pi \in \varphi_P$ means to classify all characters χ of $A(\mathbb{R})^\circ$ and representations $\widetilde{\pi}_\infty$ of $M(\mathbb{R})$ in question.
By the above, we shall therefore find all irreducible unitary representations of $M(\mathbb{R}) = SL_1(\mathbb{H}) \times SL_1(\mathbb{H})$ which have non-trivial $(\mathfrak{m}, K_M)$-cohomology with respect to a representation F_w, $w \in W^P$. This is achieved in the next

Proposition 3.1. *Let V be an irreducible unitary representation of $M(\mathbb{R})$ and F any finite dimensional, irreducible representation of $M(\mathbb{R})$. Then*

$$H^q(\mathfrak{m}, K_M, V \otimes F) = \begin{cases} \mathbb{C} & \text{if } q = 0 \text{ and } V \cong F \\ 0 & \text{else} \end{cases}$$

Proof. Since $M(\mathbb{R})$ is compact, cohomology with respect to $V \otimes F$ is one-dimensional, if $V \cong \check{F}$ and $q = 0$ and vanishes otherwise. As $M(\mathbb{C})$ is of Cartan type $\mathrm{A}_1 \times \mathrm{A}_1$, any finite dimensional representation F of $M(\mathbb{R})$ is self-dual. $\square$

A direct calculation gives us furthermore the affine action of W^P on $\lambda = \sum_{i=1}^3 c_i\alpha_i$, restricted to $\mathfrak{b}_\mathbb{C}$. It is listed in Table 1.

	$\mu_w = w(\lambda+\rho) - \rho\|_{\mathfrak{b}_\mathbb{C}}$
id	$(2c_1 - c_2)\omega_1 + (2c_3 - c_2)\omega_3$
w_2	$(c_1 + c_2 - c_3 + 1)\omega_1 + (c_3 + c_2 - c_1 + 1)\omega_3$
w_2w_1	$(2c_2 - c_1 - c_3)\omega_1 + (c_1 + c_3 + 2)\omega_3$
w_2w_3	$(c_1 + c_3 + 2)\omega_1 + (2c_2 - c_1 - c_3)\omega_3$
$w_2w_1w_3$	$(c_3 + c_2 - c_1 + 1)\omega_1 + (c_1 + c_2 - c_3 + 1)\omega_3$
$w_2w_1w_3w_2$	$(2c_3 - c_2)\omega_1 + (2c_1 - c_2)\omega_3$

TABLE 1. The affine action of W^P

We still need to classify the characters χ in question. As it follows from (3), the character χ must satisfy $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$. Its possible values are hence given in Table 2.

3.3. Definition of the Eisenstein map. In order to construct Eisenstein cohomology classes, we start from a class of type (π, w). Since we are interested in cohomology, we can assume without loss of generality that $\widetilde{\pi}_\infty = F_w$ (cf., Proposition 3.1) and by (3) that $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$. Moreover, we can assume that $d\chi$ lies inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. So Table 2 shows that we must have $w \in W^+ := \{w_2w_3, w_2w_1w_3, w_2w_1w_3w_2\}$.

	$d\chi = -w(\lambda+\rho)\vert_{\mathfrak{a}_{\mathbb{C}}}$
id	$-(c_2+2)\alpha_2 < 0$
w_2	$-(c_1-c_2+c_3+1)\alpha_2 < 0$
w_2w_1	$-(c_3-c_1)\alpha_2 \leq 0$
w_2w_3	$(c_3-c_1)\alpha_2 \geq 0$
$w_2w_1w_3$	$(c_1-c_2+c_3+1)\alpha_2 > 0$
$w_2w_1w_3w_2$	$(c_2+2)\alpha_2 > 0$

TABLE 2. The possible characters χ

We reinterpret $S_\chi(\mathfrak{a}^*)$ as the space of formal, finite $\mathbb{C}$-linear combinations of differential operators $\frac{d^n}{d\Lambda^n}$ on $\mathfrak{a}^*_{\mathbb{C}}$. It is a consequence of [26], Prop. IV.1.11 or [24] §7, that there is a non-trivial polynomial $q(\Lambda)$ such that $q(\Lambda)E(f,\Lambda)$ is holomorphic at $d\chi$ for all K-finite $f \in \mathrm{I}_{P,\widetilde{\pi}}$. Since $\mathcal{A}_\varphi(G)$ can be written as the space which is generated by the coefficient functions in the Taylor series expansion of $q(\Lambda)E(f,\Lambda)$ at $d\chi$, f running through the K-finite functions in $\mathrm{I}_{P,\widetilde{\pi}}$, we are able to define a surjective homomorphism E_π of $(\mathfrak{g}_s, K, G(\mathbb{A}_f))$-modules between the K-finite elements in $\mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*)$ and $\mathcal{A}_\varphi(G)$ by

$$f \otimes \frac{d^n}{d\Lambda^n} \mapsto \frac{d^n}{d\Lambda^n}\left(q(\Lambda)E(f,\Lambda)\right)\vert_{d\chi}. \tag{4}$$

Therefore we get a well-defined map in cohomology

$$E^*_\pi : H^*(\mathfrak{g}_s, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \longrightarrow H^*(\mathfrak{g}_s, K, \mathcal{A}_\varphi(G) \otimes E)$$

which we will call *Eisenstein map.* It is this way, how we can lift classes of type (π, w) (which, as we now recall, are the elements of the space on the left hand side) to Eisenstein cohomology.

3.4. Holomorphic Case. Suppose $[\omega] \in H^q(\mathfrak{g}, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class of type (π, w), represented by a morphism ω, such that for all elements $f \otimes \frac{d^n}{d\Lambda^n}$ in its image, $E_\pi(f \otimes \frac{d^n}{d\Lambda^n}) = \frac{d^n}{d\Lambda^n}\left(q(\Lambda)E(f,\Lambda)\right)\vert_{d\chi}$ is just the regular value $E(f, d\chi)$ of the Eisenstein series $E(f,\Lambda)$, which is assumed to be holomorphic at the point $d\chi = -w(\lambda+\rho)\vert_{\mathfrak{a}_{\mathbb{C}}}$ inside the closed, positive Weyl chamber defined by $\Delta(P, A)$. Then $E^q_\pi([\omega])$ is a non-trivial Eisenstein cohomology class

$$E^q_\pi([\omega]) \in H^q(\mathfrak{g}_s, K, \mathcal{A}_\varphi(G) \otimes E).$$

This is a consequence of [34], Thm. 4.11.

3.5. Residual Case. Suppose now that there is a K-finite $f \in \mathrm{I}_{P,\widetilde{\pi}}$ such that the Eisenstein series $E(f,\Lambda)$ has a pole at $d\chi$ and notice that $E(f,\Lambda)$ is always holomorphic at 0, by [26], Prop. IV.1.11 (b). If $[\omega] \in H^q(\mathfrak{g}_s, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ is a class represented by a morphism ω having only functions f as in the previous sentence in its image, then the residual Eisenstein cohomology class $E^q_\pi([\omega])$ defines a class

$$E^q_\pi([\omega]) \in H^{q'}(\mathfrak{g}_s, K, \mathcal{A}_\varphi(G) \otimes E),$$

with $q' = 4 - q$. This follows from [14], Thm. 2.1.

It is a rather delicate issue to describe the image of the Eisenstein map E^q_π in the residual case (in the above sense). In order to do that, we need more knowledge on the residues of Eisenstein series. Hence, we shall determine all relevant poles in the next section.

3.6. Poles of Eisenstein series. We have seen that the Eisenstein series we have to consider are meromorphic functions in the parameter $\Lambda \in \mathfrak{a}_\mathbb{C}^*$. However, we have also seen that we only need to determine the behaviour of holomorphy of the Eisenstein series at certain points $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}^*} \in \mathfrak{a}_\mathbb{C}^* \subset (\mathfrak{g}_s)_\mathbb{C}^*$ given in Table 2.

Writing $\Lambda = (s_1, s_2) \in \mathfrak{t}_\mathbb{C}^*$ with respect to the coordinates given by the functionals 2α and 2β (cf. section 1.1) then the restriction of Λ to $\mathfrak{a}_\mathbb{C}^*$ is $s_1 - s_2$ in the coordinate given by the functional 2α. In terms of the absolute roots of $\mathfrak{g}_s$ this means that if $\Lambda|_{\mathfrak{a}_\mathbb{C}^*} = s\alpha_2$, then $s = s_1 - s_2$. For our evaluation points $d\chi$, these values s are hence listed in Table 2.

As a consequence of this consideration we only need to check the behaviour of holomorphy of the Eisenstein series along the line $s_1 + s_2 = 0$ representing $\mathfrak{a}_\mathbb{C}^*$ inside $\mathfrak{t}_\mathbb{C}^*$, a point Λ in $\mathfrak{a}_\mathbb{C}^*$ being identified with its coordinate s as above. For this purpose, let us recall the following result, cf. [26], I.4.10:

Proposition 3.2. *The poles of the Eisenstein series $E(f,\Lambda)$ are the ones of its constant Fourier coefficient $E(f,\Lambda)_P$ along P.*

Hence, we analyze this constant Fourier coefficient. It can be written as

$$E(f,\Lambda)_P = f \cdot e^{\langle \Lambda, H_P(.)\rangle} + M(s,\widetilde{\pi})(f \cdot e^{\langle \Lambda, H_P(.)\rangle}), \tag{5}$$

where $M(s,\widetilde{\pi})$ is the meromorphic function in the parameter $s \leftrightarrow \Lambda$, given for $g \in G(\mathbb{A})$, w the only non-trivial element in $W_\mathbb{Q}$ and $Re(s) \gg 0$ by

$$M(s,\widetilde{\pi}) : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\widetilde{\pi} \otimes e^{\langle \Lambda, H_P(.)\rangle}] \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[w(\widetilde{\pi}) \otimes e^{\langle w(\Lambda), H_P(.)\rangle}]$$
$$M(s,\widetilde{\pi})\psi(g) = \int_{N(\mathbb{A})} \psi(w^{-1}ng)dn.$$

as in [26], II.1.6. Since $L(\mathbb{A}) = GL_1'(\mathbb{A}) \times GL_1'(\mathbb{A})$, the cuspidal automorphic representation $\widetilde{\pi}$ can be decomposed into $\widetilde{\pi} = \sigma \hat{\otimes} \tau$, σ and τ being cuspidal automorphic representations of the factors $GL_1'(\mathbb{A})$. The action of w on $\widetilde{\pi}$ then reads explicitly as $w(\widetilde{\pi}) = \tau \hat{\otimes} \sigma$. Moreover, if $\Lambda \in \mathfrak{a}_\mathbb{C}^*$ then $w(\Lambda) = -\Lambda$.

Now, let S be a finite set of places containing $S(D)$, such that $\widetilde{\pi}_p$ has got a non-trivial $L(\mathbb{Z}_p)$-fixed vector for $p \notin S$: That is, outside S, L splits and $\widetilde{\pi}_p$ is unramified. We can hence formally write $f \cdot e^{\langle \Lambda, H_P(.)\rangle} = \otimes_p' \psi_p =: \psi$ (restricted tensor product over all places), where ψ_p is a suitably normalized, $L(\mathbb{Z}_p)$-fixed function for $p \notin S$. Therefore, $M(s,\widetilde{\pi})(f \cdot e^{\langle \Lambda, H_P(.)\rangle})$ factors as $M(s,\widetilde{\pi})\psi = \otimes_p' M(s,\widetilde{\pi}_p)\psi_p$. Using the Gindikin-Karpelevich integral formula, as shown in [23], p. 27 (see also [35], p.554), we see that - again for suitably normalized, non-trivial $L(\mathbb{Z}_p)$-fixed functions $\widetilde{\psi}_p$

$$M(s,\widetilde{\pi})\psi = \bigotimes_{p \in S} M(s,\widetilde{\pi}_p)\psi_p \otimes {\bigotimes_{p \notin S}}' \frac{L(s,\widetilde{\pi}_p,\check{r})}{L(1+s,\widetilde{\pi}_p,\check{r})}\widetilde{\psi}_p. \tag{6}$$

Here, $\check{r}$ is the dual of the adjoint representation of the L-group of L on the Lie algebra of the L-group of N (see [4], 2 and 3.4). It is irreducible ([23] sec. 6, case (iii)) and the corresponding local L-function associated to $\widetilde{\pi}$ and $\check{r}$ at the place $p \notin S$ is denoted $L(s,\widetilde{\pi}_p,\check{r})$, cf. again [4], 7.2. The following proposition is crucial.

Proposition 3.3. *For all $p \in S(D)$ the operator $M(s,\widetilde{\pi}_p)$ is holomorphic and non-vanishing in the region $Re(s) > 0$. Hence, there is a K-finite $f \in \mathrm{I}_{P,\widetilde{\pi}}$ such that the Eisenstein series $E(f,\Lambda)$ has a pole at $\Lambda = s\alpha_2$, $Re(s) > 0$, if and only if the product $\prod_{p \notin S(D)} M(s,\widetilde{\pi}_p)$ has a pole at s, $Re(s) > 0$.*

Proof. Observe that for $p \in S(D)$, $\widetilde{\pi}_p$ is obviously supercuspidal, (i.e. every matrix coefficient is compactly modulo the center $T(\mathbb{Q}_p)$ of $L(\mathbb{Q}_p)$), since for these places p the quotient $L(\mathbb{Q}_p)/T(\mathbb{Q}_p)$ is compact itself. So, for any such p, $M(s, \widetilde{\pi}_p)$ is nothing but the intertwining operator whose image is the Langlands quotient associate to $P(\mathbb{Q}_p)$, the tempered representation $\widetilde{\pi}_p$ and the value s with $Re(s) > 0$. In particular, $M(s, \widetilde{\pi}_p)$ is holomorphic and non-vanishing for $Re(s) > 0$, see, e.g., [9], IV, Lemma 4.4 and XI, Cor. 2.7. Hence, the proposition follows. □

We distinguish three cases: Either (i) σ and τ are both not one-dimensional, (ii) exactly one of the representations σ and τ is one-dimensional, or (iii) $\widetilde{\pi} = \sigma\tau$ is a character of $L(\mathbb{A})$.

Case (i) is the generic one, i.e., σ_p and τ_p are both locally generic for $p \notin S(D)$. This is well-known and can be seen as follows: For a cuspidal automorphic representation ρ of $GL_1'(\mathbb{A})$, let $JL(\rho)$ be the global Jacquet-Langlands lift as described in [21] and [12]. It is known that $JL(\rho)$ is an automorphic representation of $GL_2(\mathbb{A})$ appearing in the discrete spectrum. Moreover, $JL(\rho)$ is cuspidal if and only if ρ is not one-dimensional, see [12] Thm. 8.3. At $p \notin S(D)$ the Levi subgroup L splits, i.e., $L(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p)$ and the local component of $JL(\rho)$ at such a p satisfies the (meaningful) equality $JL(\rho)_p = \rho_p$. So, assuming that ρ is not a character, $\rho_p = JL(\rho)_p$ is the local component of a cuspidal automorphic representation of $GL_2(\mathbb{A})$. By [36], corollary on p. 190, ρ_p is hence generic.
Coming back to the setup of case (i), i.e., σ and τ are both not one-dimensional, we see that for $p \notin S$,

$$L(s, \widetilde{\pi}_p, \check{r}) = L(s, \sigma_p \times \check{\tau}_p),$$

the usual local Rankin-Selberg L-function, cf. [20]. As a local Rankin-Selberg L-function has no poles and zeros in the region $Re(s) > 1$, $L(s, \sigma_p \times \check{\tau}_p)^{-1} M(s, \widetilde{\pi}_p)$ is holomorphic and non-vanishing in the region $Re(s) > 0$ for $p \notin S$, cf. (6). In [25], Prop. I.10, p.639, it is shown that for $p \in S - S(D)$, $L(s, \sigma_p \times \check{\tau}_p)^{-1} M(s, \widetilde{\pi}_p)$ is holomorphic and non-vanishing in the region $Re(s) > 0$. Hence, the poles of the product $\prod_{p \notin S(D)} M(s, \widetilde{\pi}_p)$ are the ones of the partial L-function $\prod_{p \notin S(D)} L(s, \sigma_p \times \check{\tau}_p)$. Furthermore, by [12] Thm. 8.3 the local components $JL(\sigma)_p$ and $JL(\check{\tau})_p$ are both square-integrable for all $p \in S(D)$, and therefore $L(s, JL(\sigma)_p \times JL(\check{\tau})_p)$ is holomorphic and non-zero for $Re(s) > 0$ and all $p \in S(D)$. This implies that the poles of $\prod_{p \notin S(D)} M(s, \widetilde{\pi}_p)$ in the region $Re(s) > 0$ are the poles of the global Rankin-Selberg L-function $L(s, JL(\sigma) \times JL(\check{\tau}))$. Combining this with Proposition 3.3, we finally see that in case (i), there is a (K-finite) $f \in \mathrm{I}_{P,\widetilde{\pi}}$ such that $E(f, \Lambda)$ has a pole at $\Lambda = s\alpha_2$ with $Re(s) > 0$, if and only if $L(s, JL(\sigma) \times JL(\check{\tau}))$ has a pole. By the well–known analytic properties of global Rankin-Selberg L-functions, this happens if and only if $s = 1$ and $JL(\sigma) = JL(\tau)$, i.e., by strong multiplicity one for GL_1', if $s = 1$ and $\sigma = \tau$.

For the remaining cases, we use the work of N. Grbac. He normalized the local intertwining operators for $p \notin S$ in [13], Cor. 2.2.5 using the work of C. Mœglin and J.-L. Waldspurger ([25]). Using Grabc's result and a case-by-case analysis distinguishing the cases $p \in S(D)$, $p \in S - S(D)$ and $p \notin S$ analogous to the reasoning we provided above, it turns out that in case (ii) the poles of the intertwining operator are the ones of the standard Langlands L-function $L(s - \frac{1}{2}, \sigma\check{\tau})$ attached to the representation $\sigma\check{\tau}$ of $GL_1'(\mathbb{A})$. But this L-function is entire, see [21], so there are no poles in case (ii).
If $\widetilde{\pi} = \sigma\tau$ is a character of $L(\mathbb{A})$, then the poles of an Eisenstein series are determined by the product

$$(7) \qquad L(s,\sigma\tau^{-1})L(s-1,\sigma\tau^{-1})\prod_{p\in S(D)} L(s-1,\sigma_p\tau_p^{-1})^{-1},$$

of Hecke L-functions. This was proved again in [13], Cor. 2.2.5, applying the idea of [25], Lemme I.8, i.e., via induction from generic representations of smaller parabolic subgroups. We therefore conclude that in case (iii) $M(s,\widetilde{\pi})$ has a pole at $Re(s)>0$ if and only if $s=2$ and $\sigma=\tau$. To see this, observe that the poles of the global Hecke L-functions $L(s,\sigma\tau^{-1})L(s-1,\sigma\tau^{-1})$ at $s=1$ are canceled by the zeros of the inverse of the $|S(D)|$-many local L-functions. As D is non-split over $\mathbb{Q}$, $|S(D)|\geq 2$.

We summarize the results of this section in the following proposition.

Proposition 3.4. *Let $\widetilde{\pi}=\sigma\otimes\tau$ be a unitary cuspidal automorphic representation of $L(\mathbb{A})$. There is a K-finite function $f\in I_{P,\widetilde{\pi}}$, such that $E(f,\Lambda)$ has a pole at $\Lambda=s\alpha_2$, $Re(s)>0$, if and only if $\sigma=\tau$ and either*

(1) $\dim\sigma>1$ *and* $s=1$ *or*
(2) $\dim\sigma=1$ *and* $s=2$

3.7. The image of the Eisenstein map revisited. In sections 3.4 and 3.5 we gave a (still incomplete) description of the image (and so *vice versa* of the kernel) of the Eisenstein map E^q_π. In order to complete it, we still need to understand the image of E^q_π in the case of non-holomorphic Eisenstein series (see section 3.5). Therefore observe that the residue of an Eisenstein series, as determined in Proposition 3.4, generates a residual automorphic representation of $G(\mathbb{A})$, i.e., an irreducible subrepresentation of $L^2_{res}(G(\mathbb{Q})Z(\mathbb{R})^\circ\backslash G(\mathbb{A}))$, cf. 2.2. In particular, $Z(\mathbb{R})^\circ$ acts trivially on such a representation. As $G(\mathbb{R})=Z(\mathbb{R})^\circ\times G_s(\mathbb{R})$, cf. section 1.2, we may view its archimedean component as an irreducible, unitary representation of $G_s(\mathbb{R})=SL_2(\mathbb{H})$. Clearly, we are only interested in residual automorphic representations, which have non-vanishing $(\mathfrak{g}_s,K)$-cohomology tensorised by the $G_s(\mathbb{R})$-representation E. Therefore, in order to understand the image of the Eisenstein map in the case of non-holomorphic Eisenstein series, we have to understand the cohomological unitary dual of $G_s(\mathbb{R})$. It is classified in the next proposition.

Proposition 3.5. *For each irreducible, finite-dimensional representation E of $G_s(\mathbb{R})$ of highest weight $\lambda=\sum_{i=1}^3 c_i\alpha_i$ there is an integer $j(\lambda)$, $0\leq j(\lambda)\leq 3$ such that the irreducible, unitary representations of $G_s(\mathbb{R})$ with non-trivial cohomology with respect to E are the uniquely determined representations $A_j(\lambda)$, $j(\lambda)\leq j\leq 2$ having the property*

$$H^q(\mathfrak{g}_s,K,A_j(\lambda)\otimes E)=\begin{cases}\mathbb{C} & \text{if } q=j \text{ or } q=5-j\\ 0 & \text{otherwise}\end{cases}$$

This integer is given as

$$j(\lambda)=\begin{cases}0 & \text{if } \lambda=0\\ 1 & \text{if } \lambda=k\omega_2, k\geq 1\\ 2 & \text{if } \lambda\neq k\omega_2, k\geq 0 \text{ but } c_1=c_3\\ 3 & \text{otherwise}\end{cases}$$

Let $J(F,t)$ be the Langlands' Quotient associate to the triple $(P_s(\mathbb{R}),F,t\alpha_2)$, F an irreducible representation of $M(\mathbb{R})$ and $t>0$. Then these representations read as follows:

$$A_0(\lambda)=\mathbf{1}_{G_s(\mathbb{R})}=J(F_{id},2)$$
$$A_1(\lambda)=J(F_{w_2},1)$$

$$A_2(\lambda) = \mathrm{Ind}_{P_s(\mathbb{R})}^{G_s(\mathbb{R})}[F_{w_2w_3} \otimes \mathbf{1}_{A(\mathbb{R})}]$$

Short sketch of a proof. This can be achieved using the well-known Vogan-Zuckerman classification of the cohomological unitary dual of connected semisimple Lie groups. More precisely it is a consequence of Thm.s 5.5, 5.6, 6.16 and Prop. 6.5 in [37]. The information relevant for applying these results to our specific case consists of a list of the so-called θ-stable parabolic subalgebras $\mathfrak{q}$ of $(\mathfrak{g}_s)_\mathbb{C} \cong \mathfrak{sl}_4(\mathbb{C})$ up to K-conjugacy. There are four such classes, represented by subalgebras $\mathfrak{q}_0$, $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $\mathfrak{q}_3$. In fact, the essential data (cf. Table 3) is already provided by knowing the Levi subalgebras $\mathfrak{l}_j$, $j = 0, 1, 2, 3$, of these parabolic subalgebras and the sets Δ_j of those roots which appear both, in the direct sum decomposition of the nilpotent radical of $\mathfrak{q}_j$, and the (-1)-Eigenspace of θ in $\mathfrak{g}_s$. If two such sets of roots coincide, the corresponding θ-stable parabolic subalgebras, although not K-conjugate, provide isomorphic irreducible unitary representations and may hence be identified. According to Table 3, we may hence focus on $\mathfrak{l}_0$, $\mathfrak{l}_1$ and $\mathfrak{l}_2$. Their corresponding irreducible, unitary representations are $A_0(\lambda)$, $A_1(\lambda)$ and $A_2(\lambda)$ as described in our proposition. □

	$\mathfrak{l}_j$	$(-i)\Delta_j$
$j = 0$	$\mathfrak{g}_s$	$\emptyset$
$j = 1$	$\mathbb{R} \oplus \mathfrak{sl}_2(\mathbb{C})$	$\{\varepsilon_1 + \varepsilon_3\}$
$j = 2$	$\mathbb{R} \oplus \mathbb{C}$	$\{\varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_4\}$
$j = 3$	$\mathbb{R}^2 \oplus \mathfrak{sl}_2(\mathbb{R})$	$\{\varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_4\}$

TABLE 3

Remark 3.1. If $c_1 \neq c_3$, i.e., $E \ncong \check{E}$, then the proposition says that there are no irreducible, unitary cohomological representations of $G_s(\mathbb{R})$.
As it follows from the last paragraph of [37], p.8, $A_2(\lambda)$ is the only tempered representation among the $A_i(\lambda)$, $i = 0, 1, 2$.

Now, let (π, w) be a tuple as before, i.e., $w \in W^P$ and $\pi = \chi\widetilde{\pi} \in \varphi_P$ with $\widetilde{\pi} = \sigma \otimes \tau$ and $d\chi = -w(\lambda + \rho)|_{\mathfrak{a}_\mathbb{C}} = s\alpha_2$, $Re(s) > 0$. Assume that π gives rise to Eisenstein series $E(f, \Lambda)$ which have a pole at the uniquely determined point $\Lambda = d\chi$, i.e., $\widetilde{\pi}$ and s satisfy the necessary and sufficient conditions of Proposition 3.4. Let Π be the residual automorphic representation spanned by the residues of these Eisenstein series. We know from Proposition 3.5 that its archimedean component Π_∞ must be one of the representations $A_i(\lambda)$, $i = 0, 1, 2$ (to be of interest for us). In fact, we may exclude the case $i = 2$. In order to see this, recall from remark 3.1 that $A_2(\lambda)$ is tempered. But as it is shown in [38] Thm. 4.3, any automorphic representation which appears in the discrete spectrum and has a tempered archimedean component can only appear in the cuspidal spectrum. But this would contradict the assumption that Π is residual.
In the remaining possible cases, $i = 0, 1$, the irreducible unitary representation Π_∞ is a proper Langlands quotient $J(F_w, t)$, $w \in W^P$, $t = 1, 2$. According to Proposition 3.5 the Kostant representative w and the number t are given es follows: If $\Pi_\infty = A_i(\lambda)$, then $w = w_i$ is the unique element of W^P of length $\ell(w) = i$ and $t = 2 - i$. Therefore, for $i = 0, 1$ we have an exact sequence

$$0 \to U_i(\lambda) \to \mathrm{Ind}_{P_s(\mathbb{R})}^{G_s(\mathbb{R})}[F_{w_i} \otimes \mathbb{C}_{(2-i)\alpha_2}] \to A_i(\lambda) \to 0$$

for a certain representation $U_i(\lambda)$. It is clear that the only subquotients of $U_i(\lambda)$ can be $A_1(\lambda)$ or $A_2(\lambda)$. Since the $(\mathfrak{g}_s, K)$-cohomology of $A_j(\lambda)$, $j = 1, 2$, is non-trivial in degree j, also $\mathrm{Hom}_K(\Lambda^j(\mathfrak{g}_s/\mathfrak{k}), A_j(\lambda)\otimes E)\neq 0$. But, as all K-isotypic components of $\mathrm{Ind}_{P_s(\mathbb{R})}^{G_s(\mathbb{R})}[F_{w_j}\otimes\mathbb{C}_{(2-j)\alpha_2}]$ are of multiplicity one, we get that $U_i(\lambda)=A_{i+1}(\lambda)$. This enables us to show the following result which completes the description of the image of the Eisenstein map.

Theorem 3.1. *Let (π,w) be a pair as above and suppose $\Pi_\infty=A_i(\lambda)$, $i=0,1$. Let $\Omega_{res}(s,\widetilde{\pi})$ be the span of those classes $[\omega]$ of type (π,w) such that the associated Eisenstein series have a pole at the uniquely determined point $d\chi=s\alpha_2$, $s>0$. Then the image of $\Omega_{res}(s,\widetilde{\pi})$ under the Eisenstein map E^q_π is non-trivial if and only if $q=4-i$.*

Proof. Recall from the above that we have an exact sequence

$$0\to A_{i+1}(\lambda)\to \mathrm{Ind}_{P_s(\mathbb{R})}^{G_s(\mathbb{R})}[F_{w_i}\otimes\mathbb{C}_{(2-i)\alpha_2}]\to A_i(\lambda)\to 0.$$

Hence, we are in the situation considered in [32], Lem. 1.4.1. and so formally the same arguments as presented in the proof of [32], Prop. 1.4.3 show that $E^q_\pi(\Omega_{res}(s,\widetilde{\pi}))=0$ if $q'\neq i$, cf. section 3.5. We therefore obtain that the Eisenstein map can only have non-trivial image if $q=4-q'=4-i$. However, in this degree the $G(\mathbb{A}_f)$-module $E^q_\pi(\Omega_{res}(s,\widetilde{\pi}))$ is really non-zero. This follows either by the arguments given in [32], Thm. 1.4.4 or – more directly – if we observe that the following conditions are matched in our specific case: The pole of the Eisenstein series we are looking at is obtained as a pole of the intertwining operator $M(s,\widetilde{\pi})$ at a point $s>0$. Since G_s is of $\mathbb{Q}$-rank 1, this pole is automatically of maximal possible order (namely 1, cf. [26] IV. 1.11), w is clearly the longest element in the quotient $W(A):=N_{G_s(\mathbb{Q})}(A(\mathbb{Q}))/L_s(\mathbb{Q})\cong\{id,w\}$ and the corresponding representation Π spanned by the residues of the Eisenstein series is square-integrable. Furthermore, the archimedean component Π_∞ is the image of $M(s,\widetilde{\pi}_\infty)$, which is the Langlands quotient associate to the real parabolic subgroup $P(\mathbb{R})$, the tempered representation $\widetilde{\pi}_\infty$ and the value $s>0$, cf. the proof of Proposition 3.3. The minimal degree in which $\Pi_\infty=A_i(\lambda)$ for $i\in\{0,1\}$ has non-zero $(\mathfrak{g}_s,K)$-cohomology is $q'=i$, cf. Proposition 3.5. Hence, all assumptions made in [31], Thm. III.1 are satisfied in our specific case and the non–triviality of $E^q_\pi(\Omega_{res}(s,\widetilde{\pi}))$ is a consequence of [31], Thm. III.1. □

3.8. Determination of Eisenstein cohomology. Now we are ready to prove our first main theorem.

Theorem 3.2. *Let $G=GL_2'$ and E be any finite-dimensional, irreducible, complex-rational representation of $G(\mathbb{R})$ of highest weight $\lambda=\sum_{i=1}^3 c_i\alpha_i$ and assume that $Z(\mathbb{R})^\circ$ acts trivially on E. For any tuple (π,w), $w\in W^P$ and $\pi=\chi\widetilde{\pi}\in\varphi_P$ with $\widetilde{\pi}=\sigma\otimes\tau$ and $d\chi=-w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}=s\alpha_2$, let $\Omega_{hol}(s,\widetilde{\pi})$ (resp. $\Omega_{res}(s,\widetilde{\pi})$) be the span of those classes $[\omega]$ of type (π,w) such that the associated Eisenstein series are holomorphic (resp. have a pole) at the uniquely determined point $\Lambda=s\alpha_2$, $Re(s)\geq 0$. Then the Eisenstein cohomology of G with respect to E is given as follows:*

(1) *<u>If $\lambda=k\omega_2$, $k\in\mathbb{Z}_{\geq 0}$</u>:*

$$\begin{array}{rcl}
H^0_{Eis}(G,\mathbb{C}) & = & \bigoplus\limits_{\substack{d\chi=2\alpha_2\\ \widetilde{\pi}_\infty=\mathbf{1}_{M(\mathbb{R})}\\ \sigma=\tau}} E^4_\pi(\Omega_{res}(2,\widetilde{\pi})) \quad \textit{if } k=0\\
H^1_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_1w_3}\\ \sigma=\tau}} E^3_\pi(\Omega_{res}(1,\widetilde{\pi}))\\
H^2_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=0\\ \pi_\infty=F_{w_2w_3}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\pi_f]\\
H^3_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_1w_3}\\ \sigma\neq\tau}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{\alpha_2}\otimes\widetilde{\pi}_f]\oplus\\
 & & \bigoplus\limits_{\substack{d\chi=\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_1w_3}\\ \sigma=\tau}} \Omega_{hol}(1,\widetilde{\pi})\\
H^4_{Eis}(G,\mathbb{C}) & = & \bigoplus\limits_{\substack{d\chi=2\alpha_2\\ \widetilde{\pi}_\infty=\mathbf{1}_{M(\mathbb{R})} \text{ but}\\ \sigma\neq\tau}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{2\alpha_2}\otimes\widetilde{\pi}_f]\oplus\\
 & & \bigoplus\limits_{\substack{d\chi=2\alpha_2\\ \widetilde{\pi}_\infty=\mathbf{1}_{M(\mathbb{R})}\\ \sigma=\tau}} \Omega_{hol}(2,\widetilde{\pi}) \quad \textit{if } k=0\\
H^4_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=(c_2+2)\alpha_2\\ \widetilde{\pi}_\infty=\mathbf{1}_{M(\mathbb{R})}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{(c_2+2)\alpha_2}\otimes\widetilde{\pi}_f] \quad \textit{if } k\neq 0\\
H^q_{Eis}(G,E) & = & 0 \quad \textit{else.}
\end{array}$$

Cohomology in degrees $2,3$ and 4 is entirely built up by values of holomorphic Eisenstein series. Cohomology in degree 0 and 1 consists of residual classes, which can be represented by square-integrable, residual automorphic forms. Both spaces do not vanish.

(2) *<u>If $\lambda\neq k\omega_2$, $k\in\mathbb{Z}_{\geq 0}$</u>:*

$$\begin{array}{rcl}
H^2_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=(c_3-c_1)\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_3}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{(c_3-c_1)\alpha_2}\otimes\widetilde{\pi}_f]\\
H^3_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=(c_1-c_2+c_3)\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_1w_3}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{(c_1-c_2+c_3)\alpha_2}\otimes\widetilde{\pi}_f]
\end{array}$$

$$\begin{array}{rcl}
H^4_{Eis}(G,E) & = & \bigoplus\limits_{\substack{d\chi=(c_2+2)\alpha_2\\ \widetilde{\pi}_\infty=F_{w_2w_1w_3w_2}}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\mathbb{C}_{(c_2+2)\alpha_2}\otimes\widetilde{\pi}_f]\\
H^q_{Eis}(G,E) & = & 0 \quad \textit{else.}
\end{array}$$

All of these spaces are entirely built up by values of holomorphic Eisenstein series, whence the are no residual Eisenstein cohomology classes in this case.

Proof. By [8], Cor. 11.4.3, $H^q(G,E) = 0$ if $q \geq \dim_{\mathbb{R}} X = 5$. So we may concentrate on degrees $0 \leq q \leq 4$. Let us first consider the case $q = 0, 4$. Take $\pi = \chi\widetilde{\pi} \in \varphi_P$. In order to be of cohomological interest, it must satisfy $\widetilde{\pi}_\infty = F_w$, with $w = w_2w_1w_3w_2 \in W^P$ (i.e., $\widetilde{\pi}_\infty$ is the irreducible finite-dimensional representation of $M(\mathbb{R})$ of highest weight $\mu_w = (2c_3 - c_2)\omega_1 + (2c_1 - c_2)\omega_3)$ and $d\chi = (c_2 + 2)\alpha_2$. This follows from Proposition 3.1 and Tables 1 and 2. In particular, $s = c_2 + 2 \geq 2$. So, in order to get a pole of an Eisenstein series associated to π at s, we know from Proposition 3.4 that it is necessary and sufficient that $\widetilde{\pi}$ is a character of the form $\widetilde{\pi} = \sigma \otimes \sigma$ and $c_2 = 0$. It follows that

$$1 = \dim \sigma_\infty = 2c_3 - c_2 + 1 = \dim \tau_\infty = 2c_1 - c_2 + 1,$$

i.e., $c_1 = c_3 = 0$, too, whence $\lambda = 0$ and $\widetilde{\pi}_\infty = \mathbf{1}_{M(\mathbb{R})} = F_{id}$. Recalling that under this assumption $s = 2$, Theorem 3.1 together with section 3.4 show the assertion in degrees $q = 0, 4$.

In degrees $q = 1, 3$ we have to have $\widetilde{\pi}_\infty = F_w$, with $w = w_2w_1w_3$ and $d\chi = (c_1 - c_2 + c_3 + 1)\alpha_2$. Checking with Table 1 yields $\dim \sigma_\infty \geq 2$ and $\dim \tau_\infty \geq 2$, so both factors of $\widetilde{\pi}$ are infinite-dimensional. In order to give rise to a pole, our evaluation point s must therefore be equal to $s = 1$ (cf., Proposition 3.4), which implies $c_1 - c_2 + c_3 = 0$, i.e., $\lambda = k\omega_2$, $k \in \mathbb{Z}_{\geq 0}$. Further, σ must be τ, whence the equation $c_1 = c_3$. Finally, the space $E^3_\pi(\Omega_{res}(1,\pi))$ is non-trivial because of Theorem 3.1 which together with section 3.4 shows the assertion in degrees $q = 1, 3$.

Finally, we consider degree $q = 2$. Suppose there is a pole of an Eisenstein series at the uniquely determined point $s = c_3 - c_1 \geq 0$, cf. Table 2, for a cuspidal representation $\pi \in \varphi_P$. As $\dim \sigma_\infty = c_1 + c_3 + 3 \geq 3$, Proposition 3.4 tells us that $c_3 - c_1 = 1$ and $\sigma = \tau$. But this implies $c_1 + c_3 + 2 = 2c_2 - c_1 - c_3$ (cf., Table 1), leading to $2 = c_2 - 2c_1$. As $2c_1 \geq c_2$ we end up in a contradiction. Hence, all Eisenstein series contributing to degree $q = 2$ are holomorphic at the evaluation point $s = c_3 - c_1$. Now, the proof of the theorem is complete. □

3.9. Langlands Functoriality and Eisenstein cohomology in degree $q = 1$. We want to point out that the global Jacquet-Langlands Correspondence for GL'_n, as it was recently established by I. A. Badulescu and D. Renard in [2], gives an alternative proof of the non-vanishing of $H^1_{Eis}(G,E)$:

Theorem 3.3. *For all E with highest weight $\lambda = k\omega_2$, $k \geq 0$, there is a residual automorphic representation π of $G(\mathbb{A})$ which has cohomology in degree $q = 1$. In particular, $H^1_{Eis}(G,E) \neq 0$.*

The philosophy of the following proof will be to use the classical Jacquet-Langlands Correspondence for GL'_1 to construct an appropriate cuspidal automorphic representation of $GL_2(\mathbb{A}) \times GL_2(\mathbb{A})$ and then take the unique irreducible quotient of the representation induced to $GL_4(\mathbb{A})$. Using the work of Badulescu and Renard we can therefrom construct a residual representation π of $G(\mathbb{A})$ having the claimed properties. As this was already shown in Theorem 3.2, we allow ourselves to keep the proof of this fact rather short by assuming some familiarity with the paper [2]. Its underlying idea fits very well with the idea to use functoriality in order to get cohomological automorphic representations. The interested reader may find a survey on this topic in [28], section 5.2.

Proof. By Table 1, $F_{w_2} = \mathrm{Sym}^{k+1}\mathbb{C}^2 \otimes \mathrm{Sym}^{k+1}\mathbb{C}^2$. Obviously, $\mathrm{Sym}^{k+1}\mathbb{C}^2$ is a discrete series representation of $GL'_1(\mathbb{R}) = GL_1(\mathbb{H})$. So there is a cuspidal automorphic representation ρ' of $GL'_1(\mathbb{A})$ having archimedean component $\rho'_\infty = \mathrm{Sym}^{k+1}\mathbb{C}^2$. The classical Jacquet-Langlands lift, [12] Thm. 8.3, gives us therefore a cuspidal automorphic representation $\rho := JL(\rho')$ of $GL_2(\mathbb{A})$ having only square-integrable representations ρ_v at all places $v \in S(D)$. By the description of the residual spectrum of the $\mathbb{Q}$-split group GL_n, cf. [25], the unique irreducible quotient σ of $\mathrm{Ind}^{GL_4(\mathbb{A})}_{GL_2(\mathbb{A})\times GL_2(\mathbb{A})}[|\det|^{\frac{1}{2}}\rho\otimes|\det|^{-\frac{1}{2}}\rho]$ is a residual representation of $GL_4(\mathbb{A})$. Now, we see by Prop. 15.3.(a) of [2] that for each $v\in S(D)$, σ_v is "2-compatible", i.e., the local transfer of σ_v from the level of $GL_4(\mathbb{Q}_v)$ to the level of $GL'_2(\mathbb{Q}_v)$ is non-trivial. Here we use that ρ_v is square-integrable at all places $v\in S(D)$. Hence, we might apply Thm. 18.1 of [2] and see that σ is in the image of the global Jacquet-Langlands correspondence developed in the paper [2]. It follows that there is an unique representation π of $G(\mathbb{A})$ which appears in the discrete spectrum of G and transfers to σ. It is residual by Prop. 18.2.(b) of [2]. By its very construction, the archimedean component of it satisfies $\pi_\infty|_{G_s(\mathbb{R})}=J(F_{w_2},1)$, cf. [2], Thm. 13.8. Combining this with our Proposition 3.5 and [31], Thm. III.1, we get the claim. □

3.10. Revisiting Eisenstein cohomology in degree $q=2$. For our special case $G=GL'_2$, we would now like to take up a question once raised by G. Harder. As the contents of this section will not be needed in the sequel, we allow ourselves to be rather brief. Recall the adelic Borel-Serre Compactification $\overline{X}_{\mathbb{A}}$ of $X_{\mathbb{A}}:=G(\mathbb{Q})\backslash(X\times G(\mathbb{A}_f))$, and its basic properties: (For this we refer to [8] as the original source and [30] for the adelic setting.) It is a compact space with boundary $\partial(\overline{X}_{\mathbb{A}})$ and the inclusion $X_{\mathbb{A}}\hookrightarrow\overline{X}_{\mathbb{A}}$ is a homotopy equivalence. Furthermore, there is the natural restriction morphism of $G(\mathbb{A}_f)$-modules

$$res^q: H^q(X_{\mathbb{A}},\widetilde{E})\cong H^q(\overline{X}_{\mathbb{A}},\widetilde{E})\to H^q(\partial(\overline{X}_{\mathbb{A}}),\widetilde{E}).$$

Here, $\widetilde{E}$ stands for the sheaf with espace étalé $(X\times G(\mathbb{A}_f))\times_{G(\mathbb{Q})}E$, E given the discrete topology. It is finally a consequence of [10] Thm. 18 that there is also the following isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\overline{X}_{\mathbb{A}},\widetilde{E})\cong H^q(G,E).$$

It makes therefore sense to talk about Eisenstein cohomology as a subspace of $H^q(\overline{X}_{\mathbb{A}},\widetilde{E})$ and hence to restrict Eisenstein cohomology classes to the cohomology of the boundary $\partial(\overline{X}_{\mathbb{A}})$.

Now, let $q=2$ and let E be self-dual. We know that for each $\pi=\sigma\otimes\tau$

$$H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\sigma\otimes\tau]\otimes E)\quad\text{and}\quad H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\tau\otimes\sigma]\otimes E)$$

are linear independent subspaces of $H^2(\partial(\overline{X}_{\mathbb{A}}),\widetilde{E})$, since all Eisenstein series showing up in degree 2 are holomorphic (cf. [5], Lemma 2.12). Taking the restriction $res^2([\omega])$ of a class $[\omega]\in\mathrm{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}[\pi_f]\subset H^2_{Eis}(G,E)$ to the boundary means to calculate the constant term of the Eisenstein series representing ω and then taking the corresponding class (cf. [34], Satz 1.10). Therefore, by (5)

$$\begin{aligned}res^2([\omega]) &= [\omega]\oplus[M(0,\pi)_*\omega]\\ &\in H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\sigma\otimes\tau]\otimes E)\oplus H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}[\tau\otimes\sigma]\otimes E)\end{aligned}$$

G. Harder asked (in a more general context), if $[M(0,\pi)_*\omega]\neq0$ for some Eisenstein class $[\omega]$. This is actually true in our case. We devote the next theorem to this result. Clearly, we only need to check the archimedean place on the level of cohomology.

Theorem 3.4. *The local intertwining operator $M(0,\pi_\infty)$ is an isomorphism. In particular, it induces an isomorphism of cohomologies*

$$[M(0,\pi_\infty)_*]: H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G_s(\mathbb{R})}_{P_s(\mathbb{R})}[\sigma_\infty\otimes\tau_\infty]\otimes E)\xrightarrow{\sim}H^2(\mathfrak{g}_s,K,\mathrm{Ind}^{G_s(\mathbb{R})}_{P_s(\mathbb{R})}[\tau_\infty\otimes\sigma_\infty]\otimes E)$$

Proof. The operator $M(0,\pi_\infty)$ is holomorphic, cf. [26], Prop. IV.1.11 (b). The same holds for its adjoint map $M(0,\sigma_\infty\otimes\tau_\infty)^* = M(0,\tau_\infty\otimes\sigma_\infty)$. Here we used [1], ($J_4$), p.26. *Ibidem*, line (3.5) shows that the composition $M(0,\pi_\infty)^*M(0,\pi_\infty)$ is a positive real number

$$M(0,\pi_\infty)^*M(0,\pi_\infty) = |r_{\overline{P}|P}(\pi_\infty)|^2 > 0.$$

It follows that $M(0,\pi_\infty)$ is not identically zero. Now observe that the induced representations

$$\mathrm{Ind}^{G_s(\mathbb{R})}_{P_s(\mathbb{R})}[F_{w_2w_3}\otimes \mathbf{1}_{A(\mathbb{R})}] \quad\text{and}\quad \mathrm{Ind}^{G_s(\mathbb{R})}_{P_s(\mathbb{R})}[F_{w_2w_1}\otimes \mathbf{1}_{A(\mathbb{R})}]$$

are isomorphic since $\sigma_\infty\otimes\tau_\infty = F_{w_2w_3}$ and $\tau_\infty\otimes\sigma_\infty = F_{w_2w_1}$ are conjugate by an element $m\in N_K(A(\mathbb{R})^\circ)$ which is not in $M(\mathbb{R})$. Recall that $\mathrm{Ind}^{G_s(\mathbb{R})}_{P_s(\mathbb{R})}[F_{w_2w_3}\otimes\mathbf{1}_{A(\mathbb{R})}]$ is irreducible. As $M(0,\pi_\infty)$ is not identically zero, it is hence an isomorphism and the claim holds. □

Observe that G_s is an inner form of the $\mathbb{Q}$-split group $SL_4/\mathbb{Q}$. Harder writes in [18], that he got a letter of B. Speh in which she shows that Harder's operator $T^{\mathrm{loc}}_\infty(0)$ induces zero on the level of cohomology. Here, $T^{\mathrm{loc}}_\infty(0)$ is actually the analog of our operator $M(0,\pi_\infty)$, connecting induced representations of $L_1 := SL_1\times SL_3$ and $L_3 := SL_3\times SL_1$ sitting as Levi factors of the two non self-associate maximal parabolic $\mathbb{Q}$-subgroups inside SL_4. Of course our (self-associate) Levi subgroup L_s is an inner form of the second (and hence self-associate) Levi subgroup $L_2 := SL_2\times SL_2$ of SL_4. We expect that Theorem 3.4 also holds for L_2.

We also consider Theorem 3.4 as the starting point for further investigations, whose aim is to establish the rationality of critical values of L-functions, as it was carried out by Harder in [18] for the $\mathbb{Q}$-split group $SL_3/\mathbb{Q}$. The proof of such a result would go beyond the scope of this paper and we hope to report on it in a forthcoming work.

4. Cuspidal Cohomology

4.1. Having analyzed Eisenstein cohomology in the previous sections, we still need to describe the space of cuspidal cohomology

$$H^q_{cusp}(G,E) = H^q(\mathfrak{g}_s,K,\mathcal{A}_{cusp}(G)\otimes E)$$

as defined in section 2.3, in order to know the full space $H^q(G,E)$. Recall from (1) that we have the direct sum decomposition

$$H^q_{cusp}(G,E) = \bigoplus_\pi H^q(\mathfrak{g}_s,K,\pi\otimes E),$$

the sum ranging over all (equivalence classes of) cuspidal automorphic representations

$$\pi\hookrightarrow L^2_{cusp}(G(\mathbb{Q})Z(\mathbb{R})^\circ\backslash G(\mathbb{A}))$$

of $G(\mathbb{A})$. In particular, $Z(\mathbb{R})^\circ$ acts trivially on such a representation. Since $G(\mathbb{R}) = Z(\mathbb{R})^\circ\times G_s(\mathbb{R})$, cf. section 1.2, we may again view its archimedean component as an irreducible, unitary representation of $G_s(\mathbb{R})$. The cohomological irreducible unitary representations of $G_s(\mathbb{R})$ were classified in the Proposition 3.5.

Let us start with the well-known fact that there cannot be any non-trivial cuspidal cohomology class in degrees $q=0,5$. Indeed, a cuspidal automorphic representation π giving rise to such a class would have to satisfy $\pi_\infty = \mathbf{1}_{G(\mathbb{R})}$, cf. Proposition 3.5. Hence, by [3] IX.3 Lemma, π would have to be a character of $G(\mathbb{A})$, leading to a contradiction. The question whether there are cohomological cuspidal automorphic representations of $G(\mathbb{A})$ for the remaining possible degrees $1 \leq q \leq 4$ is more delicate and we devote the next subsections to it.

4.2. After our consideration of the previous subsection a cuspidal automorphic representation π of $G(\mathbb{A})$ which has non-trivial $(\mathfrak{g}_s, K)$-cohomology tensorised by E must have a representation $A_i(\lambda)$, $i = 1, 2$ as its archimedean component. In case of the $\mathbb{Q}$-*split* general linear group GL_n, we know that

Proposition 4.1. *The archimedean component of a cohomological (unitary) cuspidal representation of $GL_n(\mathbb{A})$ is tempered.*

See, e.g., [33], Thm. 3.3. We will show in the next theorem that this does not hold for the non-split inner form G. Again, functoriality will be the key-method.

4.3. Degrees $q = 1, 4$**.**

Theorem 4.1. *For all E with highest weight $\lambda = k\omega_2$, $k \geq 0$, there is a unitary cuspidal automorphic representation π of $G(\mathbb{A})$ which has cohomology in degrees $q = 1, 4$. In particular, the non-tempered representation $A_1(\lambda)$ appears as the archimedean component of a cohomological cuspidal automorphic representation and so Proposition 4.1 cannot be generalized to inner forms of GL_n.*

Proof. Let $k \geq 0$. By our Proposition 3.5, we need to find a unitary cuspidal automorphic representation π of $GL_2'(\mathbb{A})$ which satisfies $\pi_\infty|_{G_s(\mathbb{R})} = J(F_{w_2}, 1)$ at the archimedean component. Therefore, let $\rho'_\infty := \mathrm{Sym}^{k+1}\mathbb{C}^2$ be the $k+1$-th symmetric power of the standard representation of $GL_1'(\mathbb{R})$. By the local Jacquet-Langlands correspondence JL_∞ between representations of $GL_1'(\mathbb{R}) = \mathbb{H}^*$ and $GL_2(\mathbb{R})$ (cf. [12], Thm. 8.1) it lifts to a square-integrable, irreducible, unitary representation ρ_∞ of $GL_2(\mathbb{R})$. Hence, $\rho_\infty = D_\ell$ for some integer $\ell \geq 2$, D_ℓ denoting as usual the discrete series representation of $GL_2(\mathbb{R})$ of minimal $O(2)$-type ℓ.

Now, take a non-archimedean prime $p_0 \in S(D)$ and let N be a positive integer prime to p_0. If N is big enough, which we assume, then there is a non-zero modular cuspform f of weight ℓ and level N of $GL_2(\mathbb{R})$. We may assume that f is an Eigenfunction for all Hecke operators T_p, $(p, N) = 1$, cf. [29] p.21. As in [29], (1.5) f defines a cuspidal automorphic form φ for $GL_2(\mathbb{A})$ and hence an admissible subrepresentation ρ of $\mathcal{A}_{cusp}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$. As proved in [29], Prop. 2.13, ρ is in fact irreducible, its archimedean component is $\rho_\infty = D_\ell$ and ρ_{p_0} is a spherical representation of $GL_2(\mathbb{Q}_{p_0})$, whence it is in the principal series. As a consequence, ρ_{p_0} is not square-integrable. By the characterization of the image of the global Jacquet-Langlands correspondence JL from the level of $GL_1'(\mathbb{A})$ to the level of $GL_2(\mathbb{A})$, cf. [12], Thm. 8.3, ρ can therefore not be of the form $\rho = JL(\rho')$ for any automorphic representation ρ' of $GL_1'(\mathbb{A})$ (although it transfers at the archimedean place!)

Now take σ to be the unique irreducible quotient of

$$\mathrm{Ind}^{GL_4(\mathbb{A})}_{GL_2(\mathbb{A})\times GL_2(\mathbb{A})}[|\det|^{\frac{1}{2}}\rho \otimes |\det|^{-\frac{1}{2}}\rho].$$

By [25] it is a residual automorphic representation of $GL_4(\mathbb{A})$. According to [2], Thm. 18.1 there is a unique square-integrable automorphic representation π of $GL_2'(\mathbb{A})$ which is mapped onto σ via the global Jacquet Langlands Correspondence from the level of $GL_2'(\mathbb{A})$ to the level of $GL_4(\mathbb{A})$, as developed in the aforementioned paper. As ρ is not in the image of the global Jacquet-Langlands correspondence, Prop. 18.2 of [2] ensures that π is cuspidal. But as ρ_∞ transfers via the local

Jacquet-Langlands correspondence to $\rho'_\infty = \mathrm{Sym}^{k+1}\mathbb{C}^2$ we see that $\pi_\infty|_{G_s(\mathbb{R})} = J(F_{w_2}, 1)$, cf. [2], Thm. 13.8. This proves the theorem. □

Remark 4.1. The number ℓ can easily be made explicit and equals $\ell = k+3$. In fact, complexifying the action of ρ'_∞ gives a representation of $SL'_1(\mathbb{C}) = SL_2(\mathbb{C})$ which restricts to an irreducible representation $\widetilde{\rho}_\infty$ of the split real form $SL_2(\mathbb{R})$. On the other hand, restricting D_ℓ to $SL_2(\mathbb{R})$ defines two irreducible discrete series representations D_ℓ^+ and D_ℓ^- of $SL_2(\mathbb{R})$. As the local Jacquet-Langlands correspondence at the archimedean place can be characterized as the assignment JL_∞ sending ρ'_∞ to the unique D_ℓ, which satisfies that $D_\ell^\pm$ and $\widetilde{\rho}_\infty$ appear as irreducible subquotients of the same principal series representation of $SL_2(\mathbb{R})$, we must have $\ell = \dim \widetilde{\rho}_\infty + 1 = k+3$, cf. [22], II §5.

Let us put Theorem 4.1 in a broader context. To that end, recall the notion of a CAP representation of a general connected reductive algebraic group $H/\mathbb{Q}$: Therefore, let H be an inner form of a quasi-split group $\widetilde{H}$ and $\widetilde{P}$ be a *proper* parabolic subgroup of $\widetilde{H}$ with Levi subgroup $\widetilde{L}$. A unitary cuspidal automorphic representation π of $H(\mathbb{A})$ is called CAP, if there is a unitary cuspidal automorphic representation η of $\widetilde{L}(\mathbb{A})$ such that π is nearly equivalent (i.e., locally equivalent at all but finitely many places) to an irreducible subquotient of $\mathrm{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{H}(\mathbb{A})}[\eta]$. Philosophically speaking, CAP representations typically look at almost all places like a residual representation of $\widetilde{H}(\mathbb{A})$, although they might not be nearly equivalent to a residual automorphic representation of $H(\mathbb{A})$. We obtain

Corollary 4.1. *Let π be as constructed in the proof of Theorem 4.1. Then π is a cohomological CAP-representation of $G(\mathbb{A}) = GL'_2(\mathbb{A})$. It is also an example for a cuspidal automorphic representation of $G(\mathbb{A})$, which has a non-tempered archimedean component π_∞.*

Proof. This is clear, since π is by its very construction nearly equivalent to the residual automorphic representation σ of $GL_4(\mathbb{A})$ and has archimedean component $\pi_\infty = A_1(\lambda)$. □

4.4. Degrees $q = 2, 3$**.** If the highest weight λ of E does not satisfy the equation $\lambda = k\omega_2$, $k \geq 0$, but - in view of Proposition 3.5 - we still have $E \cong \check{E}$, we can use Lefschetz numbers, resp. the twisted Arthur Trace Formula to show that there is a cuspidal automorphic representation π which has non-vanishing cohomology with respect to E, i.e., whose archimedean component is isomorphic to $A_2(\lambda)$, cf. Proposition 3.5. We prefer not to use the Jacquet–Langlands correspondence in this case, because it only transfers the difficulty of showing the existence of certain cohomological cuspidal automorphic representations to GL_4. The approach, which we would like to follow here, goes back to the paper [7], resp. work of D. Barbasch and B. Speh, [3]. We leave it to the reader to check that all the conditions of their theorem in Sect. I.1 are satisfied in our given case. As a consequence, we obtain

Theorem 4.2 ([3] I.1)**.** *For all self-dual E with highest weight $\lambda \neq k\omega_2$, $k \geq 0$, there is a cuspidal automorphic representation π of $G(\mathbb{A})$ which has cohomology in degrees $q = 2, 3$. In particular, $A_2(\lambda)$ appears as the archimedean component of a global cuspidal representation.*

4.5. Combining Theorems 4.1 and 4.2 we have proved

Theorem 4.3. *For all finite-dimensional, self-dual, irreducible complex representations E of $G(\mathbb{R})$*

$$H^*_{cusp}(G, E) \neq 0$$

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