RATIONALITY RESULTS FOR THE EXTERIOR AND THE SYMMETRIC SQUARE L-FUNCTION.

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With an appendix by Nadir Matringe

Abstract. Let $G = \mathrm{GL}_{2n}$ over a totally real number field F and $n \geq 2$. Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$, which is cohomological and a functorial lift from $\mathrm{SO}(2n+1)$. The latter condition can be equivalently reformulated that the exterior square L-function of Π has a pole at $s = 1$. In this paper, we prove a rationality result for the residue of the exterior square L-function at $s = 1$ and also for the holomorphic value of the symmetric square L-function at $s = 1$ attached to Π. As an application of the latter, we also obtain a period-free relation between certain quotients of twisted symmetric square L-functions and a product of Gauß sums of Hecke characters.

Contents

1. Introduction

1.1. **General background.** Let F be an algebraic number field and let Π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Rationality results for special values of the associate automorphic L-function $L(s, \Pi)$ have been studied by several authors over the last decades.

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For the scope of this paper, we would like to mention Y. Manin and G. Shimura, who were the first to study special values of $L(s,\Pi)$ in the particular case, when F is totally real, i.e., when Π comes from a Hilbert modular form, cf. [27] and [33], and P. F. Kurchanov, who treated the case of a CM-field F in a series of papers, cf. [24, 25]. Shortly later, G. Harder published some articles, see [15, 16], in which he described a general approach to such rationality results. In [15], Harder considered the case of an arbitrary number field F, while in [16], he extended the methods of the above authors to some automorphic representations, which do not necessarily come from cusp forms (for F imaginary quadratic). The case of GL_2 over a general number field F has also been considered independently by H. Hida, cf. [18] and later on also by Shimura, see [34].

It took some time until extensions of these results to general linear groups GL_n of higher rank n were available. Important achievements include Ash-Ginzburg, [1], Kazhdan-Mazur-Schmidt, [22] and Mahnkopf, [26].

Guided by the above methods, meanwhile, there is a growing number of results that have been proved about the rationality of special values of certain automorphic L-functions attached to GL_n. As a selection of examples, relevant to the present paper, we refer to Raghuram [29, 28], Harder-Raghuram [17], Grobner-Harris [11]; Grobner-Raghuram [14], Grobner-Harris-Lapid [12] and Balasubramanyam-Raghuram [2].

In all of these references, the corresponding rationality result is obtained by writing the special L-value at hand as an algebraic multiple of a certain period invariant[1]. This period is defined by comparison of a rational structure on a cohomology space, attached to the given automorphic representation Π, with a rational structure on a model-space of (the finite part of) Π, such as a Whittaker model or a Shalika model. (The word "rational structure" here refers to a subspace of the vector space, carrying the action of Π, which is essentially defined over the field of rationality of Π and at the same time stable under the group action.) While the first rational structure on the cohomology space is purely of geometric nature and has its origin in the cohomology of arithmetic groups (or better: the cohomology of arithmetically defined locally symmetric spaces), the latter rational structure is defined by reference to the uniqueness of the given model-space.

In this paper, we continue the above considerations. But while most of the aforementioned papers deal with special values of the Rankin–Selberg L-function (by some variation or the other), the principal L-function, or the Asai L-functions, here we would like to study the algebraicity of the exterior square L-function and the symmetric square L-function, attached to a cuspidal automorphic representation of the general linear group.

1.2. **The main results of this paper.** To put ourselves *in medias res*, let F be a totally real number field and let $G = \mathrm{GL}_N/F$, $N = 2n$ with $n \geq 2$. The restrictions on F and the index of the general linear groups under consideration are owed to the inevitable, as it will become clear below. Indeed, let Π be a cuspidal automorphic representation of $G(\mathbb{A})$ and let $\mathcal{W}^\psi(\Pi)$ be its ψ-Whittaker model. As we want to exploit the results of Bump–Friedberg, [6], we shall assume that the partial exterior square L-function $L^S(s,\Pi,\Lambda^2)$ of Π has a pole at $s=1$. (Here, S is a finite set of places of F, containing all archimedean ones, such that for a place $v \notin S$, the local components Π_v and ψ_v are unramified.) In particular, this forces

[1] The approach taken in [12], however, is a certain, basis-free variation of the latter.

$N = 2n$ to be even, see [20], Thm. 2, and furthermore Π to be self-dual, $\Pi \cong \Pi^\vee$, and to have trivial central character.

Our first main result gives a rationality statement for the residue $\mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))$ of the exterior square L-function. More precisely, we obtain the following result:

Theorem 1.1. *Let F be a totally real number field and $G = \mathrm{GL}_{2n}/F$, $n \geq 2$. Let Π be a unitary cuspidal automorphic representation of $G(\mathbb{A})$, which is cohomological with respect to an irreducible, self-contragredient, algebraic, finite-dimensional representation E_μ of G_∞. Assume that Π satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function $L^S(s,\Pi,\Lambda^2)$ has a pole at $s=1$. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a non-trivial period $p^t({}^\sigma\Pi)$, defined by a comparison of a given rational structure on the Whittaker model of ${}^\sigma\Pi_f$ and a rational structure on a realization of ${}^\sigma\Pi_f$ in cohomology in top degree t, and a non-trivial archimedean period $p^t({}^\sigma\Pi_\infty)$, such that*

$$\sigma\left(\frac{L(\frac{1}{2},\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{p^t(\Pi)\ p^t(\Pi_\infty)}\right) = \frac{L(\frac{1}{2},{}^\sigma\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{p^t({}^\sigma\Pi)\ p^t({}^\sigma\Pi_\infty)}.$$

In particular,

$$L(\tfrac{1}{2},\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2)) \sim_{\mathbb{Q}(\Pi_f)} p^t(\Pi)\ p^t(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of the right hand side by an element in the number field $\mathbb{Q}(\Pi_f)$.

This is proved in details in §7.4, see Thm. 7.4. For a precise definition of the periods $p^t({}^\sigma\Pi)$ and $p^t({}^\sigma\Pi_\infty)$, as well as for a complete list of choices which enter their involved definitions, we refer to Prop. 4.3 and Rem. 4.4, respectively (7.2) and Rem. 7.3. The non-vanishing of the archimedean period $p^t({}^\sigma\Pi_\infty)$ is shown – building on a result of B. Sun – in our Thm. 7.1. The number field $\mathbb{Q}(\Pi_f)$ in the theorem is (by Strong Multiplicity One) the aforementioned field of rationality of the cuspidal automorphic representation Π. See §2.5 and §3.3.

The key result, which we use, in order to derive the above theorem, is a certain integral-representation, obtained by Bump-Friedberg, [6], of the residue $\mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))$ of the exterior square L-function in terms of integrating over a cycle $Z(\mathbb{A})H(F)\backslash H(\mathbb{A})$. Here, Z is the centre of G and $H = \mathrm{GL}_n \times \mathrm{GL}_n$, suitably embedded into G, cf. 2.2.

More precisely, if one combines the three main results of [6], then, under the assumptions made in the theorem, one obtains the following equality, shown in our Thm. 6.1:

(1.2)

$$c_n \cdot \hat{\Phi}(0) \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(J(g,g'))\ dg\ dg' = \frac{L^S(\frac{1}{2},\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{L^S(n,\mathbf{1})^2} \cdot \prod_{v\in S} \frac{Z_v(\xi_v, f_{v,1})}{L(n,\mathbf{1}_v)}.$$

Here, Φ is a certain global Schwartz-Bruhat function on $\mathbb{A}^n$, chosen with care in §6.1, and $c_n \cdot \hat{\Phi}(0)$ is the (non-zero) residue at $s=1$ of an Eisenstein series attached to a section $f_s = \otimes_v f_{v,s}$, which is defined by Φ. See §6.1 and §6.4 for the precise definitions of the terms appearing in (1.2). What one should observe is that the value of the partial L-function $L^S(n,\mathbf{1})$ of the trivial character of $\mathbb{A}$ at n appears in the formula. In order for the pole of $L^S(s,\Pi,\Lambda^2)$ at $s=1$ not to cancel with the pole of $L^S(n,\mathbf{1})$ at $n=1$, we assumed $n \geq 2$, which explains the corresponding assumption in Thm. 1.1 (resp. Thm. 7.4). (As for the case of $n=1$, $L^S(s,\Pi,\Lambda^2) = L^S(s,\mathbf{1})$, the analogue of Thm. 1.1 would boil down to a rationality result for the central critical value of the L-function of unitary cusp forms of $\mathrm{GL}_2(\mathbb{A})$, which is known, e.g., by Harder [15]. Therefore, considering only $n \geq 2$ is not a serious restriction.)

Observe that the top degree t, mentioned in Thm. 1.1, where Π has non-trivial cohomology, equals the dimension of the locally symmetric spaces, which are associated to the cycle $Z(\mathbb{A})H(F)\backslash H(\mathbb{A})$, cf. §5.2. (Here, we necessarily have to use that F is totally real, which explains the last obstruction, set in the beginning.) As a consequence, we may use the de Rham isomorphism. Together with (1.2) and N. Matringe's equivariance-result (Thm. A) in the appendix, this finally gives Thm. 1.1.

We point out that, if Π satisfies the assumptions made in the theorem, then Π automatically satisfies the assumptions made in Grobner-Raghuram [14]. Hence, the non-zero periods $\omega^{\epsilon_0}(\Pi_f)$ and $\omega(\Pi_\infty)$ constructed in *loc. cit.* are well-defined. See our §7.5 below for details. If we define the non-zero, top-degree Whittaker-Shalika periods,

$$P^t(\Pi) := \frac{p^t(\Pi)}{\omega^{\epsilon_0}(\Pi_f)} \qquad \text{and} \qquad P^t(\Pi_\infty) := \frac{p^t(\Pi_\infty)}{\omega(\Pi_\infty)},$$

then we may get rid of the L-factor $L(\frac{1}{2}, \Pi_f)$ in Thm. 1.1, as long as it does not vanish. The following result is Cor. 7.6.

Corollary 1.3. *Let Π be as in the statement of Thm. 1.1 (resp. Thm. 7.4). If $L(\frac{1}{2}, \Pi_f)$ is non-zero, then*

$$\mathrm{Res}_{s=1}(L^S(s, \Pi, \Lambda^2)) \ \sim_{\mathbb{Q}(\Pi_f)} \ P^t(\Pi) \ P^t(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of the right hand side by an element in the number field $\mathbb{Q}(\Pi_f)$.

In order to obtain our second main theorem on the symmetric square L-function, we need a version of one of the main results of Grobner-Harris-Lapid [12] and Balasubramanyam-Raghuram [2], which is tailored to our present situation at hand. This is achieved in §8, applying [2] to our particular case. The aforementioned result reads as follows:

Theorem 1.3. *Let Π be a self-dual, unitary, cuspidal automorphic representation of $G(\mathbb{A})$ (with trivial central character), which is cohomological with respect to an irreducible, self-contragredient, algebraic, finite-dimensional representation E_μ of G_∞. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{\mathrm{Res}_{s=1}(L^S(s, \Pi \times \Pi))}{p^t(\Pi) \ p^b(\Pi) \ p(\Pi_\infty)}\right) \ = \ \frac{\mathrm{Res}_{s=1}(L^S(s, {}^\sigma\Pi \times {}^\sigma\Pi))}{p^t({}^\sigma\Pi) \ p^b({}^\sigma\Pi) \ p({}^\sigma\Pi_\infty)}.$$

In particular,

$$\mathrm{Res}_{s=1}(L^S(s, \Pi \times \Pi)) \ \sim_{\mathbb{Q}(\Pi_f)^\times} \ p^t(\Pi) \ p^b(\Pi) \ p(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)^\times}$" means up to multiplication by a non-trivial element in the number field $\mathbb{Q}(\Pi_f)$.

In the statement of the latter theorem, $p^t(\Pi)$ is the top-degree period defined above, while $p^b(\Pi)$ is defined analogously, but using the lowest degree b, where Π has non-trivial cohomology. The non-vanishing archimedean period $p(\Pi_\infty)$ is defined in (8.3). We refer to §8.2 and §8.3 for precise assertions and definitions concerning these periods, in particular Rem. 8.1 and Rem. 8.4.

The second main theorem of this paper finally deals with the value of the symmetric square L-function at $s=1$. Recall that we have

$$L^S(s,\Pi\times\Pi)=L^S(s,\Pi,\mathrm{Sym}^2)\cdot L^S(s,\Pi,\Lambda^2).$$

As by assumption $L^S(s,\Pi,\Lambda^2)$ carries the (simple) pole of $L^S(s,\Pi\times\Pi)$ at $s=1$, the symmetric square L-function $L^S(s,\Pi,\mathrm{Sym}^2)$ is holomorphic and non-vanishing at $s=1$. Our second main theorem hence follows by combining Thm. 1.1 (resp. Thm. 7.4) with Thm. 1.3 (resp. Thm. 8.5). We obtain, see Thm. 9.2,

Theorem 1.4. *Let Π be a unitary cuspidal automorphic representation of $G(\mathbb{A})$, as in the statement of Thm. 1.1. Then, for every $\sigma\in\mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{L(\frac{1}{2},\Pi_f)\ p^b(\Pi)\ p^b(\Pi_\infty)}{L^S(1,\Pi,\mathrm{Sym}^2)}\right)\ =\ \frac{L(\frac{1}{2},{}^\sigma\Pi_f)\ p^b({}^\sigma\Pi)\ p^b({}^\sigma\Pi_\infty)}{L^S(1,{}^\sigma\Pi,\mathrm{Sym}^2)}.$$

In particular,

$$L^S(1,\Pi,\mathrm{Sym}^2)\sim_{\mathbb{Q}(\Pi_f)}L(\tfrac{1}{2},\Pi_f)\ p^b(\Pi)\ p^b(\Pi_\infty)$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of $L^S(1,\Pi,\mathrm{Sym}^2)$ by an element in the number field $\mathbb{Q}(\Pi_f)$.

Similar to before, we may define bottom-degree Whittaker-Shalika periods. Set

$$P^b(\Pi):=p^b(\Pi)\cdot\omega^{\epsilon_0}(\Pi_f)\qquad\text{and}\qquad P^b(\Pi_\infty):=p^b(\Pi_\infty)\cdot\omega(\Pi_\infty).$$

Then, we have the following corollary, see Cor. 9.4, in which we may get once more rid of the L-factor $L(\frac{1}{2},\Pi_f)$, if it is non-zero.

Corollary 1.4. *Let Π be as in the statement of Thm. 1.4 (resp. Thm. 9.2). If $L(\frac{1}{2},\Pi_f)$ is non-zero, then*

$$L^S(1,\Pi,\mathrm{Sym}^2)\ \sim_{\mathbb{Q}(\Pi_f)^\times}\ P^b(\Pi)\ P^b(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)^\times}$" means up to multiplication of $L^S(1,\Pi,\mathrm{Sym}^2)$ by a non-zero element in the number field $\mathbb{Q}(\Pi_f)$.

On a final note, we may also derive a theorem for quotients of symmetric square L-functions, which is *independent of all periods* appearing in this paper. We hope that this application of Thm. 1.4 – our third main result – serves as an interesting example of the strength of the relation provided by Thm. 1.4 between the symmetric square L-function and our a priori only abstract Whittaker period $p^b(\Pi)$. More precisely, we obtain (cf. Thm. 10.1)

Theorem 1.5. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$ and let χ_1 and χ_2 be two Hecke characters of finite order, such that $\Pi\otimes\chi_i$, $i=1,2$, both satisfy the conditions of Cor. 1.4. If χ_1 and χ_2 have moreover the same infinity-type, i.e., $\chi_{1,\infty}=\chi_{2,\infty}$, then,*

$$\frac{L^S(1,\Pi\otimes\chi_1,\mathrm{Sym}^2)}{L^S(1,\Pi\otimes\chi_2,\mathrm{Sym}^2)}\ \sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times}\mathscr{G}(\chi_{1,f})^{2n^2}\mathscr{G}(\chi_{2,f})^{-2n^2},$$

where "$\sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times}$" means up to multiplication by a non-zero element in the composition of number fields $\mathbb{Q}(\Pi_f)$, $\mathbb{Q}(\chi_{1,f})$ and $\mathbb{Q}(\chi_{2,f})$.

It shall be noted that, whereas the quantities on the left hand side of the above equation all depend crucially on Π, the right hand side is not only independent of the all periods considered in this paper, but *completely independent* of Π.

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2. Notation and conventions

2.1. **Number fields.** In this paper, F denotes a totally real number field of degree $d = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}$. For a place v, let F_v be the topological completion of F at v. Let S_∞ be the set of archimedean places of F. If $v \notin S_\infty$, we let $\mathcal{O}_v$ be the local ring of integers of F_v with unique maximal ideal $\wp_v$. Moreover, $\mathbb{A}$ denotes the ring of adèles of F and $\mathbb{A}_f$ its finite part. We use the local and global normalized absolute values, the first being denoted by $|\cdot|_v$, the latter by $\|\cdot\|$. The fact that F has no complex place is crucial, see §5.2. Once and for all, we fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ as in [14], §2.7.

2.2. **Algebraic groups and real Lie groups.** Throughout this paper G denotes GL_{2n}/F, $n \geq 2$, the general linear group over F. Although much of the paper works also for GL_N with N arbitrary (e.g., the diagram 5.2), it will be crucial for the main result that $N = 2n$ is even (because only then, the exterior square L-function may have a pole, [20], Thm. 2, p. 224) and that $n \geq 2$ (because the ζ-function attached to F has a pole at n, if $n = 1$). Let H be $\mathrm{GL}_n \times \mathrm{GL}_n$ over F. We identify H with a subgroup of G, defined as the image of the homomorphism $J : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{2n}$, where

$$J(g, g')_{k,l} := \begin{cases} g'_{i,j} & \text{if } k = 2i-1 \text{ and } l = 2j-1 \\ g_{i,j} & \text{if } k = 2i \text{ and } l = 2j \\ 0 & \text{else} \end{cases}$$

The center of G/F is denoted Z/F. We write $G_\infty := R_{F/\mathbb{Q}}(G)(\mathbb{R})$ (resp., $H_\infty := R_{F/\mathbb{Q}}(H)(\mathbb{R})$ or $Z_\infty := R_{F/\mathbb{Q}}(Z)(\mathbb{R})$), where $R_{F/\mathbb{Q}}$ stands for Weil's restriction of scalars. Lie algebras of real Lie groups are denoted by the same letter, but in lower case gothics.

At an archimedean place $v \in S_\infty$ we let K_v be a maximal compact subgroup of the real Lie group $G(F_v) = \mathrm{GL}_{2n}(\mathbb{R})$. It is isomorphic to $O(2n)$. We write K_v° for the connected component of the identity of K_v, which is isomorphic to $SO(2n)$. We set $K_\infty := \prod_{v \in S_\infty} K_v$ and $K_\infty^\circ := \prod_{v\in S_\infty} K_v^\circ$. Moreover, we denote by $K_{H,v}$ the intersection $K_v \cap H(F_v)$, which is a maximal compact subgroup of $H(F_v)$, isomorphic to $O(n) \times O(n)$. As before, we write $K_{H,v}^\circ$ for the connected component of the identity and we let $K_{H,\infty} := \prod_{v\in S_\infty} K_{H,v}$ and $K_{H,\infty}^\circ := \prod_{v\in S_\infty} K_{H,v}^\circ$.

Let A_G be the multiplicative group of positive real numbers $\mathbb{R}_+$, being diagonally embedded into the center Z_∞ of G_∞. It is a direct complement of the group $G(\mathbb{A})^{(1)} := \{g \in G(\mathbb{A}) | \|\det(g)\| = 1\}$ in $G(\mathbb{A})$. According to our conventions, the Lie algebra of the real Lie group A_G is denoted $\mathfrak{a}_G$. Furthermore, we let $\mathfrak{m}_G := \mathfrak{g}_\infty/\mathfrak{a}_G$, $\mathfrak{m}_H := \mathfrak{h}_\infty/\mathfrak{a}_G$ and $\mathfrak{s} := \mathfrak{z}_\infty/\mathfrak{a}_G$. Observe that these spaces are Lie subalgebras of $\mathfrak{g}_\infty$.

2.3. **Coefficient modules.** In this paper, E_μ denotes an irreducible, algebraic representation of G_∞ on a finite-dimensional complex vector space. It is determined by its highest weight $\mu = (\mu_v)_{v\in S_\infty}$, whose local components at an archimedean place v may be identified with $\mu_v = (\mu_{1,v}, ..., \mu_{2n,v}) \in \mathbb{Z}^{2n}$, $\mu_{1,v} \geq \mu_{2,v} \geq ... \geq \mu_{2n,v}$. We assume that E_μ is self-dual, i.e., it is isomorphic to its contragredient, $E_\mu \cong E_\mu^\vee$, or, in other words, that

$$\mu_{j,v} + \mu_{2n-j+1,v} = 0, \qquad 1 \leq j \leq n$$

at all places $v \in S_\infty$. Clearly, this condition implies that $\mu_{n,v} \geq 0 \geq \mu_{n+1,v}$ for all $v \in S_\infty$. The self-duality hypothesis, hence incorporates that $\dim_\mathbb{C} \mathrm{Hom}_{H(\mathbb{C})}(E_{\mu_v}, \mathbb{C}) = 1$ for all $v \in S_\infty$. (See [14], Prop. 6.3.1.)

2.4. **Cohomology of locally symmetric spaces.** Define the orbifolds

$$\mathcal{S}_G := G(F)\backslash G(\mathbb{A})/A_G K_\infty^\circ = G(F)\backslash G(\mathbb{A})^1/K_\infty^\circ$$

and

$$\tilde{\mathcal{S}}_H := H(F)\backslash H(\mathbb{A})/A_G K_{H,\infty}^\circ.$$

A representation E_μ as in §2.3 defines a locally constant sheaf $\mathcal{E}_\mu$ on $\mathcal{S}_G$, whose espace étalé is $G(\mathbb{A})^1/K_\infty^\circ \times_{G(F)} E_\mu$ (with the discrete topology on E_μ). Along the proper map $\mathcal{J} : \tilde{\mathcal{S}}_H \to \mathcal{S}_G$, which is induced by J, §2.2, we also obtain a sheaf on $\tilde{\mathcal{S}}_H$, which we will again denote by $\mathcal{E}_\mu$. Let $H_c^q(\mathcal{S}_G, \mathcal{E}_\mu)$ (resp. $H_c^q(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu)$) be the corresponding space of sheaf cohomology with compact support. This is an admissible $G(\mathbb{A}_f)$-module (resp. $H(\mathbb{A}_f)$-module), cf. [31], Cor. 2.13. Observe that the proper map $\mathcal{J}$ from above gives rise to a non-trivial $H(\mathbb{A}_f)$-equivariant map

$$\mathcal{J}_\mu^q : H_c^q(\mathcal{S}_G, \mathcal{E}_\mu) \to H_c^q(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu).$$

2.5. **Complex automorphisms and rational structures.** For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let us define the σ-twist ${}^\sigma\nu$ of an (abstract) representation ν of $G(\mathbb{A}_f)$ (resp., $G(F_v)$, $v \notin S_\infty$) on a complex vector space W, following Waldspurger [36], I.1: If W' is a $\mathbb{C}$-vector space with a σ-linear isomorphism $\phi : W \to W'$ then we set

$${}^\sigma\nu := \phi \circ \nu \circ \phi^{-1}.$$

This definition is independent of ϕ and W' up to equivalence of representations, whence we may always take $W' := W \otimes_\sigma \mathbb{C}$, i.e., the abelian group W endowed with the scalar multiplication $\lambda \cdot_\sigma w := \sigma^{-1}(\lambda)w$.

Furthermore, if $\nu_\infty = \bigotimes_{v\in S_\infty} \nu_v$ is a representation of the real Lie group G_∞, we let

$${}^\sigma\nu_\infty := \bigotimes_{v\in S_\infty} \nu_{\sigma^{-1}v},$$

interpreting $v \in S_\infty$ as an embedding of fields $v : F \hookrightarrow \mathbb{R}$. Combining these two definitions, we may define the σ-twist on a global representation $\nu = \nu_\infty \otimes \nu_f$ of $G(\mathbb{A})$ as

$${}^\sigma\nu := {}^\sigma\nu_\infty \otimes {}^\sigma\nu_f.$$

We recall also the definition of the rationality field of a representation from [36], I.1. If ν is any of the representations considered above, then let $\mathfrak{S}(\nu)$ be the group of all automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma\nu \cong \nu$. Then the rationality field $\mathbb{Q}(\nu)$ of ν is defined as the fixed-field of $\mathfrak{S}(\nu)$ within $\mathbb{C}$, i.e.,

$$\mathbb{Q}(\nu) := \{z \in \mathbb{C} | \sigma(z) = z \text{ for all } \sigma \in \mathfrak{S}(\nu)\}.$$

We say that a representation ν on a $\mathbb{C}$-vector space W is defined over a subfield $\mathbb{F} \subset \mathbb{C}$, if there is a $\mathbb{F}$-vector subspace $W_{\mathbb{F}} \subset W$, stable under the given action, such that the canonical map $W_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C} \to W$ is an isomorphism. The following lemma is due to Clozel, [7], p. 122 and p. 128. (See also [13], Lem. 7.1.)

Lemma 2.6. *Let E_μ be an irreducible, algebraic representation as in §2.3. As a representation of the diagonally embedded group $G(F) \hookrightarrow G_\infty$, ${}^\sigma E_\mu$ is isomorphic to the abstract representation $E_\mu \otimes_\sigma \mathbb{C}$. Moreover, as a representation of $G(F)$, E_μ is defined over $\mathbb{Q}(E_\mu)$.*

We fix once and for all a $\mathbb{Q}(E_\mu)$-structure on E_μ as a representation of $G(F)$. Clearly, this also fixes a $\mathbb{Q}(E_\mu)$-structure on E_μ as a representation of $H(F)$. As a consequence, the $G(\mathbb{A}_f)$-module $H^q_c(\mathcal{S}_G, \mathcal{E}_\mu)$ and the $H(\mathbb{A}_f)$-module $H^q_c(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu)$ carry a fixed, natural $\mathbb{Q}(E_\mu)$-structure, cf. [7] p. 123. Moreover, this also pins down natural σ-linear, equivariant isomorphisms

$$\mathcal{H}^{\sigma,q}_\mu : H^q_c(\mathcal{S}_G, \mathcal{E}_\mu) \xrightarrow{\sim} H^q_c(\mathcal{S}_G, {}^\sigma\mathcal{E}_\mu) \qquad \text{and} \qquad \tilde{\mathcal{H}}^{\sigma,q}_\mu : H^q_c(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu) \xrightarrow{\sim} H^q_c(\tilde{\mathcal{S}}_H, {}^\sigma\mathcal{E}_\mu) \tag{2.1}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, cf. [7] p. 128. The following lemma is obvious.

Lemma 2.7. *For all $\sigma \in \mathrm{Aut}(\mathbb{C})$ the following diagram commutes,*

$$\begin{tikzcd}
H^q_c(\mathcal{S}_G, \mathcal{E}_\mu) \arrow[r, "\mathcal{J}^q_\mu"] \arrow[d, "\mathcal{H}^{\sigma,q}_\mu"'] & H^q_c(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu) \arrow[d, "\tilde{\mathcal{H}}^{\sigma,q}_\mu"'] \\
H^q_c(\mathcal{S}_G, {}^\sigma\mathcal{E}_\mu) \arrow[r, "\mathcal{J}^q_{\sigma_\mu}"] & H^q_c(\tilde{\mathcal{S}}_H, {}^\sigma\mathcal{E}_\mu)
\end{tikzcd}$$

3. Facts and conventions for cuspidal automorphic representations

3.1. **Cohomological cusp forms.** In this paper, we let Π be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Furthermore, we assume that Π is self-dual, i.e., $\Pi \cong \Pi^\vee$. This is no loss of generality, as the main result will only hold for such cuspidal representations. (Compare this to Prop. 3.5 below.) For convenience we will not distinguish between a cuspidal automorphic representation, its smooth automorphic LF-space completion and its (non-smooth) Hilbert space completion in the L^2-spectrum. With these conventions, recall that Π has a (unique) Whittaker model (with respect to ψ). We write

$$W^\psi : \Pi \to \mathcal{W}^\psi(\Pi)$$

for the realization of Π in its Whittaker model $\mathcal{W}^\psi(\Pi)$ by the ψ-Fourier coefficient. Recall that there is a *canonical* decomposition $\mathcal{W}^\psi(\Pi) = \otimes'_v \mathcal{W}^{\psi_v}(\Pi_v)$, in the sense that each space $\mathcal{W}^{\psi_v}(\Pi_v)$ is canonically determined by the uniqueness of local Whittaker models. We will furthermore assume that Π is cohomological: By this we understand that there is an irreducible, algebraic representation E_μ of G_∞, as in Sect. 2.3, such that the archimedean component Π_∞ of Π has non-vanishing $(\mathfrak{m}_G, K_\infty^\circ)$-cohomology with respect to E_μ, i.e.,

$$H^q(\mathfrak{m}_G, K_\infty^\circ, \Pi_\infty \otimes E_\mu) \neq 0,$$

for some degree q.

Lemma 3.2. *Let ρ_∞ be an irreducible unitary $(\mathfrak{g}_\infty, K_\infty^\circ)$-module with trivial A_G-action. Then the following assertions are equivalent:*

(1) $H^*(\mathfrak{m}_G, K_\infty^\circ, \rho_\infty \otimes E_\mu) \neq 0,$
(2) $H^*(\mathfrak{g}_\infty, K_\infty^\circ, \rho_\infty \otimes E_\mu) \neq 0,$
(3) $H^*(\mathfrak{g}_\infty, (Z_\infty K_\infty)^\circ, \rho_\infty \otimes E_\mu) \neq 0.$

Proof. This follows combining the following well-known results on relative Lie algebra cohomology :[5], I. 1.3 (the Künneth rule), I. 5.1, I. Thm. 5.3 (Wigner's lemma) and II. Prop. 3.1 (all cochains are closed and harmonic). □

As a consequence, the archimedean component Π_∞ of a cuspidal automorphic representation Π, as above, is cohomological in our sense, if and only if Π_∞ has non-vanishing $(\mathfrak{g}_\infty, K_\infty^\circ)$-cohomology or equivalently, non-vanishing $(\mathfrak{g}_\infty, (Z_\infty K_\infty)^\circ)$-cohomology with respect to the same algebraic, self-dual coefficient module E_μ (although the degrees and dimensions of non-trivial cohomology spaces may change).

The component group $\pi_0(G_\infty) \cong K_\infty/K_\infty^\circ$ acts on the cohomology groups $H^q(\mathfrak{m}_G, K_\infty^\circ, \Pi_\infty \otimes E_\mu)$ in each degree. For a character $\epsilon \in \pi_0(G_\infty)^*$, which we identify with

$$\epsilon = (\epsilon_1, \dots, \epsilon_d) \in (\mathbb{Z}/2\mathbb{Z})^d \cong \pi_0(G_\infty)^*,$$

one obtains a corresponding $\pi_0(G_\infty)$-isotypic component $H^q(\mathfrak{m}_G, K_\infty^\circ, \Pi_\infty \otimes E_\mu)[\epsilon]$. Put

$$t := dn(n+1) - 1. \tag{3.1}$$

Then,

$$\dim_\mathbb{C} H^t(\mathfrak{m}_G, K_\infty^\circ, \Pi_\infty \otimes E_\mu)[\epsilon] = 1$$

for all $\epsilon \in \pi_0(G_\infty)^*$. This is a direct consequence of the formula in Clozel [7], Lem. 3.14 (see also [26, 3.1.2] or [13, 5.5]), the Künneth rule ([5], I. 1.3) and the fact that $\mathfrak{s}$ is a $d-1$-dimensional abelian real Lie algebra, whence $H^q(\mathfrak{s}, \mathbb{C}) \cong \Lambda^q \mathbb{C}^{d-1}$.

Observe furthermore, that (for all degrees q and characters $\epsilon \in \pi_0(G_\infty)^*$) there is a natural $G(\mathbb{A}_f)$-equivariant inclusion

$$\Delta_\Pi^q : H^q(\mathfrak{m}_G, K_\infty^\circ, \Pi \otimes E_\mu)[\epsilon] \hookrightarrow H_c^q(\mathcal{S}_G, \mathcal{E}_\mu). \tag{3.2}$$

This is well-known and follows from [4], §5.

3.3. **Rational structures.** We have the following result:

Proposition 3.3. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$, which is cohomological with respect to E_μ. Then, the σ-twisted representation ${}^\sigma\Pi$ is also cuspidal automorphic and it is cohomological with respect to ${}^\sigma E_\mu$. For every $\epsilon \in \pi_0(G_\infty)^*$, the irreducible unitary $G(\mathbb{A}_f)$-module $H^t(\mathfrak{m}_G, K_\infty^\circ, \Pi \otimes E_\mu)[\epsilon]$ is defined over the rationality field $\mathbb{Q}(\Pi_f)$. This field is a number field, containing $\mathbb{Q}(E_\mu)$.*

Proof. This is essentially due to Clozel, [7]. In order to derive the above result from [7], observe that Π_∞ is "regular algebraic" in Clozel's sense, if and only if it is cohomological in our sense: This follows using Lem. 3.2 and [13] Thm. 6.3. Hence, ${}^\sigma\Pi_f$ is the non-archimedean part of a cuspidal automorphic representation, which is cohomological with respect to ${}^\sigma E_\mu$ by [7] Thm. 3.13. By uniqueness, see e.g. [13] 5.5, the archimedean part of this cuspidal automorphic representation is isomorphic to ${}^\sigma\Pi_\infty$ as defined above. By [7], Prop. 3.1, $H^t(\mathfrak{m}_G, K_\infty^\circ, \Pi\otimes E_\mu)[\epsilon]$ is defined over $\mathbb{Q}(\Pi_f)$ (See also [13], Cor. 8.7.). Finally, it is an implicit consequence of [7], Thm. 3.13 and its proof that $\mathbb{Q}(\Pi_f)$ is a number field containing $\mathbb{Q}(E_\mu)$. For a detailed exposition of the latter assertion, we refer to [13], Thm. 8. 1 and the proof of [13], Cor. 8.7. □

Definition 3.4. The $\mathbb{Q}(\Pi_f)$-structure on $H^t(\mathfrak{m}_G, K_\infty^\circ, \Pi\otimes E_\mu)[\epsilon]$ is unique up to homotheties, i.e., up to multiplication by non-zero complex numbers, cf. [7] Prop. 3.1. As $\mathbb{Q}(E_\mu)\subseteq\mathbb{Q}(\Pi_f)$, we may fix the $\mathbb{Q}(\Pi_f)$-structure on $H^t(\mathfrak{m}_G, K_\infty^\circ, \Pi\otimes E_\mu)[\epsilon]$ which is induced by Δ^t_Π, cf. (3.2), and our choice of a $\mathbb{Q}(E_\mu)$-structure on $H^t_c(\mathcal{S}_G, \mathcal{E}_\mu)$, cf §2.5.

3.4. **Lifts from** $\mathrm{SO}(2n+1)$**.** We resume the assumptions made on Π from Sect. 3.1. As a last part of notation for Π, let us introduce $S = S(\Pi,\psi)$, which is a (sufficiently large) finite set of places of F, containing S_∞ and such that outside S, both Π and ψ are unramified.

Proposition 3.5. *Let* Π *be a cuspidal automorphic representation of* $G(\mathbb{A})$ *as in Sect. 3.1 above. Then the following assertions are equivalent:*

(1) *The partial exterior square L-function,* $L^S(s,\Pi,\Lambda^2)$*, has a pole at* $s=1$*,*
(2) Π *is the lift of an irreducible unitary generic cuspidal automorphic representation of the split special orthogonal group* $\mathrm{SO}(2n+1)$ *in the sense of* [8], *§1.*

Proof. With our assumptions on Π this is [8], Thm. 7.1. □

This result is recalled for convenience, as it provides an alternative description of what it means that the exterior square L-function of Π has a pole at $s=1$. We will have to make this assumption later, in order to obtain our main theorems. See, Thm. 6.1, Thm. 7.4, Thm. 9.2, and Thm. 10.1. It is not referred to until §6.4. In any case, the above result is accompanied by

Proposition 3.6. *Let* Π *be a cuspidal automorphic representation of* $G(\mathbb{A})$ *as in §3.1. Assume that* Π *satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function,* $L^S(s,\Pi,\Lambda^2)$*, has a pole at* $s=1$*. Then, for all* $\sigma\in\mathrm{Aut}(\mathbb{C})$*,* ${}^\sigma\Pi$ *is a cuspidal automorphic representation of* $G(\mathbb{A})$ *as in §3.1, which satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function,* $L^S(s,{}^\sigma\Pi,\Lambda^2)$*, has a pole at* $s=1$*.*

Proof. The first assertion has already been proved in Prop. 3.3. For the second assertion, observe that the set S does not change under the action of $\mathrm{Aut}(\mathbb{C})$ and then combine [14], Thm. 3.6.2 and [20], Thm. 1, p. 213. □

4. Top-degree periods

4.1. **The map** W^σ**.** Recall the unique Whittaker model $\mathcal{W}^{\psi_f}(\Pi_f)=\otimes'_{v\notin S_\infty}\mathcal{W}^{\psi_v}(\Pi_v)$ of Π_f, its decomposition being canonical. Given a Whittaker function $\xi_v\in\mathcal{W}^{\psi_v}(\Pi_v)$ on $G(F_v)$,

$v \notin S_\infty$, and $\sigma \in \mathrm{Aut}(\mathbb{C})$, we may define a Whittaker function ${}^\sigma\xi_v \in \mathcal{W}^{\psi_v}({}^\sigma\Pi_v)$ by

$$ {}^\sigma\xi_v(g_v) := \sigma(\xi_v(\boldsymbol{t}_{\sigma,v} \cdot g_v)), \tag{4.1}$$

where $\boldsymbol{t}_{\sigma,v}$ is the (uniquely determined) diagonal matrix in $G(\mathcal{O}_v)$, having 1 as its last entry, which conjugates ψ_v to $\sigma \circ \psi_v$. (Observe that $\boldsymbol{t}_{\sigma,v}$ does not depend on ψ_v). See [26], 3.3 and [30], 3.2. This provides us a σ-linear intertwining operator

$$W^\sigma : \mathcal{W}^{\psi_v}(\Pi_v) \to \mathcal{W}^{\psi_v}({}^\sigma\Pi_v)$$

$$\xi_v \mapsto {}^\sigma\xi_v,$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. In particular, we get a $\mathbb{Q}(\Pi_v)$ structure on $\mathcal{W}^{\psi_v}(\Pi_v)$ by taking the subspace of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi_v))$-invariant vectors. By the same procedure, we obtain a canonical $\mathbb{Q}(\Pi_f)$ structure on $\mathcal{W}^{\psi_f}(\Pi_f)$. (Cf. [15], p. 80, [26], 3.3 or [30], Lem. 3.2.)

4.2. **The map F^t_Π.** Let $\epsilon_0 := ((-1)^{n-1}, ..., (-1)^{n-1}) \in \pi_0(G_\infty)^*$. This choice of a character of the component group is forced upon us by the proof of Thm. 7.1 and so we restrict our attention from now on to it.

Recall the $\mathbb{Q}(\Pi_f)$-rational structure on $H^t(\mathfrak{m}_G, K^\circ_\infty, \Pi \otimes E_\mu)[\epsilon_0]$, chosen in Def. 3.4 above and recall the canonical $\mathbb{Q}(\Pi_f)$-rational structure on the Whittaker model $\mathcal{W}^{\psi_f}(\Pi_f)$ of Π_f just fixed in 4.1 above. As it has been mentioned briefly in the introduction, our top-degree Whittaker period – in abbreviated symbol $p^t(\Pi)$ – will be determined by the comparison of these two rational structures. Hence, in order to actually compare them, we have to specify a concrete comparison isomorphism

$$F^t_\Pi : \mathcal{W}^{\psi_f}(\Pi_f) \xrightarrow{\sim} H^t(\mathfrak{m}_G, K^\circ_\infty, \Pi \otimes E_\mu)[\epsilon_0].$$

It is the purpose of this section to explain this choice carefully, as it is all crucial for the definition of our periods. Our choices will be guided by the ideas in [15], p. 79, [26], 3.3 & 5.1.4, [29] 3.2.5, and [14], 4.1.

The first data we will fix once and for all consists of

Choice 4.2. (1) a basis $\{X_j\}$ of $\mathfrak{m}_G/\mathfrak{k}_\infty$, which fixes the dual-basis $\{X^*_j\}$ of $(\mathfrak{m}_G/\mathfrak{k}_\infty)^*$; given a mulit-index $\underline{i} = (i_1, ..., i_t)$, we abbreviate $X^*_{\underline{i}} := X^*_{i_1} \wedge ... \wedge X^*_{i_t}$

(2) vectors $e_\alpha := \otimes_{v\in S_\infty} e_{\alpha,v} \in E_\mu = \otimes_{v\in S_\infty} E_{\mu_v}$, such that $\{e_{\alpha,v}\}_\alpha$ defines a $\mathbb{Q}(E_\mu)$-basis of E_{μ_v} for all $v \in S_\infty$

(3) for each $v \in S_\infty$, mulit-index $\underline{i} = (i_1, ..., i_t)$ and α as above a Whittaker function $\xi^{\epsilon_0}_{v,\underline{i},\alpha} \in \mathcal{W}^{\psi_v}(\Pi_v)$, such that (putting $\xi^{\epsilon_0}_{\infty,\underline{i},\alpha} := \otimes_{v\in S_\infty} \xi^{\epsilon_0}_{v,\underline{i},\alpha} \in \mathcal{W}^{\psi_\infty}(\Pi_\infty)$) the vector

$$[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t := \sum_{\underline{i}=(i_1,...,i_t)} \sum_{\alpha=1}^{\dim E_\mu} X^*_{\underline{i}} \otimes \xi^{\epsilon_0}_{\infty,\underline{i},\alpha} \otimes e_\alpha,$$

is a generator of the one-dimensional space $\mathbb{C}$-vector space $H^t(\mathfrak{m}_G, K^\circ_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)[\epsilon_0]$. (We may and will also assume that $\{X_j\}$ is the extension of a given ordered basis $\{Y_j\}$ of $\mathfrak{m}_H/\mathfrak{k}_{H,\infty}$ along our embedding $J : H \hookrightarrow G$. This assumption, however, will only be important later on. See, e.g., §5.2 and §7.1.)

By [5], II. Prop. 3.1 and the uniqueness of the archimedean Whittaker model and its canonical decomposition into local factors $\mathcal{W}^{\psi_\infty}(\Pi_\infty)=\otimes_{v\in S_\infty}\mathcal{W}^{\psi_v}(\Pi_v)$ this generator $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ is well-defined. For the sake of readability we suppress its various dependencies, listed in Choice 4.2 above, in its notation.

Next recall (e.g. from §2.5) that $\sigma\in\mathrm{Aut}(\mathbb{C})$ acts on objects at infinity, which are parameterized by S_∞, by permuting the archimedean places. Having given a generator $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ of the one-dimensional space $H^t(\mathfrak{m}_G,K^\circ_\infty,\mathcal{W}^{\psi_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_0]$ hence provides us with a natural choice of a generator $[\mathcal{W}^{\psi_\infty}({}^\sigma\Pi_\infty)]^t$ of $H^t(\mathfrak{m}_G,K^\circ_\infty,\mathcal{W}^{\psi_\infty}({}^\sigma\Pi_\infty)\otimes {}^\sigma E_\mu)[\epsilon_0]$:

$$[\mathcal{W}^{\psi_\infty}({}^\sigma\Pi_\infty)]^t:=\sum_{\underline{i}=(i_1,\dots,i_t)}\sum_{\alpha=1}^{\dim E_\mu}X^*_{\underline{i}}\otimes {}^\sigma\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}\otimes {}^\sigma e_\alpha,$$

where ${}^\sigma\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}=\otimes_{v\in S_\infty}\xi^{\epsilon_0}_{\sigma^{-1}v,\underline{i},\alpha}$ (observe that ϵ_0 does not change, when its local components are permuted) and ${}^\sigma e_\alpha=\otimes_{v\in S_\infty}e_{\alpha,\sigma^{-1}v}$.

Finally, this entails the description of the desired "comparison isomorphism" mentioned at the beginning of this subsection, i.e., of a fixed choice of isomorphism of $G(\mathbb{A}_f)$-modules defined by

$$F^t_\Pi:\mathcal{W}^{\psi_f}(\Pi_f)\xrightarrow{\sim}H^t(\mathfrak{m}_G,K^\circ_\infty,\Pi\otimes E_\mu)[\epsilon_0]$$
$$\xi_f\mapsto F^t_\Pi(\xi_f):=\sum_{\underline{i}=(i_1,\dots,i_t)}\sum_{\alpha=1}^{\dim E_\mu}X^*_{\underline{i}}\otimes\varphi_{\underline{i},\alpha}\otimes e_\alpha,$$

where $\varphi_{\underline{i},\alpha}:=(W^\psi)^{-1}(\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}\otimes\xi_f)\in\Pi$. It is important to observe that we did not have to decompose the global map W^ψ computing the ψ-Fourier coefficient, hence there are no hidden ambiguities in this definition: A complete set of dependencies of our comparison isomorphism F^t_Π is hence given by Choice 4.2. In light of Prop. 3.3 and our discussion above, we obtain isomorphisms $F^t_{{}^\sigma\Pi}$ for all $\sigma\in\mathrm{Aut}(\mathbb{C})$ with the same precise set of dependencies.

4.3. **The map $H^{\sigma,t}_\mu$.** As a last ingredient in this section, we define a σ-linear, $G(\mathbb{A}_f)$-equivariant isomorphism

$$H^{\sigma,t}_\mu:H^t(\mathfrak{m}_G,K^\circ_\infty,\Pi\otimes E_\mu)[\epsilon_0]\xrightarrow{\sim}H^t(\mathfrak{m}_G,K^\circ_\infty,{}^\sigma\Pi\otimes{}^\sigma E_\mu)[\epsilon_0].$$

To that end, recall the embedding Δ^q_Π from (3.2) and the σ-linear isomorphism $\mathcal{H}^{\sigma,q}_\mu$ from (2.1). Observe that $\mathrm{Im}(\mathcal{H}^{\sigma,t}_\mu\circ\Delta^t_\Pi)=\mathrm{Im}(\Delta^t_{{}^\sigma\Pi})$. Indeed, by Multiplicity One and Strong Multiplicity One for the discrete automorphic spectrum of $G(\mathbb{A})$, the ${}^\sigma\Pi_f$-isotypic component of the $G(\mathbb{A}_f)$-module $H^t_c(\mathcal{S}_G,{}^\sigma\mathcal{E}_\mu)$ is precisely the image of $H^t(\mathfrak{m}_G,K^\circ_\infty,{}^\sigma\Pi\otimes{}^\sigma E_\mu)$. As the natural action of $\pi_0(G_\infty)$ and of $G(\mathbb{A}_f)$ on $H^t_c(\mathcal{S}_G,\mathcal{E}_\mu)$ commute, this shows that $\mathrm{Im}(\mathcal{H}^{\sigma,t}_\mu\circ\Delta^t_\Pi)=\mathrm{Im}(\Delta^t_{{}^\sigma\Pi})$ as claimed. Since $\Delta^t_{{}^\sigma\Pi}$ is injective, the map

$$H^{\sigma,t}_\mu:=(\Delta^t_{{}^\sigma\Pi})^{-1}\circ\mathcal{H}^{\sigma,t}_\mu\circ\Delta^t_\Pi$$

is hence a well-defined σ-linear, $G(\mathbb{A}_f)$-equivariant isomorphism mapping $H^t(\mathfrak{m}_G,K^\circ_\infty,\Pi\otimes E_\mu)[\epsilon_0]$ onto $H^t(\mathfrak{m}_G,K^\circ_\infty,{}^\sigma\Pi\otimes{}^\sigma E_\mu)[\epsilon_0]$ as desired. (Shortly speaking, this amounts to say that the restriction $H^{\sigma,t}_\mu$ of $\mathcal{H}^{\sigma,t}_\mu$ to the submodule $H^t(\mathfrak{m}_G,K^\circ_\infty,\Pi\otimes E_\mu)[\epsilon_0]$ of $H^t_c(\mathcal{S}_G,\mathcal{E}_\mu)$ has image $H^t(\mathfrak{m}_G,K^\circ_\infty,{}^\sigma\Pi\otimes{}^\sigma E_\mu)[\epsilon_0]$.)

4.4. **Top-degree Whittaker Periods.** Recall the maps W^σ (§4.1), F^t_Π (§4.2) and $H^{\sigma,t}_\mu$ (§4.3). There is the following result:

Proposition 4.3. *For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a non-zero complex number $p^t({}^\sigma\Pi)$ (a "period"), uniquely determined up to multiplication by elements in $\mathbb{Q}({}^\sigma\Pi_f)^\times$, such that the normalized maps $\mathcal{F}^t_{{}^\sigma\Pi} := p^t({}^\sigma\Pi)^{-1}F^t_{{}^\sigma\Pi}$ make the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{W}^{\psi_f}(\Pi_f) \arrow[r, "\mathcal{F}^t_\Pi"] \arrow[d, "W^\sigma"'] & H^t(\mathfrak{m}_G, K^\circ_\infty, \Pi\otimes E_\mu)[\epsilon_0] \arrow[d, "H^{\sigma,t}_\mu"] \\
\mathcal{W}^{\psi_f}({}^\sigma\Pi_f) \arrow[r, "\mathcal{F}^t_{{}^\sigma\Pi}"] & H^t(\mathfrak{m}_G, K^\circ_\infty, {}^\sigma\Pi\otimes {}^\sigma E_\mu)[\epsilon_0]
\end{tikzcd}$$

Proof. This is essentially due to the uniqueness of essential vectors for Π_v, $v \notin S_\infty$: Otherwise put, the proof of Prop./Def. 3.3 in Raghuram–Shahidi [30] goes through word for word in our (slightly different) situation at hand. □

Remark 4.4. A lot of choices have been made in order to give the definition of our top-degree Whittaker periods, while (almost) none of them is reflected explicitly in our choice of notation "$p^t(\Pi)$". So, for the sake of precision, we would like to summarize comprehensively at one place on which data, i.e., fixed chosen ingredients, $p^t(\Pi)$ actually depends:

(1) Π, ψ and the cohomological degree t
(2) The fixed concrete choices of a $\mathbb{Q}(\Pi_f)$-rational structure on the canonical Whittaker model $\mathcal{W}^{\psi_f}(\Pi_f)$ (§4.1) and on $H^t(\mathfrak{m}_G, K^\circ_\infty, \Pi\otimes E_\mu)[\epsilon_0]$ (Def. 3.4)
(3) The concrete choice of a comparison isomorphism F^t_Π (§4.2), which depends itself precisely on the data fixed in Choice 4.2
(4) The σ-linear intertwining operator $W^\sigma : \mathcal{W}^{\psi_v}(\Pi_v) \to \mathcal{W}^{\psi_v}({}^\sigma\Pi_v)$ defined unambiguously in §4.1 and
(5) The σ-linear intertwining operator $H^{\sigma,t}_\mu$ defined unambiguously in §4.3.

The (Whittaker) periods $p^t(\Pi)$ defined by Prop. 4.3 are the analogues of the (Shalika) periods $\omega^\epsilon(\Pi_f)$ defined in Grobner–Raghuram Def.Prop. 4.2.1. The idea behind the construction of $p^t(\Pi)$ (as of $\omega^\epsilon(\Pi_f)$), however, goes back to [15], [26] and [30].

5. An Aut($\mathbb{C}$)-rational assignment for Whittaker functions

5.1. **The map T_μ.** Let $E_\mu = \otimes_{v\in S_\infty} E_{\mu_v}$ be an irreducible, algebraic representation as in §2.3. We have $\dim_\mathbb{C} \mathrm{Hom}_{H(\mathbb{C})}(E_{\mu_v}, \mathbb{C}) = 1$ for all $v \in S_\infty$. Let us fix $T_{\mu_v} \in \mathrm{Hom}_{H(\mathbb{C})}(E_{\mu_v}, \mathbb{C})$ and set $T_\mu := \otimes_{v\in S_\infty} T_{\mu_v} \in \mathrm{Hom}_{R_{F/\mathbb{Q}}(H)(\mathbb{C})}(E_\mu, \mathbb{C})$. For $\sigma \in$Aut$(\mathbb{C})$, we obtain $T_{{}^\sigma\mu} = \otimes_{v\in S_\infty} T_{\mu_{\sigma^{-1}v}} \in \mathrm{Hom}_{R_{F/\mathbb{Q}}(H)(\mathbb{C})}({}^\sigma E_\mu, \mathbb{C})$. The map induced on cohomology,

$$H^t_c(\tilde{\mathcal{S}}_H, {}^\sigma\mathcal{E}_\mu) \to H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C})$$

will be denoted by the same letter $T_{{}^\sigma\mu}$. Finally, we normalize T_μ such that it becomes rational, i.e., we require that T_μ is induced from a homomorphism of representations of underlying algebraic groups. This pins down T_μ up to multiples in $\mathbb{Q}^\times$.

5.2. **The de-Rham-isomorphism $\mathcal{R}$.** So far, we have not made any choice of a Haar measure on $H(\mathbb{A}_f)$. From this section on, we will restrict our possible choices on $\mathbb{Q}$-valued Haar measures on $H(\mathbb{A}_f)$. In §6.3 we will specify our concrete choice of a measure in details. (So far, this is not necessary.) Let dh_f be any $\mathbb{Q}$-valued Haar measure of $H(\mathbb{A}_f)$. It is important to notice that we have $\dim_{\mathbb{R}} \tilde{\mathcal{S}}_H = dn(n+1)-1 = t$, cf. §3.1, because we assumed that F is totally real. Knowing this, a short moment of thought shows that we obtain a surjective map $\mathcal{R} : H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C}) \to \mathbb{C}$, induced by the de Rham-isomorphism: Indeed, let K_f be a compact open subgroup of $H(\mathbb{A}_f)$ and set

$$\tilde{\mathcal{S}}_H^{K_f} := H(F)\backslash H(\mathbb{A})/A_G K^\circ_{H,\infty} K_f.$$

Then it is easy to see that $\tilde{\mathcal{S}}_H$ is homeomorphic to the projective limit

$$\tilde{\mathcal{S}}_H \cong \varprojlim_{K_f} \tilde{\mathcal{S}}_H^{K_f}$$

running over the compact open subgroups K_f of $H(\mathbb{A}_f)$, partially ordered by opposite inclusion, [31] Prop. 1.9. As $\dim_{\mathbb{R}} \tilde{\mathcal{S}}_H^{K_f} = t$ for all K_f, we may use the de-Rham-isomorphism to define a surjective map $H^t_c(\tilde{\mathcal{S}}_H^{K_f}, \mathbb{C}) \to \mathbb{C}$. More precisely, each of the (finitely many, cf. [3], Thm. 5.1) connected components of $\tilde{\mathcal{S}}_H^{K_f}$ is homeomorphic to a quotient of $H^\circ_\infty/A_G K^\circ_{H,\infty}$ by a discrete subgroup of $H(F)$. Recall the ordered basis $\{Y_j\}$ of $\mathfrak{m}_H/\mathfrak{k}_{H,\infty}$, from §4.2. It determines a choice of an orientation on $H^\circ_\infty/A_G K^\circ_{H,\infty}$, whence on each connected component of $\tilde{\mathcal{S}}_H^{K_f}$ and so finally also on $\tilde{\mathcal{S}}_H^{K_f}$. Hence, the de-Rham-isomorphism provides us a surjection

$$R^{K_f} : H^t_c(\tilde{\mathcal{S}}_H^{K_f}, \mathbb{C}) \to \mathbb{C}.$$

The normalized maps, $\mathcal{R}^{K_f} := \mathrm{vol}_{dh_f}(K_f) \cdot R^{K_f}$ form a system of compatible maps with respect to the pull-backs, given by the coverings $\tilde{\mathcal{S}}_H^{K_f} \twoheadrightarrow \tilde{\mathcal{S}}_H^{K'_f}$, $K_f \subseteq K'_f$. Hence, taking the inductive limit, we obtain a surjection $\varinjlim_{K_f} \mathcal{R}^{K_f} : \varinjlim_{K_f} H^t_c(\tilde{\mathcal{S}}_H^{K_f}, \mathbb{C}) \to \mathbb{C}$. Since $H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C}) \cong \varinjlim_{K_f} H^t_c(\tilde{\mathcal{S}}_H^{K_f}, \mathbb{C})$, see [31] Cor. 2.12, this finally defines a surjection

$$\mathcal{R} : H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C}) \to \mathbb{C},$$

as mentioned above. (Compare this also to the considerations in [14], 6.4, [11], 3.8, [12], 5.1 and [29], 3.2.3 as well as to the corresponding references therein.)

5.3. **In summary: A rational diagram.** In the following proposition, we abbreviate $H^t(\mathfrak{m}_G, K^\circ_\infty, \Pi \otimes E_\mu)[\epsilon_0]$ by $H^t(\Pi \otimes E_\mu)[\epsilon_0]$ (with analogous notation for the cohomology of the σ-twisted representations). Recollecting what we observed in §4.1 – §5.2, we find

Proposition 5.1. *The following diagram is commutative:*
(5.2)

$$\begin{tikzcd}
\mathcal{W}^{\psi_f}(\Pi_f) \arrow[r, "\mathcal{F}^t_\Pi"] \arrow[d, "W^\sigma"'] & H^t(\Pi \otimes E_\mu)[\epsilon_0] \arrow[r, hook, "\Delta^t_\Pi"] \arrow[d, "H^{\sigma,t}_\mu"] & H^t_c(\mathcal{S}_G, \mathcal{E}_\mu) \arrow[r, "\mathcal{J}^t_\mu"] \arrow[d, "\mathcal{H}^{\sigma,t}_\mu"] & H^t_c(\tilde{\mathcal{S}}_H, \mathcal{E}_\mu) \arrow[r, "T_\mu"] \arrow[d, "\tilde{\mathcal{H}}^{\sigma,t}_\mu"] & H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C}) \arrow[r, "\mathcal{R}"] \arrow[d, "\tilde{\mathcal{H}}^{\sigma,t}_0"] & \mathbb{C} \arrow[d, "\sigma"] \\
\mathcal{W}^{\psi_f}({}^\sigma\Pi_f) \arrow[r, "\mathcal{F}^t_{{}^\sigma\Pi}"] & H^t({}^\sigma\Pi \otimes {}^\sigma E_\mu)[\epsilon_0] \arrow[r, hook, "\Delta^t_{{}^\sigma\Pi}"] & H^t_c(\mathcal{S}_G, {}^\sigma\mathcal{E}_\mu) \arrow[r, "\mathcal{J}^t_{\sigma_\mu}"] & H^t_c(\tilde{\mathcal{S}}_H, {}^\sigma\mathcal{E}_\mu) \arrow[r, "T_{\sigma_\mu}"] & H^t_c(\tilde{\mathcal{S}}_H, \mathbb{C}) \arrow[r, "\mathcal{R}"] & \mathbb{C}
\end{tikzcd}$$

Its horizontal arrows are linear, whereas its vertical arrows are σ-linear.

Proof. The first square from the left is commutative by Prop. 4.3, while the second square is commutative by the definition of $H^{\sigma,t}_{\mu}$ in §4.3. Commutativity of the third square is the assertion of Lem. 2.7. The fourth square commutes by the very definition of T_{σ_μ} in §5.1, while commutativity of the last square is due to the $\mathbb{Q}$-rationality of the measure on $H(\mathbb{A}_f)$, §5.2. $\square$

6. An integral representation of the residue of the exterior square L-function

In this section, we will recapitulate some results from Jacquet–Shalika [21] and Bump–Friedberg [6].

6.1. **Eisenstein series and a result of Jacquet–Shalika.** We resume the notation and assumptions made in the previous sections. In addition, for any integer $m \geq 2$, we will now fix once and for all a Schwartz–Bruhat function $\Phi = \otimes_v \Phi_v \in \mathscr{S}(\mathbb{A}^m)$: We assume that Φ_v is the characteristic function of $\mathcal{O}_v^m$ at all $v \notin S_\infty$, while at the archimedean places $v \in S_\infty$, we assume to have chosen ($O(m)$-finite) local components Φ_v, such the global Schwartz–Bruhat function Φ satisfies $\hat{\Phi}(0) \neq 0$. Here, we wrote

$$\hat{\Phi}(x) := \int_{\mathbb{A}^m} \Phi(y)\psi({}^t y \cdot x)dy$$

for the Fourier transform of Φ (at x) with respect to the self-dual Haar measure dy on $\mathbb{A}^m$, i.e., the unique Haar measure on $\mathbb{A}^m$ which satisfies $\hat{\hat{\Phi}}(x) = \Phi(-x)$ for all $x \in \mathbb{A}^m$. Let

$$f_{v,s}(g_v) := |\det(g_v)|_v^s \int_{F_v^\times} \Phi_v(t \cdot (0,...,0,1)g_v)|t|_v^{ms} d^\times t$$

and

$$f_s(g) := \otimes_v f_{v,s}(g_v) = \|\det(g)\|^s \int_{\mathbb{A}^\times} \Phi(t \cdot (0,...,0,1)g)\|t\|^{ms} d^\times t$$

for $\Re e(s) \gg 0$. Then $f_s \in \mathrm{Ind}_{\mathrm{GL}_{m-1}(\mathbb{A})\times\mathrm{GL}_1(\mathbb{A})}^{\mathrm{GL}_m(\mathbb{A})}[\delta_P^s]$, (unnormalised parabolic induction) where

$$\delta_P\left(\begin{pmatrix} h & 0 \\ 0 & a \end{pmatrix}\right) = \|\det(h)\| \cdot \|a\|^{-(m-1)}$$

is the modulus character of the standard parabolic subgroup P of GL_m, with Levi subgroup $L = \mathrm{GL}_{m-1} \times \mathrm{GL}_1$. Clearly, the analogous assertion holds for the local components $f_{v,s}$. There is the following result due to Jacquet–Shalika, [21], Lem. 4.2

Lemma 6.2. *The Eisenstein series associated with f_s, formally defined as*

$$E(f_s, \Phi)(g) := \sum_{\gamma \in P(F)\backslash \mathrm{GL}_m(F)} f_s(\gamma g),$$

extends to a meromorphic function on $\Re e(s) > 0$. It has a simple pole at $s = 1$ with non-zero constant residue $\mathrm{Res}_{s=1}(E(f_s, \Phi))(g) = c_m\, \hat{\Phi}(0)$, only depending on $\hat{\Phi}(0)$ and the rank m.

6.3. **Measures.** When dealing with rationality results of special values of L-functions, the choice of measures is all-important. In this section, we specify our choices of measures, which will be guided by the explicit choices made in Bump–Friedberg [6].

Let $m \geq 2$ be again any integer and consider the group GL_m/F. A measure dg of $\mathrm{GL}_m(\mathbb{A})$ will be a product $dg = \prod_v dg_v$ of local Haar measures of $\mathrm{GL}_m(F_v)$. We write dg_v^{BF} for the local Haar measure of $\mathrm{GL}_m(F_v)$ chosen in Bump–Friedberg, [6]. See *loc. cit.* (3.2), p. 61. At $v \notin S_\infty$, these measures assign rational volumes to compact open subgroups of $\mathrm{GL}_m(F_v)$. Furthermore, the product measure $dg_f^{BF} := \prod_{v \notin S_\infty} dg_v^{BF}$ gives rational volumes to compact open subgroups of $\mathrm{GL}_m(\mathbb{A}_f)$.

At $v \notin S_\infty$, we define our choice of a measure to be the one of Bump–Friedberg,

$$dg_v := dg_v^{BF}$$

whereas at an archimedean place $v \in S_\infty$, we let dg_v be the local Haar measures of $\mathrm{GL}_m(\mathbb{R})$ such that $SO(m)$ has volume 1. If we let $m = 2n$ as in §2.2, we hence have chosen a measure dg on $G(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A})$.

Recall the group $H = \mathrm{GL}_n \times \mathrm{GL}_n$, §2.2. We will use the notation (g, g'), to specify an element of $H(\mathbb{A})$ (and use analogous notation locally). A measure of $H(\mathbb{A})$ will be the product of a measures dg and dg' as chosen above for $m = n$ of the two isomorphic copies of $\mathrm{GL}_n(\mathbb{A})$ inside $H(\mathbb{A})$. As $Z \subset H$, also the volume $\mathrm{vol}_{dg \times dg'}(Z(F)\backslash Z(\mathbb{A})/A_G)$ is well-defined and finite.

6.4. **A result of Bump–Friedberg.** Let U_n be the group of upper triangular matrices in GL_n, having 1 on the diagonal and let Z_n be the centre of GL_n. Recall the finite set of places $S = S(\Pi, \psi)$ from §3.4. By assumption, outside S, both Π and ψ are unramified (and ψ normalised). Let $\xi = \otimes_v \xi_v \in \mathcal{W}^\psi(\Pi) \cong \otimes_v \mathcal{W}^{\psi_v}(\Pi_v)$ be a Whittaker function, such that for $v \notin S$, ξ_v is invariant under $G(\mathcal{O}_v)$ and normalized such that $\xi_v(id_v) = 1$. Recall the section $f_s = \otimes_v f_{v,s}$ from §6.1, defined by the choice of a Schwartz-Bruhat function $\Phi = \otimes_v \Phi_v$, where we now let $m = n$. Following Bump-Friedberg [6], p. 53, we define the integral

$$Z(\xi, f_s) := \int_{U_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \int_{Z_n(\mathbb{A})U_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \xi(J(g, g'))\ f_s(g)\ dg\ dg'.$$

It factors over all places of F as $Z(\xi, f_s) = \prod_v Z_v(\xi_v, f_{v,s})$, where

$$Z_v(\xi_v, f_{v,s}) := \int_{U_n(F_v)\backslash \mathrm{GL}_n(F_v)} \int_{Z_n(F_v)U_n(F_v)\backslash \mathrm{GL}_n(F_v)} \xi_v(J(g_v, g'_v))\ f_{v,s}(g_v)\ dg_v\ dg'_v.$$

Recall the value $L^S(n, \mathbf{1})$ of the partial L-function of the trivial character $\mathbf{1}$ of $\mathbb{A}^\times$ at n. Since we assumed that $n \geq 2$, this number is well-defined and non-zero. The following result is crucial for us:

Theorem 6.1. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in Sect. 3.1. Let $\varphi := (W^\psi)^{-1}(\xi) \in \Pi$ be the inverse image of a Whittaker function ξ as in §6.4 above and assume that Π satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function, $L^S(s, \Pi, \Lambda^2)$, has a pole at $s = 1$. Then,*

$$c_n\ \hat{\Phi}(0) \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(J(g, g'))\ dg\ dg' = \frac{L^S(\frac{1}{2}, \Pi) \cdot \mathrm{Res}_{s=1}(L^S(s, \Pi, \Lambda^2))}{L^S(n, \mathbf{1})^2} \cdot \prod_{v \in S} \frac{Z_v(\xi_v, f_{v,1})}{L(n, \mathbf{1}_v)}.$$

(Here, $L^S(\frac{1}{2},\Pi)$ is the partial principal L-function of Π at $s'=\frac{1}{2}$.)

Proof. With our preparatory work, this is a direct consequence of our choice of measures in §6.3 and the three main results of Bump–Friedberg [6], Thm. 1, Thm. 2 and Thm. 3. Indeed, our Lem. 6.2 together with [6], Thm. 1 and Thm. 2, identify the left hand side with

$$c_n\ \hat{\Phi}(0)\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})}\varphi(J(g,g'))\ dg\ dg' = \mathrm{Res}_{s=1}\left(\frac{Z(\xi,f_s)}{L(n,\mathbf{1})}\right),$$

where $L(n,\mathbf{1})=\prod_v L(n,\mathbf{1}_v)$ is the global L-function of the trivial character $\mathbf{1}$ of $\mathbb{A}^\times$ at n. As by assumption $n\geq 2$, $L(n,\mathbf{1})$ is well-defined and non-zero. Factorizing $Z(\xi,f_s)$ as in §6.4, and using the description of $Z_v(\xi_v,f_{v,s})$, $v\notin S$, in [6], Thm. 3,

$$Z_v(\xi_v,f_{v,s})=\frac{L(\frac{1}{2},\Pi_v)\cdot L(s,\Pi_v,\Lambda^2)}{L(n,\mathbf{1}_v)},$$

we obtain

$$\mathrm{Res}_{s=1}\left(\frac{Z(\xi,f_s)}{L(n,\mathbf{1})}\right)=\frac{L^S(\frac{1}{2},\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{L^S(n,\mathbf{1})^2}\cdot\prod_{v\in S}\frac{Z_v(\xi_v,f_{v,1})}{L(n,\mathbf{1}_v)},$$

since by assumption $L^S(s,\Pi,\Lambda^2)$ carries the (simple) pole of the above expression. □

6.5. **Consequences for the σ-twisted case.** Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in §3.1 and assume that the partial exterior square L-function, $L^S(s,\Pi,\Lambda^2)$, has a pole at $s=1$. Then by Prop. 3.6, ${}^\sigma\Pi$ satisfies the same conditions. Hence, we see that once Π satisfies the assumptions made in Thm. 6.1, then automatically also ${}^\sigma\Pi$ satisfies them, i.e., Thm. 6.1 holds for the whole $\mathrm{Aut}(\mathbb{C})$-orbit of Π.

As we are going to use this in the proof of the main results, let us render this more precise. Let $\xi=\otimes_v\xi_v\in\mathcal{W}^\psi(\Pi)\cong\otimes_v\mathcal{W}^{\psi_v}(\Pi_v)$ be a Whittaker function, such that for $v\notin S$, ξ_v is invariant under $G(\mathcal{O}_v)$ and normalized such that $\xi_v(id_v)=1$. Given $\sigma\in\mathrm{Aut}(\mathbb{C})$, let ${}^\sigma\xi\in\mathcal{W}^\psi({}^\sigma\Pi)$ be the σ-twisted Whittaker function, cf. §4.1 (the action of σ on the archimedean part of ξ being by permutations as in §4.2), and let ${}^\sigma\varphi:=(W^\psi)^{-1}({}^\sigma\xi)\in{}^\sigma\Pi$ be the corresponding cuspidal automorphic form. Recall our Schwartz-Bruhat function $\Phi\in\mathscr{S}(\mathbb{A}^n)$ from §6.1, with $m=n$ now. For $\sigma\in\mathrm{Aut}(\mathbb{C})$ we defined $({}^\sigma\Phi)_v:=\Phi_v$ at all places v. As in §6.1, we obtain a function ${}^\sigma f_s=\otimes_v{}^\sigma f_{v,s}\in\mathrm{Ind}_{\mathrm{GL}_{n-1}(\mathbb{A})\times\mathrm{GL}_1(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}[\delta_P^s]$ and an associated Eisenstein series $E({}^\sigma f_s,{}^\sigma\Phi)$. Clearly, $E({}^\sigma f_s,{}^\sigma\Phi)$ satisfies the assertions of Lem. 6.2 with the same non-zero, constant residue c_n.

In summary, with this notation, saying that Thm. 6.1 holds for the whole $\mathrm{Aut}(\mathbb{C})$-orbit of Π, amounts to the equation

(6.2)
$$c_n\ \hat{\Phi}(0)\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})}{}^\sigma\varphi(J(g,g'))\,dg\,dg'=\frac{L^S(\frac{1}{2},{}^\sigma\Pi)\cdot\mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{L^S(n,\mathbf{1})^2}\prod_{v\in S}\frac{Z_v({}^\sigma\xi_v,{}^\sigma f_{v,s})}{L(n,\mathbf{1}_v)}.$$

7. A rationality result for the exterior square L-function

7.1. **Archimedean considerations.** The integral representation of the exterior square L-function in Thm. 6.1 allows us to combine the results of §5 and §6. Before we derive out first main result, we need a non-vanishing theorem, which is an application of Sun's main result in [35].

Recall the generator $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t = \sum_{\underline{i}=(i_1,...,i_t)} \sum_{\alpha=1}^{\dim E_\mu} X^*_{\underline{i}} \otimes \xi^{\epsilon_0}_{\infty,\underline{i},\alpha} \otimes e_\alpha$ of the cohomology space $H^t(\mathfrak{m}_G, K^\circ_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_0]$ from §4.2. Recall furthermore, that the basis $\{X^*_j\}$ of $\mathfrak{m}_G/\mathfrak{k}_\infty$ was the extension of a given ordered basis $\{Y^*_j\}_{j=1}^t$ of $\mathfrak{m}_H/\mathfrak{k}_{H,\infty}$, whence, for each multi-index $\underline{i}$ there is a well-defined complex number $s(\underline{i})$, such that the restriction of $X^*_{\underline{i}}$ to $\Lambda^t(\mathfrak{m}_H/\mathfrak{k}_{H,\infty})^*$ along the injection $\Lambda^t(\mathfrak{m}_H/\mathfrak{k}_{H,\infty})^* \hookrightarrow \Lambda^t(\mathfrak{m}_G/\mathfrak{k}_\infty)^*$, induced by J, equals $s(\underline{i})\cdot Y_1\wedge..\wedge Y_t$. As a last ingredient, before we can state the aforementioned non-vanishing theorem, we need the following lemma:

Lemma 7.2. *For all $v\in S_\infty$, and K°_v-finite $\xi_v \in \mathcal{W}^{\psi_v}(\Pi_v)$, the integrals*

$$Z_v(s',\xi_v,f_{v,1}) := \int_{U_n(F_v)\backslash \mathrm{GL}_n(F_v)} \int_{Z_n(F_v)U_n(F_v)\backslash \mathrm{GL}_n(F_v)} \xi_v(J(g_v,g'_v)) f_{v,1}(g_v) \left|\frac{\det(g'_v)}{\det(g_v)}\right|_v^{s'-1/2} dg_v dg'_v$$

are a holomorphic multiple (in s') of the local archimedean L-function $L(s',\Pi_v)$.

Proof. This follows combining Thm. 6.1 with [9], Prop. 2.3 and Prop. 3.1 *loc. cit.* . □

It follows that the factor $Z_\infty(\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}, f_{\infty,1}) := \prod_{v\in S_\infty} Z_v(\xi^{\epsilon_0}_{v,\underline{i},\alpha}, f_{v,1})$ of the product $\prod_{v\in S} Z_v(\xi_v,f_{v,1})$ is well-defined. Indeed, using [23], Thm. 2 and Thm. 3 *loc. cit.*, it is easy to see that

$$L(s',\Pi_v) = h(s')\cdot \prod_{k=1}^n \Gamma(s'+\mu_{v,k}+n-k+\tfrac{1}{2}),$$

where $h(s')$ is holomorphic and non-vanishing for all $s'\in\mathbb{C}$. Since $\mu_{v,k}\geq 0$ for all $1\leq k\leq n$, by the self-duality hypotheses, cf. §2.3, $L(s',\Pi_v)$ is holomorphic at $s'=\frac{1}{2}$, whence so is $Z_v(\xi^{\epsilon_0}_{v,\underline{i},\alpha}, f_{v,1}) = Z_v(\frac{1}{2},\xi^{\epsilon_0}_{v,\underline{i},\alpha}, f_{v,1})$ by Lem. 7.2. Finally, we let

$$c^t(\Pi_\infty) := (L^S(n,\mathbf{1})^2)^{-1}\cdot \sum_{\underline{i}=(i_1,...,i_t)} \sum_{\alpha=1}^{\dim E_\mu} s(\underline{i})\cdot T_\mu(e_\alpha)\cdot \frac{Z_\infty(\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}, f_{\infty,1})}{L(n,\mathbf{1}_\infty)}.$$

Here, both numbers $L(n,\mathbf{1}_\infty)=\prod_{v\in S_\infty} L(n,\mathbf{1}_v) = \pi^{-dn/2}\Gamma(\frac{n}{2})^d$ and $L^S(n,\mathbf{1})$ are non-zero. We claim that Sun's aforementioned result now implies the following

Theorem 7.1. *The number $c^t(\Pi_\infty)$ is non-zero.*

Proof. As a first step and in order to be able to apply Sun's result ([35], Thm. C), we reduce the problem of showing that $c^t(\Pi_\infty)$ is non-zero to showing that a similarly defined number, $d^t(\Pi_v)$ is non-zero. This latter number will only depend on one archimedean place $v\in S_\infty$, whence we find ourselves back in the setting of [35].

To this end, observe that there is a projection

$$L^t : \Lambda^t(\mathfrak{m}_G/\mathfrak{k}_\infty)^* \xrightarrow{\sim} \bigoplus_{a+b=t} \Lambda^a(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^*\otimes\Lambda^b\mathfrak{s}^* \twoheadrightarrow \Lambda^r(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^*\otimes \Lambda^{d-1}\mathfrak{s}^*,$$

where $\mathfrak{c}_\infty := \mathfrak{z}_\infty\oplus\mathfrak{k}_\infty$ and $r=t-d+1$. By reasons of degree, L^t induces an isomorphism of (one-dimensional) vector spaces

$$\mathcal{L}^t : H^t(\mathfrak{m}_G, K^\circ_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_0] \xrightarrow{\sim} H^r(\mathfrak{g}_\infty, (Z_\infty K_\infty)^\circ, \mathcal{W}^{\psi_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_0]\otimes\Lambda^{d-1}\mathfrak{s}^*_{\mathbb{C}},$$

whose effect on the generator $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ is by sending $X^*_{\underline{i}}$ to $L^t(X_{\underline{i}}) \in \Lambda^r(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \otimes \Lambda^{d-1}\mathfrak{s}^*$. Without loss of generality, we write $L^t(X_{\underline{i}}) = L_r(X_{\underline{i}}) \otimes L_{d-1}(X_{\underline{i}})$, where $L_r(X_{\underline{i}}) \in \Lambda^r(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^*$ and $L_{d-1}(X_{\underline{i}}) \in \Lambda^{d-1}\mathfrak{s}^*$. Similarly, we factor $\mathcal{L}^t = \mathcal{L}_r \otimes \mathcal{L}_{d-1}$.

As $\mathfrak{z}_\infty \subset \mathfrak{h}_\infty$, and as moreover $r = \dim_{\mathbb{R}} \mathfrak{h}_\infty/\mathfrak{c}_{H,\infty}$, where $\mathfrak{c}_{H,\infty} := \mathfrak{z}_\infty \oplus \mathfrak{k}_{H,\infty}$, we also have a canonical isomorphism

$$\Lambda^t(\mathfrak{m}_H/\mathfrak{k}_{H,\infty})^* \xrightarrow{\sim} \Lambda^r(\mathfrak{h}_\infty/\mathfrak{c}_{H,\infty})^* \otimes \Lambda^{d-1}\mathfrak{s}^*.$$

Hence, L^t and $\mathcal{L}^t$ factor over the injection $\Lambda^t(\mathfrak{m}_H/\mathfrak{k}_{H,\infty})^* \hookrightarrow \Lambda^t(\mathfrak{m}_G/\mathfrak{k}_\infty)^*$ induced by J. As a consequence, $c^t(\Pi_\infty)$ is a non-trivial multiple of

$$d^t(\Pi_\infty) := \sum_{\underline{i}=(i_1,\dots,i_t)} \sum_{\alpha=1}^{\dim E_\mu} u(\underline{i}) \cdot T_\mu(e_\alpha) \cdot \frac{Z_\infty(\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}, f_{\infty,1})}{L(n, \mathbf{1}_\infty)},$$

where $u(\underline{i})$ is the uniquely defined complex number, such that the restriction of $L_r(X^*_{\underline{i}})$ to $\Lambda^r(\mathfrak{h}_\infty/\mathfrak{c}_{H,\infty})^*$ equals $u(\underline{i}) \cdot L_r(Y_1 \wedge .. \wedge Y_t)$. The number $d({}^t\Pi_\infty)$ factors as $d^t(\Pi_\infty) = \prod_{v\in S_\infty} d^t(\Pi_v)$, where each local factor $d^t(\Pi_v)$ is defined analogously (using $\Lambda^r(\mathfrak{h}_\infty/\mathfrak{c}_{H,\infty})^* \cong \bigoplus_{\sum r_v = r} \bigotimes_{v\in S_\infty} \Lambda^{r_v}(\mathfrak{h}_v/\mathfrak{c}_{H,v})^*$). Therefore, we may finish the proof by showing that $d^t(\Pi_v)$ is non-zero for all $v \in S_\infty$ and we are in the situation considered by Sun in [35].

Let $v \in S_\infty$ be an arbitrary archimedean place. For sake of simplicity, we drop the subscript "v" now everywhere, so, e.g., $\Pi = \Pi_v$, $H = GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$, $\mu = \mu_v$, $\mathfrak{g} = \mathfrak{gl}_{2n}(\mathbb{R})$ and analogously for all other local archimedean objects. The local integrals $Z(\xi, f_1)$ define a non-zero homomorphism

$$Z(., f_1) \in \mathrm{Hom}_H(\mathcal{W}^\psi(\Pi), \mathbb{C}).$$

This follows from [6], Thm. 2 and Lem. 7.2. Hence, if we let $\chi := \mathbf{1} \times \mathbf{1}$ be the trivial character of H, then $Z(., f_1)$ can be taken as the map φ_χ in Sun's Thm. C, [35]. Next, recall $T_\mu \in \mathrm{Hom}_{H(\mathbb{C})}(E_\mu \otimes \mathbb{C})$ from §5.1. If we set $w_1 := 0 =: w_2$, then we may take T_μ to be the non-zero homomorphism φ_{w_1,w_2} from [35], Thm. C. Hence, *loc. cit.* , Thm. C, asserts that the map

$$D : \mathrm{Hom}(\Lambda^r\mathfrak{g}/\mathfrak{c}, \mathcal{W}^\psi(\Pi) \otimes E_\mu) \longrightarrow \mathrm{Hom}(\Lambda^r\mathfrak{h}/\mathfrak{c}_H, \chi \otimes \mathbb{C})$$

$$h \mapsto D(h) := (Z(., f_1) \otimes T_\mu) \circ h \circ \wedge^r j_{2n}$$

is non-zero on the one-dimensional sub-space $H^r(\mathfrak{g}, (ZK)^\circ, \mathcal{W}^\psi(\Pi) \otimes E_\mu)[\epsilon_0]$. (Here, j_{2n} is Sun's notation for the embedding $\mathfrak{h}/\mathfrak{c}_H \hookrightarrow \mathfrak{g}/\mathfrak{c}$.) By the one-dimensionality of the latter cohomology space, it is hence non-zero on $\mathcal{L}_r([\mathcal{W}^\psi(\Pi)]^t)$. But, then, D computes

$$\begin{aligned} D(\mathcal{L}_r([\mathcal{W}^\psi(\Pi)]^t)) &= (Z(., f_1) \otimes T_\mu) \circ \mathcal{L}_r([\mathcal{W}^\psi(\Pi)]^t) \circ \wedge^r j_{2n} \\ &= \sum_{\underline{i}=(i_1,\dots,i_t)} \sum_{\alpha=1}^{\dim E_\mu} u(\underline{i})\, T_\mu(e_\alpha)\, Z(\xi^{\epsilon_0}_{\underline{i},\alpha}, f_1) \\ &= L(n, \mathbf{1}) \cdot d^t(\Pi). \end{aligned}$$

Hence, reintroducing the subscript "v", and recalling that $L(n, \mathbf{1}_v) = \pi^{-n/2}\Gamma(n/2) \neq 0$, the number $d^t(\Pi_v)$ is non-zero for all archimedean places, whence so is $c^t(\Pi_\infty)$. □

7.3. **Definition of the archimedean top-degree period.** As a consequence of Prop. 3.6, we may hence define the archimedean periods

$$p^t({}^\sigma\Pi_\infty) := c_n\hat{\Phi}(0)\ (\mathrm{vol}_{dg\times dg'}(Z(F)\backslash Z_(\mathbb{A})/A_G)\ c^t({}^\sigma\Pi_\infty))^{-1} \tag{7.2}$$

for all $\sigma\in$Aut$(\mathbb{C})$.

Remark 7.3. Analogously to the case of the top-degree Whittaker period $p^t(\Pi)$, which we dealt with in Rem. 4.4, (almost) none of the various choices entering the definition of our archimedean top-degree period $p^t(\Pi_\infty)$ may be found in its notation. For the sake of precision, we would like to summarize at this place on which data, i.e., fixed chosen ingredients, $p^t(\Pi_\infty)$ actually depends:

(1) Π_∞, ψ_∞, the cohomological degree t, as well as the archimedean measures dg_v and dg'_v chosen for all $v\in S_\infty$ in §6.3
(2) The fixed generator $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ (cf. §4.2) of the one-dimensional cohomology space $H^t(\mathfrak{m}_G, K^\circ_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_0]$. We remark further that this generator depends itself precisely on the data fixed in Choice 4.2
(3) The concrete choice of an intertwining operator $T_{\mu_v}\in \mathrm{Hom}_{H(\mathbb{C})}(E_{\mu_v},\mathbb{C})$ for all $v\in S_\infty$ (§5.1), unique up to multiplication by non-zero rational numbers.
(4) The archimedean Schwartz–Bruhat function $\Phi_\infty=\otimes_{v\in S_\infty}\Phi_v$ from §6.1 with $m=n$. Introducing it once here in the notation, observe that $p^t(\Pi_\infty,\Psi_\infty)=p^t(\Pi_\infty,y\Psi_\infty)$ for all $y\in\mathbb{C}^\times$.

It is hence clear that the archimedean top-degree period $p^t(\Pi_\infty)$ depends exclusively on data, which is associated with objects at archimedean places (which explains its name); and that its definition and existence is independent of the definition and proof of existence of our global Whittaker periods $p^t(\Pi)$ from §4.4.

7.4. **Rationality of the residue of the exterior square L-function at $s=1$.** This is our first main theorem. For the precise definitions of $p^t(\Pi)$ and $p^t(\Pi_\infty)$, a comprehensive list of their individual dependencies as well as for their mutual independence, we refer to §4.4, §7.3, Rem. 4.4 and Rem. 7.3

Theorem 7.4. *Let F be a totally real number field and $G=\mathrm{GL}_{2n}/F$, $n\geq 2$. Let Π be a unitary cuspidal automorphic representation of $G(\mathbb{A})$ (self-dual and with trivial central character), which is cohomological with respect to an irreducible, self-contragredient, algebraic, finite-dimensional representation E_μ of G_∞. Assume that Π satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function $L^S(s,\Pi,\Lambda^2)$ has a pole at $s=1$. Then, for every $\sigma\in\mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{L(\frac12,\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{p^t(\Pi)\ p^t(\Pi_\infty)}\right) = \frac{L(\frac12,{}^\sigma\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{p^t({}^\sigma\Pi)\ p^t({}^\sigma\Pi_\infty)}.$$

In particular,

$$L(\tfrac12,\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))\ \sim_{\mathbb{Q}(\Pi_f)}\ p^t(\Pi)\ p^t(\Pi_\infty), \tag{7.5}$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of the right hand side by an element in the number field $\mathbb{Q}(\Pi_f)$.

Proof. Let Π be as in the statement of the theorem. We consider the commutative diagram (5.2) in Prop. 5.1: Let

$$\Omega := \mathcal{R} \circ T_\mu \circ \mathcal{J}^t_\mu \circ \Delta^t_\Pi \circ \mathcal{F}^t_\Pi$$

be the composition of the upper horizontal arrows, and analogously, let ${}^\sigma\Omega$ be the composition of the lower horizontal arrows. Let $\xi_f = \otimes_{v\notin S_\infty}\xi_v \in \mathcal{W}^{\psi_f}(\Pi_f) \cong \otimes_{v\notin S_\infty}\mathcal{W}^{\psi_v}(\Pi_v)$ be a Whittaker function, such that for $v \notin S$, ξ_v is invariant under $G(\mathcal{O}_v)$ and normalized such that $\xi_v(id_v) = 1$. Given $\sigma \in$Aut$(\mathbb{C})$, let $W^\sigma(\xi_f) = {}^\sigma\xi_f \in \mathcal{W}^{\psi_f}({}^\sigma\Pi_f)$ be the σ-twisted Whittaker function, cf. §4.1. Then, Prop. 5.1 says that

$$\sigma(\Omega(\xi_f)) = {}^\sigma\Omega({}^\sigma\xi_f). \tag{7.6}$$

In order to prove the theorem, we make both sides of this equation explicit. To that end, let $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ (resp. $[\mathcal{W}^{\psi_\infty}({}^\sigma\Pi_\infty)]^t$) the generators of the respective cohomology spaces, §4.2. For each $\underline{i}$ and α, let $\varphi_{\underline{i},\alpha} := (W^\psi)^{-1}(\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}\otimes\xi_f) \in \Pi$ (resp. ${}^\sigma\varphi_{\underline{i},\alpha} := (W^\psi)^{-1}({}^\sigma\xi^{\epsilon_0}_{\infty,\underline{i},\alpha}\otimes {}^\sigma\xi_f) \in {}^\sigma\Pi$) be the corresponding cuspidal automorphic form. Recall our Schwartz-Bruhat function $\Phi \in \mathscr{S}(\mathbb{A}^n)$ (resp. ${}^\sigma\Phi \in \mathscr{S}(\mathbb{A}^n)$) from §6.1 (resp. §6.5), with $m = n$ now. Inserting these functions into Thm. 6.1 (likewise, also into (6.2)) and recalling the definition of our archimedean periods $p^t(\Pi_\infty)$ and $p^t({}^\sigma\Pi_\infty)$ from (7.2) shows that equation (7.6), induced by our diagram (5.2), may be rewritten as

$$\sigma\left(\frac{L^S(\frac12,\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{p^t(\Pi)\ p^t(\Pi_\infty)}\prod_{v\in S\backslash S_\infty}\frac{Z_v(\xi_v,f_{v,1})}{L(n,\mathbf{1}_v)}\right)$$
$$=\frac{L^S(\frac12,{}^\sigma\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{p^t({}^\sigma\Pi)\ p^t({}^\sigma\Pi_\infty)}\prod_{v\in S\backslash S_\infty}\frac{Z_v({}^\sigma\xi_v,{}^\sigma f_{v,1})}{L(n,\mathbf{1}_v)}.$$

(Recall that Π was assumed to have trivial central character.) Invoking that $L(n,\mathbf{1}_v) = (1-|\mathcal{O}_v/\wp_v|^{-n})^{-1} \in \mathbb{Q}^\times$ for $v \in S\backslash S_\infty$, this simplifies to

$$\sigma\left(\frac{L^S(\frac12,\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{p^t(\Pi)\ p^t(\Pi_\infty)}\prod_{v\in S\backslash S_\infty}Z_v(\xi_v,f_{v,1})\right)$$
$$=\frac{L^S(\frac12,{}^\sigma\Pi)\cdot \mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{p^t({}^\sigma\Pi)\ p^t({}^\sigma\Pi_\infty)}\prod_{v\in S\backslash S_\infty}Z_v({}^\sigma\xi_v,{}^\sigma f_{v,1}).$$

Since $\sigma(L(\frac12,\Pi_v)) = L(\frac12,{}^\sigma\Pi_v) \neq 0$ for all $v \in S\backslash S_\infty$, cf. [29], Prop. 3.17, and recalling once more that $S = S(\Pi,\psi) = S({}^\sigma\Pi,\psi)$, we may rewrite this by

$$\sigma\left(\frac{L(\frac12,\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2))}{p^t(\Pi)\ p^t(\Pi_\infty)}\prod_{v\in S\backslash S_\infty}Z_v(\xi_v,f_{v,1})\right) \tag{7.7}$$
$$=\frac{L(\frac12,{}^\sigma\Pi_f)\cdot \mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi,\Lambda^2))}{p^t({}^\sigma\Pi)\ p^t({}^\sigma\Pi_\infty)}\prod_{v\in S\backslash S_\infty}Z_v({}^\sigma\xi_v,{}^\sigma f_{v,1}).$$

Observe that there is a choice of ξ_v, such that $\prod_{v\in S\backslash S_\infty} Z_v(\xi_v,f_{v,1})$ is non-zero. Indeed, if the finite product $\prod_{v\in S\backslash S_\infty} Z_v(\xi_v,f_{v,1})$ were zero for all ξ_v, then by the holomorphy of $L^S(s',\Pi)$

at $s' = \frac{1}{2}$ (see the discussion below Lem. 7.2) and the hereby implied holomorphy of the archimedean integrals $Z_v(s', \xi_v, f_{v,1})$ at $s' = \frac{1}{2}$ (see Lem. 7.2) for all (K_v°-finite) $\xi_v \in \mathcal{W}^{\psi_v}(\Pi_v)$,

$$\frac{L^S(\frac{1}{2}, \Pi) \cdot L^S(s, \Pi, \Lambda^2)}{L^S(n, \mathbf{1})^2} \cdot \prod_{v \in S} \frac{Z_v(\xi_v, f_{v,s})}{L(n, \mathbf{1}_v)}$$

would have no pole at $s = 1$ for all $\xi_v \in \mathcal{W}^{\psi_v}(\Pi_v)$ and all $v \in S$. However, by [6], Thm. 3, (see also the end of the proof of Thm. 6.1), the latter expression equals

$$\frac{Z(\xi, f_s)}{L(n, \mathbf{1})}$$

as meromorphic functions in s. By our assumption that Π is a functorial lift from $\mathrm{SO}(2n+1)$, cf. Prop. 3.5, the integral $Z(\xi, f_s)$ has a pole at $s = 1$, cf. [6], p. 54, at least for some local choices of $\xi_v \in \mathcal{W}^{\psi_v}(\Pi_v)$, $v \in S$. Hence we arrived at a contradiction.

Observe that by a simple change of variable and by our specific choice of $f_{v,1} = {}^\sigma f_{v,1}$, $Z_v({}^\sigma\xi_v, {}^\sigma f_{v,1}) = Z_v(\sigma \circ \xi_v, f_{v,1})$. Invoking N. Matringe's Thm. A from the appendix, we hence get that

$$\sigma\left(\prod_{v \in S \backslash S_\infty} Z_v(\xi_v, f_{v,1})\right) = \prod_{v \in S \backslash S_\infty} Z_v({}^\sigma\xi_v, {}^\sigma f_{v,1}).$$

By what we have just observed, the right hand side must hence be non-zero for our choice of local ξ_v, $v \in S \backslash S_\infty$. Inserting this into (7.7) finally shows the first assertion of Thm. 7.4.

The last assertion of the theorem follows by Strong Multiplicity One for cuspidal automorphic representations of $G(\mathbb{A})$. □

7.5. **Whittaker-Shalika periods and the exterior square L-function.** Theorem 7.4 above is accompanied by the following corollary. Recall the non-zero Shalika periods $\omega^\epsilon(\Pi_f)$ from Grobner–Raghuram [14]: These were defined by comparing a $\mathbb{Q}(\Pi_f)$-rational structure on a Shalika model of Π_f and a $\mathbb{Q}(\Pi_f)$-rational structure on $H^r(\mathfrak{g}_\infty, (Z_\infty K_\infty)^\circ, \Pi \otimes E_\mu)[\epsilon]$. For details, we refer to [14], Def./Prop. 4.2.1. Observe that $\omega^{\epsilon_0}(\Pi_f)$ is well-defined, if we assume that Π satisfies the assumptions made in the statement of Thm 7.4: Indeed, as these assumptions include that the partial exterior square L-function $L^S(s, \Pi, \Lambda^2)$ has a pole at $s = 1$, Π has a $(\mathbf{1}, \psi)$-Shalika model by [14], Thm. 3.1.1. (The extremely careful reader may also recall Lem. 3.2 at this place.) Moreover, by the same reasoning, also the archimedean Shalika period $\omega(\Pi_\infty) = \omega(\Pi_\infty, 0)$ from [14], Thm. 6.6.2 is well-defined (and non-zero). A complete list of all choices, which enter the definition of these Shalika periods $\omega^{\epsilon_0}(\Pi_f)$ and $\omega(\Pi_\infty)$, can be extracted (similar to our considerations leading to Rem. 4.4 and Rem. 7.3 above) from [14], Def./Prop. 4.2.1 and Thm. 6.6.2, where they have been constructed in details. We do not provide such a list here, for the reason that neither $\omega^{\epsilon_0}(\Pi_f)$ nor $\omega(\Pi_\infty)$ appear in the statement of the main theorems (but only in some corollaries).

Define the Whittaker-Shalika periods

$$P^t(\Pi) := \frac{p^t(\Pi)}{\omega^{\epsilon_0}(\Pi_f)} \qquad \text{and} \qquad P^t(\Pi_\infty) := \frac{p^t(\Pi_\infty)}{\omega(\Pi_\infty)}.$$

Obviously the left hand side of (7.5) is uninteresting, if $L(\frac{1}{2},\Pi_f) = 0$. Hence, we allow ourselves to make the strong assumption that $L(\frac{1}{2},\Pi_f)$ is non-zero in order to derive the following result:

Corollary 7.6. *Let Π be as in the statement of Thm. 7.4. If $L(\frac{1}{2},\Pi_f)$ is non-zero, then*

$$\mathrm{Res}_{s=1}(L^S(s,\Pi,\Lambda^2)) \ \sim_{\mathbb{Q}(\Pi_f)} \ P^t(\Pi)\ P^t(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of the right hand side by an element in the number field $\mathbb{Q}(\Pi_f)$.

Proof. This is obvious invoking Thm. 7.4 and [14], Thm. 7.1.2. □

8. A rationality result for the Rankin–Selberg L-function

8.1. The content of this section is very closely related to Grobner–Harris–Lapid [12], §4–§5 and a special case of Balasubramanyam–Raghuram [2], §2–§3. Indeed, the main result, Thm. 8.5, of this section is Thm. 5.3 from [12] (but with the totally imaginary field E from [12] being replaced by the totally real field F as a ground-field), respectively Thm. 3.3.11 from [2] (but with the L-value $L(1,\mathrm{Ad}^0,\pi)$ from [2] being replaced by the residue of $L^S(s,\Pi\times\Pi^\vee)$ at $s=1$). For the reason of these close analogies we allow ourselves to be rather brief, when it comes to details. Nevertheless, we think it is worthwhile writing down the following, already for reasons of notation, and in order to give precise statements of results in what follows.

8.2. Bottom-degree Whittaker periods. Let Π be as in §3.1 and let $b := dn^2$. Then,

$$\dim_{\mathbb{C}} H^b(\mathfrak{m}_G, K^\circ_\infty, \Pi_\infty\otimes E_\mu)[\epsilon] = 1$$

for all $\epsilon \in \pi_0(G_\infty)^*$. As in §3.1, this is a direct consequence of the formula in Clozel [7], Lem. 3.14 and the Künneth rule. It is hence clear that the entire discussion of §3.3 and §4.1–§4.4 carries over to $(\mathfrak{m}_G, K^\circ_\infty)$-cohomology in degree $q=b$. In particular, let $\epsilon_1 := ((-1)^n,...,(-1)^n) \in \pi_0(G_\infty)^*$, i.e., the inverse of the character ϵ_0. We obtain a $\mathbb{Q}(\Pi_f)$-structure on $H^b(\mathfrak{m}_G, K^\circ_\infty, \Pi\otimes E_\mu)[\epsilon_1]$ imposed by the $\mathbb{Q}(E_\mu)$-structure on $H^b_c(\mathcal{S}_G,\mathcal{E}_\mu)$ (exactly as in Def. 3.4) and we may fix once and for all a generator $[\mathcal{W}^{\psi^{-1}_\infty}(\Pi_\infty)]^b$ of the one-dimensional cohomology space $H^b(\mathfrak{m}_G, K^\circ_\infty, \mathcal{W}^{\psi^{-1}_\infty}(\Pi_\infty)\otimes E_\mu)[\epsilon_1]$,

$$[\mathcal{W}^{\psi^{-1}_\infty}(\Pi_\infty)]^b := \sum_{\underline{j}=(j_1,...,j_b)} \sum_{\beta=1}^{\dim E_\mu} X^*_{\underline{j}} \otimes \xi^{\epsilon_1}_{\infty,\underline{j},\beta}\otimes e_\beta,$$

in complete analogy to §4.2, replacing the degree of cohomology t by b in Choice 4.2. Observe that here we exchanged the non-trivial additive character ψ by its inverse ψ^{-1} and (for notational clearness only), also the index α by β.

Moreover, in light of Prop. 3.6, for all $\sigma \in$Aut($\mathbb{C}$), we obtain non-trivial Whittaker periods $p^b({}^\sigma\Pi)$, unique up to multiplication by elements in $\mathbb{Q}({}^\sigma\Pi_f)^\times$, such that

$$\begin{tikzcd} \mathcal{W}^{\psi_f^{-1}}(\Pi_f) \arrow[rr, "\mathcal{F}^b_\Pi"] \arrow[d, "W^\sigma"'] & & H^b(\mathfrak{m}_G, K^\circ_\infty, \Pi\otimes E_\mu)[\epsilon_1] \arrow[d, "H^{\sigma,b}_\mu"] \\ \mathcal{W}^{\psi_f^{-1}}({}^\sigma\Pi_f) \arrow[rr, "\mathcal{F}^b_{{}^\sigma\Pi}"] & & H^b(\mathfrak{m}_G, K^\circ_\infty, {}^\sigma\Pi\otimes {}^\sigma E_\mu)[\epsilon_1] \end{tikzcd}$$

commutes. This is the analogue of Prop. 4.3, whose proof goes through word for word in the current situation, i.e., for cohomology in degree b instead of t. See [30], Prop./Def. 3.3. In the above diagram, $H^{\sigma,b}_\mu := (\Delta^b_{{}^\sigma\Pi})^{-1}\circ\mathcal{H}^{\sigma,b}_\mu\circ\Delta^b_\Pi$ is the restriction of $\mathcal{H}^{\sigma,b}_\mu$ to $H^b(\mathfrak{m}_G, K^\circ_\infty, \Pi\otimes E_\mu)[\epsilon_1]$, this map being well-defined following by the same argument as in §4.3. We leave it to the reader to fill in the remaining details.

Remark 8.1. A comprehensive list of all ingredients on which our bottom-degree period $p^b(\Pi)$ depends is now easily accomplished reading through Rem. 4.4, *mutatis mutandis*, i.e., replacing t by b, ψ by ψ^{-1} and ϵ_0 by ϵ_1.

8.3. Another archimedean period. Recall the Schwartz–Bruhat function $\Phi = \otimes_v\Phi_v \in \mathscr{S}(\mathbb{A}^{2n})$ from §6.1 with $m = 2n$ in this case. Let U_{2n} be the subgroup of upper triangular matrices in $G = \mathrm{GL}_{2n}$, whose diagonal entries are all equal to 1. For each $v\in S_\infty$ we let $\xi_v\in\mathcal{W}^{\psi_v}(\Pi_v)$ (resp. $\xi'_v\in\mathcal{W}^{\psi_v^{-1}}(\Pi_v)$) be a local Whittaker function, which is $SO(2n)$-finite from the right. For such Whittaker functions, the local zeta-integrals

$$\Psi_v(s,\xi_v,\xi'_v,\Phi_v) := \int_{U_{2n}(F_v)\backslash \mathrm{GL}_{2n}(F_v)} \xi_v(g_v)\ \xi'_v(g_v)\Phi_v((0,...,0,1)g_v)|\det(g_v)|_v^s\ dg_v$$

converge for $\Re e(s)\geq 1$, cf. [21], Prop. (3.17). If $\xi_\infty = \otimes_{v\in S_\infty}\xi_v\in\mathcal{W}^{\psi_\infty}(\Pi_\infty)\cong\otimes_{v\in S_\infty}\mathcal{W}^{\psi_v}(\Pi_v)$ (resp. $\xi'_\infty = \otimes_{v\in S_\infty}\xi'_v\in\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty)\cong\otimes_{v\in S_\infty}\mathcal{W}^{\psi_v^{-1}}(\Pi_v)$) is K°_∞-finite, we abbreviate

$$\Psi_\infty(s,\xi_\infty,\xi'_\infty,\Phi_\infty) := \prod_{v\in S_\infty}\Psi_v(s,\xi_v,\xi'_v,\Phi_v).$$

Furthermore, by assumption $E_\mu\cong E^\vee_\mu$. So, the canonical pairing $E_\mu\times E^\vee_\mu\to\mathbb{C}$ induces a pairing $E_\mu\times E_\mu\to\mathbb{C}$, which we will denote by $\langle e_\alpha, e_\beta\rangle := e^\vee_\beta(e_\alpha)$. As a last ingredient, recall our generators $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ and $[\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty)]^b$ from §4.2 and §8.2. Similar to §7.1, we let $s(\underline{i},\underline{j})$ be the unique complex number, such that $X^*_{\underline{i}}\wedge X^*_{\underline{j}} = s(\underline{i},\underline{j})\cdot X_1\wedge...\wedge X_{t+b}$. Putting things together, consider

$$c(\Pi_\infty) := \sum_{\underline{i}=(i_1,...,i_t)}\ \sum_{\underline{j}=(j_1,...,j_b)}\ \sum_{\alpha=1}^{\dim E_\mu}\ \sum_{\beta=1}^{\dim E_\mu} s(\underline{i},\underline{j})\ \langle e_\alpha, e_\beta\rangle\Psi_\infty(1,\xi^{\epsilon_0}_{\infty,\underline{i},\alpha},\xi^{\epsilon_1}_{\infty,\underline{j},\beta},\Phi_\infty).$$

Then there is the following theorem, which follows from Prop. 5.0.3 in [2].

Theorem 8.2. *The number $c(\Pi_\infty)$ is non-zero.*

Proof. We may adapt the argument given at the beginning of the proof of Thm. 7.1, to see that the non-vanishing of $c(\Pi_\infty)$ may be reduced to showing the non-vanishing of a similarly defined number $d(\Pi_v)$, which only depends on one given archimedean place $v \in S_\infty$. Indeed, there is a projection

$$M_b : \Lambda^b(\mathfrak{m}_G/\mathfrak{k}_\infty)^* \xrightarrow{\sim} \bigoplus_{u+v=b} \Lambda^u(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \otimes \Lambda^v\mathfrak{s}^* \twoheadrightarrow \Lambda^b(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \otimes \Lambda^0\mathfrak{s}^*,$$

where we wrote again $\mathfrak{c}_\infty := \mathfrak{z}_\infty \oplus \mathfrak{k}_\infty$. By reasons of degrees of cohomology, M_b induces an isomorphism of (one-dimensional) vector spaces

$$\mathcal{M}_b : H^b(\mathfrak{m}_G, K_\infty^\circ, \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty) \otimes E_\mu)[\epsilon_1] \xrightarrow{\sim} H^b(\mathfrak{g}_\infty, (Z_\infty K_\infty)^\circ, \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty) \otimes E_\mu)[\epsilon_1] \otimes \Lambda^0\mathfrak{s}^*_{\mathbb{C}},$$

whose effect on the generator $[\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty)]^b$ is by mapping $X^*_{\underline{j}}$ to $M_b(X_{\underline{j}}) \in \Lambda^b(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \otimes \Lambda^0\mathfrak{s}^*$. Whence, at the cost of re-scaling $M_b(X_{\underline{j}})$ by the non-trivial factor in $\Lambda^0\mathfrak{s}^* = \mathbb{R}$, we may and will assume that $M_b(X_{\underline{j}}) \in \Lambda^b(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^*$. Recall the projection $L^t = L_r \otimes L_{d-1}$ and the isomorphism $\mathcal{L}^t = \mathcal{L}_r \otimes \mathcal{L}_{d-1}$ from the proof of 7.1[2]. Moreover, observe that there is an isomorphism

$$N^{t+b} : \Lambda^{t+b}(\mathfrak{m}_G/\mathfrak{k}_\infty)^* \xrightarrow{\sim} \Lambda^{r+b}(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \otimes \Lambda^{d-1}\mathfrak{s}^*,$$

which we factor similarly to L^t as $N^{t+b}(X_1 \wedge ... \wedge X_{t+b}) = N_{r+b}(X_1 \wedge ... \wedge X_{t+b}) \otimes N_{d-1}(X_1 \wedge ... \wedge X_{t+b})$, where $N_{r+b}(X_1 \wedge ... \wedge X_{t+b}) \in \Lambda^{r+b}(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^*$ and $N_{d-1}(X_1 \wedge ... \wedge X_{t+b}) \in \Lambda^{d-1}\mathfrak{s}^*$. It hence follows that the number $c(\Pi_\infty)$ is a non-trivial multiple of

$$d(\Pi_\infty) := \sum_{\underline{i}=(i_1,...,i_t)} \sum_{\underline{j}=(j_1,...,j_b)} \sum_{\alpha=1}^{\dim E_\mu} \sum_{\beta=1}^{\dim E_\mu} u(\underline{i}, \underline{j})\ \langle e_\alpha, e_\beta\rangle \Psi_\infty(1, \xi^{\epsilon_0}_{\infty,\underline{i},\alpha}, \xi^{\epsilon_1}_{\infty,\underline{j},\beta}, \Phi_\infty),$$

where $u(\underline{i}, \underline{j})$ is the uniquely defined complex number, such that $L_r(X^*_{\underline{i}}) \wedge M_b(X^*_{\underline{j}}) = u(\underline{i}, \underline{j}) \cdot N_{r+b}(X_1 \wedge ... \wedge X_{t+b})$. The number $d(\Pi_\infty)$ factors as $d(\Pi_\infty) = \prod_{v \in S_\infty} d(\Pi_v)$, where each local factor $d(\Pi_v)$ is defined analogously (using $\Lambda^{r+b}(\mathfrak{g}_\infty/\mathfrak{c}_\infty)^* \cong \bigotimes \Lambda^{2n^2+n-1}(\mathfrak{g}_v/\mathfrak{c}_v)^*$). Therefore, we may finish the proof by showing that $d(\Pi_v)$ is non-zero for all $v \in S_\infty$. This is the reduction to a single archimedean place $v \in S_\infty$, mentioned at the beginning of the proof. The result hence follows by [2], Prop. 5.0.3. $\square$

In view of the latter non-vanishing result and Prop. 3.6, we may define

(8.3) $$p({}^\sigma\Pi_\infty) := c_{2n}\hat{\Phi}(0)\ (\mathrm{vol}_{dg}(Z(F)\backslash Z_(\mathbb{A})/A_G)\ c({}^\sigma\Pi_\infty))^{-1}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Remark 8.4. Analogously to Rem. 7.3, let us recollect at one place the various choices which enter the definition of our archimedean period $p(\Pi_\infty)$, since (almost) none of them appear in its notation:

(1) Π_∞, ψ_∞, the cohomological degrees b and t, as well as the archimedean measures dg_v chosen for all $v \in S_\infty$ in §6.3.
(2) The fixed generators $[\mathcal{W}^{\psi_\infty}(\Pi_\infty)]^t$ and $[\mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty)]^b$ from §4.2 and §8.2. We remark further that these generators depend themselves on the data fixed in Choice 4.2.

[2]Observe the difference between the last factors in L^t and M_b: While $\Lambda^0\mathfrak{s}^* = \mathbb{R}$ by convention, the isomorphism $\Lambda^{d-1}\mathfrak{s}^* \cong \mathbb{R}$ is not canonical, for which we introduced the notational factor M_{d-1}.

(3) The concrete choice of an archimedean Schwartz–Bruhat function $\Phi_\infty = \otimes_{v\in S_\infty}\Phi_v$ from §6.1 with $m = 2n$. Again, $p(\Pi_\infty)$ is insensitive for taking non-zero complex multiples of Φ_∞.

It is hence clear that existence and definition of $p(\Pi_\infty)$ is independent of the other periods considered so far in this paper, $p^t(\Pi)$, $p^b(\Pi)$ and $p^t(\Pi_\infty)$.

8.4. Rationality of the residue of the Rankin–Selberg L-function at $s = 1$. Having set up our additional notation above, we obtain the main result of this section:

Theorem 8.5. *Let F be a totally real number field and $G = \mathrm{GL}_{2n}/F$, $n \geq 2$. Let Π be a self-dual, unitary, cuspidal automorphic representation of $G(\mathbb{A})$ (with trivial central character), which is cohomological with respect to an irreducible, self-contragredient, algebraic, finite-dimensional representation E_μ of G_∞. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{\mathrm{Res}_{s=1}(L^S(s,\Pi\times\Pi))}{p^t(\Pi)\ p^b(\Pi)\ p(\Pi_\infty)}\right) = \frac{\mathrm{Res}_{s=1}(L^S(s,{}^\sigma\Pi\times{}^\sigma\Pi))}{p^t({}^\sigma\Pi)\ p^b({}^\sigma\Pi)\ p({}^\sigma\Pi_\infty)}.$$

In particular,

$$\mathrm{Res}_{s=1}(L^S(s,\Pi\times\Pi)) \sim_{\mathbb{Q}(\Pi_f)^\times} p^t(\Pi)\ p^b(\Pi)\ p(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)^\times}$" means up to multiplication by a non-trivial element in the number field $\mathbb{Q}(\Pi_f)$.

Proof. The first assertion follows from Thm. 3.3.11 of [2]. The second assertion of Thm. 8.5 follows from the first one, applying Strong Multiplicity One for the cuspidal automorphic spectrum of $G(\mathbb{A})$ and recalling that $\mathrm{Res}_{s=1}(L^S(s,\Pi\times\Pi))$ is non-zero. In fact, $\mathrm{Res}_{s=1}(L^S(s,\Pi\times\Pi)) \neq 0$ is well-known and is a consequence of Thm. 8.2 together with [21] (5), p. 550 and Prop. (2.3) in *loc. cit.*. □

9. A rationality result for the symmetric square L-function

9.1. Definition of the archimedean bottom-degree period. Let Π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in §3.1 and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Recall the archimedean periods $p^t({}^\sigma\Pi_\infty)$ from (7.2) and $p({}^\sigma\Pi_\infty)$ from (8.3). We define our bottom-degree, archimedean period by

$$p^b({}^\sigma\Pi_\infty) := \frac{p({}^\sigma\Pi_\infty)}{p^t({}^\sigma\Pi_\infty)}.$$

Remark 9.1. Of course, by Thm. 7.1 and Thm. 8.2, $p^b({}^\sigma\Pi_\infty)$ is well-defined and non-zero. For the sake of the reader, we remark that a complete list of all concrete choices on which $p^b(\Pi_\infty)$ actually depends, is hence given by merging the precise lists provided by Rem. 7.3 and Rem. 8.4.

9.2. Rationality of the symmetric square L-function at $s = 1$. Recall our bottom-degree periods $p^b(\Pi)$ and $p^b(\Pi_\infty)$, defined in §8.2 and §9.1, respectively. For a complete list of all data, entering the respective definition, we refer to Rem. 8.1 and Rem. 9.1 above. The following result is our second main theorem.

Theorem 9.2. *Let F be a totally real number field and $G = \mathrm{GL}_{2n}/F$, $n \geq 2$. Let Π be a unitary cuspidal automorphic representation of $G(\mathbb{A})$ (self-dual and with trivial central character), which is cohomological with respect to an irreducible, self-contragredient, algebraic, finite-dimensional representation E_μ of G_∞. Assume that Π satisfies the equivalent conditions of Prop. 3.5, i.e., the partial exterior square L-function $L^S(s, \Pi, \Lambda^2)$ has a pole at $s = 1$. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{L(\frac{1}{2}, \Pi_f)\ p^b(\Pi)\ p^b(\Pi_\infty)}{L^S(1, \Pi, \mathrm{Sym}^2)}\right) = \frac{L(\frac{1}{2}, {}^\sigma\Pi_f)\ p^b({}^\sigma\Pi)\ p^b({}^\sigma\Pi_\infty)}{L^S(1, {}^\sigma\Pi, \mathrm{Sym}^2)}.$$

In particular,

$$L^S(1, \Pi, \mathrm{Sym}^2) \sim_{\mathbb{Q}(\Pi_f)} L(\tfrac{1}{2}, \Pi_f)\ p^b(\Pi)\ p^b(\Pi_\infty) \tag{9.3}$$

where "$\sim_{\mathbb{Q}(\Pi_f)}$" means up to multiplication of $L^S(1, \Pi, \mathrm{Sym}^2)$ by an element in the number field $\mathbb{Q}(\Pi_f)$.

Proof. Recall that $L^S(s, \Pi \times \Pi) = L^S(s, \Pi, \mathrm{Sym}^2) \cdot L^S(s, \Pi, \Lambda^2)$ as meromorphic functions in s, whence, by the assumptions on Π, we obtain

$$\mathrm{Res}_{s=1}(L^S(s, \Pi \times \Pi)) = L^S(1, \Pi, \mathrm{Sym}^2) \cdot \mathrm{Res}_{s=1}(L^S(s, \Pi, \Lambda^2)).$$

Since $L^S(1, \Pi, \mathrm{Sym}^2)$ is non-zero (cf. [32], Thm. 5.1), the first assertion of the theorem follows from Thm. 7.4 and Thm. 8.5. The second assertion is now again a consequence of Strong Multiplicity One for the cuspidal automorphic spectrum of $G(\mathbb{A})$. □

9.3. Whittaker-Shalika periods and the symmetric square L-function. As in the case of the exterior square L-function, we obtain a corollary of our second main theorem, Thm. 9.2, using the main results of our paper [14]. Recall the non-zero Shalika periods $\omega^\epsilon(\Pi_f)$ and $\omega(\Pi_\infty) = \omega(\Pi_\infty, 0)$ from §7.5 above, respectively from [14], Def./Prop. 4.2.1 and Thm. 6.6.2, therein, their existence being guaranteed as in §7.5. Define the Whittaker-Shalika periods

$$P^b(\Pi) := p^b(\Pi) \cdot \omega^{\epsilon_0}(\Pi_f) \qquad \text{and} \qquad P^b(\Pi_\infty) := p^b(\Pi_\infty) \cdot \omega(\Pi_\infty).$$

Analogously to the situation considered in §7.5 above, the right hand side of (9.3) is uninteresting if $L(\frac{1}{2}, \Pi_f) = 0$. Hence, we allow ourselves to make the strong assumption that $L(\frac{1}{2}, \Pi_f)$ is non-zero in order to obtain the following result.

Corollary 9.4. *Let Π be as in the statement of Thm. 9.2. If $L(\frac{1}{2}, \Pi_f)$ is non-zero, then*

$$L^S(1, \Pi, \mathrm{Sym}^2) \ \sim_{\mathbb{Q}(\Pi_f)^\times}\ P^b(\Pi)\ P^b(\Pi_\infty),$$

where "$\sim_{\mathbb{Q}(\Pi_f)^\times}$" means up to multiplication of $L^S(1, \Pi, \mathrm{Sym}^2)$ by a non-zero element in the number field $\mathbb{Q}(\Pi_f)$.

Proof. This follows directly from Thm. 9.2 and [14], Thm. 7.1.2. □

10. Applications for quotients of symmetric square L-functions

10.1. Gauß sums of algebraic Hecke characters. It is the purpose of this section to provide a result, independent of the all the periods mentioned above for certain quotients of symmetric square L-functions.

To that end, let χ be a Hecke character of finite order. We define the Gauß sum of its finite part χ_f, following Weil [37, VII, Sect. 7]: Let $\mathfrak{c}_\chi$ stand for the conductor ideal of χ_f and let $y=(y_v)_{v\notin S_\infty}\in\mathbb{A}_f^\times$ be chosen such that $\mathrm{ord}_v(y_v)=-\mathrm{ord}_v(\mathfrak{c}_\chi)-\mathrm{ord}_v(\mathfrak{D}_F)$. Here, $\mathfrak{D}_F$ stands for the absolute different of F, that is, $\mathfrak{D}_F^{-1}=\{x\in F: Tr_{F/\mathbb{Q}}(x\mathcal{O})\subset\mathbb{Z}\}$.

Recall our fixed non-trivial additive character $\psi: F\backslash\mathbb{A}\to\mathbb{C}^\times$ from §2.1. The Gauß sum of χ_f with respect to y and ψ is now defined as $\mathscr{G}(\chi_f,\psi_f,y)=\prod_{v\notin S_\infty}\mathscr{G}(\chi_v,\psi_v,y_v)$, where the local Gauß sum $\mathscr{G}(\chi_v,\psi_v,y_v)$ is defined as

$$\mathscr{G}(\chi_v,\psi_v,y_v)=\int_{\mathcal{O}_v^\times}\chi_v(u_v)^{-1}\psi_v(y_vu_v)\,du_v.$$

For almost all v, we have $\mathscr{G}(\chi_v,\psi_v,y_v)=1$, and for all v we have $\mathscr{G}(\chi_v,\psi_v,y_v)\neq 0$. (See, for example, Godement [10, Eq. 1.22].) Note that, unlike in [37], we do not normalize the Gauß sum to make it have absolute value one. For the sake of easing notation and readability we suppress its dependence on ψ and y, and denote $\mathscr{G}(\chi_f,\psi_f,y)$ simply by $\mathscr{G}(\chi_f)$.

10.2. An application of Thm. 9.2.

Theorem 10.1. *Let F be a totally real number field and $G=\mathrm{GL}_{2n}/F$, $n\geq 2$. Let Π be any cuspidal automorphic representation of $G(\mathbb{A})$ and let χ_1 and χ_2 be two Hecke characters of finite order, such that $\Pi\otimes\chi_i$, $i=1,2$, both satisfy the conditions of Cor. 9.4. If χ_1 and χ_2 have moreover the same infinity-type, i.e., $\chi_{1,\infty}=\chi_{2,\infty}$, then,*

$$\frac{L^S(1,\Pi\otimes\chi_1,\mathrm{Sym}^2)}{L^S(1,\Pi\otimes\chi_2,\mathrm{Sym}^2)}\sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times}\mathscr{G}(\chi_{1,f})^{2n^2}\mathscr{G}(\chi_{2,f})^{-2n^2},$$

where "$\sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times}$" means up to multiplication by a non-zero element in the composition of number fields $\mathbb{Q}(\Pi_f)$, $\mathbb{Q}(\chi_{1,f})$ and $\mathbb{Q}(\chi_{2,f})$.

It shall be noted that, whereas the quantities on the left hand side of the above equation all depend crucially on Π, the right hand side is not only independent of the all periods considered in this paper, but *completely independent* of Π.

Proof. Recall the Whittaker-Shalika periods $P^b(\Pi\otimes\chi_i)=p^b(\Pi\otimes\chi_i)\cdot\omega^{\epsilon_0}(\Pi_f\otimes\chi_{i,f})$ and $P^b(\Pi_\infty\otimes\chi_{i,\infty})=p^b(\Pi_\infty\otimes\chi_{i,\infty})\cdot\omega(\Pi_\infty\otimes\chi_{i,\infty})$, $i=1,2$, from §9.3. We remind the reader that since both $\Pi\otimes\chi_i$, $i=1,2$, satisfy the assumptions of Thm. 9.2, all periods appearing in their definition are well-defined and non-zero, cf. §7.5. Since it follows directly from the definition of rationality fields that $\mathbb{Q}(\Pi_f\otimes\chi_{1,f})\mathbb{Q}(\Pi_f\otimes\chi_{2,f})=\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})$, our Cor.

9.4 (or, alternatively, Thm. 9.2 together with [14], Thm. 7.1.2) implies that

$$\begin{aligned}\frac{L^S(1,\Pi\otimes\chi_1,\mathrm{Sym}^2)}{L^S(1,\Pi\otimes\chi_2,\mathrm{Sym}^2)} &\sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times} \frac{P^b(\Pi\otimes\chi_1)\ P^b(\Pi_\infty\otimes\chi_{1,\infty})}{P^b(\Pi\otimes\chi_2)\ P^b(\Pi_\infty\otimes\chi_{2,\infty})}\\ &= \frac{p^b(\Pi\otimes\chi_1)\cdot\omega^{\epsilon_0}(\Pi_f\otimes\chi_{1,f})}{p^b(\Pi\otimes\chi_2)\cdot\omega^{\epsilon_0}(\Pi_f\otimes\chi_{2,f})}\frac{p^b(\Pi_\infty\otimes\chi_{1,\infty})\cdot\omega(\Pi_\infty\otimes\chi_{1,\infty})}{p^b(\Pi_\infty\otimes\chi_{2,\infty})\cdot\omega(\Pi_\infty\otimes\chi_{2,\infty})}.\end{aligned}$$

Moreover, the infinity-types of χ_1 and χ_2 are equal by assumption, which implies that $\Pi_\infty\otimes\chi_{1,\infty}$ and $\Pi_\infty\otimes\chi_{2,\infty}$ are not only isomorphic, but literally *identical.* As a consequence, the contribution of all archimedean periods above cancels out, and we are left with

$$\frac{L^S(1,\Pi\otimes\chi_1,\mathrm{Sym}^2)}{L^S(1,\Pi\otimes\chi_2,\mathrm{Sym}^2)} \sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times} \frac{p^b(\Pi\otimes\chi_1)\cdot\omega^{\epsilon_0}(\Pi_f\otimes\chi_{1,f})}{p^b(\Pi\otimes\chi_2)\cdot\omega^{\epsilon_0}(\Pi_f\otimes\chi_{2,f})}. \tag{10.2}$$

(We remark aside that in order to see this cancellation it would also have been enough to know that the χ_i are of finite order, since then $\Pi_\infty\otimes\chi_{i,\infty}\cong\Pi_\infty$, for $i=1,2$, see [14] 5.3. However, the *equality* $\Pi_\infty\otimes\chi_{1,\infty}=\Pi_\infty\otimes\chi_{2,\infty}$ makes the cancellation even more obvious.)

It is exactly the main result of [30], Thm. 4.1., that – if χ_1 and χ_2 have the same infinity-type, which we assume – one has the relation

$$p^b(\Pi\otimes\chi_i) \sim_{\mathbb{Q}(\Pi_f,\chi_{i,f})^\times} p^b(\Pi)\cdot\mathscr{G}(\chi_{i,f})^{n(2n-1)}. \tag{10.3}$$

Furthermore, it is the main result of section 5 of [14], Thm. 5.2.1, that we obtain

$$\omega^{\epsilon_0}(\Pi_f\otimes\chi_{i,f}) \sim_{\mathbb{Q}(\Pi_f,\chi_{i,f})^\times} \omega^{\epsilon}(\Pi_f)\cdot\mathscr{G}(\chi_{i,f})^{n}. \tag{10.4}$$

where $\epsilon\in K_\infty/K_\infty^\circ$ is the same for both $i=1,2$, because we assumed that χ_1 and χ_2 have the same infinity-type. Inserting the relations (10.3) and (10.4) into (10.2), we obtain

$$\begin{aligned}\frac{L^S(1,\Pi\otimes\chi_1,\mathrm{Sym}^2)}{L^S(1,\Pi\otimes\chi_2,\mathrm{Sym}^2)} &\sim_{\mathbb{Q}(\Pi_f,\chi_{1,f},\chi_{2,f})^\times} \frac{p^b(\Pi)\cdot\mathscr{G}(\chi_{1,f})^{n(2n-1)}\cdot\omega^\epsilon(\Pi_f)\cdot\mathscr{G}(\chi_{1,f})^n}{p^b(\Pi)\cdot\mathscr{G}(\chi_{2,f})^{n(2n-1)}\cdot\omega^\epsilon(\Pi_f)\cdot\mathscr{G}(\chi_{2,f})^n}\\ &= \mathscr{G}(\chi_{1,f})^{2n^2}\mathscr{G}(\chi_{2,f})^{-2n^2}.\end{aligned}$$

This shows the claim. □

Appendix

by Nadir Matringe

In this appendix, F denotes a non-archimedean local field with valuation v and absolute value $|.|$ (normalised as usual). We will write $|g|$ for $|\det(g)|$ when g is a square matrix, $\mathcal{O}$ for the ring of integers of F and let $\wp=\varpi\mathcal{O}$ be the maximal ideal of $\mathcal{O}$.

Proposition A. *Let $\phi\in\mathscr{C}_c^\infty(F)$, χ a character of $F^\times$, and $m\geq 0$ and integer. Then,*

$$T(q^{-s},\chi,m,\phi) := \int_{F^\times}\phi(x)\ \chi(x)\ v(x)^m|x|^s\ d^\times x$$

(with $\mathrm{vol}_{d^\times x}(\mathcal{O}^\times)=1$*) converges for* $|q^{-s}|<|\chi(\varpi)|^{-1}$ *and can be extended to an element of* $L(s,\chi)^m\cdot\mathbb{C}[q^{\pm s}]$*. Moreover, if* $\sigma\in\mathrm{Aut}(\mathbb{C})$*, then*

$$\sigma(T(q^{-s},\chi,m,\phi))=T(\sigma(q^{-s}),\sigma(\chi),m,\sigma(\phi)).$$

Proof. For $k\in\mathbb{Z}$, we set

$$c_k(\chi,\phi):=\int_{\mathcal{O}^\times}\phi(\varpi^k x)\ \chi(x)\ d^\times x,$$

and

$$c(\chi):=\int_{\mathcal{O}^\times}\chi(x)\ d^\times x,$$

so that we have $c(\chi)=0$, if χ is ramified, and $c(\chi)=1$, if χ is unramified.

As $\mathcal{O}^\times$ is compact, take U an open compact subgroup of $\mathcal{O}^\times$ fixing $x\mapsto\phi(\varpi^k x)$ and χ, and let $\mathcal{O}^\times=\coprod_{i=1}^{\ell}x_iU$, then $c_k(\chi,\phi)=\sum_{i=1}^{\ell}\frac{1}{\ell}\phi(\varpi^k x_i)\chi(x_i)$, hence, as $\sigma(1/\ell)=1/\ell$, we have

$$\sigma(c_k(\chi,\phi))=c_k(\sigma(\chi),\sigma(\phi)).$$

Let a be a positive integer such that the support of ϕ is contained in $\wp^{a-1}$. Let $b\geq a$ be such that ϕ is constant on $\wp^b$. We have

$$T(q^{-s},\chi,m,\phi)=\sum_{a\leq k\leq b}c_k(\chi,\phi)k^m\chi(\varpi)^k q^{-ks}+\sum_{k\geq b}c(\chi)k^m\chi(\varpi)^k q^{-ks}.$$

We set

$$A(q^{-s},\chi,m,\phi):=\sum_{a\leq k\leq b}c_k(\chi,\phi)k^m\chi(\varpi)^k q^{-ks}$$

and

$$B(q^{-s},\chi,m,\phi):=\sum_{k\geq b}c(\chi)k^m\chi(\varpi)^k q^{-ks}.$$

Suppose that χ is ramified, i.e., non trivial on $\mathcal{O}^\times$. Then

$$T(q^{-s},\chi,m,\phi)=A(q^{-s},\chi,m,\phi).$$

In this case, we have

$$\begin{aligned}\sigma(T(q^{-s},\chi,m,\phi)) &= \sum_{a\leq k\leq b}\sigma(c_k(\chi,\phi))k^m\sigma(\chi(\varpi))^k\sigma(q^{-ks})\\ &= \sum_{a\leq k\leq b}c_k(\sigma(\chi),\sigma(\phi))k^m\sigma(\chi(\varpi))^k\sigma(q^{-ks})\\ &= A(\sigma(q^{-s}),\sigma(\chi),m,\sigma(\phi))\\ &= T(\sigma(q^{-s}),\sigma(\chi),m,\sigma(\phi)),\end{aligned}$$

which shows the claim in this case. Suppose now that χ is unramified. Then there is $P\in\mathbb{Q}[X,X^{-1}]$ (which can be determined explicitly, notice that the coefficients of P are in $\mathbb{Q}$, hence σ-invariant) such that

$$B(q^{-s},\chi,m,\phi)=\sum_{k\geq b}k^m\chi(\varpi)^k q^{-ks}=P(\chi(\varpi)q^{-s})/(1-\chi(\varpi)q^{-s})^m,$$

which implies that

$$\sigma(B(q^{-s},\chi,m,\phi)) = P(\sigma(\chi(\varpi))\sigma(q^{-s}))/(1-\sigma(\chi(\varpi))\sigma(q^{-s}))^m = B(\sigma(q^{-s}),\sigma(\chi),m,\sigma(\phi)).$$

This implies again $\sigma(T(q^{-s},\chi,\phi)) = T(\sigma(q^{-s}),\sigma(\chi),\sigma(\phi))$.

□

We denote by P_n the mirabolic subgroup of $G_n = GL(n,F)$, and by A_n the diagonal torus of G_n, which is contained in the standard Borel B_n with unipotent radical N_n. For $k \in \{1,\dots,n-1\}$, the group G_k embeds naturally in G_n, so the center Z_k of G_k embeds in A_n, and $A_n = Z_1 \dots Z_n$ (direct product). The following result follows from Proposition 2.2 of [19]. We fix a non-trivial additive character ψ of F. If z_i belongs to $Z_i \subset A_n$, we set $t(z_i)$ to be the element of F^* such that $z_i = diag(t(z_i), I_{n-i})$

Proposition B. *Let π be an irreducible generic representation of G_n, and $\xi \in \mathcal{W}^\psi(\pi)$. For each $k \in \{1,\dots,n-1\}$, there exists a finite set I_k, a string of characters $(c_{i_k})_{i_k\in I_k}$ of F^*, non-negative integers $(m^\xi_{i_k})_{i_k\in I_k}$, and functions $(\phi^\xi_{i_k})_{i_k\in I_k}$ such that*

$$\xi(z_1\dots z_{n-1}) = \sum_{k=1}^{n-1}\sum_{i_k\in I_k}\prod_{k=1}^{n-1} c_{i_k}(t(z_k))\ v(t(z_k))^{m^\xi_{i_k}}\ \phi^\xi_{i_k}(t(z_k)).$$

(The characters c_{ik}, which we allow to be equal, depend only on π.)

We denote by w_n the element of the symmetric group $\mathfrak{S}_n$ naturally embedded in G_n, defined by

$$\begin{pmatrix} 1 & 2 & \dots & m-1 & m & m+1 & m+2 & \dots & 2m-1 & 2m \\ 1 & 3 & \dots & 2m-3 & 2m-1 & 2 & 4 & \dots & 2m-2 & 2m \end{pmatrix}$$

when $n=2m$ is even, and by

$$\begin{pmatrix} 1 & 2 & \dots & m-1 & m & m+1 & m+2 & \dots & 2m & 2m+1 \\ 1 & 3 & \dots & 2m-3 & 2m-1 & 2m+1 & 2 & \dots & 2m-2 & 2m \end{pmatrix}$$

when $n=2m+1$ is odd. We denote by L_n the standard Levi subgroup of G_n which is $G_{\lfloor (n+1)/2\rfloor}\times G_{\lfloor n/2\rfloor}$ embedded by the map $(g_1,g_2)\mapsto diag(g_1,g_2)$. We denote by H_n the group $L_n^{w_n} = w_n^{-1}L_nw_n$, by $J(g_1,g_2)$ the matrix $w_n^{-1}diag(g_1,g_2)w_n$ of H_n (with $diag(g_1,g_2)\in L_n$). Let r be a positive integer. Thanks to the Iwasawa decomposition $G_r = N_r\cdot A_r\cdot G_r(\mathcal{O})$, if χ is an unramified character of A_r, then the map

$$\tilde{\chi}: n\cdot a\cdot k \mapsto \chi(a)$$

is well defined on G_r. For example, if δ_r is the modulus character of the maximal parabolic subgroup of type $(r-1,1)$ restricted to A_r, we have a map $\tilde{\delta_r}$ on G_r. Similarly, if

$$\lambda: z_1\dots z_r\in A_r\mapsto |t(z_1)\dots t(z_{r-1})|,$$

the map $\tilde{\lambda}$ is also defined on G_r, and left invariant under Z_r.

Theorem A. *Let π be an irreducible generic representation of G_n with trivial central character and $\xi\in\mathcal{W}^\psi(\pi)$. Set $m' = \lfloor (n+1)/2\rfloor$ and $m = \lfloor n/2\rfloor$. The integral*

$$Z(\xi,q^{-s}) := \int_{N_m\backslash G_m}\int_{Z_{m'}N_{m'}\backslash G_{m'}} \xi(J(h,g))\ \tilde{\delta}_{m'}(g)\ |g|^s\ \tilde{\lambda}(h)^s\ dg\ dh$$

(with the normalisations $dg = d^\times a\ dk$ with $d^\times a(A_m(\mathcal{O})) = 1$ and $dk(G_m(\mathcal{O})) = 1$, $dh = d^\times b\ dk$ with $d^\times b(Z_{m'}(\mathcal{O})\backslash A_{m'}(\mathcal{O})) = 1$ and $dk(G_{m'}(\mathcal{O})) = 1$) converges absolutely for $|q^{-s}|$ small enough. It extends to an element of $\mathbb{C}(q^{-s})$, which satisfies that for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, one has

$$\sigma(Z(\xi, q^{-s})) = Z(\sigma \circ \xi, \sigma(q^{-s})).$$

Proof. Let δ' be the modulus character of B_m and δ'' that of $B_{m'}$. Let $U' \times U$ be a compact open subgroup of $G_{m'}(\mathcal{O}) \times G_m(\mathcal{O})$ such that $J(U' \times U)$ fixes ξ on the right, and write $G_{m'}(\mathcal{O}) \times G_n(\mathcal{O}) = \coprod_{j=1}^{\ell} x_j U' \times y_j U$. We have

$$Z(\xi, q^{-s}) = \sum_{j=1}^{\ell} \frac{1}{\ell} \int_{A_m} \int_{A_{m'-1}} \xi_j(J(b,a))\ \delta_{m'}(a)\ \delta'(a)^{-1}\ \delta''(b)^{-1} |a|^s |b|^s\ d^\times a\ d^\times b,$$

where $\xi_j(g) = \xi(gJ(x_j, y_j))$. We identify $A_m \times A_{m'-1}$ with A_{n-1} by $(a,b) \mapsto J(b,a)$, and set χ the character of A_{n-1} defined by $J(b,a) \mapsto \delta_{m'}(a)\delta'(a)^{-1}\delta''(b)^{-1}$. The previous integral becomes

$$Z(\xi, q^{-s}) = \sum_{j=1}^{\ell} \frac{1}{\ell} \int_{A_{n-1}} \xi_j(z)\ \chi(z)\ |z|^s\ d^\times z,$$

We set χ_k to be the restriction of χ to Z_k. It takes values in $q^{\mathbb{Z}} \subset \mathbb{Q}$. If we now apply the second proposition of this appendix, we obtain that

$$Z(\xi, q^{-s})$$

is the sum for k between 1 and $n-1$, $i \in I_k$, and $j \in \{1, ..., \ell\}$ of

$$\frac{1}{\ell} \prod_{k=1}^{n-1} T((q^{-s})^k, c_{i_k}\chi_k, m_{i_k}^{\xi_j}, \phi_{i_k}^{\xi_j}).$$

This implies, according to the first proposition of this appendix, that

$$\sigma(Z(\xi, q^{-s}))$$

is the sum for k between 1 and $n-1$, $i \in I_k$, and $j \in \{1, ..., \ell\}$ of

$$\frac{1}{\ell} \prod_{k=1}^{n-1} T((\sigma(q^{-s}))^k, \sigma(c_{i_k})\chi_k, m_{i_k}^{\xi_j}, \sigma(\phi_{i_k}^{\xi_j})).$$

This means that $\sigma(Z(\xi, q^{-s}))$ is equal to $Z(\sigma \circ \xi, \sigma(q^{-s}))$. □

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References

[1] A. Ash, D. Ginzburg, p-adic L-functions for $\mathrm{GL}(2n)$, *Invent. Math.* **116** (1994) pp. 27–73

[2] B. Balasubramanyam, A. Raghuram, Special values of adjoint L-functions and congruences for automorphic forms on $\mathrm{GL}(n)$ over a number field, *Amer. J. Math.* **139** (2017) pp. 641–679

[3] A. Borel, Some finiteness properties of adele groups over number fields, *Publ. Math. IHES* **16** (1963) pp. 5–30

[4] A. Borel, *Stable Real Cohomology of Arithmetic Groups II*, in: *Manifolds and Lie Groups, Papers in Honor of Yozô Matsushima*, eds. J. J. Hano, A. Morimoto, S. Murakami, K. Okamoto, H. Ozeki, Progress in Mathematics **14** (Boston, 1981), pp. 21–55

[5] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies **94**, (Princeton Univ. Press, New Jersey, 1980)

[6] D. Bump, S. Friedberg, *The Exterior Square Automorphic L-Functions on GL(n)*, in: *Festschrift in Honour of Prof. I.I. Piatetski-Shapiro*, Israel Math. Conf. Proc. 3 (1990) pp. 47–65

[7] L. Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*, in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. I, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 77–159

[8] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, F. Shahidi, Functoriality for the classical groups, *Publ. Math. IHES* **99** (2004) pp. 163–233

[9] S. Friedberg, H. Jacquet, Linear periods, *J. reine angew. Math.* **443** (1993) pp. 91–139

[10] R. Godement, *Notes on Jacquet–Langlands' Theory*, mimeographed notes, I.A.S Princeton, (New Jersey, 1970)

[11] H. Grobner, M. Harris, Whittaker periods, motivic periods, and special values of tensor product L-functions, *J. Inst. Math. Jussieu* **15** (2016) 711–769

[12] H. Grobner, M. Harris, E. Lapid, Whittaker rational structures and special values of the Asai L-function, *Contemp. Math.* **664** (2016) 119–134

[13] H. Grobner, A. Raghuram, On some arithmetic properties of automorphic forms of GL_m over a division algebra, *Int. J. Number Theory* **10** (2014) pp. 963–1013

[14] H. Grobner, A. Raghuram, On the arithmetic of Shalika models and the critical values of L-functions for GL_{2n}, with an appendix by Wee Teck Gan, *Amer. J. Math.* **136** (2014) pp. 675–728

[15] G. Harder, *General aspects in the theory of modular symbols*, in: *Séminaire de Théorie des Nombres, Paris 1981-82*, ed M.-J. Bertin, Progress in Mathematics **38** (Boston, Basel, Stuttgart 1983), pp. 73–88

[16] G. Harder *Period Integrals of Eisenstein Cohomology Classes and Special Values of Some L-Functions*, in: *Number Theory Related to Fermat's Last Theorem, Proceedings of the conference sponsored by the Vaughn Foundation*, ed N. Koblitz, Progress in Mathematics **26** (Boston, 1982), pp. 103–142

[17] G. Harder, A. Raghuram, Eisenstein cohomology and ratios of critical values of Rankin-Selberg L-functions, *Comptes Rendus Math. Acad. Sci. Paris* 349 (2011) 719–724

[18] H. Hida, On the critical values of L-functions of $\mathrm{GL}(2)$ and $\mathrm{GL}(2) \times \mathrm{GL}(2)$, *Duke Math. J.* **74** (1994) pp. 431–529

[19] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, Automorphic forms on GL(3) II, *Ann. Math.* **109** (1979) pp. 213–258

[20] H. Jacquet, J. Shalika, *Exterior square L-functions*, in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. II, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 143–226

[21] H. Jacquet, J. Shalika, On Euler Products and the Classification of Automorphic Representations I, *Amer. J. Math.* **103** (1981) pp. 499–558

[22] D. Kazhdan, B. Mazur, C.-G. Schmidt, Relative modular symbols and Rankin–Selberg convolutions, *J. reine angew. Math.* **519** (2000) pp. 97–141

[23] A. W. Knapp, *Local Langlands correspondence: the Archimedean case*, in: *Motives*, Proc. Sympos. Pure Math., Vol. 55, Part 2, AMS, Providence, R.I., (1994), pp. 393–410

[24] P. F. Kurchanov, Cohomology of discrete groups and Dirichlet series connected with Jacquet-Langlands cusp forms, *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978) pp. 588–601

[25] P. F. Kurchanov, Dirichlet series of Jacquet-Langlands cusp forms over fields of CM-type, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979) pp. 67–86
[26] J. Mahnkopf, Cohomology of arithmetic groups, parabolic subgroups and the special values of automorphic L-Functions on $GL(n)$, *J. de l'Institut de Mathématiques de Jussieu* **4** (2005) pp. 553–637.
[27] Y. I. Manin, Non-archimedean integration and Jacquet–Langlands p-adic l-functions, *Russian Math. Surveys* **31** (1976) pp. 5–57
[28] A. Raghuram, Critical values of Rankin-Selberg L-functions for $GL_n \times GL_{n-1}$ and the symmetric cube L-functions for GL_2, *Forum Math.* **28** (2016) 457–489
[29] A. Raghuram, On the special values of certain Rankin-Selberg L-functions and applications to odd symmetric power L-functions of modular forms, *Int. Math. Res. Not.* Vol. (2010) 334–372, doi:10.1093/imrn/rnp127
[30] A. Raghuram, F. Shahidi, On certain period relations for cusp forms on GL_n, *Int. Math. Res. Not.* Vol. (2008), doi:10.1093/imrn/rnn077.
[31] J. Rohlfs, Projective limits of locally symmetric spaces and cohomology, *J. Reine Angew. Math.* **479** (1996) pp. 149–182
[32] F. Shahidi, On certain L-functions, *Amer. J. Math.* **103** (1981) pp. 297–355
[33] G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* **45** (1978) pp. 637–679
[34] G. Shimura, On the critical values of certain Dirichlet series and the periods of automorphic forms, *Invent. Math.* **94** (1988) pp. 245–306
[35] B. Sun, *Cohomologically induced distinguished representations and a non-vanishing hypothesis for algebraicity of critical L-values*, `arXiv:1111.2636` preprint (2011)
[36] J.-L. Waldspurger, Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2), *Comp. Math.* **54** (1985) pp. 121–171
[37] A. Weil, *Basic number theory*, Die Grundlehren der mathematischen Wissenschaften, **144**, (Springer-Verlag, New York, 1967)

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