RESIDUES OF EISENSTEIN SERIES AND THE AUTOMORPHIC COHOMOLOGY OF REDUCTIVE GROUPS

HARALD GROBNER

Abstract. Let G be a connected, reductive algebraic group over a number field F and let E be an algebraic representation of G_∞. In this paper we describe the Eisenstein cohomology $H^q_{Eis}(G,E)$ of G below a certain degree q_{res} in terms of Franke's filtration of the space of automorphic forms. This entails a description of the map $H^q(\mathfrak{m}_G,K,\Pi\otimes E)\to H^q_{Eis}(G,E)$, $q<q_{\mathsf{res}}$, for all automorphic representations Π of $G(\mathbb{A})$ appearing in the residual spectrum. Moreover, we show that below an easily computable degree q_{max}, the space of Eisenstein cohomology $H^q_{Eis}(G,E)$ is isomorphic to the cohomology of the space of square-integrable, residual automorphic forms. We discuss some more consequences of our result and apply it, in order to derive a result on the residual Eisenstein cohomology of inner forms of GL_n and the split classical groups of type B_n, C_n, D_n.

Contents

Introduction

Let G be a connected, reductive linear algebraic group over an arbitrary number field F. The cohomology of an arithmetic congruence subgroup Γ of $G(F)$ is isomorphic to a subspace of the cohomology of the space of automorphic forms. This identification was conjectured by Borel and Harder and first established in a conceptual way by Harder in the case of groups of rank one in [20], [19] and [18]. It is finally due to Franke, [12], that such an identification may also be given in the framework of an arbitrary connected, reductive algebraic group G. This makes it possible to study the cohomology of arithmetic congruence subgroups by means of automorphic representation theory.

Rendering the above more precise, let E be a finite dimensional irreducible algebraic representation of G_∞ and $\mathcal{J}$ the central ideal of $\mathfrak{U}(\mathfrak{g}_\infty)$, which annihilates the contragredient of E. For simplicity, we shall also assume that a fixed maximally F-split central torus A_G of G acts trivially on E. We view the representation E as a module under $\mathfrak{m}_G:=\mathfrak{g}_\infty/\mathfrak{a}_G$. Being given this data, we denote by $\mathcal{A}_\mathcal{J}(G)$ the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$, which are annihilated by some power of $\mathcal{J}$. It is a $(\mathfrak{m}_G,K,G(\mathbb{A}_f))$-module, where we let K denote (the connected component of the identity of) a maximal compact subgroup of G_∞. Taking up

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what we said above, the object to be studied here is hence the $G(\mathbb{A}_f)$-module structure of the relative Lie algebra cohomology

$$H^q(G,E) := H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E)$$

to be called the *automorphic cohomology* of G/F with respect to E.

As shown by Franke in [12], every automorphic form on G can be written as the sum of main values of derivatives of cuspidal or residual Eisenstein series, attached to the associate classes of parabolic F-subgroups $\{P\}$ of G. This finally amounts to a fine decomposition of the $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module $\mathcal{A}_{\mathcal{J}}(G)$, obtained by Franke-Schwermer in [13], as

$$\mathcal{A}_{\mathcal{J}}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{\{P\}} \bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G),$$

along the associate classes of parabolic F-subgroups $\{P\}$ and the various cuspidal supports φ_P. For details see [13] or Section 2.3. The space of automorphic cohomology hence inherits from the above decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ a decomposition as $G(\mathbb{A}_f)$-module:

$$H^q(G,E) = \bigoplus_{\{P\}} \bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E).$$

As $\mathcal{A}_{\mathcal{J},\{G\}}(G)$ consists of all cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$, one usually calls $H^q_{cusp}(G,E) := H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{G\}}(G) \otimes E)$ the space of cuspidal cohomology, while, by the nature of the spaces $\mathcal{A}_{\mathcal{J},\{P\}}(G)$, $P \neq G$, it is justified to call

$$H^q_{Eis}(G,E) := \bigoplus_{\{P\} \neq \{G\}} \bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E)$$

the space of *Eisenstein cohomology*. In the case when all Eisenstein series attached to a pair of supports $(\{P\}, \varphi_P)$ are holomorphic at the point of evaluation, the $G(\mathbb{A}_f)$-module structure of the summand $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E)$ is well–understood by the work of Schwermer [28],[27] and Li-Schwermer [23].

Apart from this case, the actual contribution of an arbitrary pair of supports $(\{P\}, \varphi_P)$ to Eisenstein cohomology, i.e., the $G(\mathbb{A}_f)$-module structure of $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E)$ for arbitrary $(\{P\}, \varphi_P)$, is mostly unknown. This is reflected in the fact that for a square-integrable, residual automorphic representation Π of $G(\mathbb{A})$, only very little can be said about the behaviour of the natural map

$$H^q(\mathfrak{m}_G, K, \Pi \otimes E) \to H^q_{Eis}(G,E).$$

Even less is known, when Π is an arbitrary automorphic representation supported in $(\{P\}, \varphi_P)$.

It is the aim of this article to use Franke's filtration of the space of automorphic forms $\mathcal{A}_{\mathcal{J}}(G)$ in order to overcome this problem up to a certain degree q_{res}. In other words, we want to describe the summand $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E)$ of Eisenstein cohomology in terms of Franke's filtration, where $(\{P\}, \varphi_P)$ is an arbitrary pair of supports of a reductive group G and $q < q_{\mathsf{res}}$ is smaller than a certain degree q_{res}. This includes a description of the map $H^q(\mathfrak{m}_G, K, \Pi \otimes E) \to H^q_{Eis}(G,E)$, for Π a square-integrable residual automorphic representation of $G(\mathbb{A})$ and $q < q_{\mathsf{res}}$.

In [12], Franke introduced a certain kind of filtration on $\mathcal{A}_{\mathcal{J},\{P\}}(G)$, which lies at the core of the decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ along the supports $(\{P\}, \varphi_P)$. This filtration depends on the choice of a function T, which itself depends on the automorphic exponents of $f \in \mathcal{A}_{\mathcal{J},\{P\}}(G)$. More precisely, T has to have values in the non-negative integers, such that

$$T(\lambda) > T(\theta) \quad \text{for } \lambda \in \theta - \overline{{}^{+}\mathfrak{a}_0^G}, \lambda \neq \theta.$$

For an exact definition of T, which is rather technical, we refer the reader to Section 3.2.

If we let $\mathcal{A}^{(j)}_{\mathcal{J},\{P\}}(G)$ denote the j-th filtration step of the summand $\mathcal{A}_{\mathcal{J},\{P\}}(G)$, Franke showed that each consecutive quotient is spanned by main values of the derivatives of cuspidal and residual Eisenstein series. In more precise terms, he proved that in the present setup, every consecutive quotient decomposes as a direct sum of representations, which are induced from a space of square integrable automorphic forms. These spaces of square integrable automorphic forms are indexed by certain triples $t = (R, \Lambda, \chi)$, where R is a standard parabolic F-subgroup of G containing an element of $\{P\}$, Λ is a continuous character of $A_R(\mathbb{A})$, whose derivative is compatible with the filtration, and χ dictates the infinitesimal character of the inducing module.

It is this important result which is the starting point of this article and which we are going to use, in order to describe the Eisenstein cohomology of reductive groups in low degrees of cohomology. First of all, we need to refine Franke's filtration to the level of cuspidal supports φ_P, i.e., define spaces $\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)$, and prove an analogue of his theorem on the decomposition of the resulting consecutive quotients. This is done in Theorem 4, where we pass over from triples t to quadruples (R,Π,ν,λ) of the form

(1) $R = L_R N_R$ a standard parabolic F-subgroup of G containing a representative of $\{P\}$
(2) Π is a unitary discrete series automorphic representation of $L_R(\mathbb{A})$ with cuspidal support determined by φ_P, spanned by iterated residues of Eisenstein series at the point $\nu \in \check{\mathfrak{a}}^R_{P,\mathbb{C}}$.
(3) $\lambda \in \check{\mathfrak{a}}_{R,\mathbb{C}}$ such that $\Re e(\lambda) \in \overline{\check{\mathfrak{a}}^{G+}_R}$ and such that $\lambda + \nu + \chi_{\tilde{\pi}}$ is annihilated by $\mathcal{J}$.

We let $M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$ be the set of all quadruples (R,Π,ν,λ), for which λ contributes to the j-th filtration step. This is a technical condition, made precise in Section 3.3, to which we refer the reader also for all details left out here. We obtain the following result, which takes into account the cuspidal support φ_P, cf. Theorem 4:

Theorem. *For all $j \geq 0$, there is an isomorphism of $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules*

$$\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}} I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]^{m(\Pi)},$$

where $m(\Pi)$ denotes the finite multiplicity of Π in the intersection of the discrete spectrum of $L_R(\mathbb{A})$ and $\mathcal{A}_{\mathcal{J},\{P\cap L_R\},\varphi_P}(L_R)$.

Observe that the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ is of finite length. We let $m = m(\{P\})$ be its length, which we may assume to have minimized by an appropriate choice of T, see Section 3.2.

As a next step, we need to establish a certain purity or rigidity result, see Proposition 10, on the possible values of $-w_v(\mu_v+\rho_v)|_{\mathfrak{a}_{P,v}\cap\mathfrak{m}_G}$ as v runs through the archimedean places of F. (Here, $\mu = (\mu_v)_{v\in S_\infty}$ is the highest weight of E, $w = (w_v)_{v\in S_\infty}$ is a Kostant representative of a parabolic $R \supseteq P$, and $\rho = (\rho_v)_{v\in S_\infty}$ denotes the half sum of positive roots.) Such a result was already proved by Harder for $G = GL_2/F$, see [18], and later on his arguments were used in Grbac–Grobner [15] in the case of $G = Sp_4$ over a totally real field. Here, we are going to use Clozel's "lemme de pureté", see [8], in order to obtain a general result.

This rigidity result, which is based on a rather intricate analysis, carried out in Section 5, implies a restriction on the length of the Kostant representatives, which one needs to consider, in order to obtain a valid quadruple $(R,\Pi,\nu,\lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, cf. Proposition 12. This finally gives way to the definition of the above mentioned bound q_{res}. It is the minimum over all quadruples $(R,\Pi,\nu,\lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, $0 \leq j < m$, of the values

$$\sum_{v\in S_\infty}\left(\left\lceil \tfrac{1}{2}\dim_{\mathbb{R}} N_R(F_v)\right\rceil + m(L_{R,v},\Pi_v)\right),$$

where $m(L_{R,v},\Pi_v)$ is the minimal degree, in which Π_v has non-zero cohomology. For a more precise definition of q_{res}, we refer to Section 6.1.

The main result of this paper now is

Main Theorem. *Let G be a connected, reductive group over a number field F and let E be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $m = m(\{P\})$ be the length of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$. Then, the map in cohomology, induced from the natural inclusion $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{m}_G,K,\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E) \xrightarrow[Eis^q_{\mathcal{J},\{P\},\varphi_P}]{\cong} H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E)$$

for all degrees $0 \leq q < q_{\mathsf{res}}$.
In other words, the Eisenstein cohomology supported in $(\{P\},\varphi_P)$ is entirely given by the $(\mathfrak{m}_G,K)$-cohomology of the m-th filtration step of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$ in all degrees $0 \leq q < q_{\mathsf{res}}$.

Observe that as a consequence, cf. Corollary 16, the Eisenstein cohomology supported in $(\{P\},\varphi_P)$ has a direct sum decomposition

$$H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E)\cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^{(m)}_{\mathcal{J},\{P\},\varphi_P}} H^q(\mathfrak{m}_G,K,I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]\otimes E)^{m(\Pi)},$$

for all $q<q_{\mathsf{res}}$. Hence, the following corollary, which deals with the contribution of a square-integrable, residual automorphic representations Π to Eisenstein cohomology, follows immediately from our main theorem:

Corollary. *Let G be a connected, reductive group over a number field F and let E be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let Π be a square-integrable, residual automorphic representation of $G(\mathbb{A})$ with cuspidal support $\pi\in\varphi_P$, spanned by iterated residues of Eisenstein series at a point $\nu\in\check{\mathfrak{a}}^G_{P,\mathbb{C}}$, for which $\nu+\chi_{\tilde\pi}$ is annihilated by $\mathcal{J}$. Let $m(\Pi)$ be its finite multiplicity in the intersection of the residual spectrum of $G(\mathbb{A})$ and the summand $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Then, the map in cohomology*

$$H^q(\mathfrak{m}_G,K,\Pi\otimes E)^{m(\Pi)}\longrightarrow H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E),$$

induced from the natural inclusion $\Pi^{m(\Pi)}\hookrightarrow\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is injective in all degrees $0\le q<q_{\mathsf{res}}$.

In Section 7, we analyze the consequences of our main theorem more closely and comment on its interplay with some results on Eisenstein cohomology in the literature.

First, we discuss the nature of the bound q_{res}. In Section 7.1 we show that one may always replace the rather involved bound q_{res} by the weaker and easily computable constant

$$q_{\mathsf{max}}:=\min_{\substack{R\text{ maximal}\\ R\supseteq P}}\left(\sum_{v\in S_\infty}\left\lceil\tfrac{1}{2}\dim_{\mathbb{R}}N_R(F_v)\right\rceil\right),$$

whose calculation does not invoke any quadruples and which already turns out to be a valuable bound in many cases. Moreover – underlining the profitableness of q_{max} – we show the following theorem, which says that below q_{max}, the cohomology of the space of square-integrable, residual automorphic forms exhausts the full space of Eisenstein cohomology of a reductive group:

Theorem. *Let G be a connected, reductive group over a number field F and let E be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $L^2_{\mathcal{J},\{P\},\varphi_P}(G)$ be the space of square-integrable (and hence residual) automorphic forms in $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Then, the natural inclusion $L^2_{\mathcal{J},\{P\},\varphi_P}(G)\hookrightarrow\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ induces an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{m}_G,K,L^2_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E)\xrightarrow{\ \cong\ }H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E)$$

for all degrees $0\le q<q_{\mathsf{max}}$.

We refer to Theorem 18 for this fact and to Theorem 24 for families of examples, showing the use of q_{max} in the case of the split classical groups.

Further, we show that q_{res} *is best possible.* This is meant in the sense, that there is a choice of a reductive group G/F, a coefficient system E and of a pair of supports $(\{P\},\varphi_P)$, such that $Eis^q_{\mathcal{J},\{P\},\varphi_P}$ is not an isomorphism for $q=q_{\mathsf{res}}$. We give an example in Section 7.1.2: One may simply take $G=SL_2/\mathbb{Q}$, $E=\mathbb{C}$, $P=B$ and $\varphi_B=\{\mathbf{1}_{T(\mathbb{A})}\}$.

In Section 7.2, we show that our main theorem and its corollary provide a certain generalization as well as a refinement of a recent result of Rohlfs-Speh, cf. [26]. There they show that certain square-integrable, residual automorphic representations Π have a non-trivial contribution to $H^{q_1}_{Eis}(G,\mathbb{C})$, for q_1 the minimal cohomological degree of Π. In contrast, our main theorem may be applied to all square-integrable, residual automorphic representations Π of a reductive group G/F and it says that $H^q(\mathfrak{m}_G, K, \Pi\otimes E)$ even injects into $H^q_{Eis}(G,E)$ in all degrees $0\leq q<q_{\mathsf{res}}$ with its full multiplicity $m(\Pi)$ in $L^2_{\mathcal{J},\{P\},\varphi_P}(G)$. Moreover, our coefficient module E does not need to be the trivial representation. Hence, we obtain a refined version of [26], if $q_1<q_{\mathsf{res}}$.

Next we recall a vanishing result for $H^q_{Eis}(G,E)$, proved by Li-Schwermer, [23]. They show that if E is of regular highest weight, then $H^q_{Eis}(G,E)=0$ in all degrees $0\leq q<q_0(G(\mathbb{R}))$. If one adapts the proof of our main theorem to regular coefficients, then one obtains an alternative approach to the theorem of Li-Schwermer, see Section 7.3. Indeed, our main theorem may be viewed as a generalisation of a weak version of Li-Schwermer's result, applying also to non-regular coefficient modules E. This is made precise in Theorem 20.

In Section 7.4, we discuss the interplay of our Corollary 17, which describes the contribution of square-integrable, residual automorphic representations Π to $H^q_{Eis}(G,E)$ for $q<q_{\mathsf{res}}$, with Franke's description of the contribution of $\mathbf{1}_{G(\mathbb{A})}$ to $H^*_{Eis}(G,\mathbb{C})$, given in [11]. We show that our Corollary, when applied to $\Pi=\mathbf{1}_{G(\mathbb{A})}$, is compatible with Franke's result. In fact, they coincide for the range of degrees considered. Therefore, our main theorem may also be seen as an independent way to improve Borel's classical result on the image of $H^q(\mathfrak{m}_G,K,\mathbf{1}_{G(\mathbb{A})})\to H^q_{Eis}(G,\mathbb{C})$, [3].

In the last two sections, we apply our main theorem to certain families of reductive groups, in order to give some examples. In Section 8, we consider the contribution of residual automorphic representations to the Eisenstein cohomology of inner forms of GL_n over any number field. That is, we consider this question for $G=GL'_n$, $n\geq 1$, by which we denote the general linear group over a central division algebra D over F. In this case, the associate classes of parabolic F-subgroups P of G are indexed by partitions $[n_1,...,n_k]$ of $n=\sum n_i$. In the special case that all k summands n_i are equal, we simply write $P=P_k$. Using the recent classification of the residual spectrum of $GL'_n(\mathbb{A})$, see [1], in terms of generalized Mœglin-Waldspurger quotients $MW'(\rho',k)$, we obtain the following result, see Theorem 22:

Theorem. *Let $G=GL'_n$, $n\geq 1$, and let $d\geq 1$ be the index of D over F. Let $\{P\}=\{P_{[n_1,...,n_k]}\}$ be an associate class of proper parabolic F-subgroups and φ_P an associate class of cuspidal automorphic representations π of $L(\mathbb{A})=L_{[n_1,...,n_k]}(\mathbb{A})$. If either $\{P\}\neq\{P_k\}$ or $\pi\not\cong\otimes_{i=1}^k\rho'$, then there is no residual automorphic representation $\Pi\hookrightarrow L^2_{\mathcal{J},\{P\},\varphi_P}(G)$ of $G(\mathbb{A})$ supported by $(\{P\},\varphi_P)$. If $\{P\}=\{P_k\}$ and $\pi\cong\otimes_{i=1}^k\rho'$, then the representation $\Pi=MW'(\rho',k)$ appears precisely once in the residual spectrum of $G(\mathbb{A})$ and the map in cohomology*

$$H^q(\mathfrak{m}_G,K,\Pi\otimes E)\longrightarrow H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E),$$

induced from the natural inclusion $\Pi\hookrightarrow\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is injective in all degrees

$$0\leq q<\sum_{\substack{v\in S_\infty\\ v\text{ complex}}} d^2(k-1)\frac{n^2}{k^2}+\sum_{\substack{v\in S_\infty\\ v\text{ real}}}\left\lceil d^2(k-1)\frac{n^2}{2k^2}\right\rceil.$$

If $d=1$ and $k=2$, i.e., if $G=GL_n/F$ is the split general linear group over F and P is the self-associate maximal parabolic, then this bound can be improved to

$$0\leq q<\sum_{\substack{v\in S_\infty\\ v\text{ complex}}}\frac{1}{2}(n^2-n)+\sum_{\substack{v\in S_\infty\\ v\text{ real}}}\frac{n^2}{4}.$$

In the particular case of the split general linear group, the result is complementary to Franke-Schwermer, [13]. There the authors considered residual Eisenstein cohomology classes attached to maximal parabolic subgroups of $GL_n/\mathbb{Q}$ and proved that for P self-associate, $H^q(\mathfrak{m}_G,K,L^2_{\mathcal{J},\{P\},\varphi_P}(G))$ maps surjectively onto the summand $H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G))$ in degrees $q<n^2/4+\lceil n/4\rceil$. Here, for $q<n^2/4$ we also prove injectivity.

In Section 9, we consider the case of the split classical groups $G_n = SO_{2n+1}, Sp_{2n}, SO_{2n}$ over $F = \mathbb{Q}$ and P a maximal parabolic subgroup. For split classical groups of $\mathbb{Q}$-rank n, the standard maximal parabolic $\mathbb{Q}$-subgroups are indexed by the simple roots α_k, $1 \leq k \leq n$. We obtain the following result, see Theorem 24

Theorem. *Let $G = G_n$ be a $\mathbb{Q}$-split classical group of Cartan type B_n, C_n or D_n, i.e., either the $\mathbb{Q}$-split symplectic or special orthogonal group of $\mathbb{Q}$-rank n. Let $P = P_k$, $1 \leq k \leq n$, be the standard maximal parabolic $\mathbb{Q}$-subgroup of G corresponding to the k-th simple root and let $\{P_k\}$ be the so–defined associate class of parabolic $\mathbb{Q}$-subgroups. (Here we leave out the case $k = n-1$, $G_n = SO_{2n}$.) If φ_{P_k} is an associate class of cuspidal automorphic representations of $L_k(\mathbb{A})$, then there is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{g}, K, L^2_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E) \xrightarrow{\cong} H^q(\mathfrak{g}, K, \mathcal{A}_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E),$$

for all degrees $0 \leq q < \frac{1}{2}\left((n-k)\frac{n-k+3}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor\right) + q(G_n, k)$, where

$$q(G_n, k) = \begin{cases} \lceil k(n - \frac{3k+1}{4}) \rceil & \text{if } G_n = SO_{2n} \\ \lceil k(n - \frac{3k-1}{4}) \rceil & \text{if } G_n = SO_{2n+1}, Sp_{2n}. \end{cases}$$

In the case of $G = SO_{2n+1}$ resp. Sp_{2n}, the latter theorem is complementary to the results in Gotsbacher-Grobner [14] resp. Grbac-Schwermer [16]. In these references, necessary conditions for non-trivial residual Eisenstein cohomology classes, stemming from globally generic cuspidal automorphic representations of maximal Levi subgroups were given. In contrary, the conditions provided here are sufficient for the existence of such classes. Moreover, in the range of degrees given by the above theorem, it is shown that these residual Eisenstein cohomology classes exhaust the full space $H^q(\mathfrak{g}, K, \mathcal{A}_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E)$. Also, the condition of global genericity does not enter the present assumptions.

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1. Notation and basic assumptions

1.1. **Number fields.** We let F be an algebraic number field. Its set of places is denoted $S = S_\infty \cup S_f$, where S_∞ stands for the set of archimedean places and S_f is the set of non–archimedean places. The ring of adeles of F is denoted $\mathbb{A}$, the subspace of finite adeles is denoted $\mathbb{A}_f$.

1.2. **Algebraic groups.** In this paper, G is a connected, reductive linear algebraic group over a number field F. We assume to have fixed a minimal parabolic F-subgroup P_0 with Levi decomposition $P_0 = L_0 N_0$ and let A_0 be the maximal F-split torus in the center Z_{L_0} of L_0. This choice defines the standard parabolic F-subgroups P with Levi decomposition $P = L_P N_P$, where $L_P \supseteq L_0$ and $N_P \subseteq N_0$. We let A_P be the maximal F-split torus in the center Z_{L_P} of L_P, satisfying $A_P \subseteq A_0$. If it is clear form the context, we will also drop the subscript "P". We put $\check{\mathfrak{a}}_P := X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_P := X_*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$, where X^* (resp. X_*) denotes the group of F-rational characters (resp. co-characters). These real Lie algebras are in natural duality to each other. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $\mathfrak{a}_P$ and $\check{\mathfrak{a}}_P$. The inclusion $A_P \subseteq A_0$ (resp. the restriction to P) defines $\mathfrak{a}_P \to \mathfrak{a}_0$ (resp. $\check{\mathfrak{a}}_P \to \check{\mathfrak{a}}_0$), which gives rise to direct sum decompositions $\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P$ and $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P$. We let $\mathfrak{a}_P^Q := \mathfrak{a}_P \cap \mathfrak{a}_0^Q$ and $\check{\mathfrak{a}}_P^Q := \check{\mathfrak{a}}_P \cap \check{\mathfrak{a}}_0^Q$ for parabolic F-subgroups Q and P. Furthermore, we set $\check{\mathfrak{a}}_{P,\mathbb{C}} := \check{\mathfrak{a}}_P \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{a}_{P,\mathbb{C}} := \mathfrak{a}_P \otimes_{\mathbb{R}} \mathbb{C}$. Then the analogous assertions hold for these complex Lie algebras. We denote by $H_P : L_P(\mathbb{A}) \to \mathfrak{a}_{P,\mathbb{C}}$ the standard Harish-Chandra height function, cf. [12], p. 185. The group $L_P(\mathbb{A})^1 := \ker H_P = \bigcap_{\chi \in X^*(L_P)} \ker(\|\chi\|_{\mathbb{A}})$, $\| \cdot \|_{\mathbb{A}}$ the adelic norm, admits a direct complement $A_P^{\mathbb{R}} \cong \mathbb{R}_+^{\dim \mathfrak{a}_P}$ in $L_P(\mathbb{A})$ whose Lie algebra is isomorphic to $\mathfrak{a}_P$. With respect to a maximal compact subgroup $K_{\mathbb{A}} \subseteq G(\mathbb{A})$ in good position, cf. [24] I.1.4, we obtain an extension $H_P : G(\mathbb{A}) \to \mathfrak{a}_{P,\mathbb{C}}$ to all of $G(\mathbb{A})$. The group P acts on N_P by the adjoint representation. The weights of this action with respect to the torus A_P are denoted $\Delta(P, A_P)$ and ρ_P denotes the half-sum of these weights, counted with multiplicity. We will not distinguish between ρ_P and its derivative, so we may also view ρ_P as an element of $\mathfrak{a}_P$. In particular, $\Delta(P_0, A_0)$ defines a choice of positive F-roots of G. With respect to this choice, we shall use the notation $\check{\mathfrak{a}}_P^{G+}$ and $\mathfrak{a}_P^{G+}$ (resp. ${}^+\check{\mathfrak{a}}_P^G$ and ${}^+\mathfrak{a}_P^G$) for the open positive Weyl chambers in $\check{\mathfrak{a}}_P^G$ and $\mathfrak{a}_P^G$ (resp. the open positive cones dual to them). Overlining one of these cones denotes its topological closure.

1.3. **Lie groups.** We put $G_\infty := R_{F/\mathbb{Q}}(G)(\mathbb{R})$, where $R_{F/\mathbb{Q}}$ denotes the restriction of scalars from F to $\mathbb{Q}$. We shall also write $G_v := G(F_v)$, $v \in S_\infty$ some archimedean place. The analogous notation is used for groups different from G. Lie algebras of real Lie groups are denoted by the same but lower case gothic letter, e.g., $\mathfrak{g}_\infty = Lie(G_\infty)$ or $\mathfrak{a}_{P,v} = Lie(A_{P,v})$. The Lie algebra $\mathfrak{a}_P$ of the connected Lie group $A_P^{\mathbb{R}}$ is viewed as being diagonally embedded into $\mathfrak{a}_{P,\infty}$. We let $\mathfrak{m}_L := \mathfrak{l}_{P,\infty}/\mathfrak{a}_P = Lie(L_P(\mathbb{A})^1 \cap L_{P,\infty})$ and denote by $\mathfrak{Z}(\mathfrak{g})$ the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}_{\infty,\mathbb{C}}$.

Let $K_\infty \subset G_\infty$ be a maximal compact subgroup (the archimedean factor of the maximal compact subgroup $K_{\mathbb{A}}$ of $G(\mathbb{A})$ in good position). Then K_∞ has trivial intersection with $A_G^{\mathbb{R}}$ (but might not have trivial intersection with $A_{G,\infty}$). For any open subgroup $K \subseteq K_\infty$, with $K_\infty^\circ \subseteq K \subseteq K_\infty$, one may recover the $(\mathfrak{m}_G, K)$-cohomology functor by taking the K/K_∞°-invariants in $(\mathfrak{m}_G, K_\infty^\circ)$-cohomology: One has $H^q(\mathfrak{m}_G, K, \cdot) = H^q(\mathfrak{m}_G, K_\infty^\circ, \cdot)^{K/K_\infty^\circ}$, see [7] I.6.2. We will hence focus on $(\mathfrak{m}_G, K_\infty^\circ)$-cohomology in this paper and set once and for all $K := K_\infty^\circ$. Observe that this choice of a compact subgroup of G_∞ is in accordance with Franke [12], p. 184. We refer the reader to Borel–Wallach [7], I, for the basic facts and notations concerning $(\mathfrak{m}_G, K)$-cohomology. For any Lie subgroup H of G_∞, we let $K_H := K \cap H$.

Let $\mathfrak{h}_\infty$ be a Cartan subalgebra of $\mathfrak{g}_\infty$ that contains $\mathfrak{a}_{0,\infty}$ (and hence all $\mathfrak{a}_{P,\infty}$, $\mathfrak{a}_{P,v}$ and $\mathfrak{a}_P$). The choice of positivity on the set of F-roots of G is extended to a choice of positivity on the set of absolute roots $\Delta(\mathfrak{g}_{\infty,\mathbb{C}}, \mathfrak{h}_{\infty,\mathbb{C}})$. The half sum of the positive absolute roots is denoted $\rho = (\rho_v)_{v\in S_\infty} \in \check{\mathfrak{h}}_\infty$. In this paper, we always let $E = E_\mu$ be a finite–dimensional irreducible algebraic representation of G_∞ on a complex vector space, given by its highest weight $\mu = (\mu_v)_{v\in S_\infty} \in \check{\mathfrak{h}}_\infty$. As G_∞ is viewed as a real Lie group, μ_v has two coordinate vectors μ_{ι_v} and $\mu_{\bar{\iota}_v}$ at a complex place $v \in S_\infty$, which correspond to the complex embedding $\iota_v : F_v \hookrightarrow \mathbb{C}$ and its complex conjugate $\bar{\iota}_v$. We will assume that $A_G^{\mathbb{R}}$ (and so $\mathfrak{a}_G$) acts trivially on E. There is hence no difference between the $(\mathfrak{g}_\infty, K)$–module and the $(\mathfrak{m}_G, K)$-module defined by E.

1.4. **Weyl groups.** For the various sets of roots $(\Delta(\mathfrak{g}_{\infty,\mathbb{C}}, \mathfrak{h}_{\infty,\mathbb{C}}), \Delta(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})...)$, we define the according Weyl groups $(W(\mathfrak{g}_{\infty,\mathbb{C}}, \mathfrak{h}_{\infty,\mathbb{C}}), W(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})...)$ as the groups generated by all reflections corresponding to the elements in the defining root system. Let $v \in S_\infty$ be an archimedean place and $P_v = P(F_v)$. The set of Kostant representatives W^{P_v} is the set of all elements w of the Weyl group $W(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$ such that $w^{-1}(\alpha) > 0$ for all positive roots $\alpha \in \Delta(\mathfrak{l}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$. Replacing "$v$" by "$\infty$" gives $W^P := W^{P_\infty} = \prod_{v\in S_\infty} W^{P_v}$. For $\mu \in \check{\mathfrak{h}}_{\infty,\mathbb{C}}$ we define an affine action of $w \in W^P$ by $w\cdot\mu := \mu_w := w(\mu+\rho)-\rho$. The same definition applies locally. If v is a complex place, W^{P_v} splits as a product of two sets of Kostant representatives of the same size, $W^{P_v} = W^{P_{\iota_v}} \times W^{P_{\bar{\iota}_v}}$. At such a place, we shall hence write $\mu_{w_v} = (\mu_{w_{\iota_v}}, \mu_{w_{\bar{\iota}_v}})$. Given $\mu = (\mu_v)_{v\in S_\infty}$ an algebraic, dominant weight of $\mathfrak{g}_\infty$ and $w \in W^P$, we let $E_{\mu_w} = \otimes_{v\in S_\infty} E_{\mu_{w_v}}$ be the irreducible representation of $L_{P,\infty}$ of highest weight μ_w.

1.5. **Induction.** The symbol "${}^a\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}$" denotes un-normalized, algebraic induction from $(\mathfrak{p}_\infty, K_{P,\infty}, P(\mathbb{A}_f))$- to $(\mathfrak{g}_\infty, K, G(\mathbb{A}_f))$-modules. If V is any $(\mathfrak{p}_\infty, K_{P,\infty}, P(\mathbb{A}_f))$-module and $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, we let

$$I_{P(\mathbb{A})}[V, \lambda] := {}^a\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[V \otimes e^{\langle \lambda+\rho_P, H_P(\cdot)\rangle}].$$

Similarly, locally for a $(\mathfrak{p}_v, K_{P,v})$– (resp. $P(F_v)$–)module V_v, we let $I_{P(F_v)}[V_v, \lambda] := {}^a\mathrm{Ind}_{P(F_v)}^{G(F_v)}[V_v \otimes e^{\langle \lambda+\rho_P, H_{P_v}(\cdot)\rangle}]$ be the induced $(\mathfrak{g}_v, K_v^\circ)$– (resp. $G(F_v)$)–module. If V factors as restricted tensor product, $V \cong \otimes_{v\in S} V_v$, then we have $I_{P(\mathbb{A})}[V, \lambda] \cong \otimes_{v\in S} I_{P(F_v)}[V_v, \lambda]$.

2. Spaces of automorphic forms

2.1. **Generalities.** In this section we would like to summarize some known results from the theory of automorphic forms. Standard references for the facts presented in this section are Borel–Jacquet [5], Mœglin–Waldspurger [24], Langlands [22], Franke [12] and Franke–Schwermer [13].

Our notion of an *automorphic form* $f : G(\mathbb{A}) \to \mathbb{C}$ and our notion of an *automorphic representation* of $G(\mathbb{A})$ is the one of Borel-Jacquet [5], 4.2 and 4.6, to which we refer. Let $\mathcal{A}(G)$ be the space of all automorphic forms $f : G(\mathbb{A}) \to \mathbb{C}$ which are constant on the real Lie subgroup $A_G^{\mathbb{R}}$. We recall that by its very definition, every automorphic form is annihilated by some power of an ideal $\mathcal{J}$ of $\mathfrak{Z}(\mathfrak{g})$ of finite codimension. Let us now – once and for all – fix such an ideal $\mathcal{J}$: As we will only be interested in cohomological automorphic forms, we take $\mathcal{J}$ to be the ideal which annihilates the contragredient representation $E_\mu^{\vee}$ of E_μ, cf. Sect.1.3, and denote by

$$\mathcal{A}_{\mathcal{J}}(G) \subset \mathcal{A}(G)$$

the space consisting of those automorphic forms which are annihilated by some power of $\mathcal{J}$. With this notation, both spaces $\mathcal{A}(G)$ and $\mathcal{A}_{\mathcal{J}}(G)$ carry commuting $(\mathfrak{g}_\infty, K_\infty)$ and $G(\mathbb{A}_f)$-actions and hence define a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module. The $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-submodule of all square integrable automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$ is denoted $\mathcal{A}_{dis,\mathcal{J}}(G)$. An irreducible subquotient of $\mathcal{A}_{dis,\mathcal{J}}(G)$ will be called a *discrete series automorphic* representation, cf. Borel [2] 9.6.

To have the notation ready at hand, recall that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is called *cuspidal*, if its constant term $f_P(g) := \int_{N_P(F)\backslash N_P(\mathbb{A})} f(ng)dn = 0$ for all $g \in G(\mathbb{A})$ and along all proper parabolic F-subgroups P. Let $\mathcal{A}_{cusp,\mathcal{J}}(G)$ be the space of all cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$. As $G(F)A_G^{\mathbb{R}}\backslash G(\mathbb{A})$ has finite volume, $\mathcal{A}_{cusp,\mathcal{J}}(G)$ coincides with the space of all smooth, K_∞-finite functions in $L^2_{cusp}(G(F)A_G^{\mathbb{R}}\backslash G(\mathbb{A}))$ which are annihilated by a power of $\mathcal{J}$. It is a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module and a submodule of $\mathcal{A}_{dis,\mathcal{J}}(G)$. Its complement in $\mathcal{A}_{dis,\mathcal{J}}(G)$ is denoted $\mathcal{A}_{res,\mathcal{J}}(G)$. An irreducible subquotient of $\mathcal{A}_{cusp,\mathcal{J}}(G)$ (resp. $\mathcal{A}_{res,\mathcal{J}}(G)$) will be called a *cuspidal automorphic* representation (resp. *residual automorphic representation*). See also [5], 4.6.

2.2. **Parabolic supports.** Let $\{P\}$ be the associate class of the parabolic F-subgroup P: It consists by definition of all parabolic F-subgroups $Q = L_Q N_Q$ of G for which L_Q and L_P are conjugate by an element in $G(F)$. We denote by $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ the space of all $f \in \mathcal{A}_{\mathcal{J}}(G)$ which are negligible along every parabolic F-subgroup $Q \notin \{P\}$. We recall that the latter condition means that for all $g \in G(\mathbb{A})$, the function $L_Q(\mathbb{A}) \to \mathbb{C}$ given by $l \mapsto f_Q(lg)$ is orthogonal to the space of cuspidal functions on $L_P(F)A_G^{\mathbb{R}}\backslash L_P(\mathbb{A})$. Then there is the following decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ as a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module, cf. [6] Thm. 2.4 or [2] 10.3:

$$\mathcal{A}_{\mathcal{J}}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}(G).$$

2.3. **Cuspidal supports.** The various summands $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ can be decomposed even further. To this end, recall from [13], 1.2, the notion of an *associate class* φ_P of cuspidal automorphic representations of the Levi subgroups of the elements in the class $\{P\}$. Therefore, let $\{P\}$ be represented by $P = LN$. Then the associate classes φ_P may be parameterized by pairs of the form $(\Lambda, \tilde{\pi})$, where

(1) $\tilde{\pi}$ is a unitary cuspidal automorphic representation of $L(\mathbb{A})$, whose central character vanishes on the group $A_P^{\mathbb{R}}$.
(2) $\Lambda : A_P^{\mathbb{R}} \to \mathbb{C}^*$ is a Lie group character and
(3) the infinitesimal character $\chi_{\tilde{\pi}}$ of $\tilde{\pi}_\infty$ and the derivative $d\Lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ of Λ are compatible with the action of $\mathcal{J}$ (cf. [13], 1.2).

Each associate class φ_P may hence be represented by a cuspidal automorphic representation

$$\pi := \tilde{\pi} \otimes e^{\langle d\Lambda, H_P(\cdot)\rangle}$$

of $L(\mathbb{A})$. Given such a representative, let $W_{P,\tilde{\pi}}$ be the space of all smooth, K_∞-finite functions

$$f : L(F)N(\mathbb{A})A_P^{\mathbb{R}}\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $l \mapsto f(lg)$ on $L(\mathbb{A})$ is contained in the $\tilde{\pi}$-isotypic component $\tilde{\pi}^{m(\tilde{\pi})}$ of $L^2_{cusp}(L(F)A_P^{\mathbb{R}}\backslash L(\mathbb{A}))$ ($m(\tilde{\pi})$ being the finite multiplicity of $\tilde{\pi}$). For a function $f \in W_{P,\tilde{\pi}}$, $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}^G$ and $g \in G(\mathbb{A})$ an *Eisenstein series* is formally defined as

$$E_P(f, \lambda)(g) := \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma g) e^{\langle \lambda + \rho_P, H_P(\gamma g)\rangle}.$$

We will also view $f \cdot e^{\langle \lambda+\rho_P, H_P(\cdot)\rangle}$ as an element of $I_{P(\mathbb{A})}[\tilde{\pi},\lambda]^{m(\tilde{\pi})}$. The so-defined Eisenstein series converges absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times \{\lambda \in \check{\mathfrak{a}}^G_{P,\mathbb{C}} | \Re e(\lambda) \in \rho_P + \check{\mathfrak{a}}^{G+}_P\}$. It is known that $E_P(f,\lambda)$ is an automorphic form there and that the map $\lambda \mapsto E_P(f,\lambda)(g)$ can be analytically continued to a meromorphic function on all of $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$, cf. [24] or [22] §7. It is known that the singularities λ_0 (i.e poles) of $E_P(f,\lambda)$ lie along certain affine hyperplanes of the form $R_{\alpha,t} := \{\xi \in \check{\mathfrak{a}}^G_{P,\mathbb{C}} | \langle \xi, \alpha\rangle = t\}$ for some constant t and some root $\alpha \in \Delta(P, A_P)$, called "root-hyperplanes" ([24] Prop. IV.1.11 (a) or [22] p.131). Choose a normalized vector $\nu \in \check{\mathfrak{a}}^G_{P,\mathbb{C}}$ orthogonal to $R_{\alpha,t}$ and assume that λ_0 is on no other singular hyperplane of $E_P(f,\lambda)$. Then define $\lambda_0(u) := \lambda_0 + u\nu$ for $u \in \mathbb{C}$. If c is a positively oriented circle in the complex plane around zero which is so small that $E_P(f, \lambda_0(.))(g)$ has as no singularities on the interior of the circle with double radius, then

$$\mathrm{Res}_{\lambda_0}(E_P(f,\lambda)(g)) := \frac{1}{2\pi i}\int_c E_P(f,\lambda_0(u))(g)du$$

is a meromorphic function on $R_{\alpha,t}$, called the *residue* of $E_P(f,\lambda)$ at λ_0. Its poles lie on the intersections of $R_{\alpha,t}$ with the other singular hyperplanes of $E_P(f,\lambda)$. Iterating this process, one gets a function, which is holomorphic at a given λ_0, in finitely many steps by taking successive residues as explained above.

Given φ_P, represented by a cuspidal representation π of the above form, a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-submodule

$$\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$$

of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ was defined in [13], 1.3 as follows: It is the span of all possible holomorphic values or residues of all Eisenstein series attached to $\tilde{\pi}$, evaluated at the point $\lambda = d\Lambda$, together with all their derivatives. This definition is independent of the choice of the representatives P and π, thanks to the functional equations satisfied by the Eisenstein series considered. For details, we refer the reader to [13] 1.2-1.4.

The following refined decomposition as $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules of the spaces $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ of automorphic forms was obtained in Franke–Schwermer [13], Thm. 1.4:

Theorem 1 (Franke–Schwermer). *There is an isomorphism of* $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$*-modules*

$$\mathcal{A}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G).$$

This gives rise to the following

Definition 2. Let Π be an automorphic representation of $G(\mathbb{A})$, whose central character is trivial on $A^{\mathbb{R}}_G$. If Π is an irreducible subquotient of the space $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, we call the associate class $\{P\}$ a *parabolic support* and the associate class φ_P a *cuspidal support* of Π.

2.4. The above construction of the spaces $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ entails the following assertion: We let $S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})$ be the symmetric algebra

$$S(\check{\mathfrak{a}}^G_{P,\mathbb{C}}) := \bigoplus_{n\geq 0} \mathrm{Sym}^n \check{\mathfrak{a}}^G_{P,\mathbb{C}},$$

endowed with a $(\mathfrak{p}_\infty, K_{P,\infty}, P(\mathbb{A}_f))$-module structure as follows. Since $S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})$ can be viewed as the space of polynomials on $\mathfrak{a}^G_{P,\mathbb{C}}$, an element $Y \in \mathfrak{a}^G_P$ acts on $X \in S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})$ by translation and we extend this action trivially to all of $\mathfrak{p}_\infty$. The action of $P(\mathbb{A}_f)$ is trivial. We may also view $S(\check{\mathfrak{a}}^G_{P,\mathbb{C}})$ as the algebra of differential operators $\frac{\partial^n}{\partial\lambda^n}$ ($n = (n_1, ..., n_{\dim \mathfrak{a}^G_P})$ being a multi-index with respect to a fixed basis of $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$) on $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$. Furthermore, one may choose a non-trivial holomorphic function $q(\lambda)$ such that for a given associate class φ_P, represented by a cuspidal automorphic representation π, the function $q(\lambda)E_P(f,\lambda)$ is holomorphic in a neighborhood of $\lambda = d\Lambda$. Hence, having said this, by the construction of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ there is a surjective homomorphism of $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules

$$Eis_{\mathcal{J},\{P\},\varphi_P} : I_{P(\mathbb{A})}[\tilde{\pi} \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}}), d\Lambda]^{m(\tilde{\pi})} \longrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \tag{2.1}$$

given explicitly by

$$f \otimes \frac{\partial^n}{\partial\lambda^n} \mapsto \frac{\partial^n}{\partial\lambda^n}\left(q(\lambda)E_P(f,\lambda)\right)|_{\lambda=d\Lambda}.$$

3. Franke's filtration

3.1. The spaces $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ and $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ can be filtered in a certain way, which, together with our specific choice of $\mathcal{J}$, allows one to express the consecutive filtration quotients as a direct sum of induced representations $I_{R(\mathbb{A})}[\Pi^{m(\Pi)}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]$, Π now being a discrete series automorphic representation of some parabolic F-subgroup $R=L_RN_R$ containing a representative of $\{P\}$ and $\lambda\in\check{\mathfrak{a}}^G_{R,\mathbb{C}}$. This result is a direct consequence of the main result of Franke [12], Thm. 14. As this will be crucial for what follows, we recall Franke's filtration in this section.

3.2. Let $f\in\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Then the constant term along a standard parabolic F-subgroup Q has the form

$$f_Q(g)=\sum_{\lambda\in\check{\mathfrak{a}}_{Q,\mathbb{C}}} f_{Q,\lambda}(g,H_Q(g))\cdot e^{\langle\lambda+\rho_Q,H_Q(g)\rangle},$$

where $f_{Q,\lambda}$ is in the second variable a polynomial on $\mathfrak{a}_Q$ with values in the space of automorphic forms $f:G(\mathbb{A})\to\mathbb{C}$, which are constant on $Q(F)N_Q(\mathbb{A})A^{\mathbb{R}}_Q$; this automorphic form can then be evaluated in the first variable $g\in G(\mathbb{A})$, which explains $f_{Q,\lambda}(g,H_Q(g))\in\mathbb{C}$. The set of $\lambda\in\check{\mathfrak{a}}_{Q,\mathbb{C}}$, for which $f_{Q,\lambda}\neq 0$ for some f, is finite, cf. [12], p. 233. Let $\Lambda(Q,\mathcal{J})$ be this set. For $\lambda\in\Lambda(Q,\mathcal{J})$, the notion $\Re e(\lambda)_+$ was defined in [12] p. 233. Now, let T be a function

$$T:\{\Re e(\lambda)_+|\lambda\in\bigcup_Q\Lambda(Q,\mathcal{J})\}\to\mathbb{N},$$

with the property

$$T(\lambda)>T(\theta)\quad\text{for }\lambda\in\theta-\overline{{}^+\mathfrak{a}^G_0},\lambda\neq\theta.$$

Definition 3. (1) The j-th filtration step of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ is defined as

$$\mathcal{A}^{(j)}_{\mathcal{J},\{P\}}(G):=\{f\in\mathcal{A}_{\mathcal{J},\{P\}}(G)|f_{Q,\lambda}=0\quad\forall Q\in\{P\}\text{ and }\forall\lambda\in\Lambda(Q,\mathcal{J}):T(\Re e(\lambda)_+)<j\}.$$

(2) The j-th filtration step of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ is defined as

$$\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G):=\{f\in\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)|f_{Q,\lambda}=0\quad\forall Q\in\{P\}\text{ and }\forall\lambda\in\Lambda(Q,\mathcal{J}):T(\Re e(\lambda)_+)<j\}.$$

Observe that we suppressed the choice of T in the notation of the j-th filtration step. In any case, the length of the filtration is finite, cf. [12], p. 233. We assume to have chosen T once and for all such that for every associate class $\{P\}$, the length $m=m(\{P\})$ of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ is minimal. By the very definition, we obtain

$$\mathcal{A}^{(0)}_{\mathcal{J},\{P\}}(G)=\mathcal{A}_{\mathcal{J},\{P\}}(G)\qquad\text{and}\qquad\mathcal{A}^{(0)}_{\mathcal{J},\{P\},\varphi_P}(G)=\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G).$$

3.3. Given a cuspidal support φ_P, we will need the following collection of data: Let $M_{\mathcal{J},\{P\},\varphi_P}$ be the set of quadruples (R,Π,ν,λ) of the form

(1) R a standard parabolic F-subgroup of G containing a representative of $\{P\}$
(2) Π is a unitary discrete series automorphic representation of $L_R(\mathbb{A})$ with cuspidal support determined by φ_P, spanned by iterated residues of Eisenstein series at the point $\nu\in\check{\mathfrak{a}}^R_{P,\mathbb{C}}$. Let $m(\Pi)$ be its finite multiplicity in $\mathcal{A}_{dis,\mathcal{J}}(L_R)\cap\mathcal{A}_{\mathcal{J},\{P\cap L_R\},\varphi_P}(L_R)$.
(3) $\lambda\in\check{\mathfrak{a}}_{R,\mathbb{C}}$ such that $\Re e(\lambda)\in\overline{\check{\mathfrak{a}}^{G+}_R}$ and such that $\lambda+\nu+\chi_{\tilde{\pi}}$ is annihilated by $\mathcal{J}$.

We point out that with this definition, although not entirely obvious, one can show that T is well-defined on $\Re e(\lambda)_+$. Therefore, taking this for granted, it makes sense to define

$$M^{(j)}_{\mathcal{J},\{P\},\varphi_P}:=\{(R,\Pi,\nu,\lambda)|T(\Re e(\lambda)_+)=j\}.$$

These sets of quadruples $M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$ originate from [12], p. 218, 233–234. There, however, only the parabolic support $\{P\}$ and not the cuspidal support φ_P was taken into account. Doing so, there is the following theorem, which is a slight refinement of [12] Thm. 14.

Theorem 4. *For all $j \geq 0$, there is an isomorphism of $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules*

$$\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}} I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda]^{m(\Pi)}.$$

Proof. In the notation given in [12] Thm. 14, if we take the direct limit $\tau \to \infty$ in the positive Weyl chamber, then computing the main value $\mathbf{MW}$, cf. [12], (13), yields an isomorphism of $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules

$$\mathcal{A}^{(j)}_{\mathcal{J},\{P\}}(G)\Big/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{k=0}^{\mathrm{rank}(P)} \lim_{t\in\mathcal{M}^{k,T,j}_{\mathcal{J},\{P\},\infty}} M(t). \tag{3.1}$$

Here, $\mathcal{M}^{k,T,j}_{\mathcal{J},\{P\},\infty}$ is a goupoid, whose elements are by definition, cf. [12] p. 218, triples $t = (R, \Lambda, \chi)$, where

1. $R = L_R N_R$ a standard parabolic F-subgroup of G containing an element of $\{P\}$, such that $\dim \mathfrak{a}^G_P = \dim \mathfrak{a}^G_R + k$.
2. $\Lambda : A_R(F)A^{\mathbb{R}}_G\backslash A_R(\mathbb{A}) \to \mathbb{C}^*$ is a continuous character such that $d\Lambda_\infty$ defines an element $\lambda_t \in \check{\mathfrak{a}}_{R,\mathbb{C}}$ with the property $\Re e(\lambda_t) \in \overline{\check{\mathfrak{a}}^{G+}_R}$ and $T(\Re e(\lambda_t)_+) = j$
3. $\chi : \mathfrak{Z}(\mathfrak{m}_R) \to \mathbb{C}$ is a unitary character such that $\lambda_t + \chi$ is annihilated by $\mathcal{J}$.

Attached to this datum, a space $V(u_t)$ is defined on [12], p. 218, as follows: It is the space of all smooth, $K_{\mathbb{A}}$-finite functions

$$f \in L^2(R(F)N_R(\mathbb{A})A^{\mathbb{R}}_R\backslash R(\mathbb{A}), \mathbb{C})$$

which satisfy

1. f_Q is orthogonal to the space of cusp forms of L_Q, for all $Q \subseteq R$ which are not in $\{P\}$.
2. If $\tilde{\Lambda} := \Lambda \cdot e^{\langle -\lambda_t, H_R(.)\rangle}$, then $f(ag) = \tilde{\Lambda}(a)f(g)$ for all $a \in A_R(\mathbb{A})$ and $g \in R(\mathbb{A})$
3. $Xf(\cdot g) = \chi(X)f(\cdot g)$ for all $X \in \mathfrak{Z}(\mathfrak{m}_R)$ and $f(\cdot g) : L_{R,\infty} \to \mathbb{C}$.

Finally, the space $M(t)$ was defined as

$$M(t) = I_{R(\mathbb{A})}[V(u_t) \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda_t],$$

[12], (11) p. 234.

Since by our choice, $\mathcal{J}$ annihilates a finite–dimensional, irreducible algebraic representation of G_∞, $\mathcal{J}$ consists of regular elements of $\check{\mathfrak{h}}_\infty$ and so no element of the groupoid $\mathcal{M}^{k,T,j}_{\mathcal{J},\{P\},\infty}$ has non–trivial automorphisms, cf. [12], Thm. 19.I. Therefore, the direct limit of (3.1) becomes a direct sum over the (isomorphism classes) of the elements t of $\mathcal{M}^{k,T,j}_{\mathcal{J},\{P\},\infty}$. Letting $\lambda = \lambda_t$ and $\nu = \chi - \chi_{\tilde{\pi}}$, then our result follows from the definition of $V(u_t)$ and the well–known fact that the discrete spectrum $\mathcal{A}_{dis,\mathcal{J}}(L_R)$ of L_R decomposes discretely with finite multiplicities, or – more generally – by Franke–Schwermer [13], Thm. 1.4. Compare this also to [12], Prop. 1, p. 245 and the comment below it, *ibidem.* □

Remark 5 (Sp_4/F)**.** For a non–trivial case-study, where the above description of the successive quotients of the filtration of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ was made explicit, the reader may have a look at Grbac–Grobner, [15], Theorems 3.3 and 3.6. There the case $G = Sp_4$ over a totally real field F was considered.

Remark 6 (The deepest step)**.** We would like to point out that Theorem 4 trivially implies that

$$\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^{(m)}_{\mathcal{J},\{P\},\varphi_P}} I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda]^{m(\Pi)}.$$

Here, $m = m(\{P\})$ is the deepest step in the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$.

4. Automorphic Cohomology

4.1. We recall the following

Definition 7. The cohomology space

$$H^q(G, E) := H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E),$$

endowed with its natural $G(\mathbb{A}_f)$-module structure, is called the *the space of automorphic cohomology of G.*

This $G(\mathbb{A}_f)$-module inherits from Thm. 1 a direct sum decomposition. This was established in Franke–Schwermer, [13] Thm. 2.3.

4.2. Cuspidal cohomology. We recall that the summand $\mathcal{A}_{\mathcal{J},\{G\}}(G)$ in Thm. 1, indexed by the associate class of G itself, is precisely the space $\mathcal{A}_{cusp,\mathcal{J}}(G)$ of all cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$. This motivates the following definition: The $G(\mathbb{A}_f)$-submodule

$$\begin{aligned} H^q_{cusp}(G,E) &:= H^q(\mathfrak{m}_G, K, \mathcal{A}_{cusp,\mathcal{J}}(G)\otimes E) \\ &= H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{G\}}(G)\otimes E) \end{aligned}$$

of $H^q(G,E)$ is called the *cuspidal cohomology of* G. An associate class φ_G degenerates to a singleton, represented by a unitary cuspidal automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$, trivial on $A^{\mathbb{R}}_G$. Hence, by Thm. 1 and [7] XIII, resp. more generally by [13] Thm. 2.3, we obtain the following well–known infinite direct sum decomposition as $G(\mathbb{A}_f)$-module

$$H^q_{cusp}(G,E) \cong \bigoplus_{\tilde{\pi}} H^q(\mathfrak{m}_G, K, \tilde{\pi}_\infty \otimes E)\otimes \tilde{\pi}_f^{m(\tilde{\pi})},$$

the sum ranging over all (isomorphism classes of) unitary cuspidal automorphic representations $\tilde{\pi}$ of $G(\mathbb{A})$.

4.3. Eisenstein cohomology. As it follows from Thm. 1, there is a $G(\mathbb{A}_f)$-invariant complement of $\mathcal{A}_{cusp,\mathcal{J}}(G)$ in $\mathcal{A}_{\mathcal{J}}(G)$, given by

$$\mathcal{A}_{Eis,\mathcal{J}}(G) := \bigoplus_{\{P\}\neq\{G\}} \mathcal{A}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{\{P\}\neq\{G\}}\bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G).$$

The subscript "Eis" shall allude to the fact, that each summand $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ may be constructed by means of Eisenstein series. In this regard, we define the *Eisenstein cohomology of* G to be the $G(\mathbb{A}_f)$-module

$$\begin{aligned} H^q_{Eis}(G,E) &:= H^q(\mathfrak{m}_G, K, \mathcal{A}_{Eis,\mathcal{J}}(G)\otimes E) \\ &\cong \bigoplus_{\{P\}\neq\{G\}}\bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E). \end{aligned}$$

See again Franke–Schwermer, [13] Thm. 2.3.

Remark 8. The Eisenstein cohomology, as defined above, differs in general from the "cohomology at infinity", a notion coined by G. Harder, cf. e.g., [19], [20]. This is due to the fact that there might be residual automorphic representations of G, which contribute non–trivially to Eisenstein cohomology, but restrict trivially to the boundary of the Borel–Serre compactification of $G(F)A^{\mathbb{R}}_G\backslash G(\mathbb{A})/K$. In this paper, we prefer to take the above "transcendental" point of view.

4.4. The final goal. It is the main aim of this article to identify a certain range of degrees q of cohomology, in which we can give a general description of the summands $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E)$ appearing in the decomposition of Eisenstein cohomology, by use of maximally residual Eisenstein series. Thus, serving as a general construction principle of residual Eisenstein cohomology for reductive groups.
To this end, it will be necessary to understand the cohomology of the consecutive filtration quotients $\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G)$, whose $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module structure was already described in Theorem 4. This needs a few preparatory results, which make up the contents of the next section.

5. Cohomology of filtration quotients

5.1. As a first step, we shall prove the following proposition. Its proof essentially consists in a careful exercise in using Wigner's Lemma.

Proposition 9. *Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be a cuspidal support. For $0\leq j\leq m$ and $(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, let $I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]$ be the attached induced representation. If $H^q(\mathfrak{m}_G, K, I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]\otimes E_\mu)$ is nonzero for some degree q, then $A^\circ_{R,\infty}$ acts trivially on*

$$\Pi\otimes E_{\mu_w}\otimes e^{\langle\lambda+\rho_R, H_R(.)\rangle},$$

where $w\in W^R$ is a uniquely determined Kostant representative.

Proof. By Borel–Wallach [7] III Thm. 3.3, $H^q(\mathfrak{m}_G, K, I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]\otimes E_\mu)$ being nonzero, implies that

$$H^{q-\ell(w)}(\mathfrak{l}_{R,\infty}\cap\mathfrak{m}_G, K_{L_{R,\infty}}, \Pi_\infty\otimes e^{\langle\lambda+\rho_R,H_R(.)|_{L_{R,\infty}}\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes E_{\mu_w})\neq 0,$$

for a uniquely determined Kostant representative $w\in W^R$. See also Franke [12], (2), p. 256. In general, $K_{L_{R,\infty}}$ will not be connected. However, by [7] I.5.1, also the $(\mathfrak{l}_{R,\infty}\cap\mathfrak{m}_G, K^\circ_{L_{R,\infty}})$-cohomology of the above coefficient module is non-vanishing in degree $q-\ell(w)$ and so, using [7] I.5.1 again, we see that under the present assumptions, also the relative Lie algebra cohomology

$$H^{q-\ell(w)}(\mathfrak{l}_{R,\infty}\cap\mathfrak{m}_G, \mathfrak{k}_{L_{R,\infty}}, \Pi_\infty\otimes e^{\langle\lambda+\rho_R,H_R(.)|_{\mathfrak{l}_{R,\infty}}\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes E_{\mu_w})$$

is non–zero. Now, let us write $\mathfrak{l}_{R,\infty}\cap\mathfrak{m}_G=\mathfrak{l}^{ss}_{R,\infty}\oplus(\mathfrak{a}_{R,\infty}\cap\mathfrak{m}_G)$ and set $\mathfrak{k}^{ss}_{L_R}:=\mathfrak{k}_{L_{R,\infty}}\cap\mathfrak{l}^{ss}_{R,\infty}$. Then the decomposition $\mathfrak{k}_{L_{R,\infty}}=\mathfrak{k}^{ss}_{L_R}\oplus(\mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}})$ is direct and hence, we may use the Künneth rule to obtain that the $(\mathfrak{a}_{R,\infty}\cap\mathfrak{m}_G, \mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}})$-cohomology of

$$\left(\omega_\Pi\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes\mathbb{C}_{\mu_w}\right)|_{\mathfrak{a}_{R,\infty}\cap\mathfrak{m}_G}$$

is non–zero in some degree. Here, ω_Π is the central character of Π and $\mathbb{C}_{\mu_w}$ is the one–dimensional representation of $\mathfrak{a}_{R,\infty}$ given by the weight $\mu_w\in\check{\mathfrak{h}}_\infty$. Since $\mathfrak{a}_G$ acts trivially on E_μ and Π, also

$$H^*\left(\mathfrak{a}_{R,\infty}, \mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}}\oplus\mathfrak{a}_G, \left(\omega_\Pi\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes\mathbb{C}_{\mu_w}\right)|_{\mathfrak{a}_{R,\infty}}\right)\neq 0 \tag{5.1}$$

Using the Künneth rule once more, we obtain that

$$H^*\left(\mathfrak{a}^G_R, \left(e^{\langle\lambda+\rho_R,H_R(.)\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes\mathbb{C}_{\mu_w}\right)|_{\mathfrak{a}^G_R}\right)\neq 0,$$

Since $S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$ is a polynomial algebra, this implies that

$$\mathrm{pr}_{\check{\mathfrak{h}}_\infty\to\check{\mathfrak{a}}^G_R}(\mu_w)=-\rho_R-\lambda, \tag{5.2}$$

cf. [12], p. 256, or otherwise put that $\lambda=-\mathrm{pr}_{\check{\mathfrak{h}}_\infty\to\check{\mathfrak{a}}^G_R}(w(\mu+\rho))$. In particular, we see that $\mathfrak{a}^G_R$ acts trivially on $\omega_\Pi\otimes E_{\mu_w}\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}$. Next, we observe that by (5.1), $\mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}}$ has to act trivially on the coefficients $\omega_\Pi\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes\mathbb{C}_{\mu_w}$. Therefore, any Lie algebra complement $\mathfrak{a}^{\mathsf{cpl}}_R$ of $\mathfrak{a}_R\oplus(\mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}})$ in $\mathfrak{a}_{R,\infty}$ has to act trivially, too, because $\mathfrak{a}_{R,\infty}$ is abelian. Collecting all that we obtain so far, we see that

$$\mathfrak{a}_{R,\infty}=\mathfrak{a}_G\oplus\mathfrak{a}^G_R\oplus(\mathfrak{a}_{R,\infty}\cap\mathfrak{k}_{L_{R,\infty}})\oplus\mathfrak{a}^{\mathsf{cpl}}_R$$

acts trivially on $\omega_\Pi\otimes E_{\mu_w}\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}$. This implies the assertion. □

5.2. A purity result. Proposition 9 implies a certain purity or rigidity result on the possible values of $-w_v(\mu_v+\rho_v)|_{\mathfrak{a}_{R,v}\cap\mathfrak{m}_G}$. Such a result was already proved by Harder for $G=GL_2/F$, see [18], and later on his arguments were used in Grbac–Grobner [15] for the case of $G=Sp_4$ over a totally real field. Here, we are going to use Clozel's "lemme de pureté", see [8], in order to derive the following result.

Proposition 10. *Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be a cuspidal support. For $0\leq j\leq m$ and $(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, let $I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]$ be the attached induced representation. If $H^q(\mathfrak{m}_G,K,I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]\otimes E_\mu)$ is nonzero for some degree q, then the attached, uniquely determined Kostant representative $w=(w_v)_{v\in S_\infty}\in W^R$ satisfies*

$$\mathrm{pr}_{\check{\mathfrak{h}}_v\to\check{\mathfrak{a}}^G_R}(w_v(\mu_v+\rho_v))=\mathrm{pr}_{\check{\mathfrak{h}}_{v'}\to\check{\mathfrak{a}}^G_R}(w_{v'}(\mu_{v'}+\rho_{v'}))$$

for all archimedean places $v,v'\in S_\infty$.

Proof. Since A_R is F-split, we may write $A_R=\prod_{k=1}^s GL_1$ as algebraic group over F. In this decomposition, we will also write $GL_1^{[k]}$ for the k-th factor $A_R^{[k]}$ of A_R. Similarly, $\mathbb{C}_{\mu_w}|_{A^\circ_{R,\infty}}$ breaks as a tensor product

$$\mathbb{C}_{\mu_w}|_{A^\circ_{R,\infty}}\cong\bigotimes_{v\in S_\infty}\bigotimes_{k=1}^{s}\mathbb{C}^{[k]}_{\mu_{w,v}},$$

$\mathbb{C}^{[k]}_{\mu_{w,v}}$ being the representation space of the character of $GL_1^{[k]}(F_v)^\circ$ given by its highest weight

$$\mu^{[k]}_{w,v}:=\mathrm{pr}_{\check{\mathfrak{h}}_v\to\check{\mathfrak{a}}^{[k]}_{R,v}}(\mu_{w_v}).$$

Recall that if v is complex, then $\mu^{[k]}_{w,v}=(\mu^{[k]}_{w,\iota_v},\mu^{[k]}_{w,\bar\iota_v})$. As μ is the highest weight of an algebraic representation, for all k, $1\le k\le s$, there is a positive integer $r(k)\in\mathbb{Z}_{>0}$, such that $(\mathbb{C}^{[k]}_{\mu_w})^{r(k)}=\mathbb{C}^{[k]}_{r(k)\mu_w}$ is algebraic as well. Let ω_Π be the central character of Π. It defines a character $\omega_\Pi: A_R(F)\backslash A_R(\mathbb{A})\to\mathbb{C}^*$, which we factor as $\omega_\Pi=\otimes_{k=1}^s\omega_\Pi^{[k]}$, where $\omega_\Pi^{[k]}$ is a Hecke character $GL_1^{[k]}(F)\backslash GL_1^{[k]}(\mathbb{A})\to\mathbb{C}^*$. Similarly, we may write

$$e^{\langle\lambda+\rho_R,H_R(.)\rangle}=\bigotimes_{k=1}^{s}\mathbb{C}^{[k]}_{\lambda+\rho_R},$$

where $\mathbb{C}^{[k]}_{\lambda+\rho_R}$ is a Hecke character $GL_1^{[k]}(F)\backslash GL_1^{[k]}(\mathbb{A})\to\mathbb{C}^*$. By Proposition 9, we obtain that

$$H^0(\mathfrak{a}_{R,\infty},A^\circ_{R,\infty},\omega_{\Pi_\infty}\otimes e^{\langle\lambda+\rho_R,H_R(.)\rangle}\otimes\mathbb{C}_{\mu_w})\neq 0.$$

This implies, using the Künneth rule, that

$$H^0(\mathfrak{gl}^{[k]}_{1,\infty},(GL^{[k]}_{1,\infty})^\circ,\omega^{[k]}_{\Pi_\infty}\otimes\mathbb{C}^{[k]}_{\lambda+\rho_R}\otimes\mathbb{C}^{[k]}_{\mu_w})\neq 0$$

and hence also

$$H^0\left(\mathfrak{gl}^{[k]}_{1,\infty},(GL^{[k]}_{1,\infty})^\circ,\left(\omega^{[k]}_{\Pi_\infty}\otimes\mathbb{C}^{[k]}_{\lambda+\rho_R}\right)^{r(k)}\otimes\mathbb{C}^{[k]}_{r(k)\mu_w}\right)\neq 0$$

for all $1\le k\le s$. Therefore, by [8] Lemme 3.14, $(\omega^{[k]}_\Pi\otimes\mathbb{C}^{[k]}_{\lambda+\rho_R})^{r(k)}$ is a regular algebraic cuspidal automorphic representation of $GL_1^{[k]}/F$ in the sense of [8] Def. 3.12. Let $\{\sigma_1,...,\sigma_r\}$ be the set of real places and $\{\tau_1,...,\tau_c\}$ the set of complex places. Clozel's "Lemme de pureté", [8] Lemme 4.9 now implies that for all k

$$2\mu^{[k]}_{w,\sigma_1}=2\mu^{[k]}_{w,\sigma_2}=...=2\mu^{[k]}_{w,\sigma_r}=\mu^{[k]}_{w,\iota_{\tau_1}}+\mu^{[k]}_{w,\bar\iota_{\tau_1}}=\mu^{[k]}_{w,\iota_{\tau_2}}+\mu^{[k]}_{w,\bar\iota_{\tau_2}}=...=\mu^{[k]}_{w,\iota_{\tau_c}}+\mu^{[k]}_{w,\bar\iota_{\tau_c}}.$$

Here, we already divided by $r(k)\neq 0$. In particular, recalling that $\mu^{[k]}_{w,v}=\mathrm{pr}_{\check{\mathfrak{h}}_v\to\check{\mathfrak{a}}^{[k]}_{R,v}}(\mu_{w_v})$, we obtain

$$\mathrm{pr}_{\check{\mathfrak{h}}_v\to\check{\mathfrak{a}}^G_R}(\mu_{w_v})=\mathrm{pr}_{\check{\mathfrak{h}}_{v'}\to\check{\mathfrak{a}}^G_R}(\mu_{w_{v'}})$$

for all archimedean places $v,v'\in S_\infty$. Since $\rho=(\rho_v)_{v\in S_\infty}$ has repeating coordinates in the real and complex places respectively, the result follows. □

Corollary 11. *With the assumptions of Proposition 10, there are the following identities over all places $v\in S_\infty$:*

$$\lambda=\begin{cases}-w_v(\mu_v+\rho_v)|_{\mathfrak{a}_{R,v}\cap\mathfrak{m}_G} & \forall v\quad\text{real}\\ -\frac{1}{2}(w_{\iota_v}(\mu_{\iota_v}+\rho_{\iota_v})|_{\mathfrak{a}_{R,\iota_v}\cap\mathfrak{m}_G}+w_{\bar\iota_v}(\mu_{\bar\iota_v}+\rho_{\bar\iota_v})|_{\mathfrak{a}_{R,\bar\iota_v}\cap\mathfrak{m}_G}) & \forall v\quad\text{complex}\end{cases}$$

Recall that by the definition of the quadruples $(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, the parameter $\Re e(\lambda)$ is in the closure of the positive Weyl chamber $\overline{\check{\mathfrak{a}}^{G+}_R}$. Since by Corollary 11, λ is necessarily real valued in order to give rise to a quadruple (R,Π,ν,λ) whose attached induced representation has non-trivial $(\mathfrak{m}_G,K)$-cohomology with respect to E_μ, we obtain that $\lambda\in\overline{\check{\mathfrak{a}}^{G+}_R}$. This, together with the purity property of the coordinates of $\mathrm{pr}_{\check{\mathfrak{h}}_\infty\to\check{\mathfrak{a}}^G_R}(w(\mu+\rho))$ in the archimedean places v, cf. Proposition 10, yields serious restrictions on the Kostant representatives $w=(w_v)_{v\in S_\infty}\in W^R$. This will be made precise in the next

Proposition 12. *Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be a cuspidal support. For $0\le j\le m$ and $(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, let $I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]$ be the attached induced representation. If $H^q(\mathfrak{m}_G,K,I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}),\lambda]\otimes E_\mu)$ is nonzero for some degree q, then the attached, uniquely determined $w=(w_v)_{v\in S_\infty}\in W^R$ satisfies*

$$\ell(w)\ge\sum_{v\in S_\infty}\left\lceil\tfrac{1}{2}\dim_{\mathbb{R}}N_R(F_v)\right\rceil,$$

where $\lceil x\rceil$ denotes the smallest integer greater or equal to x.

Proof. It is enough to show that locally $\ell(w_v)\ge\frac{1}{2}\dim_{\mathbb{R}}N_R(F_v)$ for all archimedean places $v\in S_\infty$. Therefore, recall that in the course of the proof of Proposition 9 we have shown that

$$H^{q-\ell(w)}(\mathfrak{l}_{R,\infty}\cap\mathfrak{m}_G,\mathfrak{k}_{L_{R,\infty}},\Pi_\infty\otimes e^{\langle\lambda+\rho_R,H_R(.)|_{\mathfrak{l}_{R,\infty}}\rangle}\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})\otimes E_{\mu_w})\neq 0.$$

Writing $\mathfrak{l}_{R,\infty} \cap \mathfrak{m}_G = \mathfrak{l}^{ss}_{R,\infty} \oplus (\mathfrak{a}_{R,\infty} \cap \mathfrak{m}_G)$ and $\mathfrak{k}^{ss}_{L_R} = \mathfrak{k}_{L_{R,\infty}} \cap \mathfrak{l}^{ss}_{R,\infty}$ and using the Künneth rule, it follows that

$$H^r(\mathfrak{l}^{ss}_{R,\infty}, \mathfrak{k}^{ss}_{L_R}, \Pi_\infty|_{\mathfrak{l}^{ss}_{R,\infty}} \otimes E_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}}) \neq 0$$

for some degree r. The Lie algebra $\mathfrak{l}^{ss}_{R,\infty}$ is reductive, Π_∞ defines a unitary representation of $L^\circ_{R,\infty}$ by restriction and $E_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}}$ is the finite multiple of an irreducible representation (since $L_{R,\infty}$ needs not to be connected). Hence, by Borel–Wallach [7] I, Cor. 4.2 and Borel–Casselman [4], Lem. 1.3, we must have

$$E_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}} \cong \bar{E}^{\vee}_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}},$$

where $\bar{E}^{\vee}_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}}$ denotes the complex conjugate, contragredient representation of the $\mathfrak{l}^{ss}_{R,\infty}$-module $E_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}}$. In particular, we obtain

$$E_{\mu_{w_v}}|_{\mathfrak{l}^{ss}_{R,v}} \cong \bar{E}^{\vee}_{\mu_{w_v}}|_{\mathfrak{l}^{ss}_{R,v}} \tag{5.3}$$

for all archimedean places $v \in S_\infty$. Without loss of generality, we may assume that the latter representations are irreducible. Furthermore, by the definition of λ and Corollary 11,

$$\lambda_{\mu_{w_v}} := -w_v(\mu_v + \rho_v)|_{\mathfrak{a}_{R,v} \cap \mathfrak{a}^G_R} \in \overline{\check{\mathfrak{a}}^{G+}_R} \tag{5.4}$$

where we identify $\mathfrak{a}^G_R$ and $\overline{\check{\mathfrak{a}}^{G+}_R}$ with its image in $\mathfrak{a}_{R,v}$ and $\check{\mathfrak{a}}_{R,v}$, respectively. As a last ingredient, recall the involution $w_v \mapsto w'_v$ on W^{R_v} from [7], V.1.4: If w_{G_v} (resp. $w_{L_{R,v}}$) is the longest element of the Weyl group $W(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$ (resp. $W(\mathfrak{l}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$) then

$$w'_v := w_{L_{R,v}} w_v w_{G_v} \quad \text{and} \quad \ell(w_v) + \ell(w'_v) = \dim_{\mathbb{R}} N_R(F_v). \tag{5.5}$$

Let $\mu^{\vee}_v$ be the highest weight of the representation contragredient to E_μ, i.e., $\mu^{\vee}_v = -w_{G_v}(\mu_v)$. Hence, the first line of p. 153 in [7] implies that

$$\lambda_{\mu_{w_v}} = -\lambda_{\mu^{\vee}_{w'_v}}. \tag{5.6}$$

In particular, $\lambda_{\mu_{w_v}}$ is in the closure of a Weyl chamber C, if and only if $\lambda_{\mu^{\vee}_{w'_v}}$ is in the closure of $-C$.

We claim that (5.3) and (5.4) imply the result. Indeed, if we let $\Psi_{w_v} := w_v(-\Delta^+(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})) \cap \Delta^+(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$, then it is well–known that $\ell(w_v) = |\Psi_{w_v}|$. By (5.5) it is hence enough to show that $|\Psi_{w_v}| \geq |\Psi_{w'_v}|$. Therefore, let $\alpha \in \Delta^+(\mathfrak{g}_{v,\mathbb{C}}, \mathfrak{h}_{v,\mathbb{C}})$ be a positive absolute root. Then $w_v^{-1}(\alpha)$ is again a root and since $\mu_v + \rho_v$ is a regular dominant weight, it is straight forward to see that $\alpha \in \Psi_{w_v}$ if and only if $\langle w_v(\mu_v + \rho_v), \alpha \rangle \leq 0$. Of course, the same holds for w_v being replaced by w'_v and $\mu_v + \rho_v$ being replaced by the regular dominant weight $\mu^{\vee}_v + \rho_v$. We may decompose the latter inner product for w'_v as

$$\langle w'_v(\mu^{\vee}_v + \rho_v), \alpha \rangle = \langle w'_v(\mu^{\vee}_v + \rho_v)|_{\mathfrak{l}^{ss}_{R,v}}, \alpha \rangle + \langle w'_v(\mu^{\vee}_v + \rho_v)|_{\mathfrak{a}_{R,v} \cap \mathfrak{a}^G_R}, \alpha \rangle + \langle w'_v(\mu^{\vee}_v + \rho_v)|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}}, \alpha \rangle.$$

By (5.4) and (5.6), the second summand on the right hand side is non-negative. Therefore, if $\alpha \in \Psi_{w'_v}$, then

$$\langle w'_v(\mu^{\vee}_v + \rho_v)|_{\mathfrak{l}^{ss}_{R,v}}, \alpha \rangle + \langle w'_v(\mu^{\vee}_v + \rho_v)|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}}, \alpha \rangle \leq 0.$$

Since both of these summands are real-valued, the left hand side of the latter inequality equals

$$\langle \overline{w'_v(\mu^{\vee}_v + \rho_v)}|_{\mathfrak{l}^{ss}_{R,v}}, \overline{\alpha} \rangle + \langle \overline{w'_v(\mu^{\vee}_v + \rho_v)}|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}}, \overline{\alpha} \rangle.$$

By (5.3) and the definition of w'_v,

$$\overline{w'_v(\mu^{\vee}_v + \rho_v)}|_{\mathfrak{l}^{ss}_{R,v}} = w_v(\mu_v + \rho_v)|_{\mathfrak{l}^{ss}_{R,v}}. \tag{5.7}$$

Moreover,

$$\begin{aligned}
\overline{w'_v(\mu^{\vee}_v + \rho_v)}|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}} &= \overline{w_{L_{R,v}} w_v w_{G_v}(\mu^{\vee}_v + \rho_v)}|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}} \\
&= -\overline{w_{L_{R,v}} w_v(\mu_v + \rho_v)}|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}} \\
&= -\overline{w_v(\mu_v + \rho_v)}|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}} \\
&= w_v(\mu_v + \rho_v)|_{\mathfrak{a}_{R,v} \cap \mathfrak{t}_{L_{R,v}}}.
\end{aligned}$$

Here, the first line is the definition of w'_v; the second line follows from the equalities $\mu^{\vee}_v = -w_{G_v}(\mu_v)$ and $\rho_v = -w_{G_v}(\rho_v)$; the third line is a consequence of the fact that $w_{L_{R,v}}$ operates trivially on $\mathfrak{a}_{R,v}$; and the forth line follows from $\mathfrak{a}_{R,v} \cap \mathfrak{k}_{L_{R,v}}$ being compact, cf. [4] 1.2. Hence, summarizing what we obtained so far, if $\alpha \in \Psi_{w'_v}$, then

$$\langle w_v(\mu_v + \rho_v)|_{\mathfrak{l}^{ss}_{R,v}}, \overline{\alpha}\rangle + \langle w_v(\mu_v + \rho_v)|_{\mathfrak{a}_{R,v}\cap\mathfrak{k}_{L_{R,v}}}, \overline{\alpha}\rangle \leq 0.$$

But since $\langle w_v(\mu_v + \rho_v)|_{\mathfrak{a}_{R,v}\cap\mathfrak{a}^G_R}, \overline{\alpha}\rangle \leq 0$ by (5.4), we have proved that

$$\alpha \in \Psi_{w'_v} \Rightarrow \overline{\alpha} \in \Psi_{w_v}.$$

As a consequence,

$$\ell(w'_v) = |\Psi_{w'_v}| \leq |\Psi_{w_v}| = \ell(w_v)$$

and by what we observed above, cf. (5.5), this implies the result. □

5.3. Proposition 12 implies the following proposition on the potential degrees where an induced representation $I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda]$, attached to a quadruple (R, Π, ν, λ), may have non-trivial $(\mathfrak{m}_G, K)$-cohomology. Therefore, given an irreducible, unitary $L_{R,v}$–representation Π_v, let $m(L_{R,v}, \Pi_v)$ be the smallest degree, in which Π_v has non–trivial $(\mathfrak{l}^{ss}_{R,v}, \mathfrak{k}^{ss}_{L_{R,v}})$-cohomology, twisted by an irreducible, finite-dimensional, algebraic representation of $L_{R,v}$. If there is no such coefficient module, then we let $m(L_{R,v}, \Pi_v) = 0$. Then we obtain

Proposition 13. *Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be a cuspidal support. For $0 \leq j \leq m$ and $(R, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, let $I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda]$ be the attached induced representation. If $H^q(\mathfrak{m}_G, K, I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda] \otimes E_\mu)$ is non-zero in degree q, then*

$$q \geq \sum_{v\in S_\infty} \left(\left\lceil \tfrac{1}{2}\dim_{\mathbb{R}} N_R(F_v)\right\rceil + m(L_{R,v}, \Pi_v)\right),$$

where $\lceil x\rceil$ denotes the smallest integer greater or equal to x.

Proof. In the course of the proof of Proposition 9 we have shown that

$$H^{q-\ell(w)}(\mathfrak{l}_{R,\infty} \cap \mathfrak{m}_G, \mathfrak{k}_{L_{R,\infty}}, \Pi_\infty \otimes e^{\langle\lambda+\rho_R, H_R(.)|_{\mathfrak{l}_{R,\infty}}\rangle} \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}) \otimes E_{\mu_w}) \neq 0.$$

Moreover, we have for each archimedean place $v \in S_\infty$, $H^{r_v}(\mathfrak{l}^{ss}_{R,v}, \mathfrak{k}^{ss}_{L_{R,v}}, \Pi_v|_{\mathfrak{l}^{ss}_{R,v}} \otimes E_{\mu_{w_v}}|_{\mathfrak{l}^{ss}_{R,v}}) \neq 0$ for some degree r_v. By its definition, necessarily $r_v \geq m(L_{R,v}, \Pi_v)$. In particular, the Künneth rule implies that

$$\sum_{v\in S_\infty} m(L_{R,v}, \Pi_v) \leq q - \ell(w).$$

The assertion now follows from Proposition 12. □

In contrast to the computation of the dimensions $\dim_{\mathbb{R}} N_R(F_v)$, in practise it may be tedious to calculate the numbers $m(L_{R,v}, \Pi_v)$. The next corollary, which is a direct consequence of the last proposition, provides an alterative lower bound, which is weaker than the one given in Prop. 13, but may be more convenient in calculations. See also Theorem 18 later on.

Corollary 14. *Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be a cuspidal support. For $0 \leq j \leq m$ and $(R, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, let $I_{R(\mathbb{A})}[\Pi\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda]$ be the attached induced representation. If $H^q(\mathfrak{m}_G, K, I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda] \otimes E_\mu)$ is non-zero in degree q, then*

$$q \geq \sum_{v\in S_\infty} \left\lceil \tfrac{1}{2}\dim_{\mathbb{R}} N_R(F_v)\right\rceil.$$

Proof. By definition, $m(L_{R,v}, \Pi_v) \geq 0$. Therefore, the corollary follows from Proposition 13. □

6. The main result

6.1. Definition of the bound q_{res}. In order to state the main theorem of this paper, we need a certain constant $q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$, depending on a pair of supports $(\{P\}, \varphi_P)$, as a last ingredient. So, let $\{P\}$ be a given associate class of proper parabolic F-subgroups $\{P\}$ of G, and φ_P an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$ and let $0 \leq j \leq m = m(\{P\})$. As a first step, for a quadruple $(R, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, we define

$$q_{\mathsf{res},j}((R, \Pi, \nu, \lambda)) := \sum_{v \in S_\infty} \left(\left\lceil \tfrac{1}{2} \dim_{\mathbb{R}} N_R(F_v) \right\rceil + m(L_{R,v}, \Pi_v) \right).$$

and set

$$q_{\mathsf{res},j}(\{P\}, \varphi_P) := \min_{(R,\Pi,\nu,\lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}} q_{\mathsf{res},j}((R, \Pi, \nu, \lambda)).$$

Finally, the constant $q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$, mentioned above, is defined as

$$q_{\mathsf{res}} := \min_{0 \leq j < m} q_{\mathsf{res},j}(\{P\}, \varphi_P).$$

Observe that we assume j to be strictly smaller than m. We have now accomplished the preparatory work in order to prove the main result of this paper.

Theorem 15. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $m = m(\{P\})$ be the length of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ as defined in Section 3.2. Then, the map in cohomology, induced from the natural inclusion $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow[Eis^q_{\mathcal{J},\{P\},\varphi_P}]{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$$

for all degrees $0 \leq q < q_{\mathsf{res}}$.
In other words, the Eisenstein cohomology supported in $(\{P\}, \varphi_P)$ is entirely given by the $(\mathfrak{m}_G, K)$-cohomology of the m-th filtration step of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$ in all degrees $0 \leq q < q_{\mathsf{res}}$.

Proof. For each $0 \leq j < m$, we obtain a short exact sequence of $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules

$$0 \longrightarrow \mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \longrightarrow \mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G) \longrightarrow \mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \longrightarrow 0.$$

It induces an long exact sequence of $G(\mathbb{A}_f)$-modules in $(\mathfrak{m}_G, K)$-cohomology:

$$\ldots \to H^q(\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \longrightarrow H^q(\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \longrightarrow H^q(\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \to \ldots,$$

where we abbreviated $H^q(V \otimes E_\mu) := H^q(\mathfrak{m}_G, K, V \otimes E_\mu)$ for V a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module. By Thm. 4, there is an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)/\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \cong \bigoplus_{(R,\Pi,\nu,\lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}} H^q(I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda] \otimes E_\mu)^{m(\Pi)}. \tag{6.1}$$

Now, by our Proposition 13, the right hand side of (6.1) vanishes if $q < q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$. Therefore, for all $0 \leq j < m$ and $q < q_{\mathsf{res}}$, there is an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow{\sim} H^q(\mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu).$$

By construction, it is induced from the natural inclusion $\mathcal{A}^{(j+1)}_{\mathcal{J},\{P\},\varphi_P}(G) \hookrightarrow \mathcal{A}^{(j)}_{\mathcal{J},\{P\},\varphi_P}(G)$. As $\mathcal{A}^{(0)}_{\mathcal{J},\{P\},\varphi_P}(G) = \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, the result follows. $\square$

Before we comment on the consequences of our main theorem at length, let us state the following immediate two corollaries:

Corollary 16. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $m = m(\{P\})$ be the length of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ as defined in Section 3.2. Then, there is an isomorphism of $G(\mathbb{A}_f)$-modules,*

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \cong \bigoplus_{(R,\Pi,\nu,\lambda)\in M^{(m)}_{\mathcal{J},\{P\},\varphi_P}} H^q(\mathfrak{m}_G, K, I_{R(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \lambda] \otimes E_\mu)^{m(\Pi)},$$

in all degrees $q < q_{\mathsf{res}}$, giving rise to a direct sum decomposition of the Eisenstein cohomology supported in $(\{P\}, \varphi_P)$. If $m(\{P\}) = 0$, then the above decomposition even holds for all degrees q.

Proof. This is a direct consequence of Theorems 4 and 15. See also Remark 6. □

Corollary 17. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $\Pi \hookrightarrow \mathcal{A}_{res,\mathcal{J}}(G)$ be a residual automorphic representation of $G(\mathbb{A})$ with cuspidal support $\pi \in \varphi_P$, spanned by iterated residues of Eisenstein series at a point $\nu \in \check{\mathfrak{a}}^G_{P,\mathbb{C}}$, for which $\nu + \chi_{\tilde{\pi}}$ is annihilated by $\mathcal{J}$. Let $m(\Pi)$ be its multiplicity in $\mathcal{A}_{dis,\mathcal{J}}(G) \cap \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Then, the map in cohomology*

$$H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)^{m(\Pi)} \longrightarrow H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu),$$

induced from the natural inclusion $\Pi^{m(\Pi)} \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is injective in all degrees $0 \leq q < q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$.

Proof. By our assumptions, $(G, \Pi, \nu, 0)$ is an element of $M^{(m)}_{\mathcal{J},\{P\},\varphi_P}$, $m = m(\{P\})$ the length of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$. Hence, the corollary follows from Theorems 4 and 15 (or directly from Corollary 16). □

7. Consequences and Comments

7.1. The bound q_{res} and L^2-cohomology.

7.1.1. *A simplification.* Our first remark deals with the constant $q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$. Although it maybe seems to be rather complicated in its nature, since it involves quite refined data attached to the quadruples (R, Π, ν, λ), it is not too difficult to make it explicit in many cases. See, e.g., Grbac–Grobner [15], Sections 3 and 4, for the case $G = Sp_4$ over a totally real number field; or Franke–Schwermer [13], Sect. 5, for the case $G = GL_n/\mathbb{Q}$ and $\{P\}$ being represented by a maximal parabolic $\mathbb{Q}$-subgroup.
In the general case, q_{res} can always be bounded from below by the weaker bound

$$q_{\mathsf{alt}} := \min_{0\leq j<m} \min_{(R,\Pi,\nu,\lambda)\in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}} \left\lceil \tfrac{1}{2} \dim_{\mathbb{R}} N_R(F_v) \right\rceil.$$

This is clear be the definition of q_{res} and Corollary 14. Even simpler, the following weaker, but more feasible version of our main theorem holds:

Theorem 18. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $L^2_{\mathcal{J},\{P\},\varphi_P} := \mathcal{A}_{res,\mathcal{J}}(G) \cap \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$ be the space of square-integrable (and hence necessarily residual) automorphic forms in $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$. Then, the inclusion $L^2_{\mathcal{J},\{P\},\varphi_P}(G) \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ induces an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{m}_G, K, L^2_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow[Eis^q_{\mathcal{J},\{P\},\varphi_P}]{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu),$$

in all degrees

$$q < q_{\mathsf{max}} := \min_{\substack{R \text{ maximal} \\ R\supseteq P}} \left(\sum_{v\in S_\infty} \left\lceil \tfrac{1}{2} \dim_{\mathbb{R}} N_R(F_v) \right\rceil \right).$$

Proof. Let q_{max} be as in the statement of the theorem. We first show that $R = G$ never appears as the first component of a $(R, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$ for $j \neq m$: Arguing by contraposition, if $R = G$, then by its very definition $\lambda = 0$, since it has to be in $\check{\mathfrak{a}}^G_{G,\mathbb{C}} = 0$. As we also have $j = T(\Re e(\lambda)_+) = T(0)$, we must have $j = m$ be the definition of T. This is a contradiction. Hence, if $m = 0$, then the theorem follows directly from Theorem 4, applied to $j = m$, and Proposition 12. If $m > 0$, then by what we just saw, $0 < q_{\mathsf{max}} \leq q_{\mathsf{res}}$. So, our main theorem, Theorem 15, shows that

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$$

for $q < q_{\mathsf{max}}$. The result now follows again from Theorem 4, applied to $j = m$, and Proposition 12. □

Remark 19. Let r_G be the constant, introduced in Vogan–Zuckerman [29], 8, Kumaresan [21] and Enright [10]. *In most cases, already q_{max} is strictly greater than r_G.* See Theorem 24 for a family of examples. This shall underline the profitableness of Theorem 18 in practical use.

7.1.2. *The bound is "sharp".* As another important fact on q_{res}, let us point out that – in this generality – q_{res} establishes a sharp upper bound for the range of degrees q, where Eisenstein cohomology supported in $(\{P\}, \varphi_P)$ is entirely given by the $(\mathfrak{m}_G, K)$-cohomology of the deepest filtration step of $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Here "sharp" is meant in the way that there is a choice of a reductive group G/F, a coefficient system E_μ and of a pair of supports $(\{P\}, \varphi_P)$, such that $Eis^q_{\mathcal{J},\{P\},\varphi_P}$ is not an isomorphism for $q = q_{\mathsf{res}}$.
As an example, this can already be seen by taking $G = SL_2/\mathbb{Q}$, $E = \mathbb{C}$, $P = B$ the Borel subgroup and $\varphi_B = \{\mathbf{1}_{T(\mathbb{A})}\}$ the associate class represented by the trivial character of the torus T. Then $m = 1$ and $q_{\mathsf{res}} = \dim_{\mathbb{R}} U(\mathbb{R}) = 1$. It is well-known (but can also be seen directly by considering the long exact sequence in the proof of Thm. 15) that $H^1_{Eis}(G, \mathbb{C})$ is spanned by so-called regular Eisenstein cohomology classes. One has

$$H^1(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{B\},\varphi_B}(G)) \hookrightarrow H^1(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathcal{A}_{\mathcal{J},\{B\},\varphi_B}(SL_2)/\mathbf{1}_{G(\mathbb{A})}).$$

For a more complicated example in this direction, the reader may have a look at Grbac-Grobner [15], Thm. 5.1 and 5.4, which deal with the Eisenstein cohomology of $G = Sp_4$ over a totally real number field F (having made Franke's filtration explicit before).

7.2. A theorem of Rohlfs-Speh.

7.2.1. In [26], Rohlfs–Speh considered the contribution of certain automorphic subrepresentation Π of $\mathcal{A}_{res,\mathcal{J}}(G)$ for a semisimple algebraic group over $\mathbb{Q}$ to $H^*_{Eis}(G, \mathbb{C})$. They show that under certain constraints on Π, to be made precise below, the inclusion $\Pi \hookrightarrow \mathcal{A}_{Eis,\mathcal{J}}(G)$ induces a non-trivial map

$$H^{q_1}(\mathfrak{m}_G, K, \Pi) \to H^{q_1}_{Eis}(G, \mathbb{C})$$

in the lowest degree q_1 where $H^q(\mathfrak{g}, K, \Pi_\infty)$ is non-zero. See [26] Thm. I.1 and Thm. III.1. To explain their assumptions on Π, let $M(w, \pi)$ be the intertwining operator defined in Mœglin-Waldspurger [24] II.1.6 attached to a cuspidal support $\pi \in \varphi_P$ and $w \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$. As usual, the latter space is the set of all linear maps $\mathfrak{a}_P \to \mathfrak{a}_{P'}$ which are given by conjugation by an element $\tilde{w} \in G(F)$. In order to obtain their result, Rohlfs-Speh have to assume that Π_∞ is the image of the archimedean component of the normalized intertwining operator $N(w_0, \pi) = r(w_0, \pi)^{-1} M(w_0, \pi)$ attached to the longest element $w_0 \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$ and a cuspidal automorphic representation π of $L_P(\mathbb{A})$ whose unitary factor $\tilde{\pi}$ is tempered at the archimedean component.

7.2.2. In view of what we said above, our Theorem 15 and its Corollary 17 provide a generalization as well as a refinement of the result of Rohlfs-Speh, if $q_1 < q_{\mathsf{res}}$. Indeed, our main result, Theorem 15, and its Corollary 17 may be applied to all residual automorphic representations $\Pi \hookrightarrow \mathcal{A}_{res,\mathcal{J}}(G)$ of a reductive group G/F and say that the cohomology $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)^{m(\Pi)}$ even injects into $H^q_{Eis}(G, E_\mu)$ in all degrees $q_1 \leq q < q_{\mathsf{res}}$. In particular, the restriction that Π_∞ is the image of a residual Eisenstein intertwining operator attached to a pair (π, w_0), π_∞ being tempered and w_0 being the longest element in $\Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$, can be dropped. Moreover, we allow general coefficient modules E_μ.

7.3. A theorem of Li-Schwermer.

7.3.1. In [23], Li-Schwermer proved a vanishing result for the Eisenstein cohomology of a reductive group $G/\mathbb{Q}$ in the case of a regular coefficient system E_μ. More precisely, let G be a connected reductive group over $\mathbb{Q}$ and suppose that E is an irreducible, finite-dimensional, algebraic representation of $G(\mathbb{C})$ on a complex vector space, whose highest weight is regular. Let $q_0(G(\mathbb{R})) = \frac{1}{2}(\dim_\mathbb{R}(G(\mathbb{R})) - \dim_\mathbb{R}(K) - (rk_\mathbb{C}(G(\mathbb{R})) - rk_\mathbb{C}(K)))$, $rk_\mathbb{C}$ being the absolute rank of the group in question. Then Li-Schwermer show that for any pair of supports $(\{P\}, \varphi_P)$, $P \neq G$,

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) = 0$$

for all degrees $0 \leq q < q_0(G(\mathbb{R}))$. See [23], Thm. 5.5.
If one adapts the proof of our main theorem to regular coefficients, then one obtains an alternative approach to the theorem of Li-Schwermer. Indeed, if E_μ is a regular highest weight representation as in Section 1.3, then the archimedean component Π_∞ of any discrete series automorphic representation Π appearing in a quadruple $(R, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$ for $0 \leq j \leq m$, which satisfies

$$H^*(\mathfrak{l}^{ss}_{R,\infty}, \mathfrak{k}^{ss}_{L_R}, \Pi_\infty|_{\mathfrak{l}^{ss}_{R,\infty}} \otimes E_{\mu_w}|_{\mathfrak{l}^{ss}_{R,\infty}}) \neq 0$$

must be essentially tempered. This follows from the regularity of E_{μ_w} (which is a consequence of the regularity of E_μ, see [27] Lem. 4.9) and Vogan–Zuckerman's condition [29] (5.1), p. 73, together with the last paragraph on p. 58, *ibidem*. Hence, by Wallach [30], Thm. 4.3, resp. Clozel [9], Prop. 4.10, Π is cuspidal and so $R = P$.

On the other hand, for Π_∞ being essentially tempered, the bound of vanishing in Proposition 12 may be improved to $\sum_{v \in S_\infty} \lceil \frac{1}{2}(\dim_\mathbb{R} N_{P,v} + rk_\mathbb{C}(K_v) - rk_\mathbb{C}(K_{L^{ss}_{P,v}})) \rceil$, see [23] (4.1). This, together with an easy calculation using the Cartan decomposition of G_∞ and [7] III, Prop. 5.3, shows that for all quadruples $(P, \Pi, \nu, \lambda) \in M^{(j)}_{\mathcal{J},\{P\},\varphi_P}$, $0 \leq j \leq m$,

$$H^q(\mathfrak{m}_G, K, I_{P(\mathbb{A})}[\Pi \otimes S(\check{\mathfrak{a}}^G_{P,\mathbb{C}}), \lambda] \otimes E_\mu) = 0 \quad \text{for } q < q_0(G_\infty).$$

In particular, the proof of Theorem 15 now shows that

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$$

in all degrees $q < q_0(G_\infty)$. However, by the description of the $(\mathfrak{m}_G, K)$-cohomology of the deepest filtration step $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G)$ provided by our Theorem 4, we must have

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) = 0$$

in degrees $q < q_0(G_\infty)$ as well. See also Remark 6. Hence, the claim follows.

7.3.2. As a consequence, our Theorem 15 may also be viewed as a generalisation of a weak version of the vanishing theorem of Li–Schwermer, applying to all coefficient systems E_μ. More precisely, we obtain

Theorem 20. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of proper parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let m be the length of the filtration of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ as defined in Section 3.2 and assume that $H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) = 0$ in degrees $0 \leq q < q'$. Then also*

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) = 0$$

for all degrees $0 \leq q < \min(q', q_{\mathsf{res}})$.

7.4. A theorem of Franke and a theorem of Borel.

7.4.1. In [11], Franke described the contribution of the trivial residual automorphic representation $\mathbf{1}_{G(\mathbb{A})}$ of $G(\mathbb{A})$ to Eisenstein cohomology for a connected, reductive algebraic group $G/\mathbb{Q}$, improving a result of Borel, [3] Thm. 7.5. Implicit in his general construction is the fact that the natural inclusion $\mathbf{1}_{G(\mathbb{A})} \hookrightarrow \mathcal{A}_{Eis,\mathcal{J}}(G)$ defines an injective map

$$J^q : H^q(\mathfrak{m}_G, K, \mathbf{1}_{G(\mathbb{A})}) \to H^q_{Eis}(G, \mathbb{C})$$

in all degrees $q \leq \min_{R \text{ maximal}}(\dim_\mathbb{R} N_{R,\infty})$. This follows from [11] (7.2), p. 59. In particular, J^q is injective for all degrees $q < q_{\mathsf{max}}$. Since in this special case of the trivial residual representation, $q_{\mathsf{max}} = q_{\mathsf{res}}$, our

Theorem 15 – or more explicitly Corollary 17 – applied to $\Pi = \mathbf{1}_{G(\mathbb{A})}$ is compatible with Franke's theorem. As a remark, let us also point out that Corollary 17 independently improves Borel's above mentioned result: This can already be seen for $G = Sp_4$ over a totally real number field, cf. [15] Cor. 6.1.

8. Applications I: Eisenstein Cohomology of Inner Forms of GL_n

8.1. In this section we would like to apply our Theorem 15, in order to derive a result on the contribution of the residual automorphic representations to the Eisenstein cohomology of inner forms of the general linear group over a number field F.

Let D be a central division-algebra over a number field F of index d, i.e., $d^2 = \dim_F D$. The local algebras $D_v = D \otimes_F F_v$ are central simple algebras over F_v and hence isomorphic to a matrix algebra $M_{r_v}(A_v)$, for some integer $r_v \geq 1$ and a central division algebra A_v over F_v. The algebra D is said to be split at v if $A_v = F_v$ and non-split at v otherwise, i.e., A_v is not a field. Analogous to the global situation, let d_v be the index of D_v, i.e., $d_v^2 = \dim_{F_v} A_v$. Then $r_v d_v = d$ for all v. If $v \in S_\infty$ is real then $d_v \in \{1, 2\}$, i.e., $A_v = \mathbb{R}$ or $\mathbb{H}$ and $D_v = M_d(\mathbb{R})$ if v is split and $M_{d/2}(\mathbb{H})$ is v is non-split (in which case d is even). Given any $n \geq 1$ we set $\ell := nd/2$.

The determinant $\det'$ of an $n \times n$-matrix $X \in M_n(D)$, $k \geq 1$, is the generalization of the reduced norm to matrices: $\det'(X) := \det(\varphi(X \otimes 1))$, for some isomorphism $\varphi : M_n(D) \otimes_F \overline{\mathbb{Q}} \xrightarrow{\sim} M_{dn}(\overline{\mathbb{Q}})$. It is independent of φ and is an F-rational polynomial in the coordinates of the entries of X. So the group

$$G(F) := GL'_n(F) := \{X \in M_n(D) | \det'(X) \neq 0\}$$

defines an algebraic group GL'_n over F. It is reductive and is an inner F-form of the split group GL_{dn}/F. At a real place $v \in S_\infty$ we hence obtain $G_v = GL_{dn}(\mathbb{R})$ if v is split and $G_v = GL_\ell(\mathbb{H})$ if v is not split. At a complex place $G_v = GL_{dn}(\mathbb{C})$. Hence, for the connected compact subgroup $K \subset G_\infty$ we may choose locally

$$K_v^\circ = \begin{cases} Sp(\ell) & \text{if } v \text{ non-split} \\ SO(dn) & \text{if } v \text{ split and real} \\ U(dn) & \text{if } v \text{ complex.} \end{cases}$$

Here, $Sp(\ell)$ is the compact real form of the syplectic group of split-rank ℓ.

8.2. The associate classes of parabolic F-subgroups $P = LN$ of $G = GL'_n$ are in one-to-one correspondence with unordered partitions $[n_1, ..., n_k]$ of n, i.e., $n = \sum_{i=1}^k n_i$, $n_i \geq 1$. We will write $\{P_{[n_1,...,n_k]}\}$ for the associate class corresponding to $[n_1, ..., n_k]$. A Levi subgroup $L_{[n_1,...,n_k]}$ of an element of $\{P_{[n_1,...,n_k]}\}$ is always isomorphic to

$$L_{[n_1,...,n_k]} \cong GL'_{n_1} \times ... \times GL'_{n_k}.$$

In the special case that all n_i are equal, the partition is determined by k and we shall abbreviate our notation to $L_{[n_1,...,n_k]} = L_k$, if this is the case.

The following theorem, classifying the residual spectrum of $G(\mathbb{A})$, was obtained in Badulescu-Renard, [1] Prop. 18.2. For the special case $D = F$, i.e. $G = GL_n/F$, this result is a theorem of Mœglin–Waldspurger, cf. [25].

Theorem 21. *Every residual automorphic representation* Π *of* $G(\mathbb{A}) = GL'_n(\mathbb{A})$ *is given by a pair* (ρ', k)*, where* $k|n$*,* $k \neq 1$*, and* ρ' *is a unitary cuspidal automorphic representation of* $GL'_r(\mathbb{A})$ *with* $r = n/k$*.*

More precisely, let $\pi = \otimes_{j=1}^k \rho'$ *be the product representation of* $L_k(\mathbb{A})$ *and let* $\lambda_k(\pi) = k_\rho(\frac{k-1}{2}, ..., -\frac{k-1}{2}) \in \mathfrak{a}^G_{P_k}$*. Here,* k_ρ *is the uniquely determined integer of* [1] *Prop. 18.2.(i) and the coordinates are in the projections on the* GL'_r*-factors of* L_k*. Then,* Π *is the unique irreducible quotient* $MW'(\rho', k)$ *of the induced representation*

$$I_{L_k(\mathbb{A})}[\pi, \lambda_k(\pi)].$$

As a direct consequence of this result, only those associate classes $\{P\}$ of parabolic F-subgroups of G matter for the description of the residual spectrum $\mathcal{A}_{res,\mathcal{J}}(G)$, which are parameterized by a partition $[r, ..., r]$, $n = kr$, $k \neq 1$.

8.3. We obtain the following theorem on the contribution of the residual automorphic representations of $G = GL'_n/F$ to Eisenstein cohomology .

Theorem 22. *Let $G = GL'_n/F$, $n \geq 1$, $\{P\} = \{P_{[n_1,...,n_k]}\}$ an associate class of parabolic F-subgroups, $k \geq 2$, and φ_P an associate class of cuspidal automorphic representations π of $L(\mathbb{A}) = L_{[n_1,...,n_k]}(\mathbb{A})$. If either $\{P\} \neq \{P_k\}$ or $\pi \not\cong \otimes_{i=1}^k \rho'$, then there is no residual automorphic representation $\Pi \hookrightarrow \mathcal{A}_{res,\mathcal{J}}(G)$ of $G(\mathbb{A})$ supported by $(\{P\}, \varphi_P)$. If $\{P\} = \{P_k\}$ and $\pi \cong \otimes_{i=1}^k \rho'$, then the representation $\Pi = MW'(\rho', k)$ appears precisely once in $\mathcal{A}_{res,\mathcal{J}}(G)$ and the map in cohomology*

$$H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \longrightarrow H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu),$$

induced from the natural inclusion $\Pi \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$, is injective in all degrees

$$0 \leq q < \sum_{\substack{v \in S_\infty \\ v \text{ complex}}} d^2(k-1)\frac{n^2}{k^2} + \sum_{\substack{v \in S_\infty \\ v \text{ real}}} \left\lceil d^2(k-1)\frac{n^2}{2k^2} \right\rceil.$$

If $d = 1$ and $k = 2$, i.e., if $G = GL_n/F$ is the split general linear group over F and P is the self-associate maximal parabolic subgroup, then this bound can be improved to

$$0 \leq q < \sum_{\substack{v \in S_\infty \\ v \text{ complex}}} \frac{1}{2}(n^2 - n) + \sum_{\substack{v \in S_\infty \\ v \text{ real}}} \frac{n^2}{4}.$$

Proof. The first assertions follow from Thm. 21 together with Multiplicity One for discrete series automorphic representations of $G(\mathbb{A})$, cf. Badulescu–Renard [1] Thm. 18.1.(b). A direct calculation gives

$$q_{\mathsf{max}} = \sum_{\substack{v \in S_\infty \\ v \text{ complex}}} d^2(k-1)\frac{n^2}{k^2} + \sum_{\substack{v \in S_\infty \\ v \text{ real}}} \left\lceil d^2(k-1)\frac{n^2}{2k^2} \right\rceil,$$

see also Thm. 18. Hence, the first part of the theorem is a consequence of Corollary 17 and Theorem 18. Next, recall that a cohomological cuspidal automorphic representation of $GL_r(\mathbb{A})$ has necessarily an essentially tempered archimedean component. Hence, Borel–Wallach [7] III, Prop. 5.3 provides a lower bound for $m(L_{P_k,v}, \Pi_v)$ for all $v \in S_\infty$ and all Π_v appearing as a local archimedean component of a representation Π showing up in a quadruple (P_k, Π, ν, λ). Distinguishing the cases of complex and real v, the result follows from a direct computation of this lower bound and Corollary 17, respectively Theorem 15. □

Remark 23. In the case of $G = GL_n/\mathbb{Q}$ and for maximal parabolic subgroups P, Franke–Schwermer considered the contribution of residual automorphic representations to $H^q_{Eis}(G, \mathbb{C})$ in [13], Thm. 5.6. Our Theorem 22 improves their result in the sense, that for n even and P being the self-associate maximal parabolic subgroup, the map

$$H^q(\mathfrak{m}_G, K, L^2_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \longrightarrow H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$$

is not only an epimorphism, but also injective in all degrees $0 \leq q < n^2/4$. Compare this result also to Rohlfs-Speh [26], Thm. IV.3.
In the case $d = n = 2$, $F = \mathbb{Q}$, Theorem 22 is essentially contained in Grobner [17] Thm. 3.2.

9. Applications II: Eisenstein Cohomology of split classical groups

9.1. In this last section we would like to apply our Theorem 15 to families of split classical groups over $\mathbb{Q}$, in order to obtain another series of examples. Therefore, let $n \geq 2$ be an integer and define $G_n/\mathbb{Q}$ to be one of the following groups

$$G_n := \begin{cases} SO_{2n+1}/\mathbb{Q} \\ Sp_{2n}/\mathbb{Q} \\ SO_{2n}/\mathbb{Q} \end{cases}$$

Here, SO_k denotes the $\mathbb{Q}$-split special orthogonal group of $\mathbb{Q}$-rank $\lfloor \frac{k}{2} \rfloor$ and Sp_{2k} denotes the $\mathbb{Q}$-split symplectic group of $\mathbb{Q}$-rank k. In view of Section 8, we left out the general linear group. Furthermore, we let $\{P\}$ be an associate class of maximal parabolic $\mathbb{Q}$-subgroups of G_n.

9.2. The standard maximal parabolic $\mathbb{Q}$-subgroups $P = LN$ of $G = G_n$ are parameterized by the n simple roots α_k, $1 \leq k \leq n$. None of them are associate, except in the case $G_n = SO_{2n}$, n odd and the standard parabolic subgroups P_{n-1} and P_n. A Levi subgroup L_k of an element of $\{P_k\}$ is always isomorphic to

$$L_k \cong GL_k \times G_{n-k},$$

where G_{n-k} is the $\mathbb{Q}$-split classical group of rank $n-k$ of the same type as $G = G_n$.

Theorem 24. *Let $G = G_n$ be a $\mathbb{Q}$-split classical group of Cartan type B_n, C_n or D_n, i.e., either the $\mathbb{Q}$-split symplectic or special orthogonal group of $\mathbb{Q}$-rank n. Let $P = P_k$, $1 \leq k \leq n$, be the standard maximal parabolic $\mathbb{Q}$-subgroup of G corresponding to the k-th simple root and let $\{P_k\}$ be the so–defined associate class of parabolic $\mathbb{Q}$-subgroups. (Here we leave out the case $k = n-1$, $G_n = SO_{2n}$.) If φ_{P_k} is an associate class of cuspidal automorphic representations of $L_k(\mathbb{A})$, then there is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(\mathfrak{g}, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E_\mu) \xrightarrow{\cong} H^q(\mathfrak{g}, K, \mathcal{A}_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E_\mu),$$

for all degrees $0 \leq q < \frac{1}{2}\left((n-k)\frac{n-k+3}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor\right) + q(G_n, k)$, where

$$q(G_n,k) = \begin{cases} \left\lceil k(n - \frac{3k+1}{4}) \right\rceil & \text{if } G_n = SO_{2n} \\ \left\lceil k(n - \frac{3k-1}{4}) \right\rceil & \text{if } G_n = SO_{2n+1}, Sp_{2n}. \end{cases}$$

Proof. Without loss of generality, $m \neq 0$. One directly computes that $\dim_{\mathbb{R}} N_k(\mathbb{R})$ equals $2k(n - \frac{3k+1}{4})$ for $G_n = SO_{2n}$ and $k \neq n-1$ and $2k(n - \frac{3k-1}{4})$ for $G_n = SO_{2n+1}, Sp_{2n}$ and any k. Furthermore, as a cohomological cuspidal automorphic representation of $GL_k(\mathbb{A})$ has necessarily an essentially tempered archimedean component, combining Borel–Wallach [7] III, Prop. 5.3 and Vogan–Zuckerman [29], Table 8.2, shows that $\frac{1}{2}\left((n-k)\frac{n-k+3}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor\right)$ is a lower bound for $m(L_k(\mathbb{R}), \Pi_\infty)$ for all Π_∞ appearing as the archimedean component of a representation Π showing up in a quadruple (P_k, Π, ν, λ). The claim now follows from Theorem 15. $\square$

Remark 25. If $G_n = SO_{2n+1}, Sp_{2n}$, $n \geq 2$, then the bound $q(G_n, k) = q_{\mathsf{max}}$ of Thm. 24 serves as an example where Vogan–Zuckerman's constant r_G is smaller than q_{max}, and hence, in particular, smaller than q_{res}. The same holds true for $G_n = SO_{2n}$, $n \geq 5$ and $k \neq 1$.

Remark 26. In the case of $G = SO_{2n+1}$ resp. Sp_{2n}, the latter theorem is complementary to the results in Gotsbacher-Grobner [14] resp. Grbac-Schwermer [16]. In these references, necessary conditions for non-trivial residual Eisenstein cohomology classes, stemming from globally generic cuspidal automorphic representations of maximal Levi subgroups were given. In contrast, the conditions provided in Theorem 24 are sufficient for the existence of such classes. Moreover, in the range of degrees given by the above theorem, it is shown that these residual Eisenstein cohomology classes exhaust the full space $H^q(\mathfrak{g}, K, \mathcal{A}_{\mathcal{J},\{P_k\},\varphi_{P_k}}(G) \otimes E_\mu)$, cf. Thm. 18. Also, the condition of global genericity does not enter the present assumptions.

References

[1] I. A. Badulescu, D. Renard, Unitary Dual of GL(n) at archimedean places and global Jacquet-Langlands correspondence, *Comp. Math.* **145** (2010) 1115–1164

[2] A. Borel, *Automorphic Forms on Reductive Groups*, in: *Automorphic forms and Applications*, IAS/Park City Mathematics Series, vol. 12, eds. P. Sarnak and F. Shahidi, (Utah, 2002) AMS, 2007, pp. 5–40.

[3] A. Borel, Stable real cohomology of arithmetic groups, *Ann. scient. Éc. Norm. Sup.* **7** (1974), 235–272

[4] A. Borel, W. Casselman, L^2-Cohomology of locally symmetric manifolds of finite volume *Duke Math. J.* **50** (1983) pp. 625-647

[5] A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979), pp. 189-202

[6] A. Borel, J.-P. Labesse, J. Schwermer, On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields, *Comp. Math.*, **102** (1996), pp. 1-40

[7] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies **94**, (Princeton Univ. Press, New Jersey, 1980)

[8] L. Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*, in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. I, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 77–159.

[9] L. Clozel, On the cohomology of Kottwitz's arithmetic varieties, *Duke Math. J.* **72** (1993) pp. 757–795

[10] T. J. Enright, Relative Lie algebra cohomology and unitary representations of complex Lie groups, *Duke Math. J.* **46** (1979) 513–525

[11] J. Franke, *A topological model for some summand of the Eisenstein cohomology of congruence subgroups*, in: *Eisenstein series and applications*, W.T. Gan, S.S. Kudla, Y. Tschinkel eds., *Progr. Math.* **258**, Birkhäuser Boston, Boston, 2008, pp. 27–85

[12] J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. de l'ENS* **2** (1998) 181–279

[13] J. Franke, J. Schwermer, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.*, **311** (1998), pp. 765-790

[14] G. Gotsbacher, H. Grobner, On the Eisenstein cohomology of odd orthogonal groups, Forum Math. **25** (2013) pp. 283–311

[15] N. Grbac, H. Grobner, The residual Eisenstein Cohomology of Sp_4 over a totally real number field, to appear in *Trans. Amer. Math. Soc.* (2013)

[16] N. Grbac, J. Schwermer, On residual cohomology classes attached to relative rank one Eisenstein series for the symplectic group, *I.M.R.N.* **7** (2011), pp. 1654–1705

[17] H. Grobner, Automorphic Forms, Cohomology and CAP Representations. The Case GL_2 over a definite quaternion algebra, J. Ramanujan Math. Soc. **28** (2013) pp. 19–43

[18] G. Harder, Eisenstein cohomology of arithmetic groups. The case GL_2, *Invent. Math.* **89** (1987), 37–118

[19] G. Harder, *On the cohomology of* $SL(2, \mathfrak{O})$, in: Lie groups and their representations, Proc. of the summer school on group representations, I. M. Gelfand ed., London (1975), pp. 139-150

[20] G. Harder, *On the cohomology of discrete arithmetically defined groups*, in: Discrete Subgroups of Lie Groups and Applications to Moduli, Papers presented at the Bombay Colloquium, 1973, Oxford University Press, (1975), pp. 129-160

[21] S. Kumaresan, On the Canocical k-Types in the Irreducible Unitary g-Modules with Non-Zero Relative Cohomology, *Invent. Math.* **59** (1980) 1–11

[22] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, No. **544**, Springer-Verlag (1976)

[23] J.–S. Li, J. Schwermer, On the Eisenstein cohomology of arithmetic groups, *Duke Math. J.* **123** (2004) 141–169

[24] C. Mœglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Univ. Press (1995)

[25] C. Mœglin, J.-L. Waldspurger, Le Spectre Résiduel de $GL(n)$, *Ann. Sci. de l'ENS* 4^e série tome 22 **4** (1989) 605–674

[26] J. Rohlfs, B. Speh, Pseudo Eisenstein Forms and the Cohomology of Arithmetic Groups III: Residual Cohomology Classes, in *On certain L-functions: Conference in Honor of Freydoon Shahidi, July 23-27, 2007, Purdue Univerisity, West Lafayette, Indiana*, eds J. Arthur, J.W. Codgell, S. Gelbart, D. Goldberg, D. Ramakrishnan, J.-K. Yu, AMS, Providence, R.I., (2011), pp. 501–524

[27] J. Schwermer, Eisenstein series and cohomology of arithmetic groups: The generic case, *Invent. Math.* **116** (1994), 481-511

[28] J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, Lecture Notes in Mathematics, **988**, Springer-Verlag (1983)

[29] D. A. Vogan Jr., G. J. Zuckerman, Unitary representations with nonzero cohomology, *Comp. Math.* **53** (1984) 51–90

[30] N. Wallach, *On the constant term of a square integrable automorphic form*, in: Operator algebras and group representations, Vol. II (Neptun, 1980), Monographs Stud. Math., vol. **18**, Pitman, Boston, MA, London (1984), pp. 227-237

Harald Grobner: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: `harald.grobner@univie.ac.at`

URL: `http://homepage.univie.ac.at/harald.grobner`