# ON THE EISENSTEIN COHOMOLOGY OF ODD ORTHOGONAL GROUPS

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Abstract. The paper investigates a significant part of the automorphic, in fact of the so-called Eisenstein cohomology of split odd orthogonal groups over $\mathbb{Q}$. The main result provides a description of residual and regular Eisenstein cohomology classes for *maximal* parabolic $\mathbb{Q}$-subgroups in case of *generic* cohomological cuspidal automorphic representations of their Levi subgroups. That is, such identifying necessary conditions on these latter representations as well as on the complex parameters in order for the associated Eisenstein series to possibly yield non-trivial classes in the automorphic cohomology.

## Introduction

. The main objective of the effort to be unfolded here is to study the Eisenstein cohomology of the $\mathbb{Q}$-split odd orthogonal groups $G = SO_{2n+1}$. Specifically, it is the contribution to the latter stemming from maximal parabolic $\mathbb{Q}$-subgroups that is dealt with.

To put ourselves in medias res let $\mathfrak{g}$ be the Lie algebra of the group of real points of $G$, $K$ a maximal compact subgroup therein and $\mathcal{A}(G)$ the $(\mathfrak{g}, K)$-module of adèlic automorphic forms on $G$. If $E$ is a finite dimensional irreducible rational representation of $G(\mathbb{R})$, the automorphic cohomology of $G$ twisted by $E$ is defined to be $H^*(\mathfrak{g}, K; \mathcal{A}(G) \otimes E)$. Let then $P \subset G$ be a maximal parabolic $\mathbb{Q}$-subgroup with Levi subgroup $L$ and maximal central $\mathbb{Q}$-split torus $A \subseteq L$. We consider the Eisenstein series $E_P(f, \Lambda)$ associated to a cohomological, globally generic, cuspidal automorphic representation $\pi$ of $L(\mathbb{A})$, resp. an element $f$ in the representation induced from $\pi$ and a complex parameter $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Langlands' theory of Eisenstein series ensures convergence of $E_P(f, \Lambda)$ on a right half plane with respect to $\Lambda$, the existence of a meromorphic continuation to the entire complex plane $\mathfrak{a}_{\mathbb{C}}^*$ and gives a restriction on the location of its possible poles in terms of affine lines in it. Moreover, $E_P(f, \Lambda)$ (resp. its residue) gives rise to an element of $\mathcal{A}(G)$ for fixed but arbitrary $\Lambda$. In order for it to represent a class in the automorphic cohomology $\Lambda$ has to be of a specific form $\Lambda_w$ involving what is called the set $W^P$ of Kostant representatives $w$ for $P$. At this point we are presented with the key task which is to determine the cohomological cuspidal automorphic representations of $L(\mathbb{A})$ and the Kostant representatives for $P$. With these data at hand it remains to decide whether the given Eisenstein series is holomorphic or has a pole at the point $\Lambda_w$. In solving the latter the general Theorems available in the theory of Eisenstein cohomology provide us with a closing answer to the initial question, which roughly

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assumes the following form: (see Theorems 25 & 26 as well as section 6 for the precise statements)

**Result.** *Let $P$ be the standard maximal parabolic $\mathbb{Q}$-subgroup of $G = SO_{2n+1}$ with Levi subgroup $L \cong GL_k \times SO_{2(n-k)+1}$. Further, let $\alpha$ be the only simple root of $G$ which does not vanish identically on $A$. Suppose given a non-trivial class in $H^*(\mathfrak{g}, K, \mathcal{A}(G) \otimes E)$ associated to a choice $(\pi, w)$ of a cohomological, globally generic, cuspidal automorphic representation $\pi$ of $L(\mathbb{A})$ and a Kostant representative $w \in W^P$. As $\pi = \chi(\sigma \hat{\otimes} \tau)$, with $\chi$ being equal to the central character of $\pi$ on $A(\mathbb{R})^\circ$ and $\sigma$ (resp. $\tau$) being a cohomological, globally generic, cuspidal automorphic representation of $GL_k$ (resp. $SO_{2(n-k)+1}$), in the* residual case *$\pi$ is either of the form*

(1) *$d\chi = \Lambda_w = \frac{k}{2}\alpha$, $\sigma$ self-dual and such that the Rankin-Selberg $L$-function of $\sigma \times \tau$ does not vanish at $\frac{1}{2}$ (a condition, which we set empty if $k = n$). If in addition $k$ is even and the central character of $\sigma$ is trivial, then $k \geq 4$ and $\sigma$ is a self-dual Weak Langlands functorial lift of a globally generic cuspidal automorphic representation of $SO_k(\mathbb{A})$. Or*
(2) *$d\chi = \Lambda_w = k\alpha$, $k \neq n$ is even and $\sigma$ self-dual such that the Rankin-Selberg $L$-function of $\sigma \times \tau$ has a pole at 1.*

*The* regular case *pertaining to the situation that $\pi$ either doesn't meet either one of conditions* (1) *and* (2) *or it does while producing regular values for the Eisenstein series is settled in light of the description of regular Eisenstein classes available in general.*

*Finally, we find lower and upper bounds for the degree of the Eisenstein class constructible from the above in both the residual and the regular case.*

The structure of the paper mirrors closely the steps mentioned in the above outline:

Section 1 introduces the automorphic cohomology of a reductive algebraic $\mathbb{Q}$-group and provides a brief outline of Eisenstein cohomology. In particular, it sketches the decomposition of the automorphic cohomology along the cuspidal support and gathers the main Theorems pertaining to the construction of cohomology classes by means of Eisenstein series.

Section 2 gives a terse description of so-called Kostant data by virtue of a method the idea behind which we learnt from N. Grbac. Its main advantage is to be seen in a formulaic description of the action of Kostant representatives on arbitrary weights on the Cartan subalgebra of $\mathfrak{g}$, and consequently of the Kostant data relevant to the aforementioned construction.

Section 3 lists the cuspidal representations of the Levi subgroups with non-trivial cohomology and provides lower and upper bounds for their contribution to cohomology drawing from well-known results by a number of people (for precise references cf. section 3).

Section 4 recounts H. Kim's determination of the part of the residual spectrum of $G$ accounted for by the maximal parabolic $\mathbb{Q}$-subgroups in order to derive the possible poles of Eisenstein series.

Finally, sections 5 and 6 present our results on residual and regular Eisenstein cohomology classes for maximal parabolic $\mathbb{Q}$-subgroups of $G = SO_{2n+1}$.

On a final note, we want to remark that at the time of composing this work, N. Grbac and J. Schwermer were addressing similar questions for split $Sp_{2n}$. An analogous result exists for split $SL_n$, see [34]. The interested reader may also consult the first author's paper [15], where regular Eisenstein cohomology classes were

constructed for the inner forms $SO(n,2)$ of $SO_{n+2}$ or the second author's papers [17] and [18] on residual and regular Eisenstein cohomology of $Sp(2,2)$ and $Sp(1,1)$.

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**Notation and Conventions.** Throughout this paper $G$ will denote a connected, semisimple algebraic group over $\mathbb{Q}$[1] of rank $rk_{\mathbb{Q}}(G) \geq 1$ with finite center. Lie algebras of groups of real points of algebraic groups will be denoted by the same but gothic letter, e.g. $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$. The complexification of a Lie algebra will be denoted by subscript "$\mathbb{C}$", e.g. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

We use the standard terminology and hypotheses concerning algebraic groups and their subgroups to be found in [32] I.1.4-I.1.12. In particular, we assume that a minimal parabolic subgroup $P_0$ has been fixed and that $K_{\mathbb{A}} = K_{\mathbb{R}} \times K_{\mathbb{A}_f}$ is a maximal compact subgroup of the group $G(\mathbb{A})$ of adelic points of $G$ which is in *good position* with respect to $P_0$ ([32], I.1.4). Then $K = K_{\mathbb{R}}$ is maximal compact in $G(\mathbb{R})$, hence has an associate Cartan involution $\theta$. If $H$ is a subgroup of $G$, we let $K_H = K \cap H(\mathbb{R})$.

Assume that $L_0$ is the unique Levi subgroup of $P_0$ which is invariant under $\theta$ and $N_0$ is an unipotent radical of $P_0$ such that we have the Levi decomposition $P_0 = L_0 N_0$. If we additionally denote by $A_0$ a maximal, central $\mathbb{Q}$-split torus in $L_0$ then we also get a Langlands decomposition $P_0 = M_0 A_0 N_0$. Let $P$ be a standard parabolic $\mathbb{Q}$-subgroup of $G$ with respect to $P_0$. It has a unique Levi decomposition $P = L_P N_P$, with $L_P \supseteq L_0$ and also a unique Langlands decomposition $P = M_P A_P N_P$ with unique $\theta$-stable split component $A_P \subseteq A_0$. If it is clear from the context we will also omit the subscript "$P$". We write $\Delta(P, A)$ for the set of weights of the adjoint action of $P$ with respect to $A_P$. $\rho_P$ denotes the half-sum of these weights. In particular, $\rho = \rho_{P_0}$ is the half sum of positive $\mathbb{Q}$-roots of $G$ with respect to $A_0$.
Extend the Lie algebra $\mathfrak{a}$ of $A(\mathbb{R})$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by adding a $\theta$-stable Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. The absolute root system of $\mathfrak{g}$ is denoted $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, a simple subsystem (compliant with the requirement that positivity on the system of absolute roots shall be compatible with the positivity on the set $\Delta_{\mathbb{Q}}$ of $\mathbb{Q}$-roots given by the choice of the minimal pair $(P_0, A_0)$) is denoted $\Delta^{\circ}$. We also write $\Delta_M^{\circ}$ for the set of absolute simple roots of $\mathfrak{m}$ with respect to $\mathfrak{b}$ (so $\Delta^{\circ} = \Delta_G^{\circ}$). The Weyl group associated to $\Delta$ is denoted $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. We let $W^P = \{w \in W | w^{-1}(\alpha) > 0 \quad \forall \alpha \in \Delta_M^{\circ}\}$. The elements of $W^P$ are called *Kostant representatives*.
Using the fact that $K_{\mathbb{A}}$ is in good position, we can extend the standard *Harish-Chandra height-function* $H_P : P(\mathbb{A}) \to \mathfrak{a}^*$ given by $\prod_p |\chi(p)|_p = e^{\langle \chi, H_P(p) \rangle}$, for all $\mathbb{Q}$-characters $\chi$ of $L$ (viewed as an element of $\mathfrak{a}_{\mathbb{C}}^*$), to a function on all of $G(\mathbb{A})$ by setting $H_P(g) := H_P(p)$, $g = kp$.

Let $G$ be a connected, *reductive* group over $\mathbb{Q}$ and $\chi$ a central character. As usual $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$) denotes the discrete spectrum

[1] Which for us includes the (only technical) assumption that $G$ is not obtained from restriction of scalars $Res_{F/\mathbb{Q}}$ with $F \neq \mathbb{Q}$.

of $G$ (resp. the part of it consisting of functions with central character $\chi$). It can be written as the direct sum of the cuspidal spectrum $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (resp. $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi)$) and the residual spectrum $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A})$ (resp. $L^2_{res}(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi)$). By [12] the space $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi)$, decomposes as direct Hilbert sum over all irreducible, admissible representations $\pi$ of $G(\mathbb{A})$ with central character $\chi$, each of which occurring with finite multiplicity $m_{dis}(\pi)$. The same is therefore true for the cuspidal (resp. residual) spectrum, if we replace the multiplicity by $m(\pi)$ (resp. $m_{res}(\pi)$). Every $\pi$ can be written as a restricted tensor product $\pi = \otimes'_p \pi_p$, where $p$ is a place of $\mathbb{Q}$. i.e. either a rational prime or $\infty$ and $\pi_p$ a local irreducible, admissible representation $\pi_p$ of $G(\mathbb{Q}_p)$, [9]. Further, $\pi$ is (and therefore all $\pi_p$ are, simultaneously) unitary if and only if $\chi$ is. Then $\pi$ is the completed restricted tensor product $\pi = \hat{\otimes}'_p \pi_p$.

For any $G(\mathbb{A})$-representation $\sigma$, we will write $\sigma^\infty$ for the space of its smooth vectors and $\sigma_{(K)}$ for the space of $K$-finite vectors. Clearly, if $\sigma$ is unitary, then $\sigma^\infty_{(K)}$ is a unitary $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module.

## 1. Generalities on Automorphic forms and Cohomology

1.1. We start our study with the space of automorphic forms $\mathcal{A}(G) = \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ on $G(\mathbb{A})$. It is a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module and hence it makes sense to talk about its $(\mathfrak{g}, K)$-cohomology (to be called the *automorphic cohomology of* $G$) which we may also twist by an irreducible rational representation $E$ of $G(\mathbb{R})$ of highest weight $\lambda$ on a finite-dimensional, complex vector space:

$$H^q(G,E) := H^q(\mathfrak{g}, K, \mathcal{A}(G)\otimes E).$$

Clearly, $H^q(G,E)$ carries a $G(\mathbb{A}_f)$-module structure, induced from the action of $G(\mathbb{A}_f)$ on $\mathcal{A}(G)$, which we shall now investigate.

In order to do so, we shall analyze the cohomological automorphic representations $\pi$ of $G$. Recall ([31], Prop. 2) that a representation $\pi$ of $G(\mathbb{A})$ is automorphic if and only if it is an irreducible constituent of a globally (normalized) induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\sigma]$, $P$ being a parabolic subgroup of $G$ and $\sigma$ a cuspidal automorphic representation of the Levi $L_P(\mathbb{A})$. Hence, cuspidal automorphic representations of Levi subgroups $L_P$ will play a crucial role in the determination of $H^q(G,E)$. In fact, we may divide the space of automorphic forms into two parts, $\mathcal{A}(G) = \mathcal{A}_{cusp}(G) \oplus \mathcal{A}_{Eis}(G)$, where $\mathcal{A}_{cusp}(G)$ is the space of cuspidal automorphic forms and $\mathcal{A}_{Eis}(G)$ a natural complement spanned as a representation by all irreducible subquotients of induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}[\sigma]$ as above, but with $P$ proper. Therefrom the automorphic cohomology of $G$ inherits a natural decomposition as $G(\mathbb{A}_f)$-module:

$$H^q(G,E) = H^q_{cusp}(G,E) \oplus H^q_{Eis}(G,E). \tag{1}$$

1.2. Let us refine this decomposition even further. As one may guess from the characterization of automorphic representations as subquotients of parabolically induced representations, there should be somehow a refinement of (1) which involves all parabolic subgroups $P$ of $G$ and cuspidal automorphic representations $\pi$ of $L_P(\mathbb{A})$. This is, indeed, true and we will briefly discuss this refined decomposition as it will serve as the starting-point of our further investigations.

First of all, $\mathcal{A}(G)$ admits a certain decomposition as a direct sum with respect to the classes $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups $P \subseteq G$. This relies on such

a decomposition of the space $V_G$ of $K$-finite, left $G(\mathbb{Q})$-invariant, smooth functions $f : G(\mathbb{A}) \to \mathbb{C}$ of uniform moderate growth, first proved by Langlands in a letter to Borel, [29]. See also [3] Thm. 2.4: $V_G = \bigoplus_{\{P\}} V_G(\{P\})$, where $V_G(\{P\})$ denotes the space of elements $f$ in $V_G$ which are negligible along $Q$ for every parabolic $\mathbb{Q}$-subgroup $Q \subseteq G$, $Q \notin \{P\}$. Putting $\mathcal{A}_P(G) = V_G(\{P\}) \cap \mathcal{A}(G)$ we made the first of two steps in the decomposition of $\mathcal{A}(G)$ alluded to above:

$$\mathcal{A}(G) = \bigoplus_{\{P\}} \mathcal{A}_P(G).$$

Observe that $\mathcal{A}_G(G) \subset V_G(\{G\}) = L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^\infty_{(K)}$. Hence, we see that the following holds:

**Proposition 1.**

$$H^q_{cusp}(G, E) = H^q(\mathfrak{g}, K, \mathcal{A}_G(G) \otimes E)$$

*and*

$$H^q_{Eis}(G, E) = \bigoplus_{\{P\}, P \neq G} H^q(\mathfrak{g}, K, \mathcal{A}_P(G) \otimes E).$$

1.3. **Eisenstein series.** We still want to take the second step in refining (1), meaning that we want to decompose the summands $H^q(\mathfrak{g}, K, \mathcal{A}_P(G) \otimes E)$ involving cuspidal automorphic representations $\pi$ of $L_P(\mathbb{A})$. We refer the reader to [11] for details concerning this section.

We need some technical assumptions: For $Q = LN = MAN$ associate to the standard parabolic $P$, $\varphi_Q$ is a finite set of irreducible representations $\pi = \chi\widetilde{\pi}$ of $L(\mathbb{A})$, with $\chi : A(\mathbb{R})^\circ \to \mathbb{C}^*$ a continuous character and $\widetilde{\pi}$ an irreducible, unitary subrepresentation of $L^2_{cusp}(L(\mathbb{Q})A(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$ whose central character induces a continuous homomorphism $A(\mathbb{Q})A(\mathbb{R})^\circ\backslash A(\mathbb{A}) \to U(1)$ and whose infinitesimal character matches the one of the dual of an irreducible subrepresentation of $H^*(\mathfrak{n}, E)$. This just means that $\widetilde{\pi}$ is a unitary, cuspidal automorphic representation of $L(\mathbb{A})$ whose central and infinitesimal character satisfy the above conditions. Finally, three further "compatibility conditions" have to be satisfied between these sets $\varphi_Q$, skipped here and listed in [11], 1.2. The family of all collections $\varphi = \{\varphi_Q\}$ of such finite sets is denoted $\Psi_P$.

Now denote $\mathrm{I}_{Q,\widetilde{\pi}} = \mathrm{Ind}_{Q(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \mathrm{Ind}_{(\mathfrak{l}, K_L)}^{(\mathfrak{g}, K)} \left[\widetilde{\pi}^\infty_{(K_L)}\right]^{m(\widetilde{\pi})}$ (unnormalized induction). For a function $f \in \mathrm{I}_{Q,\widetilde{\pi}}$, $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $g \in G(\mathbb{A})$ we consider the Eisenstein series (formally) defined as

$$E_Q(f, \Lambda)(g) := \sum_{\gamma \in Q(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) e^{\langle \Lambda + \rho_Q, H_Q(\gamma g)\rangle}.$$

If we set $(\mathfrak{a}^*)^+ := \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} | \Re e(\Lambda) \in \rho_Q + C\}$, where $C$ denotes the open, positive Weyl-chamber with respect to $\Delta(Q, A)$, the series converges absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times (\mathfrak{a}^*)^+$. It is known that for fixed $\Lambda$ the function $E_Q(f, \Lambda)$ on $G(\mathbb{A})$ is an automorphic form there and that the map $\Lambda \mapsto E_Q(f, \Lambda)(g)$ can be analytically continued to a meromorphic function on all of $\mathfrak{a}^*_{\mathbb{C}}$, cf. [32] p. 140 or [30] §7. It is known that the singularities $\Lambda_0$ (i.e., poles) of $E_Q(f, \Lambda)$ lie along certain affine hyperplanes of the form $R_{\alpha,t} := \{\xi \in \mathfrak{a}^*_{\mathbb{C}} | \langle \xi, \alpha\rangle = t\}$ for some constant $t$ and some root $\alpha \in \Delta(Q, A)$, called "root-hyperplanes" ([32] Prop. IV.1.11 (a) or [30] p.131). Choose a normalized vector $\eta \in \mathfrak{a}^*_{\mathbb{C}}$ orthogonal to $R_{\alpha,t}$ and assume that $\Lambda_0$ lies on no other singular hyperplane of $E_Q(f, \Lambda)$. Then define $\Lambda_0(u) := \Lambda_0 + u\eta$ for $u \in \mathbb{C}$. If $c$ is a positively oriented circle in the complex plane around zero which

is so small that $E_Q(f,\Lambda_0(.))(g)$ has no singularities on the interior of the circle with double radius, then

$$\mathrm{Res}_{\Lambda_0}(E_Q(f,\Lambda)(g)) := \frac{1}{2\pi i}\int_c E_Q(f,\Lambda_0(u))(g)du$$

is a meromorphic function on $R_{\alpha,t}$, called the *residue* of $E_Q(f,\Lambda)$ at $\Lambda_0$. Its poles lie on the intersections of $R_{\alpha,t}$ with the other singular hyperplanes of $E_Q(f,\Lambda)$. By this procedure one gets a function holomorphic at $\Lambda_0$ in finitely many steps by taking successive residues as explained above.

Now we are able to take the desired second step in the decomposition of the $G(\mathbb{A}_f)$-module summand $H^q(\mathfrak{g},K,\mathcal{A}_P(G)\otimes E)$: For $\pi=\chi\widetilde{\pi}\in\varphi_P\in\varphi\in\Psi_P$ let $\mathcal{A}_{P,\varphi}(G)$ be the space of functions, spanned by all possible residues and derivatives of Eisenstein series defined via all $f\in \mathrm{I}_{P,\widetilde{\pi}}$ at the value $d\chi$. It is a $(\mathfrak{g},K,G(\mathbb{A}_f))$-module. Thanks to the functional equations (see [32] IV.1.10) satisfied by the Eisenstein series considered, this is well defined, i.e., independent of the choice of a representative for the class of $P$ (whence we took $P$ itself) and the choice of a representation $\pi\in\varphi_P$. Finally, we get

**Proposition 2** ([11] Thm.s 1.4 & 2.3; [32] III, Thm. 2.6)**.** *There is a direct sum decomposition as $G(\mathbb{A}_f)$-module*

$$H^q_{Eis}(G,E)=\bigoplus_{\{P\},P\neq G}\bigoplus_{\varphi\in\Psi_P} H^q(\mathfrak{g},K,\mathcal{A}_{P,\varphi}(G)\otimes E). \tag{2}$$

**Remark 3.** Notice, that the statement entails the claim that the $G(\mathbb{A}_f)$-module $H^q_{Eis}(G,E)$ is generated by derivatives and residues of cuspidal Eisenstein series associated to $\Lambda\in\overline{C}$.

1.4. How does this refined decomposition (2) help us in determining the $G(\mathbb{A}_f)$-module $H^q_{Eis}(G,E)$? It allows us to construct classes in the Eisenstein cohomology by lifting classes associated to cuspidal automorphic representations $\widetilde{\pi}$. In order to have this procedure readily available we will now recall the notion of classes of type $(\pi,w)$, $\pi\in\varphi_P$, $w\in W^P$.

Therefore, let $\pi=\chi\widetilde{\pi}\in\varphi_P$ and consider the symmetric tensor algebra

$$S_\chi(\mathfrak{a}^*)=\bigoplus_{n\geq 0}\bigodot^n\mathfrak{a}^*_{\mathbb{C}},$$

$\bigodot^n\mathfrak{a}^*_{\mathbb{C}}$ being the symmetric tensor product of $n$ copies of $\mathfrak{a}^*_{\mathbb{C}}$, as module under $\mathfrak{a}$: Since $S_\chi(\mathfrak{a}^*)$ can be viewed as the space of polynomials on $\mathfrak{a}_{\mathbb{C}}$, we let $\xi\in\mathfrak{a}$ act via translation followed by multiplication with $\langle\xi,\rho_P+d\chi\rangle$. This explains the subscript "$\chi$". We extend this action trivially on $\mathfrak{l}$ and $\mathfrak{n}$ to get an action of the Lie algebra $\mathfrak{p}$ on the space $S_\chi(\mathfrak{a}^*)$. We may also define a $P(\mathbb{A}_f)$-module structure via the rule

$$q\cdot X=e^{\langle d\chi+\rho_P,H_P(q)\rangle}X,$$

for $q\in P(\mathbb{A}_f)$ and $X\in S_\chi(\mathfrak{a}^*)$. There is a linear isomorphism

$$\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\mathrm{Ind}_{(\mathfrak{l},K_L)}^{(\mathfrak{g},K)}\left[\widetilde{\pi}^\infty_{(K_L)}\otimes S_\chi(\mathfrak{a}^*)\right]^{m(\widetilde{\pi})}\xrightarrow{\sim}\mathrm{I}_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*),$$

so in particular one can view the right hand side as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module by transport of structure. Doing this, it is shown in [10], pp. 256-257, that

$$H^q(\mathfrak{g}, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \cong$$

(3)

$$\bigoplus_{\substack{w \in W^P \\ -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}} = d\chi}} \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \left[ H^{q-l(w)}(\mathfrak{m}, K_M, (\widetilde{\pi}_\infty)_{(K_M)} \otimes {}^\circ F_w) \otimes \mathbb{C}_{d\chi+\rho_P} \otimes \widetilde{\pi}_f^{\infty_f} \right]^{m(\widetilde{\pi})}.$$

Here ${}^\circ F_w$ is the irreducible, finite dimensional representation of $M(\mathbb{C})$ with highest weight $\mu_w := w(\lambda+\rho) - \rho|_{\mathfrak{b}_\mathbb{C}}$ and $\mathbb{C}_{d\chi+\rho_P}$ the one-dimensional, complex $P(\mathbb{A}_f)$-module on which $q \in P(\mathbb{A}_f)$ acts by multiplication by $e^{\langle d\chi+\rho_P, H_P(q)\rangle}$. A non-trivial class in a summand of the right hand side is called a cohomology class *of type* $(\pi, w)$, $\pi \in \varphi_P$, $w \in W^P$ (this notion was first introduced in [35] p. 56).
Further, since $L(\mathbb{R}) \cong M(\mathbb{R}) \times A(\mathbb{R})^\circ$, $\widetilde{\pi}_\infty$ can be viewed as an irreducible, unitary representation of $M(\mathbb{R})$. Therefore, a $(\pi, w)$ type consists of an irreducible representation $\pi = \chi\widetilde{\pi}$ whose unitary part $\widetilde{\pi} = \widetilde{\pi}_\infty \hat{\otimes} \widetilde{\pi}_f$ has at the infinite place an irreducible, unitary representation $\widetilde{\pi}_\infty$ of the semisimple group $M(\mathbb{R})$ with non-trivial $(\mathfrak{m}, K_M)$-cohomology with respect to ${}^\circ F_w$.

1.5. **The Eisenstein map.** In order to construct Eisenstein cohomology classes, we start from a class of type $(\pi, w)$. By (1.4) we can assume that $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_\mathbb{C}}$ and that this point lies inside the closed, positive Weyl chamber defined by $\Delta(P, A)$.

Reinterpret $S_\chi(\mathfrak{a}^*)$ as the space of formal, finite $\mathbb{C}$-linear combinations of differential operators $\frac{\partial^\nu}{\partial\Lambda^\nu}$ on the complex, $\ell$-dimensional vector space $\mathfrak{a}^*_\mathbb{C}$. It is understood that some choice of Cartesian coordinates $z_1(\Lambda), ..., z_\ell(\Lambda)$ on $\mathfrak{a}^*_\mathbb{C}$ has been fixed and $\nu = (n_1, ..., n_\ell) \in \mathbb{N}_0^\ell$ denotes a multi-index with respect to these. As a consequence of [32] Prop. IV.1.11, there exists a polynomial $0 \neq q(\Lambda)$ on $\mathfrak{a}^*_\mathbb{C}$ such that for every $f \in \mathrm{I}_{P,\widetilde{\pi}}$ the function

$$\Lambda \mapsto q(\Lambda) E_P(f, \Lambda)$$

is holomorphic at $d\chi$. Since $\mathcal{A}_{P,\varphi}(G)$ can be written as the space which is generated by the coefficient functions in the Taylor series expansion of $q(\Lambda)E_P(f,\Lambda)$ at $d\chi$, $f$ running through $\mathrm{I}_{P,\widetilde{\pi}}$, (cf. [11]) we are able to define a surjective homomorphism of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules $E_{P,\pi}$

$$\mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \xrightarrow{E_{P,\pi}} \mathcal{A}_{P,\varphi}(G)$$

$$f \otimes \frac{\partial^\nu}{\partial\Lambda^\nu} \mapsto \frac{\partial^\nu}{\partial\Lambda^\nu} \left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}.$$

and hence get a well-defined homomorphism in cohomology:

$$H^q(\mathfrak{g}, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E) \xrightarrow{E^q_\pi} H^*(\mathfrak{g}, K, \mathcal{A}_{P,\varphi}(G) \otimes E). \tag{4}$$

There are the following results. The first one deals with the regular (i.e. holomorphic) case:

**Theorem 4** ([35], Thm. 4.11)**.** *Suppose* $[\beta] \in H^q(\mathfrak{g}, K, \mathrm{I}_{P,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*) \otimes E)$ *is a class of type* $(\pi, w)$*, represented by a homomorphism* $\beta$*, such that for all elements* $f \otimes \frac{\partial^\nu}{\partial\Lambda^\nu}$ *in its image,* $E_{P,\pi}(f \otimes \frac{\partial^\nu}{\partial\Lambda^\nu}) = \frac{\partial^\nu}{\partial\Lambda^\nu}\left(q(\Lambda)E_P(f,\Lambda)\right)|_{d\chi}$ *is just the regular value* $E_P(f, d\chi)$ *of the Eisenstein series* $E_P(f,\Lambda)$*, which is assumed to be holomorphic*

*at the point* $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}}$ *inside the closed, positive Weyl chamber defined by* $\Delta(P,A)$. *Then* $E^q_\pi([\beta])$ *is a non-trivial Eisenstein cohomology class*

$$E^q_\pi([\beta]) \in H^q(\mathfrak{g}, K, \mathcal{A}_{P,\varphi}(G)\otimes E).$$

Parts of the residual case are treated in [17]. For sake of simplicity we also assume that $P$ is self-associate. Put

$$\mathrm{I}_{P,\widetilde{\pi},\Lambda} := \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\mathrm{Ind}_{(\mathfrak{l},K_L)}^{(\mathfrak{g},K)}\left[\widetilde{\pi}^\infty_{(K_L)}\otimes\mathbb{C}_{\Lambda+\rho_P}\right]^{m(\widetilde{\pi})} = \mathrm{I}_{P,\widetilde{\pi}}\otimes\mathbb{C}_{\Lambda+\rho_P}$$

and recall the standard intertwining operators $M(\Lambda,\widetilde{\pi},v) : \mathrm{I}_{P,\widetilde{\pi},\Lambda} \to \mathrm{I}_{P,v(\widetilde{\pi}),v(\Lambda)}$, see [32], II, associated to $\Lambda\in\mathfrak{a}^*_{\mathbb{C}}$, $\widetilde{\pi}$ and certain Weyl group elements $v\in W(A) := N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$. If $f\in\mathrm{I}_{P,\widetilde{\pi}}$, we write $f_\Lambda = fe^{\langle\Lambda+\rho_P,H_P(.)\rangle}\in\mathrm{I}_{P,\widetilde{\pi},\Lambda}$. If $M(\Lambda,\widetilde{\pi},v)$ has a pole at $\Lambda=\Lambda_0$, then we assume to have normalized it to a function $N(\Lambda,\widetilde{\pi},v)$, which is holomorphic and non-vanishing in a region containing $\Lambda_0$. Put

$$W(A)_{\mathrm{res}} = \{v\in W(A) | M(\Lambda,\widetilde{\pi},v) \text{ has a pole of order } \ell = \dim\mathfrak{a}_{\mathbb{C}} \text{ at } \Lambda = d\chi\}.$$

This means that the order of the pole is maximal and implies that the longest element $w_0$ of $W(A)$ (as a reduced word in the simple reflections generating $W(A)$) will be inside $W(A)_{\mathrm{res}}$. We have the following

**Theorem 5** ([17], Thm. 2.1)**.** *Let* $[\beta]\in H^q(\mathfrak{g},K,\mathrm{I}_{P,\widetilde{\pi}}\otimes S_\chi(\mathfrak{a}^*)\otimes E)$ *be a class of type* $(\pi,w)$. *If all Eisenstein series* $E_P(f,\Lambda)$, $f\otimes 1$ *in the image of* $\beta$, *have a pole of maximal possible order* $\ell = \dim\mathfrak{a}_{\mathbb{C}}$ *at* $d\chi = -w(\lambda+\rho)|_{\mathfrak{a}_{\mathbb{C}}}$ *inside the closed, positive Weyl chamber defined by* $\Delta(P,A)$ *and if* $\mathrm{Im}N(d\chi,\widetilde{\pi},w_0)$ *is a direct summand of* $\sum_{v\in W(A)_{res}}\mathrm{Im}N(d\chi,\widetilde{\pi},v)$, *then* $E^q_\pi([\beta])$ *contributes at least in degree* $q' := q+\dim N(\mathbb{R})-2l(w)$, $l(w)$ *the length of* $w$.

**Remark 6.** Theorem 5 can always be applied to non-holomorphic Eisenstein series coming from self-associate *maximal* parabolic subgroups $P$, since then $W(A)$ has exactly one non-trivial element. We recall further that Eisenstein series associated to non-self-associate maximal parabolic subgroups are always holomorphic in the region $Re(\Lambda)\geq 0$. See also [32].

Theorem 5 is complemented by the following result

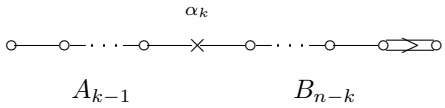
**Theorem 7** ([33], Thm. III. 1)**.** *Let* $\sigma$ *be a residual, cohomological (with respect to a non-regular coefficient module* $E$*) representation of* $G(\mathbb{A})$ *which equals (via the constant term map) the image* $\mathrm{Im}N(d\chi,\widetilde{\pi},w_0)$ *at* $\Lambda = d\chi$ *of the normalization of an intertwining operator* $M(\Lambda,\widetilde{\pi},w_0)$ *which has a pole of maximal order at* $\Lambda = d\chi$. *Suppose further that* $d\chi$ *is inside the open, positive Weyl chamber defined by* $\Delta(P,A)$ *and that* $\widetilde{\pi}_\infty$ *is a tempered representation of* $L(\mathbb{R})$. *If* $r$ *is the lowest degree in which* $\sigma$ *has non-trivial* $(\mathfrak{g},K)$*-cohomology, then the image of* $H^r(\mathfrak{g},K,\sigma\otimes E)$ *in* $H^r_{Eis}(G,E)$ *is non-trivial and consists of residual Eisenstein cohomology classes.*

## 2. Odd orthogonal groups and their maximal parabolic $\mathbb{Q}$-subgroups

2.1. The main objective of this paper is to calculate Eisenstein cohomology of the odd orthogonal group, thus pursuing the above ideas for this case. We will focus on the subspaces $H^q(\mathfrak{g},K,\mathcal{A}_{P,\varphi}(G)\otimes E)$ coming from a *maximal* parabolic $\mathbb{Q}$-subgroup $P$. In this section we provide the algebraic group data for the latter.

From now on $G$ denotes the split odd orthogonal group $SO_{2n+1}$ over $\mathbb{Q}$ of $\mathbb{Q}$-rank $n\geq 3$. The algebra of diagonal matrices $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ determines a Cartan subalgebra as before and we denote by $\Delta^\circ = \{\alpha_1,...,\alpha_n\}$ the set of simple roots of $\mathfrak{g}_{\mathbb{C}}$. If we write $\varepsilon_i$ for the linear functional extracting the $i$-th entry of $\mathfrak{h}_{\mathbb{C}}$, the set $\Delta^\circ$ is given

by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n = \varepsilon_n$. Now, the simple root $\alpha_k$ determines the unique crossed Dynkin diagram




with the $k$-th node replaced by a cross, which in turn corresponds to the unique standard maximal parabolic $\mathbb{Q}$-subgroup $P_k \subset G$, $P_k = L_k N_k = M_k A_k N_k$ with Levi factor $L_k \cong GL_k \times SO_{2l+1}$, $l = n - k$. The correspondence is by means of the requirement that $\alpha_k$ is the only simple root that does not vanish identically on $(\mathfrak{a}_k)_{\mathbb{C}}$. Furthermore, since $\dim N_k = |\Delta^+| - |\Delta^+_{M_k}|$, we see that $\dim N_k = k(2n-k) - \binom{k}{2}$. It is easy to check that associate classes and conjugacy classes of maximal parabolic $\mathbb{Q}$-subgroups $P_k \subset G$ coincide in this case, hence all $n$ maximal parabolic subgroups are self-associate.

There is a canonical isomorphism $\mathfrak{h}^* \cong \mathfrak{a}_k^* \oplus \mathfrak{b}_k^*$, which allows to restrict weights on $\mathfrak{h}_{\mathbb{C}}$ in a canonical way to its direct summands. If $\beta = \sum_{i=1}^n \beta_i \varepsilon_i$, we see by the very definition of $\mathfrak{a}_k$ and $\mathfrak{b}_k$ that $\beta|_{(\mathfrak{a}_k)_{\mathbb{C}}} = \frac{1}{k}\sum_{i=1}^k \left(\sum_{i=1}^k \beta_i\right)\varepsilon_i$ and hence $\beta|_{(\mathfrak{b}_k)_{\mathbb{C}}} = \sum_{i=1}^n \beta_i\varepsilon_i - \frac{1}{k}\sum_{i=1}^k \left(\sum_{i=1}^k \beta_i\right)\varepsilon_i$. Moreover, we get $\rho_{P_k} = \frac{k(2n-k)}{2}\alpha_k|_{(\mathfrak{a}_k)_{\mathbb{C}}}$.

2.2. **Kostant data.** Let us now turn to the Kostant representatives $w \in W^{P_k}$. In particular, we shall calculate the evaluation points $\Lambda_w := -w(\lambda+\rho)|_{(\mathfrak{a}_k)_{\mathbb{C}}}$ of Eisenstein series $E_{P_k}(f,\Lambda)$. Rereading Lemma 4.3 in [42] in view of this latter calculation we prefer to identify the elements $w \in W^{P_k}$ in the form set forth by the following

**Proposition 8.** *For each $k$, $1 \le k \le n$, the Kostant representatives $W^{P_k}$ are parameterized by the set $\mathcal{S}_k$ of all ordered pairs $(I,J)$ of disjoint subsets $I$, $J$ of $\mathbb{N}_{\le n} = \{1,2,...,n\}$ satisfying $|I|+|J| = k$. A parametrization is given as follows: Let $\mathbb{i} = |I|$ and $\mathbb{j} = |J|$, so we can write $I = \{i_1,...,i_{\mathbb{i}}\}$, $J = \{j_1,...,j_{\mathbb{j}}\}$ and $R := \mathbb{N}_{\le n}\backslash I \cup J = \{r_1,...,r_{n-k}\}$. Then the element $w_{(I,J)} \in W^{P_k}$ corresponding to the pair $(I,J)$ is given by*
*$w_{(I,J)}(\varepsilon_{i_l}) := -\varepsilon_{k+1-l}$ for $i_l \in I$,*
*$w_{(I,J)}(\varepsilon_{j_l}) := \varepsilon_l$ for $j_l \in J$ and*
*$w_{(I,J)}(\varepsilon_{r_l}) := \varepsilon_{k+l}$ for $r_l \in R$.*

*Proof.* First of all we notice that $|W^{P_k}| = 2^k\binom{n}{k} = |\mathcal{S}_k|$. So we only need to show that $w_{(I,J)} \in W^{P_k}$. But since

$$w^{-1}_{(I,J)}(\alpha_l) = \begin{cases} \varepsilon_{j_l} - \varepsilon_{j_{l+1}} & 1 \le l \le \mathbb{j}-1 \\ \varepsilon_{j_{\mathbb{j}}} - \varepsilon_{i_{\mathbb{i}}} & l = \mathbb{j} \\ \varepsilon_{i_{k-l}} - \varepsilon_{i_{k-l+1}} & \mathbb{j}+1 \le l \le k-1 \\ \varepsilon_{r_{l-k}} - \varepsilon_{r_{l-k+1}} & k+1 \le l \le n-1 \\ \varepsilon_{r_{n-k}} & l = n \end{cases}$$

$w_{(I,J)} \in W^{P_k}$ by the very definition of $W^{P_k}$. □

**Remark 9.** The description of $w \in W^{P_k}$ as in the Proposition is seen to amount to the one given in [42] by observing that $X_j^k = \{w_{(I,J)} | \mathbb{j} = j\}$ (in the notation of [42]), where the $k$ here corresponds to the $i$ there.

Writing $w_{(I,J)}$ as a word in the simple reflections, the next Lemma is immediate.

**Lemma 10.** *Let* $m := \max(\{l : j_l < i \quad \forall i \in I\} \cup \{0\})$. *Then the length of* $w_{(I,J)}$ *is*

$$l(w_{(I,J)}) = \sum_{l=1}^{\mathbb{i}}(2n-k-i_l+1) + \sum_{l=1}^{\mathbb{j}}(j_l - l) - \sum_{l=m+1}^{\mathbb{j}} |\{i \in I : i < j_l\}|.$$

As announced in the beginning of this section, we want to determine the evaluation points $\Lambda_w = -w(\lambda+\rho)|_{(\mathfrak{a}_k)_{\mathbb{C}}}$. In what follows, we will write $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$. Using our parametrization of the Kostant representatives a straight forward computation shows

**Proposition 11.**

$$-w_{(I,J)}(\lambda+\rho)|_{(\mathfrak{a}_k)_{\mathbb{C}}} = \left(\sum_{l=1}^{\mathbb{i}}(\lambda_{i_l} - i_l) - \sum_{l=1}^{\mathbb{j}}(\lambda_{j_l} - j_l) + (\mathbb{i}-\mathbb{j})(n+\frac{1}{2})\right)\alpha_k|_{(\mathfrak{a}_k)_{\mathbb{C}}}$$

Let us write $t_{(I,J)}$ for the above coefficient of $\alpha_k$ in $-w_{(I,J)}(\lambda+\rho)|_{(\mathfrak{a}_k)_{\mathbb{C}}}$. Then $t_{(I,J)}$ is always a half-integer. Now we compute the highest weights $\mu_{w_{(I,J)}} = w_{(I,J)}(\lambda+\rho) - \rho|_{(\mathfrak{b}_k)_{\mathbb{C}}}$ of the irreducible $M(\mathbb{C})$-modules ${}^\circ F_{w_{(I,J)}}$ by subtracting $w_{(I,J)}(\lambda+\rho) - \rho|_{(\mathfrak{a}_k)_{\mathbb{C}}} = -\left(\frac{1}{k}t_{(I,J)} + n - \frac{k}{2}\right)\sum_{l=1}^k \varepsilon_l$ from $w_{(I,J)}(\lambda+\rho) - \rho$, cf. section 2.1.

**Proposition 12.** *We have*

$$\begin{aligned}\mu_{w_{(I,J)}} \quad = \quad & \sum_{l=1}^{\mathbb{j}}(\lambda_{j_l} - j_l + l + \frac{1}{k}t_{(I,J)} + n - \frac{k}{2})\varepsilon_l \\ & - \sum_{l=1}^{\mathbb{i}}(\lambda_{i_{\mathbb{i}-l+1}} - i_{\mathbb{i}-l+1} - \mathbb{j} - l + n + 1 - \frac{1}{k}t_{(I,J)} + \frac{k}{2})\varepsilon_{\mathbb{j}+l} \\ & + \sum_{l=1}^{n-k}(\lambda_{r_l} - r_l + k + l)\varepsilon_{k+l}\end{aligned}$$

Next, we recall that we may assume that ${}^\circ F_w$ is isomorphic to its contragredient representation ${}^\circ \check{F}_w$. This is due to [2], where it is proved that the existence of a square-integrable automorphic representation of $L_k(\mathbb{A})$ which is cohomological with respect to ${}^\circ F_w$ implies that ${}^\circ F_w$ is self-dual. In particular, if ${}^\circ F_w$ is not self-dual, then there is no cuspidal automorphic representation $\widetilde{\pi}$ which has non-trivial cohomology when twisted by ${}^\circ F_w$.

The finite-dimensional representation ${}^\circ F_w$ being self-dual is equivalent to

$$-w_{L_k}(\mu_w) = \mu_w, \tag{5}$$

where we wrote $w_{L_k}$ for the longest element of the Weyl group of $L_k(\mathbb{C})$. By prop. 12, we see that

$$\begin{aligned}-w_{L_k}(\mu_{w_{(I,J)}}) \quad = \quad & -\sum_{l=1}^{\mathbb{j}}(\lambda_{j_l} - j_l + l + \frac{1}{k}t_{(I,J)} + n - \frac{k}{2})\varepsilon_{k-l+1} \\ & + \sum_{l=1}^{\mathbb{i}}(\lambda_{i_{\mathbb{i}-l+1}} - i_{\mathbb{i}-l+1} - \mathbb{j} - l + n + 1 - \frac{1}{k}t_{(I,J)} + \frac{k}{2})\varepsilon_{k-\mathbb{j}-l+1} \\ & + \sum_{l=1}^{n-k}(\lambda_{r_l} - r_l + k + l)\varepsilon_{k+l}\end{aligned}$$

Assume that $\mathbb{i} < \mathbb{j}$ and (5) holds. Then comparing the coefficient of $\varepsilon_{\mathbb{i}+1}$ in $-w_{L_k}(\mu_w)$ to the coefficient of $\varepsilon_{\mathbb{i}+1}$ in $\mu_w$ leads to the equality

$$\lambda_{j_{\mathbb{i}+1}} + \lambda_{j_{\mathbb{j}}} + \frac{2}{k} t_{(I,J)} + 2n + 1 = j_{\mathbb{i}+1} + j_{\mathbb{j}}. \tag{6}$$

As remarked at the end of section 1.3, we may assume by the work of J. Franke that $t_{(I,J)} \geq 0$. But then the left hand side in (6) is greater or equal to $2n+1$, while the right hand side is at most $2n$. This is a contradiction. So we may assume from now on that $\mathbb{i} \geq \mathbb{j}$.

In order to make the determination of poles of Eisenstein series as simple and efficient as possible, we shall try to find restrictions on the range of evaluation points. In this sense, the following Proposition will be crucial for us.

**Proposition 13.** *There is no $w = w_{(I,J)} \in W^{P_k}$ satisfying $-w_{L_k}(\mu_w) = \mu_w$ and $0 \leq t_{(I,J)} < \frac{k}{2}$.*

*Proof.* First assume $\mathbb{j} > 0$. Suppose that we found a $w = w_{(I,J)} \in W^{P_k}$ satisfying $-w_{L_k}(\mu_w) = \mu_w$ and $0 \leq t_{(I,J)} < \frac{k}{2}$. Then equation (5) implies that the coefficient of $\varepsilon_1$ in $-w_{L_k}(\mu_w)$ and the coefficient of $\varepsilon_1$ in $\mu_w$ must be equal and since $\mathbb{j} > 0$, this reads as

$$(\lambda_{i_1} - \lambda_{j_1}) - (i_1 - j_1) = \frac{2}{k} t_{(I,J)},$$

implying

$$0 \leq (\lambda_{i_1} - \lambda_{j_1}) - (i_1 - j_1) < 1.$$

If $j_1 > i_1$ then $\lambda_{i_1} \geq \lambda_{j_1}$, leading to $(\lambda_{i_1} - \lambda_{j_1}) - (i_1 - j_1) \geq 1$. But if $j_1 < i_1$ then $\lambda_{i_1} \leq \lambda_{j_1}$ and this yields $(\lambda_{i_1} - \lambda_{j_1}) - (i_1 - j_1) \leq -1$. A contradiction.
Now assume $\mathbb{j} = 0$. Then $\mathbb{i} = k$ and by prop. 12 we see that

$$t_{(I,J)} = \sum_{l=1}^{k} (\lambda_{i_l} - i_l) + k(n + \frac{1}{2}) \geq \frac{k}{2}.$$

This proves the claim. □

We shall also see now that there are only very few Kostant representatives $w = w_{(I,J)}$ giving rise to the lowest possible, positive point $\Lambda_w = \frac{k}{2}\alpha|_{(\mathfrak{a}_k)_{\mathbb{C}}}$.

**Proposition 14.** *Suppose $-w_{L_k}(\mu_{w_{(I,J)}}) = \mu_{w_{(I,J)}}$ and $t_{(I,J)} = \frac{k}{2}$. Then, depending on the parity of $k$,*
*$I = \{i_1, ..., i_{\mathbb{i}} = n\}$, $J = \{i_1 + 1, ..., i_{\mathbb{i}-1} + 1\}$, $\lambda_{i_l} = \lambda_{i_l+1}$, $1 \leq l \leq \mathbb{i} - 1$ and $\lambda_n = 0$ if $k$ is odd,*
*$I = \{i_1, ..., i_{\frac{k}{2}}\}$, $J = \{i_1 + 1, ..., i_{\frac{k}{2}} + 1\}$, $\lambda_{i_l} = \lambda_{i_l+1}$, $1 \leq l \leq \frac{k}{2}$ if $k$ is even. In particular the length of such an $w_{(I,J)}$ is unique and given by*

$$l(w_{(I,J)}) = \begin{cases} \frac{k-1}{2}(2n - \frac{3(k-1)}{2}) + (n - k + 1) & \text{if } k \text{ is odd} \\ \frac{k}{2}(2n - \frac{3k}{2} + 1) & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* Recall $\mathbb{i} \geq \mathbb{j}$. Comparing the coefficients of $\varepsilon_l$, $1 \leq l \leq \mathbb{j}$, in $-w_{L_k}(\mu_{w_{(I,J)}})$ and in $\mu_{w_{(I,J)}}$ gives us as in the proof of Proposition 13

$$(\lambda_{i_l} - \lambda_{j_l}) - (i_l - j_l) = \frac{2}{k} t_{(I,J)} = 1.$$

Therefore $j_l = i_l + 1$ and $\lambda_{i_l} = \lambda_{i_l+1}$ for $1 \leq l \leq \mathbb{j}$. This shows the claim, if $\mathbb{i} = \mathbb{j}$ (which forces $k$ to be even). So assume that $\mathbb{i} > \mathbb{j}$. Inserting $j_l = i_l + 1$ and

$\lambda_{i_l} = \lambda_{i_l+1}$ into the formula for $t_{(I,J)}$ given in Proposition 11, yields

$$\sum_{l=\mathbb{j}+1}^{\mathbb{i}} i_l - n(\mathbb{i} - \mathbb{j}) = \sum_{l=\mathbb{j}+1}^{\mathbb{i}} \lambda_{i_l} \geq 0.$$

But the left hand side of this equation is less or equal to 0, with equality if and only if $\mathbb{i} = \mathbb{j} + 1$ and $i_{\mathbb{i}} = n$.
The formula for the length of $w_{(I,J)}$ is now a direct consequence of Lemma 10. Hence the claim. □

Assume for the rest of this section that $k < n$ and that $k$ is even. As we will see in section 4.2, we need to know which $w = w_{(I,J)} \in W^{P_k}$ give rise to $\Lambda_w = k\alpha|_{(\mathfrak{a}_k)_{\mathbb{C}}}$ for such $k$. There are more possibilities than in the case of $\frac{k}{2}$ and we classify them in the next Proposition. We omit the technical proof, as it is completely analogous to the proof of Proposition 14

**Proposition 15.** *Suppose $-w_{L_k}(\mu_{w_{(I,J)}}) = \mu_{w_{(I,J)}}$ and $t_{(I,J)} = k < n$ is even. Then, one of the following holds,*

(i) $I = \{i_1, ..., i_{\mathbb{i}-1} = n-1, i_{\mathbb{i}} = n\}$, $J = \{i_1 + 1, ..., i_{\mathbb{i}-2} + 1\}$, $\lambda_{i_l} = \lambda_{i_l+1} + 1$, $1 \leq l \leq \mathbb{i} - 2$, $\lambda_{n-1} = \lambda_n = 0$ *and*

$$l(w_{(I,J)}) = k(n - \frac{3k}{4} + \frac{1}{2}) + 1,$$

(ii) $I = \{i_1, ..., i_{\mathbb{i}-1} = n-1, i_{\mathbb{i}} = n\}$, $J = \{i_1 + 2, ..., i_{\mathbb{i}-2} + 2\}$, $\lambda_{i_l} = \lambda_{i_l+1}$, $1 \leq l \leq \mathbb{i} - 2$ *and*

$$k(n - \frac{3k}{4} + 1) - \lfloor \frac{k-2}{4} \rfloor \leq l(w_{(I,J)}) \leq k(n - \frac{3k}{4} + 1),$$

(iii) $I = \{i_1, ..., i_{\frac{k}{2}}\}$, $J = \{i_1 + 1, ..., i_{\frac{k}{2}} + 1\}$, $\lambda_{i_l} = \lambda_{i_l+1} + 1$, $1 \leq l \leq \frac{k}{2}$ *and*

$$l(w_{(I,J)}) = k(n - \frac{3k}{4} + \frac{1}{2}),$$

(iv) $I = \{i_1, ..., i_{\frac{k}{2}}\}$, $J = \{i_1 + 2, ..., i_{\frac{k}{2}} + 2\}$, $\lambda_{i_l} = \lambda_{i_l+1}$, $1 \leq l \leq \frac{k}{2}$ *and*

$$k(n - \frac{3k}{4} + 1) - \lfloor \frac{k}{4} \rfloor \leq l(w_{(I,J)}) \leq k(n - \frac{3k}{4} + 1),$$

*In any case,*

$$k(n - \frac{3k}{4} + \frac{1}{2}) \leq l(w_{(I,J)}) \leq k(n - \frac{3k}{4} + 1) \tag{7}$$

## 3. Cohomological cuspidal representations

3.1. As a second ingredient to Eisenstein cohomology we shall determine the $(\mathfrak{m}_k, K_{M_k})$-*cohomological* (unitary) cuspidal automorphic representations $\widetilde{\pi} \in \varphi_{P_k}$ of $L_k(\mathbb{A})$. It is clear that $\widetilde{\pi} = \sigma \hat{\otimes} \tau$, where $\sigma$ (resp. $\tau$) is a cohomological, unitary cuspidal automorphic representation of $GL_k(\mathbb{A})$ (resp. $SO_{2l+1}(\mathbb{A})$). Further, a representation is cohomological if and only if its infinite component is. Let us first consider the $GL_k$-factor.

3.2. Recall that $\sigma_\infty$ is actually a representation of the semisimple part of $GL_k(\mathbb{R})$, which is $SL_k^{\pm}(\mathbb{R})$. If $k = 1$ then $\sigma_\infty$ must be the same character as the one of the coefficient module in cohomology and if $k = 2$, it must be a discrete series representation. So suppose $k > 2$. The cohomological irreducible, unitary representations $\sigma_\infty$ of $SL_k^{\pm}(\mathbb{R})$, $k > 2$, are implicitly classified in [41] Thm. 4.2.2. Let us recall this result shortly for the case of generic representations. This is no restriction, since cuspidal automorphic representations of $GL_k(\mathbb{A})$, $k \geq 2$, are all globally generic (cf. [39] corollary on p. 190). Let $a = \lfloor \frac{k}{2} \rfloor$, $b = k - 2a$ and put $M(\mathbb{R}) = \prod_{i=1}^{a} SL_2^{\pm}(\mathbb{R}) \times \{\pm 1\}^b$. Then we can say

**Proposition 16** ([41] Thm. 4.2.2)**.** *If $\sigma_\infty$ is a generic, cohomological, irreducible, unitary representation of $SL_k^{\pm}(\mathbb{R})$, $k > 2$, then*

$$\sigma_\infty = \mathrm{Ind}_{P(\mathbb{R})}^{SL_k^{\pm}(\mathbb{R})}[\sigma_1 \otimes ... \otimes \sigma_{a+1} \otimes \mathbb{C}_{\rho_P}].$$

*Here $P(\mathbb{R}) = M(\mathbb{R})A(\mathbb{R})^\circ N(\mathbb{R})$ is a parabolic subgroup of $SL_k^{\pm}(\mathbb{R})$ with semisimple part $M(\mathbb{R})$ as above and $\sigma_i$ is either a power of the sign character or a discrete series representation of $SL_2^{\pm}(\mathbb{R})$.*

This Proposition together with our above considerations implies the following remarkable fact: A generic, cohomological, irreducible, unitary representation $\sigma_\infty$ of $SL_k^{\pm}(\mathbb{R})$, $k \geq 2$, is necessarily tempered (because it is fully and unitarily induced from a discrete series representation). Hence, by [4] III, Prop. 5.3, we can conclude

**Proposition 17.** *Let $\sigma$ be a cuspidal automorphic representation of $GL_k(\mathbb{A})$, $k \geq 1$, as above. If $\sigma$ has non-zero $(\mathfrak{sl}_k(\mathbb{R}), SO(k))$-cohomology in degree $q$ then*

$$\frac{1}{2}\left(\frac{k(k-1)}{2} + \lfloor \frac{k}{2} \rfloor\right) \leq q \leq \frac{1}{2}\left(\frac{(k-1)(k+4)}{2} - \lfloor \frac{k}{2} \rfloor\right).$$

See also [34] Thm. 3.3 for a similar result.

3.3. Let us now turn to the $SO_{2l+1}$-factor and a cohomological, cuspidal automorphic representation $\tau$ of it. We suppose that $\tau_\infty$ is locally generic. Combining Kostant's characterization of generic Harish-Chandra modules in [27] with Vogan's description of large Harish-Chandra modules in [43] Thm. 6.2, we see that again $\tau_\infty$ must be induced from a discrete series representation. Furthermore, the Gelfand-Kirillov dimension (cf. [43] for a definition) of $\tau_\infty$ must be maximal; that is equal to $l^2$ in our present case. On the other hand $\tau_\infty$ is cohomological, whence it is an $A_{\mathfrak{q}}(\lambda)$-module in the sense of Vogan and Zuckerman ([44]). By checking which $A_{\mathfrak{q}}(\lambda)$-modules of $SO(l+1,l)^\circ(\mathbb{R})$ are actually of Gelfand-Kirillov dimension $l^2$, we see that $\tau_\infty$ must be a discrete series representation. Hence we have proved

**Proposition 18.** *If $\tau_\infty$ is a generic, cohomological, irreducible unitary representation of $SO(l+1,l)(\mathbb{R})$ then it is in the discrete series.*

By [4] II, Thm. 5.4 we therefore have

**Proposition 19.** *Let $\tau$ be a cuspidal automorphic representation of $SO_{2l+1}(\mathbb{A})$ as above. If $\tau$ has non-zero $(\mathfrak{so}(l+1,l), S(O(l+1) \times O(l)))$-cohomology in degree $q$ then*

$$q = \frac{l^2 + l}{2}.$$

3.4. Putting our Propositions 17 and 19 together we can finally conclude by the Künneth rule ([4] 1.3) the following

**Theorem 20.** *Let $\widetilde{\pi} = \sigma\hat{\otimes}\tau \in \varphi_{P_k}$ be a cuspidal automorphic representation of $L_k(\mathbb{A})$, with $\tau$ having a generic archimedean component $\tau_\infty$. If $\widetilde{\pi}$ has non-trivial $(\mathfrak{m}_k, K_{M_k})$-cohomology in degree $q$ then*

$$\frac{1}{2}\left(\frac{k(k-1)}{2} + \lfloor\frac{k}{2}\rfloor + l^2 + l\right) \leq q \leq \frac{1}{2}\left(\frac{(k-1)(k+4)}{2} - \lfloor\frac{k}{2}\rfloor + l^2 + l\right).$$

## 4. Poles of Eisenstein series

4.1. We will now calculate the possible poles of our Eisenstein series. By the Langlands "Square-Integrability Criterion", [32], Lemma I.4.11, this amounts to determining the part of the residual spectrum of $G(\mathbb{A}) = SO_{2n+1}(\mathbb{A})$ which is given by the maximal parabolic subgroups $P_k$, a task that was achieved by H. Kim in [25]. For the convenience of the reader and for sake of completeness of our presentation we repeat his arguments briefly. Still our results here will be in a somewhat different guise.

Let now $\pi = \hat{\otimes}'\pi_p$ be a (cohomological) globally *generic* cuspidal automorphic representation of a maximal Levi subgroup $L_k(\mathbb{A}) = GL_k(\mathbb{A}) \times SO_{2l+1}(\mathbb{A})$, $1 \leq k \leq n$, $l = n - k$. Such a representation is necessarily of the form $\pi = \sigma\hat{\otimes}\tau$, where $\sigma$ is a generic cuspidal representation of $GL_k(\mathbb{A})$ and $\tau$ a generic, cuspidal representation of $SO_{2l+1}(\mathbb{A})$. It enjoys Strong Multiplicity One, combining the results Thm. 4.4 in [19], and Thm. 9 in [13]. Hence, by the multiplicity one Theorem for $GL_k$, $m(\pi) = m(\tau)$ holds and following Arthur's Conjecture, even $m(\tau) = 1$ should be true. We will not assume this. We identify $\Lambda = t\alpha_k \in (\mathfrak{a}_k)^*_{\mathbb{C}}$ with $s = \frac{t}{k} \in \mathbb{C}$ if $k < n$ and with $s = \frac{2t}{n} \in \mathbb{C}$ if $k = n$, following [38], p. 552. We will also omit the subscript "$k$".

Let $f \in \mathrm{I}_{P,\pi}$ then $f_s = fe^{(s+\rho_P)H_P(.)} \in \mathrm{I}_{P,\pi,s}$. The holomorphic behavior of the Eisenstein series $E_P(f,s)$ is the same as the one of its constant term along $P$ (cf. [30], [32], IV.1.10), which can be rewritten as

$$E_P(f,s)_P = f_s + M(s,\pi)f_s.$$

Here $M(s,\pi)$ is the standard intertwining operator (cf. section 1.5) of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules

$$M(s,\pi) : \mathrm{I}_{P,\pi,s} \to \mathrm{I}_{P,w_0(\pi),-s}$$

$$M(s,\pi)f_s(g) = \int_{N(\mathbb{Q})\cap w_0N(\mathbb{Q})w_0^{-1}\backslash N(\mathbb{A})} f_s(w_0^{-1}ng)dn,$$

$w_0$ the only non-trivial element in $W(A) = N_{G(\mathbb{Q})}(A(\mathbb{Q}))/L(\mathbb{Q})$. Therefore the poles of $E_P(f,s)$ are determined by the interplay of the poles and zeros of $M(s,\pi)$. By twisting $\pi$ by an appropriate imaginary power of the absolute value of the determinant we may and will assume that all poles are real, that is $s = \Re e(s)$ in the sequel.

Let $S$ be the finite set of places containing the archimedean one and the places where $\pi$ ramifies. Using the Langlands-Shahidi method (cf. [28, 37, 38]) we see that for suitably normalized, $L(\mathbb{Z}_p)$-fixed functions $\widetilde{f}_{s,p}$

$$M(s,\pi)f_s = \bigotimes_{p\in S} A(s,\pi_p)f_{s,p} \otimes \prod_{p\notin S} \frac{L(s,\sigma_p\times\tau_p)L(2s,\sigma_p,\mathrm{Sym}^2)}{L(1+s,\sigma_p\times\tau_p)L(1+2s,\sigma_p,\mathrm{Sym}^2)}\widetilde{f}_{s,p}.$$

if $k < n$ and

$$M(s,\pi)f_s = \bigotimes_{p\in S} A(s,\pi_p)f_{s,p} \otimes \prod_{p\notin S} \frac{L(s,\sigma_p,\mathrm{Sym}^2)}{L(1+s,\sigma_p,\mathrm{Sym}^2)} \widetilde{f}_{s,p}.$$

if $k=n$. Here $L(s,\sigma_p\times\tau_p)$ is the (local) Rankin-Selberg $L$-function associated to $\pi_p=\sigma_p\hat{\otimes}\tau_p$ and $L(s,\sigma_p,\mathrm{Sym}^2)$ is the (local) symmetric square $L$-function of $\sigma_p$. For convenience, we distinguish the cases $k<n$ and $k=n$ in what follows.

4.2. $k<n$**.** Shahidi defined the $L$-functions $L(s,\sigma_p\times\tau_p)$ and $L(s,\sigma_p,\mathrm{Sym}^2)$ also for the places $p\in S$. By [25], Prop. 4.1,

$$N(s,\pi_p) = \frac{L(1+s,\sigma_p\times\tau_p)L(1+2s,\sigma_p,\mathrm{Sym}^2)}{L(s,\sigma_p\times\tau_p)L(2s,\sigma_p,\mathrm{Sym}^2)} A(s,\pi_p)$$

is holomorphic and non-zero for $s\geq 0$. Therefore

**Proposition 21.** *There is an $f\in \mathrm{I}_{P,\pi}$ such that the Eisenstein series $E_P(f,s)$ has a pole at $s=s_0$, $s_0>0$, if and only if*

$$\frac{L(s,\sigma\times\tau)L(2s,\sigma,\mathrm{Sym}^2)}{L(1+s,\sigma\times\tau)L(1+2s,\sigma,\mathrm{Sym}^2)}$$

*has a pole at $s=s_0$.*

It is well-known ([25, 24]) that $L(s,\sigma\times\tau)$ is meromorphic with possible poles only at $s=0,1$ and non-vanishing for $s>1$. Similarly, $L(s,\sigma,\mathrm{Sym}^2)$ is holomorphic for $s\geq 1$ (except possibly at $s=1$) and non-zero there. Hence, the poles of $M(s,\pi)$ in $s>0$ are the ones of

$$L(s,\sigma\times\tau)L(2s,\sigma,\mathrm{Sym}^2) \tag{8}$$

and so - by what we just observed - we conclude that the only possible poles of Eisenstein series $E_P(f,s)$ in the region $s\geq\frac{1}{2}$ are at $s=\frac{1}{2},1$. This is enough for us, as we will only need to consider Eisenstein series at $s\geq\frac{1}{2}$, cf. Proposition 13. Just for completeness, let us remark that it is not known, if $L(2s,\sigma,\mathrm{Sym}^2)$ has poles in the remaining region $0<s<\frac{1}{2}$. But it is shown in Cor. 3.2 of [16] that this is not the case, i.e., $L(2s,\sigma,\mathrm{Sym}^2)$ is holomorphic for $0<s<\frac{1}{2}$, if Arthur's Conjecture as formulated in section 30 of [1] (see also section 2 of [16] for a precise reformulation adapted to this purpose) on the discrete spectrum holds. However, we will not need this.

Let us render the above more precise. If $\sigma$ is not self-dual, then both $L$-functions in (8) are entire. So let us from now on assume that $\sigma$ is self-dual.

Let us first consider the case $s=\frac{1}{2}$. Then the pole can originate only from the symmetric square $L$-function. If either $k$ is odd or the central character $\omega_\sigma$ of $\sigma$ is non-trivial, then this is the case, i.e., $L(2s,\sigma,\mathrm{Sym}^2)$ has a pole at $s=\frac{1}{2}$. This is well-known and can be seen as follows: The statement for odd $k$ is a consequence of the Rankin-Selberg convolutions of either [5] or [23]. If $k$ is even, but $\omega_\sigma\neq\mathbf{1}$, then the corresponding assertion is proved in Prop. 3.7 of [25]. So, let now $k=2$ and $\omega_\sigma\equiv\mathbf{1}$. Then $L(2s,\sigma,\mathrm{Sym}^2)$ is holomorphic at $s=\frac{1}{2}$. This is clear by the following easy consideration: Recall that

$$L(s,\sigma\times\sigma) = L(s,\sigma,\mathrm{Sym}^2)L(s,\sigma,\wedge^2) \tag{9}$$

has a simple pole at $s=1$, because $\sigma$ is supposed to be self-dual. We also may replace all $L$-functions by their partial analogues with respect to the set $S$ without

changing this assertion, since $\sigma_\infty$ is as cohomological representation of discrete series, whence all $\sigma_p$ are tempered ([8], Thm. I.6, together with [7], Thm. 5.6) and so the local $L$-functions of $\sigma_p$ are holomorphic and non-vanishing for $s > 0$. But for $p \notin S$, $L(s, \sigma_p, \wedge^2) = L(s, \omega_{\sigma_p})$, which has a pole at $s = 1$ as $\omega_\sigma \equiv \mathbf{1}$ by assumption and so together with (9) we see that $L(s, \sigma, \mathrm{Sym}^2)$ must be holomorphic at $s = 1$.

In order to understand the situation better in the case of even $k \geq 4$ and $\omega_\sigma \equiv \mathbf{1}$, we reformulate it in terms of the Weak Langlands Functoriality. The references for this are [6], [13], [14] and [40]. Suppose that $k \geq 4$ is even and that the central character of $\sigma$ is trivial. In this situation we know by automorphic descent that if $L(2s, \sigma, \mathrm{Sym}^2)$ has a pole at $s = \frac{1}{2}$, then $\sigma$ is the Weak Langlands Functorial Lift from a globally generic cuspidal automorphic representation of $SO_k(\mathbb{A})$. On the other hand, if $L(2s, \sigma, \mathrm{Sym}^2)$ is holomorphic at $s = \frac{1}{2}$, then automorphic descent tells us that $\sigma$ is the Weak Langlands Functorial Lift from a globally generic cuspidal automorphic representation of $SO_{k+1}(\mathbb{A})$. We may conclude that $\sigma$ will therefore not be a Weak Langlands Functorial Lift from a cuspidal automorphic representation of $SO_k(\mathbb{A})$. We remark that $s = \frac{1}{2}$ means $t = \frac{k}{2}$ and get the following

**Proposition 22** (Poles for $s = \frac{1}{2}$)**.** *There is an $f \in \mathrm{I}_{P,\pi}$ such that the Eisenstein series $E_P(f, s)$ has a pole at $s = \frac{1}{2}$ if and only if*

1. *(In case $k \geq 4$ is even and $\omega_\sigma \equiv \mathbf{1}$): $\sigma$ is a self-dual Weak Langlands functorial lift of a globally generic cuspidal automorphic representation of $SO_k(\mathbb{A})$ and $L(\frac{1}{2}, \sigma \times \tau) \neq 0$.*
2. *(In case $k$ is odd or $\omega_\sigma \neq \mathbf{1}$): $\sigma$ is self-dual and $L(\frac{1}{2}, \sigma \times \tau) \neq 0$.*

The case $s = 1$ is similar. There the pole can originate only from the Rankin-Selberg $L$-function. It is entire, if $k$ is not even, [25]. But we know that $L(2, \sigma, \mathrm{Sym}^2) \neq 0$. Hence we get the following

**Proposition 23** (Poles for $s = 1$)**.** *There is an $f \in \mathrm{I}_{P,\pi}$ such that the Eisenstein series $E_P(f, s)$ has a pole at $s = 1$ if and only if $\sigma$ is self-dual, $k$ is even and $L(s, \sigma \times \tau)$ has a pole at $s = 1$.*

Observe that $s = 1$ corresponds to $t = k$.

4.3. $k = n$**.** The remaining case $k = n$ is treated in complete analogy to the previous one. We only have to observe that there is no Rankin-Selberg $L$-function appearing in the normalization factor. Again,

$$N(s, \pi_p) = \frac{L(1 + s, \sigma_p, \mathrm{Sym}^2)}{L(s, \sigma_p, \mathrm{Sym}^2)} A(s, \pi_p)$$

is holomorphic and non-vanishing for $s > 0$, see [25], Prop. 4.1, so the poles of $M(s, \pi)$ with $P = P_n$ and $s > 0$ are determined by the holomorphic behavior of the quotient

$$\frac{L(s, \sigma, \mathrm{Sym}^2)}{L(1 + s, \sigma, \mathrm{Sym}^2)}.$$

As observed before, $L(1 + s, \sigma, \mathrm{Sym}^2)$ is holomorphic and non-vanishing at $s > 0$, so we have to analyze $L(s, \sigma, \mathrm{Sym}^2)$. If $k = n$, it will be enough for us to consider the region $s \geq 1$, see Proposition 13. Hence we can deduce that the only possible pole of an Eisenstein series which interests us can be at $s = 1$. As our parameter $s$ comes from $\Lambda = \frac{ns}{2}\alpha_n$ in the case $k = n$, we see that actually $t$ must be $\frac{n}{2}$. We have therefore proved

**Proposition 24** (Poles for $s=1$)**.** *There is an $f \in \mathrm{I}_{P,\pi}$ such that the Eisenstein series $E_P(f,s)$ has a pole at $s=1$ if and only if*

(1) *(In case $n \geq 4$ is even and $\omega_\sigma \equiv \mathbf{1}$): $\sigma$ is a self-dual Weak Langlands functorial lift of a globally generic cuspidal automorphic representation of $SO_n(\mathbb{A})$.*
(2) *(In case $n$ is odd or $\omega_\sigma \neq \mathbf{1}$): $\sigma$ is self-dual.*

4.4. **Beyond genericity.** Let us remark on the case of *non*-generic cuspidal automorphic representations $\pi$ of $L_k$. First of all, we know that such representations really exist. This was proved in [20] by showing that $SO_{2l+1}$ has CAP-representations, i.e. cuspidal automorphic representations $\tau$ which are nearly equivalent to a subquotient of $\mathcal{A}_{Eis}(G)$. CAP representations are expected to give counterexamples to the naively generalized Ramanujan Conjecture, which says that for each cuspidal automorphic representation $\tau$ all local components $\tau_p$ are tempered. On the other hand, reinterpreting Shahidi's conjecture in [36] on the holomorphy of local $L$-functions associated to tempered local representations, each tempered local L-packet should contain a locally generic irreducible admissible representation. Jiang and Soudry have proved this conjecture for $SO_{2l+1}$ by virtue of [21] and [22]. So vaguely speaking in terms of L-packets, cuspidal automorphic forms should disintegrate into the generic ones and the other part containing the CAP representations - each world being non-empty. Still, generic cuspidal representations should "generate" the whole cuspidal spectrum in the following way: It is conjectured (cf. [20] Conj. 1.1) that for each cuspidal automorphic representation $\tau$ of $SO_{2l+1}(\mathbb{A})$ there is a (possibly not proper) parabolic subgroup $P' = L'N'$ and a generic cuspidal automorphic representation $\tau'$ of $L'$ such that $\tau$ is nearly equivalent to an irreducible constituent of $\mathrm{Ind}_{P'(\mathbb{A})}^{SO_{2l+1}(\mathbb{A})}[\tau']$. For $SO_{2l+1}$ the cuspidal datum $(P',\tau')$ is an invariant for $\tau$ (up to near equivalence), see [20] Cor. 3.3.(3). In this setup, generic cuspidal representations $\tau$ should be characterized by $P'$ being the whole group $SO_{2l+1}$ and CAP representations by $P'$ being proper.

## 5. Residual Eisenstein Cohomology

5.1. **Residual Eisenstein Classes for $k<n$.** We are ready to state the first of our two main Theorems.

**Theorem 25.** *Let $0 \neq [\beta]$ be a class of type $(\pi,w)$, $\pi = \chi\widetilde{\pi} \in \varphi_{P_k}$ with $\widetilde{\pi} = \sigma\hat{\otimes}\tau$ a globally generic cuspidal automorphic representation of $L_k(\mathbb{A})$ and $w \in W^{P_k}$ represented by a homomorphism the image of which contains only functions $f \otimes 1 \in \mathrm{I}_{P_k,\widetilde{\pi}}$ for which $E_{P_k}(f,\Lambda)$ has a pole at the point $\Lambda_w = -w(\lambda+\rho)|_{(\mathfrak{a}_k)_\mathbb{C}} \in C$. Then:*

(1) *$\pi = \chi(\sigma\hat{\otimes}\tau)$ is either of the form*
   (a) *$d\chi = \Lambda_w = \frac{k}{2}\alpha|_{(\mathfrak{a}_k)_\mathbb{C}}$, $\sigma$ a self-dual cuspidal automorphic representation of $GL_k(\mathbb{A})$ and such that $L(\frac{1}{2},\sigma\times\tau) \neq 0$. If in addition $k$ is even and $\omega_\sigma \equiv \mathbf{1}$, then $k \geq 4$ and $\sigma$ is a self-dual Weak Langlands functorial lift of a globally generic cuspidal automorphic representation of $SO_k(\mathbb{A})$. Or*
   (b) *$d\chi = \Lambda_w = k\alpha|_{(\mathfrak{a}_k)_\mathbb{C}}$, $k$ even and $\sigma$ a self-dual cuspidal automorphic representation of $GL_k(\mathbb{A})$ such that $L(s,\sigma\times\tau)$ has a pole at $s=1$.*
(2) *The degree $q'$ of the residual Eisenstein cohomology class constructible from $E^q_\pi([\beta])$ as in Theorem 5 is necessarily in the following range*
   (a) *If $d\chi = \frac{k}{2}\alpha|_{(\mathfrak{a}_k)_\mathbb{C}}$*

$$\frac{1}{2}(n^2+n) - \lceil\frac{k}{2}\rceil \leq q' \leq \frac{1}{2}(n^2+n) - 1.$$

(b) *If* $d\chi = k\alpha|_{(\mathfrak{a}_k)_{\mathbb{C}}}$

$$\frac{1}{2}(n^2+n)-k \leq q' \leq \frac{1}{2}(n^2+n)-1 \quad .$$

*Proof.* Part (1) follows directly from our Propositions 22 and 23. For (2) we insert the formula from Proposition 14 for the length of $w = w_{(I,J)} \in W^{P_k}$ giving rise to $t_{(I,J)} = \frac{k}{2}$ into the equation $q' = q - 2l(w_{(I,J)}) + \dim N_k(\mathbb{R})$ and then use the bounds for $q - l(w)$ established in Theorem 20. We do the same for $w = w_{(I,J)} \in W^{P_k}$ giving rise to $t_{(I,J)} = k$, using Proposition 15 (7). This then proves (2). □

5.2. **Residual Eisenstein Classes for $k = n$.** It remains to settle the case of the Siegel parabolic subgroup. We prove

**Theorem 26.** *Let $0 \neq [\beta]$ be a class of type $(\pi, w)$, $\pi = \chi\widetilde{\pi} \in \varphi_{P_n}$ with $\widetilde{\pi} = \sigma$ a cuspidal automorphic representation of $L_n(\mathbb{A}) = GL_n(\mathbb{A})$ and $w \in W^{P_n}$ represented by a homomorphism the image of which contains only functions $f \otimes 1 \in \mathrm{I}_{P_n,\widetilde{\pi}}$ for which $E_{P_n}(f,\Lambda)$ has a pole at the point $\Lambda_w = -w(\lambda+\rho)|_{(\mathfrak{a}_n)_{\mathbb{C}}} \in C$. Then:*

(1) *$\pi = \chi\sigma$ is of the form*
*$d\chi = \Lambda_w = \frac{n}{2}\alpha|_{(\mathfrak{a}_n)_{\mathbb{C}}}$ and $\sigma$ is self-dual. If in addition $n$ is even and $\omega_\sigma \equiv \mathbf{1}$, then $n \geq 4$ and $\sigma$ is a self-dual Weak Langlands functorial lift of a globally generic cuspidal automorphic representation of $SO_n(\mathbb{A})$.*

(2) *The degree $q'$ of the residual Eisenstein cohomology class constructible from $E^q_\pi([\beta])$ as in Theorem 5 is necessarily in the following range*

$$\frac{1}{2}(n^2+n) - \lceil\frac{n}{2}\rceil \leq q' \leq \frac{1}{2}(n^2+n)-1.$$

*Proof.* As before, part (1) follows already from Proposition 24. For (2) we again insert the formula from Proposition 14 for the length of $w = w_{(I,J)} \in W^{P_n}$ giving rise to $t_{(I,J)} = \frac{n}{2}$ into the equation $q' = q - 2l(w_{(I,J)}) + \dim N_n(\mathbb{R})$ and then use the bounds for $q - l(w)$ established in Theorem 20, respectively in Proposition 17. □

5.3. **A remark on the lower bound.** Recall that by [44], Table 8.2, $n$ is the lowest possible degree in which there could be non-trivial, square-integrable, residual Eisenstein cohomology other than that coming from the trivial representation. In fact, $n \leq \frac{1}{2}(n^2+n) - \lceil\frac{k}{2}\rceil$ for all $n \geq 2$ and $k$, resp. $n \leq \frac{1}{2}(n^2+n) - k$ for all $n \geq 3$ and even $k$. We do not know if the lower bounds established by us in Theorems 25 and 26 are in fact sharp.

## 6. Regular Eisenstein Cohomology

6.1. We conclude the paper discussing regular Eisenstein cohomology classes. Therefore, let $0 \neq [\beta]$ be a class of type $(\pi, w)$, $\pi = \chi\widetilde{\pi} \in \varphi_{P_k}$ with $\widetilde{\pi} = \sigma\hat{\otimes}\tau$ a globally generic cuspidal automorphic representation of $L_k(\mathbb{A})$ and $w \in W^{P_k}$.

Obviously, if either $d\chi$ is neither $\frac{k}{2}\alpha|_{(\mathfrak{a}_k)_{\mathbb{C}}}$ nor $k\alpha|_{(\mathfrak{a}_k)_{\mathbb{C}}}$ or if $\widetilde{\pi}$ is not of the form described in Theorem 25 (1) or 26 (1), then for any $f \in \mathrm{I}_{P_k,\widetilde{\pi}}$ the associated Eisenstein series $E_{P_k}(f,\Lambda)$ will be holomorphic at $\Lambda = d\chi$. This is the content of our section 4. In particular, in this case, the image of a homomorphism $\beta$ representing the class $[\beta]$ can contain only tensors $f \otimes \frac{d^\nu}{d\Lambda^\nu}$ for which the associated Eisenstein series $E_{P_k}(f,\Lambda)$ is holomorphic at $\Lambda = d\chi$. If in addition $\nu = 0$, i.e. $\frac{d^\nu}{d\Lambda^\nu} = 1$, then $E^q_\pi([\beta])$ is a non-trivial regular Eisenstein cohomology class in degree $q$ with

$$\frac{1}{2}\left(\frac{k(k-1)}{2} + \lfloor\frac{k}{2}\rfloor + l^2 + l\right) + l(w) \leq q \leq \frac{1}{2}\left(\frac{(k-1)(k+4)}{2} - \lfloor\frac{k}{2}\rfloor + l^2 + l\right) + l(w),$$

see Theorem 4 and Theorem 20.

6.2. Now let $d\chi$ and $\widetilde{\pi}$ be as in Theorem 25 (1) or 26 (1), i.e., there are $f \in \mathrm{I}_{P_k,\widetilde{\pi}}$ such that the associated Eisenstein series $E_{P_k}(f,\Lambda)$ has a pole at $\Lambda = d\chi$. Still, if $f$ gives rise to a local component $f_{s,p} \in \ker A(s,\widetilde{\pi}_p)$, where $s$ is according to $d\chi$ either $\frac{1}{2}$ or 1, then the zero $A(s,\widetilde{\pi}_p)f_{s,p} = 0$ will cancel the simple pole of the global operator $M(s,\widetilde{\pi})$ and again $E_{P_k}(f,\Lambda)$ will be holomorphic at $\Lambda = d\chi$. Let $0 \neq [\beta]$ be a class of type $(\pi, w)$, represented by a homomorphism $\beta$ whose image in $\mathrm{I}_{P_k,\widetilde{\pi}} \otimes S_\chi(\mathfrak{a}^*)$ consists of tensors $f \otimes \frac{d^\nu}{d\Lambda^\nu}$ with $f$ as above. That is, $f$ gives rise to a local component $f_{s,p} \in \ker A(s,\widetilde{\pi}_p)$, $s = \frac{1}{2}, 1$. Then (cf. section 1.5)

$$E_{P_k,\pi}(f \otimes \frac{d^\nu}{d\Lambda^\nu}) = E_{P_k}(f, d\chi)$$

if $\nu = 1$. Assuming this, it follows now again from Theorem 4 that $E^q_\pi([\beta])$ is a non-trivial regular Eisenstein cohomology class. Its degree $q$ is bounded by

$$\frac{1}{2}\left(\frac{k(k-1)}{2} + \lfloor\frac{k}{2}\rfloor + l^2 + l\right) + l(w) \leq q \leq \frac{1}{2}\left(\frac{(k-1)(k+4)}{2} - \lfloor\frac{k}{2}\rfloor + l^2 + l\right) + l(w),$$

see Theorem 20.

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