DELIGNE'S CONJECTURE FOR AUTOMORPHIC MOTIVES OVER CM-FIELDS, PART I: FACTORIZATION

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ABSTRACT. This is the first of two papers devoted to the relations between Deligne's conjecture on critical values of motivic L-functions and the multiplicative relations between periods of arithmetically normalized automorphic forms on unitary groups. The present paper combines the Ichino-Ikeda-Neal Harris (IINH) formula with an analysis of cup products of coherent cohomological automorphic forms on Shimura varieties to establish relations between certain automorphic periods and critical values of Rankin-Selberg and Asai L-functions of GL n GL m over CM fields. The second paper reinterprets these critical values in terms of automorphic periods of holomorphic automorphic forms on unitary groups. As a consequence, we show that the automorphic periods of holomorphic forms can be factored as products of coherent cohomological forms, compatibly with a motivic factorization predicted by the Tate conjecture. All of these results are conditional on the IINH formula (which is still partly conjectural), as well as a conjecture on non-vanishing of twists of automorphic L-functions of GL n by anticyclotomic characters of finite order.

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Introduction

This is the first of two papers devoted to the relations between two themes. The first theme is Deligne's conjecture on critical values of motivic L-functions. This will be the main topic of the second paper: the main result will be an expression of the critical values of the L-functions of tensor products of motives attached to cohomological automorphic forms on unitary groups, in terms of periods of arithmetically normalized automorphic forms on the Shimura varieties attached to these unitary groups. We refer to these periods as *automorphic periods* for the remainder of the introduction. The full result, like most of the results contained in these two papers, is conditional on a short list of conjectures that will be described below, as well as a relatively mild list of restrictions on the local components of the L-functions considered.

The second theme concerns relations among the automorphic periods mentioned in the previous paragraph. This is the main topic of the present paper. Following an approach pioneered by Shimura over 40 years ago, we combine special cases of Deligne's conjecture with comparisons of distinct expressions for critical values of automorphic L-functions to relate automorphic periods on different groups. These periods are attached to motives (for absolute Hodge cycles) that occur in the cohomology of different Shimura varieties. In view of Tate's conjecture on cycle classes in ℓ-adic cohomology, the relations obtained are consistent with the determination of the representations of Galois groups of appropriate number fields on the ℓ-adic cohomology of the respective motives. The paper [Lin17] used arguments of this type to show how to factor automorphic periods on Shimura varieties attached to a CM field F as products of automorphic periods of holomorphic modular forms, each attached to an embedding $\iota_0 : F \hookrightarrow \mathbb{C}$. The main result of the present paper is a factorization of the latter periods in terms of periods of coherent cohomology classes on Shimura varieties attached to the unitary group $H^{(0)}$ of a hermitian spaces over F with signature $(n-1,1)$ at ι_0 and definite at embeddings that are distinct from ι_0 and its complex conjugate.

The main results of the two papers are based on two kinds of expressions for automorphic L-functions. The first derives from the Rankin-Selberg method for $GL(n) \times GL(n-1)$. The paper [Gro-Har15] applied this method over F, when F is imaginary quadratic, to prove some cases of Deligne's conjecture in the form derived in [Har13b]. A principal innovation of that paper was to take the automorphic representation on $GL(n-1)$ to define an Eisenstein cohomology class. This has subsequently been extended to general CM fields in [Lin15b], [Gro17] and [Gro-Lin17]. The basic structure of the argument is the same in all cases. Let G_r denote the algebraic group $GL(r)$ for any $r \geq 1$, over the base CM-field F; let $K_{G_r,\infty}$ denote a maximal connected subgroup of

$G_{r,\infty} := \mathrm{GL}_r(F \otimes_{\mathbb{Q}} \mathbb{R})$ which is compact modulo the center, and and let $\mathfrak{g}_{r,\infty}$ denote the Lie algebra of $G_{r,\infty}$. Let Π be a cuspidal automorphic representation of $G_n(\mathbb{A}_F)$ and let Π' be an isobaric automorphic representation of $G_{n-1}(\mathbb{A}_F)$,

$$\Pi' = \Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_r \tag{0.1}$$

where each Π_i is a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ and $\sum_i n_i = n-1$. We assume that both Π and Π' are cohomological, in the sense that there exist finite-dimensional, algebraic representations $\mathcal{E}$ and $\mathcal{E}'$ of $G_{n,\infty}$ and $G_{n-1,\infty}$, respectively, such that the relative Lie algebra cohomology spaces

$$H^*(\mathfrak{g}_{n,\infty}, K_{G_{n,\infty}}, \Pi_\infty \otimes \mathcal{E}), \qquad H^*(\mathfrak{g}_{n-1,\infty}, K_{G_{n-1,\infty}}, \Pi'_\infty \otimes \mathcal{E}')$$

are both non-trivial. Here Π_∞ and Π'_∞ denote the archimedean components of Π and Π', respectively. Although this is not strictly necessary at this stage we also assume

Hypothesis 0.2. *The cuspidal automorphic representations Π_i and Π are all conjugate self-dual.*

Then it is known that Π and all summands Π_i are tempered locally everywhere [Har-Tay01, Shi11, Car12] because each Π_i and Π is (up to a twist by a half-integral power of the norm, which we ignore for the purposes of this introduction) a cuspidal *cohomological* representation. Suppose there is a non-trivial $G_{n-1}(F \otimes_{\mathbb{Q}} \mathbb{C})$-invariant pairing

$$\mathcal{E} \otimes \mathcal{E}' \to \mathbb{C}. \tag{0.3}$$

Then the central critical value of the Rankin-Selberg L-function $L(s, \Pi \times \Pi')$ can be expressed as a cup product in the cohomology, with twisted coefficients, of the locally symmetric space attached to G_{n-1}. These cohomology spaces have natural rational structures over number fields, and the cup product preserves rationality. From this observation we obtain the following relation for the *central* critical value $s = s_0$:

$$L^S(s_0, \Pi \times \Pi') \sim p(s_0, \Pi_\infty, \Pi'_\infty)\, p(\Pi)\, p(\Pi') \tag{0.4}$$

where the notation $\sim$, here and below, means "equal up to specified algebraic factors", $p(\Pi)$ and $p(\Pi')$ are the *Whittaker periods* of Π and Π', respectively, and $p(s_0, \Pi_\infty, \Pi'_\infty)$ is an archimedean factor depending only on s_0, Π_∞ and Π'_∞. We point out that this archimedean period $p(s_0, \Pi_\infty, \Pi'_\infty)$ can in fact be computed as a precise integral power of $2\pi i$, see [Gro-Lin17] Cor. 4.30. It turns out that this power is precisely the one predicted by Deligne's conjecture. Furthermore, Thm. 2.5 of [Gro-Lin17] provides the expression

$$p(\Pi') \sim \prod_{i=1}^{r} p(\Pi_i) \prod_{i<j} L^S(1, \Pi_i \times \Pi_j^{\vee}). \tag{0.5}$$

Finally, the cuspidal factors $p(\Pi)$ and $p(\Pi_i)$ can be related, as in [Gro-Har15], [Gro-Har-Lap16], [Lin15b], and most generally in [Gro-Lin17], to the critical L-values of the Asai L-function $L^S(1, \Pi, As^{(-1)^n})$, $L^S(1, \Pi_i, As^{(-1)^{n_i}})$.

Under Hypothesis 0.2, it is shown in [Gro-Har15] and [Lin15b] that all the terms in (0.5) can be expressed in terms of automorphic periods of arithmetically normalized holomorphic modular forms on Shimura varieties attached to unitary groups of various signatures. So far we have only considered the central critical value s_0, but variants of (0.3) allow us to treat other critical values of $L(s, \Pi \times \Pi')$ in the same way. Non-central critical values, when they exist, do not vanish, and in this way the expressions (0.4) and (0.5) give rise to non-trivial relations among these automorphic periods, including the factorizations proved in [Lin17].

Of course
$$L(s, \Pi \otimes \Pi') = \prod_{i=1}^{r} L(s, \Pi \otimes \Pi_i).$$
When $n_i = 1$, the critical values of $L(s, \Pi \otimes \Pi_i)$ were studied in [Har97] and subsequent papers, especially [Gue16, Gue-Lin16]. Thus, provided $n_1 = m < n - 1$, it is possible to analyze the critical values of $L(s, \Pi \otimes \Pi_1)$ using (0.4) and (0.5), provided Π_1 can be completed to an isobaric sum as above, with $n_i = 1$ for $i = 2, \ldots, r$, such that (0.3) is satisfied. This argument is carried out in detail in [Lin15b]. See also [Gro-Sac17].

In the above discussion, we need to assume that abelian twists $L(s, \Pi \otimes \Pi_i)$, $i > 1$, have non-vanishing critical values; this can be arranged automatically under appropriate regularity hypotheses but requires a serious *non-vanishing hypothesis* in general – we return to this point later. For the moment, we still have to address the restriction on the method imposed by the requirement (0.3). For this, it is convenient to divide critical values of $L(s, \Pi \otimes \Pi_1)$, with $n_1 = m$ as above, into two cases. We say the weight of the L-function $L(s, \Pi \otimes \Pi_1)$ is *odd* (resp. *even*) if the integers n and m have opposite (resp. equal) parity. In the case of even parity, the left-most critical value – corresponding to $s = 1$ in the unitary normalization of the L-function – was treated completely in Lin's thesis [Lin15b]. In the second paper, we treat the remaining critical values in the even parity case by applying a method introduced long ago by Harder, and extended recently by Harder and Raghuram [Har-Rag17] for totally real fields, to compare successive critical values of a Rankin-Selberg L-function for $\mathrm{GL}(n) \times \mathrm{GL}(m)$. Under different assumptions, even more refined results for successive critical values have been established in the odd parity case in [Lin15b], [Gro-Lin17]and [Gro-Sac17], extending [Har-Rag17] to CM-fields. This reduces the analysis of critical values in the odd case to the central critical value – provided the latter does not vanish, which we now assume.

In order to treat the central critical value when we cannot directly complete Π_1 to satisfy (0.3), we need a second expression for automorphic L-functions: the Ichino-Ikeda-N. Harris formula (henceforward: the IINH formula) for central values of automorphic L-functions of $U(N) \times U(N-1)$, stated below as Conjecture 3.3. Here the novelty is that we complete both Π on $G_n(\mathbb{A}_F)$ and Π_1 on $G_m(\mathbb{A}_F)$ to isobaric cohomological representations $\tilde{\Pi}$ and $\tilde{\Pi}'$ of $G_N(\mathbb{A}_F)$ and $G_{N-1}(\mathbb{A}_F)$, respectively, for sufficiently large N, adding 1-dimensional representations χ_i and χ_j in each case, so that the pair $(\tilde{\Pi}, \tilde{\Pi}')$ satisfies (0.3). At present we have no way of interpreting the critical values of $L(s, \tilde{\Pi} \otimes \tilde{\Pi}')$ as cohomological cup products, for the simple reason that both $\tilde{\Pi}$ and $\tilde{\Pi}'$ are Eisenstein representations and the integral of a product of Eisenstein series is divergent. However, we can replace the Rankin-Selberg integral by the IINH formula, provided we assume

Hypothesis 0.6. *For all i, j we have*
$$L(s_0, \Pi \otimes \chi_j) \neq 0; \; L(s_0, \chi_i \otimes \Pi_1) \neq 0; \; L(s_0, \chi_i \otimes \chi_j) \neq 0.$$
Here s_0 denotes the central value in each case ($s_0 = \frac{1}{2}$ in the unitary normalization).

We have already assumed that the central value of interest, namely $L(s_0, \Pi \otimes \Pi_1)$, does not equal zero. Assuming Hypothesis 0.6, an argument developed in [Har13b, Gro-Har15], based on the IINH formula, allows us to express the latter central value in terms of automorphic periods of arithmetically normalized holomorphic automorphic forms on unitary groups. In order to relate the values in Hypothesis 0.6 to the IINH formula, which is a relation between periods and central values of L-functions of pairs of unitary groups, we apply Hypothesis 0.2 and the theory of stable base change for unitary groups, as developed in [KMSW14, Shi14], to identify the L-functions in the IINH formula with automorphic L-functions on general linear groups.

This argument, which is carried out completely in the second paper, is taken as the starting point for the present paper, which is concerned with the factorization of periods of a single arithmetically normalized holomorphic automorphic form $\omega\ {}_{r_{\iota_0},s_{\iota_0}}\ \Pi$ on the Shimura variety attached to the unitary group H of an n-dimensional hermitian space over F with signature r_{ι_0}, s_{ι_0} at the place ι_0 mentioned above, and definite at embeddings that are distinct from ι_0 and its complex conjugate. The notation indicates that $\omega\ {}_{r_{\iota_0},s_{\iota_0}}\ \Pi$ belongs to an automorphic representation of H whose base change to G_n is our original cuspidal automorphic representation Π. The period in question, denoted $P^{\ s_{\iota_0}}\ \Pi, \iota_0$, is essentially the normalized Petersson inner product of $\omega\ {}_{r_{\iota_0},s_{\iota_0}}\ \Pi$ with itself. It was already explained in [Har97] that Tate's conjecture implies a relation of the following form:

$$P^{\ s_{\iota_0}}\ \Pi, \iota_0 \qquad \prod_{0 i s_{\iota_0}} P_i\ \Pi, \iota_0 \tag{0.7}$$

where $P_i\ \Pi, \iota_0$ is the normalized Petersson norm of a form on the Shimura variety attached to the specific unitary group $H^{\ 0}$. *The main theorem of the present paper (Theorem 5.2) is a version of* (0.7). Like most of the other theorems already mentioned, this one is conditional on a list of conjectures and local conditions, to which we now turn.

Conjectures assumed in the proofs of the main theorems. Theorem 5.2 is conditional on the following list of conjectures:

(a) Conjecture 3.3 (the IINH-Conjecture) for pairs of cohomological representations of totally definite unitary groups and of unitary groups, which are indefinite at precisely one place (an there of real rank 1), cf. Ass. 4.7. In fact, in the latter case, we only need Conj. 3.3 for pairs of cohomological representation, which base change to a cuspidal automorphic representation of the corresponding general linear group over F, i.e., in much less generality.
(b) Conjecture 3.7 (non-vanishing of twisted central critical values).
(c) Conjecture 4.16 (rationality of certain archimedean integrals).

The first of these conjectures has been proved by W. Zhang under local hypotheses at archimedean places as well as at non-archimedean places in [Zhang14] and his results have then even been strengthened by Beuzard-Plessis. The former hypotheses were local in nature and have been removed by Hang Xue [Xue17], but the hypotheses at non-archimedean places can only be removed once the comparison of relative trace formulas used in [Zhang14] can be extended to include all parabolic as well as endoscopic terms. This is the subject of work in progress by Chaudouard and Zydor, and one can reasonably expect that this problem will be resolved in the foreseeable future.

Conjecture 4.16 can only be settled by a computation of the integrals in question. The conjecture is natural because its failure would contradict the Tate conjecture; it is also known to be true in the few cases where it can be checked. Methods are known for computing these integrals but they are not simple. In the absence of this conjecture, the methods of this paper provide a weaker statement: the period relation in Theorem 5.2 is true up to a product of factors that depend only on the archimedean component of Π. Such a statement had already been proved in [Har07] using the theta correspondence, but the proof is much more complicated.

Everyone seems to believe Conjecture 3.7, but it is clearly very difficult. In fact, the proof of general non-vanishing theorems for character twists of L-functions of $GL(n)$, with $n \geq 2$, seemed completely out of reach until recently. In the last year, however, there has been significant progress in the cases $n = 3$ and $n = 4$, by two very different methods [Jia-Zha17, Blo-Li-Mil17], and one can hope that there will be more progress in the future.

In addition to the conjectures above, we repeatedly refer to the manuscript [KMSW14] of Kaletha, Mínguez, Shin, and White, which contains a complete classification of the discrete spectrum of unitary groups over number fields in terms of the automorphic spectra of general linear groups. In particular, it contains statements of all the results we need regarding stable base change from unitary groups to general linear groups, and descent in the other direction. Some of the proofs given in [KMSW14] are not complete, however; the authors are in the process of preparing two additional instalments, which will complete the missing steps. Some readers may therefore prefer to view Theorem 5.2 as conditional on the completion of the project of [KMSW14].

Although the main theorems are conditional on conjectures as well as on results that have been announced but whose complete proofs have not yet appeared, we still believe that the methods of this paper are of interest: they establish clear relations between important directions in current research on automorphic forms and a motivic version of Deligne's conjecture in the most important cases accessible by automorphic methods. Moreover, the most serious condition is the non-vanishing Conjecture 3.10 above. The proofs, however, remain valid whenever the non-vanishing can be verified for a given automorphic representation Π and all the automorphic representations Π' that intervene in the successive induction steps as in section 6.3.

Local hypotheses. The conclusion of Theorem 5.2 is asserted for Π that satisfy a list of conditions:

Π *is a* $(n-1)$*-regular, cuspidal, conjugate self-dual, cohomological automorphic representation of* $G_n(\mathbb{A}_F)$ *such that for each* I*,* Π *descends to a cohomological cuspidal representation of a unitary group* $U_I(\mathbb{A}_{F^+})$ *of signature* I*, with tempered archimedean component.*

The conditions that Π be conjugate self-dual and cohomological are necessary in order to make sense of the periods that appear in the statement of the theorem. Assuming these two conditions, the theorem for cuspidal Π implies the analogous statement for more general Π. The other two conditions are local hypotheses. The $(n-1)$-regularity condition is a hypothesis on the archimedean component of Π that is used in some of the results used in the proof, notably in the repeated use of the results of [Lin15b]. The final condition about descent is automatic if n is odd but requires a local hypothesis at some non-archimedean place if n is even. Using quadratic base change, as in work of Yoshida and others, one can probably obtain a weaker version of Theorem 5.2 in the absence of this assumption, but we have not checked the details.

About the proofs. The main theorems of the second paper relate special values of L-functions to automorphic periods, and rely on the methods described above: the analysis of Rankin-Selberg L-functions using cohomological cup products, in particular Eisenstein cohomology, and the results of [Gro-Har15, Lin15b, Har-Rag17, Gro-Lin17] on the one hand, and the IINH conjecture on the other. The present paper obtains the factorization of periods (0.7) by applying the IINH conjecture to the results obtained in the second paper, and by using a result on non-vanishing of cup products of coherent cohomology proved in [Har14]. In fact, the case used here had already been treated in [Har-Li98], assuming properties of stable base change from unitary groups to general linear groups that were recently proved by [KMSW14].

The results of [Har14] are applied by induction on n, and each stage of the induction imposes an additional regularity condition. This explains the regularity hypothesis in the statement of Theorem 5.2. It is not clear to us whether the method can be adapted in the absence of the regularity hypothesis.

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1. The setup

1.1. **Number fields.** We let F be any CM-field of dimension $2d \quad \dim_{\mathbb{Q}} F$ and set of real places $S \quad S \; F$. Each place $v \quad S$ refers to a fixed pair of conjugate complex embeddings $\iota_v, \bar{\iota}_v$ of F, where we will drop the subscript "v" if it is clear from the context. This fixes a choice of a CM-type $\Sigma \quad \iota_v : v \quad S$. The maximal totally real subfield of F is denoted F . Its set of real places will be identified with S , identifying a place v with its first component embedding $\iota_v \quad \Sigma$ and we let Gal $F \; F \qquad 1, c$. The ring of adeles over F (resp. over F) is denoted $\mathbb{A}_F$ (resp. $\mathbb{A}_F$) and $\mathbb{A}_{\mathbb{Q}}$ for $\mathbb{Q}$. We write $\mathcal{O}_F$ (resp. $\mathcal{O}_F$) for the respective ring of integers and drop the subscript, if no confusion is possible.

1.2. **Algebraic groups and real Lie groups.** We abbreviate $G_n : \; \mathrm{GL}_n \; F$. Let $V_n, \quad ,$ be an n-dimensional non-degenerate c-hermitian space over F, $n \quad 2$, we denote the corresponding unitary group by $H : \; H_n : \; U \; V_n$ over F .

We define the *rational similitude group* $\tilde{H} : \; \tilde{H}_n : \; GU \; V_n$ over $\mathbb{Q}$ as follows: If $GU_F \quad V_n$ is the subgroup of $\mathrm{GL} \; V_n$ that preserves the hermitian form up to a scalar multiple $\nu \; g \quad \mathbb{G}_{m,F}$,

$$GU_F \quad V_n \quad : \quad g \quad \mathrm{GL} \; V_n \qquad gv, gw \quad \nu \; g \qquad v, w$$

we let $GU \; V_n$ denote the fiber product

$$GU \; V_n \; : GU_F \quad V_n \qquad {}_{R_F \quad {}_{\mathbb{Q}}\mathbb{G}_{m,F}} \; \mathbb{G}_{m,\mathbb{Q}}$$

where the map $GU_F \quad V_n \quad R_F \quad {}_{\mathbb{Q}}\mathbb{G}_{m,F}$ is the similitude map ν and $\mathbb{G}_{m,\mathbb{Q}} \; R_F \quad {}_{\mathbb{Q}}\mathbb{G}_{m,F}$ is the natural inclusion. If V_k is some non-degenerate subspace of V_n, we view $U \; V_k$ (resp. $GU \; V_k$) as a natural F -subgroup of $U \; V_n$ (resp. $\mathbb{Q}$-subgroup of $GU \; V_n$). If $n \quad 1$ we write $U \; 1 \quad U \; V_1$: As an algebraic group $U \; 1$ is isomorphic to the kernel of the norm map $N_{F \; F} \; R_{F \; F} \quad \mathbb{G}_{m \quad F}$ $\mathbb{G}_{m \quad F}$, and is thus independent of V_1.

If G is any reductive algebraic group over a number field $\mathbb{F}$, we write $Z_G \; \mathbb{F}$ for its center, $G \quad : Res_{\mathbb{F} \; \mathbb{Q}} \; G \quad \mathbb{R}$ for the Lie group of real points of the Weil-restriction of scalars from $\mathbb{F} \; \mathbb{Q}$ and denote by $K_{G,} \quad G$ the product of Z_G and the connected component of the identity of a maximal compact subgroup of G . We remark that since $\tilde{H} \;\; H, \tilde{H} \quad \tilde{H} \; \mathbb{R} \quad H \qquad {}_{v \; S} \; U \; r_v, s_v$ for some signatures $0 \;\; r_v, s_v \;\; n$. We have $K_{G_n,} \qquad {}_{v \; S} \; K_{G_n,v}$, each factor being isomorphic to $K_{G_n,v} \quad \mathbb{R} \;\; U \;\; n$; $K_{H,} \qquad {}_{v \; S} \; K_{H,v}$, with $K_{H,v} \quad U \;\; r_v \qquad U \;\; s_v$; and $K_{\tilde{H},} \quad \mathbb{R} \;\; K_{H,}$.

Lower case gothic letters denote the Lie algebra of the corresponding real Lie group (e.g., $\mathfrak{g}_{n,v} = Lie(G_n(F_v))$, $\mathfrak{k}_{H,v} = Lie(K_{H,v})$, $\mathfrak{h}_v = Lie(H(F_v))$, etc. ...).

1.3. Highest weight modules and cohomological representations.

1.3.1. *Finite-dimensional representations.* We let $\mathcal{E}_\mu$ be an irreducible finite-dimensional representation of the real Lie group $G_{n,\infty}$ on a complex vector-space, given by its highest weight $\mu = (\mu_v)_{v \in S_\infty}$. Throughout this paper such a representation will be assumed to be algebraic: In terms of the standard choice of a maximal torus and a basis of its complexified Lie algebra, consisting of the functionals which extract the diagonal entries, this means that the highest weight of $\mathcal{E}_\mu$ has integer coordinates, $\mu = (\mu_{\iota_v}, \mu_{\bar\iota_v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$. We say that $\mathcal{E}_\mu$ is *m-regular*, if $\mu_{\iota_v,i} - \mu_{\iota_v,i+1} \geq m$ and $\mu_{\bar\iota_v,i} - \mu_{\bar\iota_v,i+1} \geq m$ for all $v \in S_\infty$ and $1 \leq i \leq n$. Hence, μ is regular in the usual sense (i.e., inside the open positive Weyl chamber) if and only if it is 1-regular.

Similarly, given a unitary group $H = U(V_n)$ (resp. rational similitude group $\tilde H = GU(V_n)$), we let $\mathcal{F}_\lambda$ (resp. $\tilde{\mathcal{F}}_\lambda$) be an irreducible finite-dimensional representation of the real Lie group H_∞ (resp. $\tilde H_\infty = \tilde H(\mathbb{R})$) on a complex vector-space, given by its highest weight $\lambda = (\lambda_v)_{v \in S_\infty}$ (resp. $\lambda = ((\lambda_v)_{v \in S_\infty}; \lambda_0)$, λ_0 being the exponent of $z \in \mathbb{R}$ in diag$(z, z, ..., z) \in \tilde H_\infty$), denoted c in [Har97]). Again, every such representation is assumed to be algebraic, which means that each component $\lambda_v \in \mathbb{Z}^n$ (resp. $\lambda_v \in \mathbb{Z}^n$ and $\lambda_0 \equiv \sum_{v,i} \lambda_{v,i}$ mod 2). Interpreted as a highest weight for $K_{H,\infty}$ (resp. $K_{\tilde H,\infty}$), in which case we denote our tuple of integers by $\Lambda = (\lambda_v)_{v \in S_\infty}$ (resp. $\Lambda = ((\lambda_v)_{v \in S_\infty}, \Lambda_0)$, one obtains an irreducible algebraic representation $\mathcal{W}_\Lambda$ of $K_{H,\infty}$ (resp. $\tilde{\mathcal{W}}_\Lambda$ of $K_{\tilde H,\infty}$).

1.3.2. *Cohomological representations.* A representation Π_∞ of $G_{n,\infty}$ is said to be *cohomological* if there is a highest weight module $\mathcal{E}_\mu$ as above such that $H^*(\mathfrak{g}_{n,\infty}, K_{G_{n,\infty}}, \Pi_\infty \otimes \mathcal{E}_\mu) \neq 0$. In this case, we say Π_∞ is m-regular if $\mathcal{E}_\mu$ is.

Analogously, a representation π_∞ of H_∞ (resp. $\tilde\pi_\infty$ of $\tilde H_\infty$) is said to be *cohomological* if there is a highest weight module $\mathcal{F}_\lambda$ (resp. $\tilde{\mathcal{F}}_\lambda$) as above such that $H^*(\mathfrak{h}_\infty, K_{H,\infty}, \pi_\infty \otimes \mathcal{F}_\lambda)$ (resp. $H^*(\tilde{\mathfrak{h}}_\infty, K_{\tilde H,\infty}, \tilde\pi_\infty \otimes \tilde{\mathcal{F}}_\lambda)$) is non-zero. See [Bor-Wal80],§I, for details.

If π is a unitary automorphic representation of $H(\mathbb{A}_F)$, which is tempered and cohomological with respect to $\mathcal{F}_{\lambda}$, then each of its archimedean component-representations π_v of $H_v = U(r_v, s_v)$ is isomorphic to one of the $d_v := \binom{n}{r_v}$ inequivalent discrete series representations denoted $\pi_{\lambda,q}$, $0 \leq q < d_v$, having infinitesimal character $\chi_{\lambda_v + \rho_v}$, [Vog-Zuc84]. As it is well-known, [Bor-Wal80], II Thm. 5.4, the cohomology of each $\pi_{\lambda,q}$ is centered in the middle-degree

$$H^p(\mathfrak{h}_v, K_{H,v}, \pi_{\lambda,q} \otimes \mathcal{F}_{\lambda_v}) = \begin{cases} \mathbb{C} & \text{if } p = r_v s_v \\ 0 & \text{else} \end{cases}$$

In turn, an essentially discrete series representation $\tilde\pi_\infty$ of $\tilde H_\infty$ (i.e., by definition a character-multiple of a discrete series representation) restricts to a sum of discrete series representations of H_∞. In fact, there is no ambiguity unless $r_v = s_v$ for all v: in this case, $\tilde H_\infty$ has two connected components, whereas H_∞ is connected, whence the restriction of $\tilde\pi_\infty$ is the sum of two irreducible representations, and we choose one of the two. We thus obtain an $|S_\infty|$-tuple of Harish-Chandra parameters $(A_v)_{v\in S}$, and $\pi_S = \otimes_{v\in S}\pi(A_v)$ where $\pi(A_v)$ denotes the discrete series representation of H_v with parameter A_v.

1.3.3. *Base change, L-packets.* Let π be a cohomological square-integrable automorphic representation of $H(\mathbb{A}_F)$. It was first proved by Labesse [Lab11] (see also [Har-Lab09, Kim-Kri04, Kim-Kri05, Mor10, Shi14]) that π admits a base change $BC(\pi) = \Pi$ to $G_n(\mathbb{A}_F)$.

The resulting representation Π is an isobaric sum $\Pi = \Pi_1 \boxplus ... \boxplus \Pi_k$ of conjugate self-dual square-integrable automorphic representations Π_i of $G_{n_i}(\mathbb{A}_F)$, uniquely determined by the following: for every non-archimedean place $v \notin S$ such that π_v is unramified, the Satake parameter of Π_v is obtained from that of π_v by the formula for base change, see for example [Ming11]. The assumption that π_∞ is cohomological implies moreover that Π_∞ is cohomological: This was proved in [Lab11] §5.1 for discrete series representations π_∞ but follows in complete generality using [Clo91], §3.4; alternatively, one may recall that Π_∞ has regular dominant integral infinitesimal character and hence is necessarily cohomological combining [Enr79], Thm. 6.1 and [Bor-Wal80], III.3.3. It is then known [Clo90, Har-Tay01, Shi11, Car12] that, if all isobaric summands Π_i of $\Pi = BC(\pi)$ are cuspidal, all of their local components $\Pi_{i,v}$ are tempered. We henceforth assume that Π is of this type.

For purposes of reference in the present paper, we then define the global L-packet $\Pi_\Phi(H,\Pi)$ to be the set of tempered, square-integrable cohomological automorphic representations π of $H(\mathbb{A}_F)$ such that $BC(\pi) = \Pi$. This is consistent with the formalism in [Mok14, KMSW14], in which (as in Arthur's earlier work) the representation Π plays the role of the Langlands parameter for the square-integrable automorphic representation π of $H(\mathbb{A}_F)$. We recall that temperedness together with square-integrability imply that π is necessarily cuspidal, [Clo93], Prop. 4.10, [Wal84], Thm. 4.3.

The analogous assertions remain valid for a cohomological essentially square-integrable automorphic representation $\tilde{\pi}$ of $\tilde{H}(\mathbb{A}_\mathbb{Q})$ (i.e., a Hecke-character multiple of a square-integrable automorphic representation) see [Shi14]. If no confusion is possible, we will identify $BC(\tilde{\pi})$ with the base change of any irreducible summand π in the restriction of $\tilde{\pi}$ to $H(\mathbb{A}_F)$, all of these being in the same L-packet.

Sticking to this rule, we denote the global L-packet of essentially tempered, essentially square-integrable cohomological automorphic representations lifting to $BC(\tilde{\pi})$ simply by $\Pi_\Phi(\tilde{H},\Pi)$. As for essentially discrete series representations, essentially temperedness refers to being a character-multiple of a tempered representation, or, equivalently that the restriction to $H(\mathbb{A}_F)$ is tempered.

2. Periods

2.1. **CM-periods.** Let (T,h) be a Shimura datum where T is a torus defined over $\mathbb{Q}$ and $h : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to T_\mathbb{R}$ a homomorphism satisfying the axioms defining a Shimura variety. Such pair is called a *special* Shimura datum. Let $Sh(T,h)$ be the associated Shimura variety and let $E(T,h)$ be its reflex field.

For χ an algebraic Hecke character of $T(\mathbb{A}_\mathbb{Q})$, we let $E(\chi)$ be the number field generated by the values of χ_f, $E(T,h)$ and F^{Gal}, i.e., the composition of the rationality field $\mathbb{Q}(\chi)$ of χ_f, cf. [Wal85], and $E(T,h)F^{Gal}$. We may define a non-zero complex number $p(\chi,(T,h))$ as in [MHar-Kud91], called a *CM-period*. It is well defined modulo $E(\chi)^\times$.

In particular, if Ψ a subset of embeddings of F in $\mathbb{C}$ such that $\Psi \quad \Psi^c$, one can define a special Shimura datum T_Ψ, h_Ψ where $T_\Psi : Res_{F\ \mathbb{Q}}\ \mathbb{G}_m$ and $h_\Psi : Res_{\mathbb{C}\ \mathbb{R}}\ \mathbb{G}_{m,\mathbb{C}} \quad T_{\Psi,\mathbb{R}}$ is a homomorphism such that over $\sigma \quad \Sigma_F$, the Hodge structure induced on F by h_Ψ is of type $1, 0$ if $\sigma \quad \Psi$, of type $0, \ 1$ if $\sigma \quad \Psi^c$, and of type $0, 0$ otherwise. In this case, for χ an algebraic Hecke character of F, we write $p\ \chi, \Psi$ for $p\ \chi,\ T_\Psi, h_\Psi$. If ι is an element in Σ_F, we also write $p\ \chi, \iota$ for $p\ \chi,\ \iota$.

Proposition 2.1. *Let T and T be two tori defined over $\mathbb{Q}$ both endowed with a Shimura datum T, h and T, h . Let $u : \ T, h \quad T, h$ be a map between the Shimura data. Let χ be an algebraic Hecke character of $T\ \mathbb{A}_\mathbb{Q}$. We put $\chi : \chi \quad u$, which is an algebraic Hecke character of $T\ \mathbb{A}_\mathbb{Q}$. Then we have:*

$$p\ \chi,\ T, h \quad {}_{E\ \chi}\ p\ \chi,\ T, h$$

In fact, the CM-period is defined as the ratio between a certain deRham-rational vector and a certain Betti-rational vector inside the cohomology of the Shimura variety with coefficients in a local system. The above proposition is due to the fact that both the Betti-structure and the deRham-structure commute with the pullback map on cohomology. We refer to [Har93], in particular relation $1.4.1$ for details.

2.2. **Arithmetic automorphic periods.** Let Π be a conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n\ \mathbb{A}_F$, which is cohomological with respect to $\mathcal{E}_\mu$. For any tuple $I \quad I_\iota\ {}_{\iota\ \Sigma}$ $0, 1, \quad, n\ ^\Sigma$, we assume that there exists a unitary group U_I over F with respect of $F\ F$ of signature $n \quad I_\iota, I_\iota$ at $v \quad \iota, \bar{\iota} \quad S$, such that Π descends to a cohomological representation π of $U_I\ \mathbb{A}_F$ with tempered archimedean component (i.e., π is in the discrete series). This condition is satisfied, if μ is regular and if in addition Π is square-integrable at a finite place of F which is split in F (cf. [KMSW14]).

An *arithmetic automorphic period* P^I can then be defined as the Petersson inner product of an arithmetic holomorphic automorphic form as in [Har97], [Lin15b] and [Gue16]. The following result is proved in [Lin17].

Theorem 2.2 (Local arithmetic automorphic periods)**.** *If Π is cohomological with respect to a 5-regular coefficient module $\mathcal{E}_\mu$, then there are local arithmetic automorphic periods $P\ ^s\ \Pi, \iota$, for $0\ s\ n$ and $\iota \quad \Sigma$. These are unique up to multiplication by elements in $E\ \Pi$ and one has the relation*

$$P^I\ \Pi \quad {}_{E\ \Pi} \quad \underset{\iota\ \Sigma}{} \ P\ ^{I_\iota}\ \Pi, \iota \tag{2.3}$$

which is equivariant under the action of Gal $\bar{\mathbb{Q}}\ F^{Gal}$ *. Moreover,*

$$P\ ^0\ \Pi, \iota \quad {}_{E\ \Pi}\ p\ \check{\xi}_\Pi, \bar{\iota} \qquad \text{and} \qquad P\ ^n\ \Pi, \iota \quad {}_{E\ \Pi}\ p\ \check{\xi}_\Pi, \iota\ , \tag{2.4}$$

where $\check{\xi}_\Pi : \xi_\Pi^{c,\ 1}$ is the conjugate inverse of ξ_Π, the central character of Π.

Here, and in what follows, $E\ \Pi$ denotes the composition of the rationality field $\mathbb{Q}\ \Pi$ of Π_f, cf. [Wal85], and F^{Gal}, a Galois closure of $F\ \mathbb{Q}$ in $\bar{\mathbb{Q}}$.

2.3. **Motivic periods.** Let M : M Π be a motive over F with coefficients in E Π (conjecturally) attached to Π (cf. [Clo90]).

For 0 i n and ι Σ, we have defined *motivic periods* Q_i M, ι in [Har13b] [Har-Lin16]. Since M is conjugate self-dual, the period Q_i M, ι is equivalent to the inner product of a vector in M_ι, the Betti realisation of M at ι, whose image via the comparison isomorphism is inside the i-th bottom degree of the Hodge filtration for M.

2.4. **Short interlude: The main goal of this paper.** As predicted in [Har97] (see Conjecture 2.8.3 and Corollary 2.8.5 of [Har97]), we expect the following factorization:

$$P^{\ i} \ \Pi, \iota \quad {}_{E\ \Pi}\ Q_0 \ M, \iota \ Q_1 \ M, \iota \qquad Q_i \ M, \iota \tag{2.5}$$

Recall by Lemma 3.17 and section 4.4 of [Har-Lin16] we have

$$Q_0 \ M, \iota \quad \delta \ \det M \ \Pi \ , \iota \quad 2\pi i^{\ n \ n \ 1 \ 2} \ \delta \ M \ \xi_\Pi \ , \iota \quad {}_{E\ \Pi}\ p \ \check{\xi}_\Pi, \bar{\iota} \ .$$

Hence the motivic period Q_0 M, ι is equivalent to the automorphic period P_0 Π, ι . Therefore the previous factorization statement is equivalent to the following:

$$Q_{q\ 1} \ M, \iota \quad {}_{E\ \Pi} \frac{P^{\ q\ 1} \ \ \Pi, \iota}{P^{\ q} \ \ \Pi, \iota} \text{ for all } 0 \ q \ n \quad 1. \tag{2.6}$$

In this paper, we shall first define another automorphic period P_i Π, ι (observe the difference in notation: i is now in the subindex) for a unitary group of a particular choice of signatures r_v, s_v ${}_{v\ S}$ and then prove (2.6) (over a potentially slightly bigger field) when Q_i M, ι is replaced by its automorphic analogue P_i Π, ι . We shall construct a motive M associated to Π by realisations in our forthcoming [Gro-Har-Lin18]. Form this it will follow that P_i Π, ι and Q_i M, ι are in fact the same, showing (2.5) (over a slightly bigger field).

This result is a key step to prove a conjecture of Deligne for such M (cf. [Del79]) which will, in fact, be the main theorem of [Gro-Har-Lin18].

2.5. **Gross-Prasad periods, Petersson norms and pairings for unitary groups.** Let V V_n be a n-dimensional hermitian space over F with respect to the extension F F as in §1.2. Let V V be a non-degenerate subspace of V of codimension 1. We write V as the orthogonal direct sum V V_1 and consider the unitary groups H : U V , H : U V and H : U V U V_1 over F . Obviously, there are natural inclusions H H H.

The usual Ichino-Ikeda-N. Harris conjecture considers the inclusion H H. However, it is sometimes more convenient to consider the inclusion H H instead, see [Har13a] and [Har14], and we are going to use both points of view in this paper. In this section we take the chance to discuss the relations of the associated periods for the two inclusions H H and H H. We warn the reader that our notation differs slightly from [Har13a] and [Har14].

Let π (resp. π) be a tempered cuspidal cohomological representation of H $\mathbb{A}_F$ (resp. H $\mathbb{A}_F$). Let ξ be a Hecke character on U V_1 $\mathbb{A}_F$ (recall that U V_1 is independent of the hermitian structure on V_1, §1.2). We write π : π ξ, which is a tempered cuspidal automorphic representation of H $\mathbb{A}_F$.

For $f_1, f_2 \quad \pi$ the *Petersson inner product* on π is defined as

$$f_1, f_2 \ : \quad {}_{H\ F \quad Z_H\ \mathbb{A}_F \quad H\ \mathbb{A}_F} \quad f_1\ h\ \overline{f_2\ h}\ dh,$$

where dh denotes the Tamagawa measure on $H\ \mathbb{A}_F$ normalized as in [Har14]. In the same way we may define the Petersson inner product on π and π . Next we put

$$I^{can}\ f, f \ : \quad {}_{H\ F \quad H\ \mathbb{A}_F} \quad f\ h\ f\ h\ dh \text{ for } f \quad \pi, f \quad \pi .$$

Then it is easy to see that $I^{can} \quad \mathrm{Hom}_{H\ \mathbb{A}_F} \quad \pi \quad \pi, \mathbb{C}$.

With this notation the *Gross-Prasad period* for the pair π, π of $H\ \mathbb{A}_F \quad H\ \mathbb{A}_F$ is defined as

$$\mathcal{P}\ f, f \ : \frac{I^{can}\ f, f}{f, f \quad f, f}^{2}.$$

We can similarly define a $H\ \mathbb{A}_F$ -invariant linear form on $\pi \quad \pi$, still denoted by I^{can}, as

$$I^{can}\ f, f \ : \quad {}_{H\ F \quad H\ \mathbb{A}_F} \quad f\ h\ f\ h\ dh \text{ for } f \quad \pi, f \quad \pi ,$$

leading to a definition of the *Gross-Prasad period* for the pair π, π of $H\ \mathbb{A}_F \quad H\ \mathbb{A}_F$ as

$$\mathcal{P}\ f, f \ : \frac{I^{can}\ f, f}{f, f \quad f, f}^{2}.$$

Let ξ_π, resp. ξ_π ,be the central characters of π, resp. π . We assume that

$$\xi_\pi^{\ 1} \quad \xi_\pi\ \xi \quad {}_{Z_H\ \mathbb{A}_F} \tag{2.7}$$

(resembling equation Ξ on page 2039 of [Har13a]). Then for any $f \quad \pi$, we can write f as $f \quad f_0$. Since $f_0 \quad f_0\ id \quad \xi$, one verifies easily that $I^{can}\ f, f \quad f_0\ id \quad I^{can}\ f, f \quad \xi \quad f_0\ id \quad I^{can}\ f, f$ and $f_0, f_0 \quad f_0\ id \ ^{2} \quad \xi, \xi \quad f_0\ id \ ^{2}$. We conclude that:

Lemma 2.8.

$$\mathcal{P}\ f, f \quad \mathcal{P}\ f, f \quad {}_{H\ \mathbb{A}_F} \ .$$

It is also clear that equation (2.7) implies $\mathrm{Hom}_{H\ \mathbb{A}_F} \quad \pi \quad \pi, \mathbb{C} \quad \mathrm{Hom}_{H\ \mathbb{A}_F} \quad \pi \quad \pi, \mathbb{C}$.

2.6. **Q-periods, Petersson norms and pairings for unitary similitude groups.** In the proof of our main theorem, we are going to use results in [Har14], where certain automorphic periods for unitary similitude groups are shown to be algebraic. In this section, we introduce and discuss these periods for unitary similitude groups.

Resume the notation $V \quad V \quad V_1$ from §2.5. Let $\tilde{H} : \ GU\ V$ and $\tilde{H} \ : \ GU\ V$ be the rational similitude unitary groups over $\mathbb{Q}$, associated with V and V and define

$$\tilde{H} \ : \quad GU\ V \quad GU\ V_0 \quad GU\ V \ .$$

In other words, en element $g, t \quad GU \; V \quad GU \; V_0$ is in $\tilde{H}$ if and only if g and t have the same rational similitude. Again we have natural inclusions $\tilde{H} \quad \tilde{H} \quad \tilde{H}$.

Let $\tilde{\pi}$ (resp. $\tilde{\pi}$) be an essentially tempered cuspidal cohomological representation of $\tilde{H} \; \mathbb{A}_{\mathbb{Q}}$ (resp. $\tilde{H} \; \mathbb{A}_{\mathbb{Q}}$) of central character $\xi_{\tilde{\pi}}$ (resp. $\xi_{\tilde{\pi}}$). Let $\tilde{\xi}$ be a Hecke character of $GU \; V_1 \; \mathbb{A}_{\mathbb{Q}}$ such that $\xi_{\tilde{\pi}}{}^{1} \quad \xi_{\tilde{\pi}} \tilde{\xi} \quad {}_{Z_{\tilde{H}} \; \mathbb{A}_{\mathbb{Q}}}$. We define $\tilde{\pi} \; : \; \tilde{\pi} \quad \tilde{\xi} \quad {}_{\tilde{H} \; \mathbb{A}_{\mathbb{Q}}}$, which is a cuspidal automorphic representation of $\tilde{H} \; \mathbb{A}_{\mathbb{Q}}$.

The positive real character $\quad \xi_{\tilde{\pi}} \quad {}^{2}$ may be extended in a unique way to a character of $\tilde{H} \; \mathbb{Q} \quad \tilde{H} \; \mathbb{A}_{\mathbb{Q}}$, denoted $\nu_{\tilde{\pi}}$.

Let ϕ_1, ϕ_2 be two elements in $\tilde{\pi}$. As in section 3.4.2 of [Har13a], we define their *Petersson inner product* by:

$$\phi_1, \phi_2 \quad : \quad \int_{\tilde{H} \; \mathbb{Q} \; Z_{\tilde{H}} \; \mathbb{A}_{\mathbb{Q}} \quad \tilde{H} \; \mathbb{A}_{\mathbb{Q}}} \phi_1 \; g \; \overline{\phi_2 \; g} \, \nu_{\tilde{\pi}} \; g \quad dg,$$

where dg denotes the Tamagawa measure on $\tilde{H} \; \mathbb{A}_{\mathbb{Q}}$. Analogously, the Petersson inner product may be defined on $\tilde{\pi}$ and on $\tilde{\pi}$.

We define the linear form I^{can} on $\tilde{\pi} \quad \tilde{\pi}$ as the integral

$$I^{can} \; \phi, \phi \quad : \quad \int_{H \;\; F \quad H \;\; \mathbb{A}_F} \phi \; h \quad \phi \quad h \quad dh \;\; \text{for } \phi \quad \pi, \phi \quad \pi \; ;$$

Observe that we integrate over $H \quad F \quad H \quad \mathbb{A}_F$ *but not over* $\tilde{H} \quad \mathbb{Q} \quad \tilde{H} \quad \mathbb{A}_{\mathbb{Q}}$.

2.6.1. *Restriction to unitary groups.* We write $\tilde{\pi} \quad {}_{H \; \mathbb{A}_F} \quad {}_{\alpha \; A} \pi_\alpha$ and $\tilde{\pi} \quad {}_{H \; \mathbb{A}_F} \quad {}_{\beta \; A} \pi_\beta$ as sum of irreducible tempered cuspidal automorphic representations, cf. Lem. 4.1.2 of [Gue-Lin16]. The summands π_α (resp. π_β) belong all to the same L-packet and appear there with multiplicity one. Hence, for $\phi \quad \tilde{\pi}$, there is a canonical decomposition $\phi \quad {}_{H \; \mathbb{A}_F} \quad {}_{\alpha} \phi_{\pi_\alpha}$ and, similarly, $\phi \quad {}_{H \; \mathbb{A}_F} \quad {}_{\beta} \phi_{\pi_\beta}$ for $\phi \quad \tilde{\pi}$. The following theorem is Conjecture 5.2 of [Har13a], which has finally been proved by [Beu-Ple1], [Beu-Ple2] and [He17].

Theorem 2.9. $\displaystyle \sum_{\alpha,\beta} \dim_{\mathbb{C}} \mathrm{Hom}_{H \;\; \mathbb{A}_F} \quad \pi_\alpha \quad \pi_\beta, \mathbb{C} \quad 1.$

As a direct consequence, we obtain the following lemma

Lemma 2.10. *If there exist* $\pi \quad \pi_\alpha$ *and* $\pi \quad \pi_\beta$ *such that* $\dim_{\mathbb{C}} \mathrm{Hom}_{H \;\; \mathbb{A}_F} \quad \pi \quad \pi , \mathbb{C} \quad 1$, *then for any* $\pi_\alpha, \pi_\beta \quad \pi, \pi$ *, the canonical linear form* I^{can} *is identically zero on* $\pi_\alpha \quad \pi_\beta$. *Hence* $I^{can} \;\; \phi, \phi \quad I^{can} \;\; \phi_\pi, \phi_\pi$.

2.6.2. *Automorphic Q-periods.* Recall that the $\tilde{\mathfrak{h}} \;\; , K_{\tilde{H},} \;\; , \tilde{H} \;\; \mathbb{A}_{\mathbb{Q},f}$ -module of $K_{\tilde{H},}$ -finite vectors in $\tilde{\pi}$ may be defined over a number field $E \;\; \tilde{\pi}$, cf. [Har13a] Cor. 2.13 & Prop. 3.17. We assume that $E \;\; \tilde{\pi}$ contains F^{Gal}. In the same way, the space of $K_{\tilde{H} \;,}$ -finite vectors in $\tilde{\pi}$ is defined over a number field $E\ \tilde{\pi}$ containing F^{Gal} and refer to these rational structures as the *deRham-rational structures* on $\tilde{\pi}$ and $\tilde{\pi}$. A function inside these deRham-rational structures is said to be *deRham-rational*.

Let us also introduce a number field $E\ \pi$ attached to an irreducible summand $\pi\ \ \pi_\alpha$ of $\tilde{\pi}$. As for $\tilde{\pi}$ it denotes the composition of F^{Gal} and any (fixed choice of a) number field, over which the $\mathfrak{h}\ \ , K_{H,}\ \ , H\ \mathbb{A}_{F\ ,f}$ -module of $K_{H,}$ -finite vectors in π has a rational model, cf. [Har13a] Cor. 2.13 & Cor. 3.8 (and its correction in the Erratum). In the same way, we define number fields $E\ \pi$, where π is an irreducible summand $\pi\ \ \pi_\beta$ of $\tilde{\pi}$.

So far our definition of $E\ \pi$ (and $E\ \pi$) leaves us some freedom to include in it any appropriate choice of a number field. We will specify such a choice in Conj. 4.16, by choosing a suitable finite extension of a certain concrete number field, constructed and denoted $E_Y\ \eta$ in [Har13a]. So far, any choice (subject to the above conditions) works.

Definition-Lemma 2.11. Let π be a tempered, cohomological, cuspidal automorphic representation of $H\ \mathbb{A}_F$ and let $\tilde{\pi}$ be any essentially tempered, cohomological, cuspidal automorphic representation of $\tilde{H}\ \mathbb{A}_{\mathbb{Q}}$, extending π. Let $\phi\ \ \tilde{\pi}$ be deRham-rational. If π is 2-regular, then up to multiplication by elements in $E\ \pi$, the Petersson inner product $\ \phi_\pi, \phi_\pi$ is independent of the extension $\tilde{\pi}$ and ϕ. We define it as the automorphic Q-period $Q\ \pi\ :\ \ \phi_\pi, \phi_\pi$.

The independence is due to the fact that $\ \phi_\pi, \phi_\pi$ can be related to critical values of the L-function for the base change of π twisted by a suitable Hecke character. When π is 2-regular, one can always choose a critical point in the absolutely convergent range and hence with non-zero L-value. We shall provide more details on the proof in paper II.

3. The Ichino-Ikeda-Neil Harris conjecture for unitary groups

3.1. **Statement of the conjecture.** We resume the notation of §2.5. The base change of the tempered cuspidal cohomological automorphic representation $\pi\ \ \pi$ of $H\ \mathbb{A}_F\ \ \ H\ \mathbb{A}_F$ to $G_n\ \mathbb{A}_F\ \ \ G_{n\ 1}\ \mathbb{A}_F$ is denoted $\Pi\ \ \Pi$. Define

$$\mathcal{L}^S\ \Pi, \Pi\ \ :\ \frac{L^S\ \frac{1}{2}, \Pi\ \ \Pi}{L^S\ 1, \Pi, As\ ^{1\ n}\ L^S\ 1, \Pi\ , As\ ^{1\ n\ 1}}. \tag{3.1}$$

Choose $f\ \ \pi$, $f\ \ \pi$, and assume they are factorizable as $f\ \ \ f_v, f\ \ \ f_v$ with respect to tensor product factorizations

$$\pi\ \ \ \ _v\,\pi_v;\ \pi\ \ \ \ _v\,\pi_v. \tag{3.2}$$

We assume π_v and π_v are unramified, and f_v and f_v are the normalized spherical vectors, outside a finite set S including all archimedean places (and which we allow to include some unramified places). We define inner products on π and π exactly as in §2.5 and we choose inner products $\ ,\ \ _{\pi_v},\ \ ,\ \ _{\pi_v}$ on each of the π_v and π_v such that at an unramified place v, the local spherical vector in π_v or π_v taking value 1 at the identity element has norm 1. For each $v\ \ S$, let

$$c_{f_v}\ h_v\ :\ \ \pi_v\ h_v\ f_v, f_v\ _{\pi_v}\qquad c_{f_v}\ h_v\ :\ \ \pi_v\ h_v\ f_v, f_v\ _{\pi_v},\ h_v\ \ H_v, h_v\ \ H_v,$$

and define

$$I_v\ f_v, f_v\ :\ \ _{H_v} c_{f_v}\ h_v\ c_{f_v}\ h_v\ dh_v\qquad I_v\ f_v, f_v\ :\ \frac{I_v\ f_v, f_v}{c_{f_v}\ 1\ c_{f_v}\ 1}.$$

Neil Harris proves that these integrals converge since π and π are locally tempered at all places. Let Δ_H be the value at L-function of the *Gross motive* of H:

$$\Delta_H \quad L \quad 1, \eta_{F \ F} \quad \zeta \ 2 \ L \ 3, \eta_{F \ F} \quad \ldots . L \ n, \eta_F^n \ _F \quad .$$

The Ichino-Ikeda conjecture for unitary groups is then [NHar14]

Conjecture 3.3. *Let $f \quad \pi$, $f \quad \pi$ be factorizable vectors as above. Then there is an integer β, depending on the L-packets containing π and π , such that*

$$\mathcal{P} \ f, f \qquad 2^{\beta} \Delta_H \mathcal{L}^S \ \pi, \pi \qquad \underset{v \ S}{} I_v \ f_v, f_v \ .$$

Remark 3.4. The conjecture is known – up to a sign – by the work of Beuzart-Plessis when $\pi_v \quad \pi_v$ is supercuspidal at one non-archimedean place v of F . We point out that his latter assumption is due to the limitation of the current state of the Flicker-Rallis trace formulae and hence shall soon be removed by the forthcoming work of Chaudouard–Zydor. If $\pi_v \quad \pi_v$ is not supercuspidal, but Π is cuspidal, then Conj. 3.3 is still known for totally definite unitary groups, up to a certain algebraic number, by [Gro-Lin17].

Remark 3.5. Both sides of Conjecture 3.3 depend on the choice of factorizable vectors f, f , but the dependence is invariant under scaling. In particular, the statement is independent of the choice of factorizations (3.2), and the assertions in the following section on the nature of the local factors $I_v \quad f_v, f_v$ are meaningful.

3.2. Some instances of rationality based on the I.-I.-N.H-conjecture.

3.2.1. *Rationality of the non-archimedean local terms I_v.* The algebraicity of local terms I_v of Conjecture 3.3 was proved in [Har13b] when v is non-archimedean. More precisely, we have the following:

Lemma 3.6. *Let v be a non-archimedean place of F . Let π and π be tempered cohomological cuspidal automorphic representations as in the statement of Conjecture 3.3. Let E be a number field over which π_v and π_v both have rational models. Then for any E-rational vectors $f_v \quad \pi_v$, $f_v \quad \pi_v$, we have*

$$I_v \quad f_v, f_v \qquad E.$$

Proof. The algebraicity of the local zeta integrals $I_v \quad f_v, f_v$ is proved in [Har13b], Lemma 4.1.9, when the local inner products $\quad , \quad _{\pi_v}$ and $\quad , \quad _{\pi_v}$ are taken to be rational over E. Since f_v and f_v are rational vectors, this implies the assertion for the normalized integrals I_v as well. □

We will state an analogous result for the archimedean local factors as an expectation of ours in section 6.2.

3.2.2. *Rationality of special values of tensor product L-functions of odd weight.*

Recall that the existence of local arithmetic automorphic periods is proved in [Lin17] when Π is 5-regular (see section 2.2). This regularity condition can be removed if we assume

(i) Conjecture 3.3 (the IINH-Conjecture) for pairs of cohomological representations of totally definite unitary groups and unitary groups, which are indefinite at precisely one place (an there of real rank 1), see Ass. 4.7. Indeed, in the latter case we only need to assume Conjecture 3.3 for cohomological representations π, π lifting to cuspidal representations Π, Π .

(ii) Conjecture 3.7 (non-vanishing of twisted central critical values) below:

Conjecture 3.7. *Let S be a finite set of non-archimedean places of F and for each v S let $\alpha_v : \mathrm{GL}_1$ $\mathcal{O}_F$ $_{,v}$ $\mathbb{C}$ be a continuous character. Let χ be an algebraic character of* GL_1 F $_{\mathbb{Q}}\mathbb{R}$. *Given Π as in §2.2 there is an algebraic Hecke character χ of* GL_1 $\mathbb{A}_F$ *, with conjugate self-dual archimedean component χ , such that χ $_{\mathrm{GL}_1}$ $_{\mathcal{O}_F}$ $_{,v}$ α_v and*

$$L \ \tfrac{1}{2}, \Pi \quad \chi \qquad 0. \tag{3.8}$$

We obtain

Proposition 3.9. *Let Π be as in §2.2 and assume 3.2.2 (i) and (ii). If Π is (1-)regular then the assertion of Thm. 2.2 holds verbatim.*

We refer to the forthcoming completion of [Lin17] for a proof.

The infinity type of Π at ι_v Σ is of the form of the form $z^{a_{v,i}}\bar{z}$ $^{a_{v,i}}$ $_{1 i n}$ such that for each v, the numbers $a_{v,i}$ $\mathbb{Z}$ $\frac{n \ \ 1}{2}$ are all different for 1 i n which are arranged in decreasing order, i.e. $a_{v,1}$ $a_{v,2}$ $a_{v,n}$.

Let now 1 m n be any integer and let Π be a conjugate self-dual, cohomological cuspidal automorphic representation of GL_m $\mathbb{A}_F$ as in section 2.2. Then, similarly, the infinity type of Π at ι_v Σ is of the form of the form $z^{b_{v,j}}\bar{z}$ $^{b_{v,j}}$ $_{1 j m}$ with $b_{v,j}$ $\mathbb{Z}$ $\frac{m \ \ 1}{2}$ and $b_{v,1}$ $b_{v,2}$ $b_{v,m}$.

Definition 3.10. For 0 i n and ι_v Σ, we define the **split indices** (cf. [Lin15b], [Har-Lin16])

$$sp \ \ i, \Pi; \Pi \ , \iota_v \ \ : \# \ 1 \ \ j \ \ m \qquad a_{v,n \ \ 1 \ \ i} \ \ b_j \quad a_{v,n \ \ i}$$

and

$$sp \ \ i, \Pi; \Pi \ , \bar{\iota}_v \ \ : \# \ 1 \ \ j \ \ m \quad a_{v,i} \quad b_j \ \ a_{v,i \ \ 1} \ \ .$$

Here we put formally $a_{v,0}$ and $a_{v,n \ \ 1}$. It is easy to see that

$$sp \ \ i, \Pi^c; \Pi \ ^c, \iota_v \quad sp \ \ i, \Pi; \Pi \ , \bar{\iota_v} \quad sp \ \ n \quad i, \Pi; \Pi \ , \iota_v \ \ . \tag{3.11}$$

Similarly, for 0 j m, we define $sp \ \ j, \Pi \ ; \Pi, \iota_v \ \ : \# \ 1 \ \ i \ \ n \qquad b_{v,m \ \ 1 \ \ j} \ \ a_{v,i} \quad b_{v,m \ \ j}$ and $sp \ \ j, \Pi \ ; \Pi, \bar{\iota}_v \ \ : \# \ 1 \ \ i \ \ n \quad b_{v,j} \quad a_{v,i} \ \ b_{v,j \ \ 1} \ \ .$

The following is the main result of this subsection.

Theorem 3.12. *Let Π (resp. Π) be a conjugate self-dual cohomological cuspidal automorphic representation of* GL_n $\mathbb{A}_F$ *(resp.* GL_m $\mathbb{A}_F$ *) as in section 2.2. Assume 3.2.2 (i) and (ii). If n m* mod 2 *and if Π and Π are both (1-)regular, then*
(3.13)

$$L^S \ \tfrac{1}{2}, \Pi \quad \Pi \qquad _{E \ \Pi \ E \ \Pi} \quad 2\pi i \ ^{dnm \ \ 2} \prod_{\iota_v \ \ \Sigma \quad 0 i n} P^{\ i} \ \ \Pi, \iota_v \ ^{sp \ \ i, \Pi; \Pi \ , \iota_v} \prod_{0 j m} P^{\ j} \ \ \Pi \ , \iota_v \ ^{sp \ \ j, \Pi \ ; \Pi, \iota_v}$$

which is equivariant under the action of Gal $\bar{\mathbb{Q}}$ F^{Gal} .

We refer to our forthcoming [Gro-Har-Lin18] for a proof.

4. Shimura varieties and coherent cohomology

4.1. **Review of coherent cohomology.** Recall $V \quad V \quad V_1$ and the attached unitary similitude groups $\tilde{H} \quad \tilde{H} \quad \tilde{H}$ from §2.5. Introduce $\tilde{H}$ and $\tilde{H}$ -homogeneous hermitian symmetric domains $\tilde{X} \quad X \; \tilde{H}$ and $\tilde{X} \quad X \; \tilde{H}$ so that $\tilde{H} \; , \tilde{X} \qquad \tilde{H}, \tilde{X}$ is a morphism of Shimura data. The signature of V (resp. V) at an element $v \quad S$ is denoted r_v, s_v (resp. r_v, s_v). It determines a holomorphic structure (see [Har14, 4.1]) on the the symmetric space $\tilde{X} \; \tilde{H} \qquad H \quad K_{H,}$ and therefore we can assign to each $v \quad S$ a reference parameter

$$A_{\lambda,v} \quad A_{v,1} \qquad A_{v,n} \qquad \frac{n \; 1}{2} \quad \mathbb{Z} \; {}^{n}$$

which is the infinitesimal character of an irreducible finite-dimensional representation $\mathcal{F}_{\lambda_v}$ of H_v.

Let $\tilde{\pi}$ be an essentially discrete series representation of $\tilde{H}$ and $\pi \qquad {}_{v \; S} \; \pi \; A_v$ its restriction to H as in §1.3.2. The reference parameter is a permutation of A_v, which we now write

$$A_v \quad \alpha_{1,v} \qquad \alpha_{r_v,v}; \beta_{1,v} \qquad \beta_{s_v,v} \; .$$

Let $\Lambda_v \quad A_v \quad \rho_v$ be the v-component of the coherent parameter of π and let $\Lambda \qquad \Lambda_v \; {}_{v \; S}$. We denote by $\mathcal{W}_\Lambda$ the corresponding irreducible representation of $K_{H,}$ and by $\tilde{\mathcal{W}}_\Lambda$ the irreducible representation of $K_{\tilde{H},}$ of highest weight $\Lambda; \Lambda_0$. Here Λ_0 can be any integer of the same parity as ${}_{v,i} \Lambda_{v,i}$, according to §1.3.1: It plays no role in the final result.

The representation $\tilde{\mathcal{W}}_\Lambda$ defines an automorphic vector bundle $\tilde{\mathcal{W}}_\Lambda$ on the Shimura variety $Sh \;\; \tilde{H}, \tilde{X}$, where $\tilde{X}$ is the (disconnected) hermitian symmetric space as above and $\tilde{X} \qquad \tilde{X}$ is a fixed H -invariant component (also invariant under the identity component $\tilde{H}^0 \quad \tilde{H}$).

We write $\mathfrak{h}_{v,\mathbb{C}} \quad \mathfrak{k}_{H,v,\mathbb{C}} \quad \mathfrak{p}_v \quad \mathfrak{p}_v$ (Harish-Chandra decomposition), and let

$$\mathfrak{p}_h \; : \;\; {}_v\mathfrak{p}_v \qquad \text{and} \qquad \mathfrak{p}_h \; : \;\; {}_v\mathfrak{p}_v$$

so that

$$\mathfrak{h} \; {}_{,\mathbb{C}} \quad \mathfrak{k}_{H, \;,\mathbb{C}} \quad \mathfrak{p}_h \quad \mathfrak{p}_h \, .$$

Here $\mathfrak{p}_h$ and $\mathfrak{p}_h$ identify naturally with the holomorphic and antiholomorphic tangent spaces to $\tilde{X}$ at the fixed point $h \quad h_{K_{H, \;,}}$ of $K_{H, \;,}$ in the symmetric space $\tilde{X}$.

Algebraicity of λ implies that the canonical and sub-canonical extensions of the $\tilde{H} \;\; \mathbb{A}_{\mathbb{Q},f}$ -homogeneous vector bundle $\tilde{\mathcal{W}}_\Lambda$ give rise to coherent cohomology theories which are both defined over a finite extension of the reflex field. We let $E \;\; \Lambda$ denote a number field over which there is such a rational structure. (In general, there is a Brauer obstruction to realizing $\tilde{\mathcal{W}}_\Lambda$ over the fixed field of its stabilizer in Gal $\bar{\mathbb{Q}} \;\; \mathbb{Q}$, and we can take $E \;\; \Lambda$ to be some finite, even abelian extension of the latter.)

As in [Har14, Har90] we denote by $\bar{H} \qquad \tilde{\mathcal{W}}_\Lambda$ the interior cohomology of $\tilde{\mathcal{W}}_\Lambda$. This has a natural rational structure over $E \;\; \Lambda$. Let $H_{cute} \quad \tilde{\mathcal{W}}_\Lambda \qquad \bar{H} \quad \tilde{\mathcal{W}}_\Lambda$ denote the subspace represented by *cu*spidal essentially *te*mpered forms, more precisely by forms contained in cuspidal representations that are essentially tempered at all unramified non-archimedean places.

4.2. Rationality for cute coherent cohomology and corresponding rationality-fields.

Proposition 4.1. *The subspace* $H_{cute}\ \ \tilde{\mathcal{W}}_\Lambda$ *of* $\bar{H}\ \ \tilde{\mathcal{W}}_\Lambda$ *is rational over* $E\ \Lambda$ *.*

Proof. Let $H_t\ \ \tilde{\mathcal{W}}_\Lambda\ \ \ \bar{H}\ \ \tilde{\mathcal{W}}_\Lambda$ denote the subspace represented by forms that are essentially tempered at all unramified non-archimedean places. The condition of essentially temperedness at unramified places is equivalent to the condition that the eigenvalues of Frobenius all be the same multiple of q-numbers, all of them of the same weight, hence is a equivariant under Aut $\mathbb{C}$. Therefore $H_t\ \ \tilde{\mathcal{W}}_\Lambda$ is an $E\ \Lambda$ -rational subspace, and it suffices to show that it coincides with $H_{cute}\ \ \tilde{\mathcal{W}}_\Lambda$. Now the (graded) space $\bar{H}\ \ \tilde{\mathcal{W}}_\Lambda$ is represented entirely by essentially square-integrable automorphic forms, so it suffices to show that any essentially square-integrable representation that contributes to $H_t\ \ \tilde{\mathcal{W}}_\Lambda$ contributes to $H_{cute}\ \ \tilde{\mathcal{W}}_\Lambda$. But it follows from [Clo93] Prop. 4.10 that the non-cuspidal (residual) constituents of $\bar{H}\ \ \tilde{\mathcal{W}}_\Lambda$ are not essentially tempered, so we are done. □

Remark 4.2. As far as we know, it has not been proved in general that the cuspidal subspace of $\bar{H}\ \ \tilde{\mathcal{W}}_\Lambda$ is rational over $E\ \Lambda$.

Let $\mathcal{A}_{cute}\ \tilde{H}$ be the space of cuspidal automorphic forms on $\tilde{H}\ \ \mathbb{A}_\mathbb{Q}$, which give rise to representations which are essentially tempered at all unramified non-archimedean places.

Proposition 4.3. *If* $\tilde{\pi}\ \ \mathcal{A}_{cute}\ \ \tilde{H}$ *contributes non-trivially to* $H_{cute}\ \ \tilde{\mathcal{W}}_\Lambda$ *, then* $\tilde{\pi}$ *is essentially tempered at all places and* $\tilde{\pi}$ *is in the cohomological essentially discrete series.*

Proof. Let $\tilde{\pi}$ denote a cuspidal automorphic representation of $\tilde{H}\ \ \mathbb{A}_\mathbb{Q}$ that contributes to $H_{cute}\ \ \tilde{\mathcal{W}}_\Lambda$, hence is essentially tempered at all unramified non-archimedean places. The degeneration of the Hodge deRham spectral sequence for the cohomology of the local system $\tilde{\mathcal{F}}_\lambda$ implies that cute coherent cohomology of $\tilde{\mathcal{W}}_\Lambda$ injects into $\ \tilde{\mathfrak{h}}\ \ , K_{\tilde{H},}$ -cohomology of $\tilde{\mathcal{F}}_\lambda$, i.e., the converse of [Gro-Seb17], Thm. A.1 is true. Hence, $\tilde{\pi}$ is necessarily cohomological. Its base change $\Pi\ \ BC\ \ \tilde{\pi}$, cf. §1.3.3, for our conventions, is hence a cohomological isobaric sum $\Pi\ \ \Pi_1\ \ ...\ \ \Pi_r$ of conjugate self-dual square-integrable automorphic representations Π_i of $G_{n_i}\ \ \mathbb{A}_F$. As $\tilde{\pi}\ \ \mathcal{A}_{cute}\ \ \tilde{H}$, Π_v is tempered at infinitely many places v. It follows from the classification of Moeglin-Waldspurger [MW89] that each Π_i is cuspidal and hence everywhere tempered, see §1.3.3, in particular at the archimedean places. By Theorem 1.7.1 of [KMSW14] we know that $\tilde{\pi}$ is attached to a global Arthur parameter $\psi\ \ \psi_1\ \ ...\ \ \psi_r$, each ψ_i corresponding to the globally tempered Π_i from above, see again §1.3.3. Thus, $\tilde{\pi}$ is essentially tempered at all places and so $\tilde{\pi}$, being an essentially tempered cohomological representation of $\tilde{H}\ \ \mathbb{R}$, must be in the essentially discrete series by [Vog-Zuc84]. □

Following §4.6 of [Har90] we obtain

Corollary 4.4. *There is a unique* $q\ \ q\ \Lambda\ \ \ \ _{v\ S}\ q\ \Lambda_v$ *such that* $H^{q\ \Lambda}_{cute}\ \ \tilde{\mathcal{W}}_\Lambda\ \ \ \ 0.$

Indeed, there are the following isomorphisms for the graded ring

$$H_{cute} \quad \tilde{\mathcal{W}}_\Lambda \qquad \underset{\tilde\pi \mathcal{A}_{cute} \ \tilde H}{} \quad H \quad \mathfrak{k}_{\tilde H, \ ,\mathbb{C}} \quad \mathfrak{p}_h , K_{\tilde H,} \ , \tilde\pi \quad \tilde{\mathcal{W}}_\Lambda \quad \tilde\pi_f$$

$$\underset{\substack{\tilde\pi \mathcal{A}_{cute} \ \tilde H \quad v \ S \\ \tilde\pi \quad \mathbb{R} \ \tilde{\mathcal{W}}_\Lambda \ \mathbb{R}}}{} \quad H \quad \mathfrak{k}_{H,v,\mathbb{C}} \quad \mathfrak{p}_v , K_{H,v}, \ \tilde\pi \quad H \quad v \quad \mathcal{W}_{\Lambda_v} \quad \tilde\pi_f$$

$$(4.5) \qquad \underset{\substack{\tilde\pi \mathcal{A}_{cute} \ \tilde H \quad v \ S \\ \tilde\pi \quad \mathbb{R} \ \tilde{\mathcal{W}}_\Lambda \ \mathbb{R} \\ \pi_v \ \pi \ A_v}}{} \quad H^{q \ \Lambda_v} \quad \mathfrak{k}_{H,v,\mathbb{C}} \quad \mathfrak{p}_v , K_{H,v}, \pi_v \quad \mathcal{W}_{\Lambda_v} \quad \tilde\pi_f$$

$$\underset{\substack{\tilde\pi \mathcal{A}_{cute} \ \tilde H \\ \tilde\pi \quad \mathbb{R} \ \tilde{\mathcal{W}}_\Lambda \ \mathbb{R} \\ \pi_v \ \pi \ A_v}}{} \quad \tilde\pi_f .$$

for unique degrees $q \ \Lambda_v \quad q \ A_v \quad q \ \pi_v$. So, $H^q_{cute} \quad \tilde{\mathcal{W}}_\Lambda \quad 0$ unless $q \quad q \ \Lambda \quad {}_{v \ S} \ q \ \Lambda_v$, in which case $H^{q \ \Lambda}_{cute} \quad \tilde{\mathcal{W}}_\Lambda$ is described by (4.5).

As there is now quite a number of "fields of rationality" floating around, each of them being defined in a slightly different manner, we include the following lemma, which explains the relation of these various fields, for the convenience of the reader.

Lemma 4.6. *$E \ \Lambda \quad E \ \tilde\pi \quad E \ \pi$ and $E \ BC \ \pi \quad E \ BC \ \pi \quad E \ \pi$.*

Proof. By Prop. 4.1, (4.5) and Thm. A.2.4 of [Gro-Seb17], $\tilde\pi_f$ is defined over a finite extension of $E \ \Lambda$. This shows $E \ \Lambda \quad E \ \tilde\pi$. By assumption $\tilde\pi \ K_{\tilde H,}$ is defined over $E \ \tilde\pi$ as a $\tilde{\mathfrak{h}} \ , K_{\tilde H,} \ , \tilde H \ \mathbb{A}_{\mathbb{Q},f}$ -module, so it is in particular as a $\mathfrak{h} \ , K_{H,} \ , H \ \mathbb{A}_{F \ ,f}$ -module. As the latter, it breaks as an algebraic direct sum of irreducible $\mathfrak{h} \ , K_{H,} \ , H \ \mathbb{A}_{F \ ,f}$ -modules, containing $\pi \ K_{H,}$, cf. Lem. 4.1.2 of [Gue-Lin16]. So $\pi \ K_{H,}$ can be defined over a finite extension of $E \ \tilde\pi$ by the argument given in the proof of Thm. A.2.4 of [Gro-Seb17], suitably generalized at the archimedean places. Hence, $E \ \tilde\pi \quad E \ \pi$. Finally, strong multiplicity one implies that $E \ BC \ \pi \quad E \ BC \ \pi^{\ S}$, where S is any finite set of places containing S and the places where $BC \ \pi$ ramifies. Hence, $E \ BC \ \pi \quad E \ BC \ \pi^{\ S} \quad E \ BC \ \pi^{\ S,} \quad E \ \pi^S$, where the last inclusion is due to [Gan-Rag13], Lem. 9.2 and the definition of base change. Invoking strong multiplicity one once more, $E \ BC \ \pi \quad E \ BC \ \pi \quad E \ \pi$.

□

4.3. **Review of the results of [Har14].** From now on, we will make the following running assumption on the signatures of our unitary groups:

Assumption 4.7. *There is $v_0 \quad S$ such that $r_{v_0}, s_{v_0} \quad n \quad 1, 1$, $r_{v_0}, s_{v_0} \quad n \quad 2, 1$. For $v \quad v_0$ in S , the signature is $n, 0$ (resp. $n \quad 1, 0$).*

Remark 4.8. If we want to emphasize this local obstruction on our hermitian space, we will write $V \quad V^{\ 0}$ (resp. $V \quad V^{\ 0}$) and $H^{\ 0} \quad U \ V^{\ 0}$ (resp. $H^{\ 0} \quad U \ V^{\ 0}$) for the corresponding unitary groups, see in particular §5.1 onwards.

We may assume that $K_{H,}$ fixes a point $x \quad \tilde{X}$ that is defined over the image of F under v_0. The anti-holomorphic tangent space to X at x corresponds to a $K_{H,}$ -invariant unipotent subalgebra $\mathfrak{p}_x \quad \mathfrak{h}_{\ ,\mathbb{C}}$; we let $\mathfrak{q} \quad \mathfrak{q}_x$ denote the parabolic subalgebra $\mathfrak{q}_x : \ \mathfrak{k}_{H, \ ,\mathbb{C}} \quad \mathfrak{p}_x \quad \mathfrak{h}_{\ ,\mathbb{C}}$.

Let now $\mathcal{F}_\lambda \quad {}_{v \ S} \ \mathcal{F}_{\lambda_v}$ be any irreducible finite-dimensional algebraic representation of H as in §1.3.1. Under our Assumption 4.7, following §1.3.2 there are exactly n inequivialent discrete series representations of H , denoted $\pi_{\lambda,q}$, $0 \ q \ n \quad 1$, for which $H^p \ \mathfrak{h} \ , K_{H,} \ , \pi \quad \mathcal{F}_\lambda \quad 0$ for some degree p (which necessarily equals $p \quad n \quad 1$). Moreover, the representations $\pi_{\lambda,q}$, $0 \ q \ n \quad 1$, are distinguished by the property that,

$$\dim H^q \ \mathfrak{q}, K_{H,} \ , \pi_{\lambda,q} \quad \mathcal{W}_{\Lambda \ q} \quad 1$$

and all other $H^p \ \mathfrak{q}, K_{H,} \ , \pi_{\lambda,q} \quad \mathcal{W}$ vanish as $\mathcal{W}$ runs over all irreducible representations of $K_{H,}$ and p the cohomological degrees. Here $\Lambda \ q \quad A \ q \quad \rho_n$, where $A \ q$ is the Harish-Chandra parameter of $\pi_{\lambda,q}$ and ρ_n is the half-sum of positive, absolute roots of H . We can determine $A \ q$ explicitly: let

$$A_\lambda \quad A_{\lambda,v \ \ v \ S} \ ; A_{\lambda,v} \quad A_{v,1} \qquad A_{v,n}$$

be the infinitesimal character of $\mathcal{F}_\lambda$, as in 4.2 of [Har14]. Then $A \ q \quad A \ q \ {}_{v \ \ v \ S}$ where $A \ q \ {}_v$ $A_{\lambda,v}$ for $v \quad v_0$ and

$$A \ q \ {}_{v_0} \quad A_{v_0,1} \qquad A_{v_0,q \ 1} \qquad A_{v_0,n}; A_{v_0,q \ 1}$$

(the parameter marked by is deleted from the list). The following is obvious.

Remark 4.9. For $0 \ q \ n \quad 2$ the parameter $A \ q$ satisfies Hypothesis 4.8 of [Har14].

Hypothesis 4.10. *The Harish-Chandra parameter A_{λ,v_0} satisfies the regularity condition $A_{v_0,i}$ $A_{v_0,i \ 1} \ \ 2$ for $i \quad 1, \ldots, n \quad 1$. Equivalently, the highest weight λ_{v_0} is a regular.*

For $0 \ q \ n \quad 2$ define a Harish-Chandra parameter $A \quad q \quad A \quad q \ {}_{v \ \ v \ S}$ by the formula (4.5) of [Har14]:

$$A \quad q \ {}_{v_0} \quad A_{v_0,1} \quad \tfrac{1}{2} \qquad A_{v_0,q \ 1} \quad \tfrac{1}{2} \qquad A_{v_0,n \ 1} \quad \tfrac{1}{2}; A_{v_0,q \ 1} \quad \tfrac{1}{2} \ . \tag{4.11}$$

For $v \quad v_0$, $A \quad q \ {}_v \quad A_{v,1} \quad \tfrac{1}{2} \qquad A_{v,n \ 1} \quad \tfrac{1}{2}$.

It follows from Hypothesis 4.10 and [Har14], Lem. 4.7, that $A \quad q$ is the Harish-Chandra parameter for a unique discrete series representation $\pi_{A \quad q}$ of H . Indeed, Hypothesis 4.10 is the version of Hypothesis 4.6 of [Har14], where the condition is imposed only at the place v_0 where the local unitary group is indefinite. Observe that there is no need for a regularity condition at the definite places: for $v \quad v_0$ the parameter $A \quad q \ {}_v$ is automatically the Harish-Chandra parameter of an irreducible representation. We can thus adapt Thm. 4.12 of [Har14] to the notation of the present paper:

Theorem 4.12. *Suppose A_{λ,v_0} satisfies Hypothesis 4.10. For $0 \ q \ n \quad 2$ let $\tilde{\pi} \ q \quad \tilde{\pi} \ A \ q$ and $\tilde{\pi} \quad q \quad \tilde{\pi} \quad A \quad q$ be an essentially tempered cuspidal automorphic representations of $\tilde{H} \ \mathbb{A}_{\mathbb{Q}}$ and $\tilde{H} \quad \mathbb{A}_{\mathbb{Q}}$, respectively, whose archimedean components restrict to $\pi_{A \ q}$ and $\pi_{A \quad q}$. Let $\tilde{\xi}$ be a Hecke character of $GU \ V_1 \quad \mathbb{A}_{\mathbb{Q}}$, as in §2.6, such that $\xi_{\tilde{\pi}}^{\ 1} \quad \xi_{\tilde{\pi}} \tilde{\xi} \quad {}_{Z_{\tilde{H}} \ \mathbb{A}_{\mathbb{Q}}}$ and let $\tilde{\pi} \quad q \quad \tilde{\pi} \quad q \quad \tilde{\xi} \quad {}_{\tilde{H} \quad \mathbb{A}_{\mathbb{Q}}}$. Then for any deRham-rational elements in $\phi \ q \quad \tilde{\pi} \ q$, $\phi \quad q \quad \tilde{\pi} \quad q$*

$$I^{can} \ \phi \ q \ , \phi \quad q \qquad E \ \tilde{\pi} \ q \quad E \ \tilde{\pi} \quad q \qquad E \ \tilde{\pi} \ q \quad E \ \tilde{\pi} \quad q \quad .$$

The statement in [Har14] has two hypotheses: the first one is the regularity of the highest weight, while the second one (Hypothesis 4.8 of [Har14]) follows as in Remark 4.9 from the assumption that $q \quad n \quad 1$. We remark that the assumption in *loc.cit* on the Gan-Gross-Prasad multiplicity one conjecture for real unitary groups has been proved by H. He in [He17].

A cuspidal automorphic representation $\tilde{\pi}$ of $\tilde{H} \quad \mathbb{A}_{\mathbb{Q}}$ that satisfies the hypothesis of Theorem 4.12 contributes to interior cohomology of the corresponding Shimura variety $Sh \quad \tilde{H}, \tilde{X}$ with coefficients in the local system defined by the representation $\tilde{\mathcal{W}}_{\Lambda \quad q}$, which arises from $\mathcal{W}_{\Lambda \quad q}$ by adding a suitable last coordinate $\Lambda_0 \quad \mathbb{Z}$ in the highest weight as sketched in §4.1. This coefficient carries a (pure) Hodge structure of weight $n \quad 1$, with Hodge types corresponding to the infinitesimal character of $\tilde{\mathcal{W}}_{\Lambda \quad q}$. Ignoring the last coordinate Λ_0, note that this infinitesimal character is given by

$$A \quad q \qquad A \quad q \quad _{v,n} \quad \dots \quad A \quad q \quad _{v,1} \quad _{v \quad S} \quad .$$

The Hodge numbers corresponding to the place v_0 are

$$p_i \quad A_{v_0,n \quad 1 \quad i} \quad \tfrac{n \quad 1}{2}, q_i \quad n \quad 1 \quad p_i \ ; \qquad p_i^c \quad q_i, q_i^c \quad p_i \quad . \tag{4.13}$$

Analogously, a cuspidal automorphic representation $\tilde{\pi}$ of $\tilde{H} \quad \mathbb{A}_{\mathbb{Q}}$ as in Theorem 4.12 contributes to interior cohomology of the corresponding Shimura variety $Sh \quad \tilde{H} \ , X$ with coefficients in the local system defined by a representation $\tilde{\mathcal{W}}_{\Lambda \quad q}$, whose parameters are obtained from those of $A \quad q$ by placing them in decreasing order and substracting $\rho_{n \quad 1}$. In particular, it follows from (4.11) that the infinitesimal character of $\tilde{\mathcal{W}}_{\Lambda \quad q}$ at v_0 is given by

$$A_{v_0,1} \quad \tfrac{1}{2} \qquad A_{v_0,q} \quad \tfrac{1}{2} \ A_{v_0,q \quad 1} \quad \tfrac{1}{2} \ A_{v_0,q \quad 2} \quad \tfrac{1}{2} \qquad A_{v_0,n \quad 1} \quad \tfrac{1}{2} \ ,$$

with strict inequalities due to the regularity of A_{λ,v_0} and with corresponding Hodge numbers

$$p_i \quad A_{v_0,i} \quad \tfrac{n \quad 3}{2}, \quad \text{for} \quad i \quad q \quad 1; \qquad p_{q \quad 1} \quad A_{v_0,q \quad 1} \quad \tfrac{n \quad 1}{2} \tag{4.14}$$

and $q_i \quad n \quad 2 \quad p_i$, etc.

Here is a consequence of the main result of [Har14].

Theorem 4.15. *Let $\tilde{\pi} \quad q$ be as before. Then there exists a cuspidal automorphic representation $\pi \quad q$ of $H \quad \mathbb{A}_F$ with archimedean component $\pi_{A \quad q}$, such that*

1. *$\pi \quad q$ extends to a cuspidal automorphic representation $\tilde{\pi} \quad q$ of $\tilde{H} \quad \mathbb{A}_{\mathbb{Q}}$ which satisfies the assumptions of Thm. 4.12*
2. *$BC \quad \pi \quad q$ is cuspidal automorphic and supercuspidal at one finite place of F which is split in F, and*
3. *there are vectors $f \quad \pi \quad q$, $f \quad \pi \quad q$, that are factorizable, as in the statement of Conjecture 3.3, so that $I^{can} \quad f, f \quad 0$ with f_v (resp. f_v) in the minimal $K_{H,v}$- (resp. $K_{H \ ,v}$-type) of $\pi \quad q \quad _v$ (resp. $\pi \quad q \quad _v$) for all $v \quad S$.*

In particular, the Gross-Prasad period $\mathcal{P} \quad f, f$ does not vanish.

Proof. Although this is effectively the main result of [Har14], it is unfortunately nowhere stated in that paper. Thus we explain why this is a consequence of the results proved there, and recent results on base change, see [Shi11, KMSW14].

First, we claim that the discrete series representation $\pi_{A \quad q}$ is isolated in the (classical) automorphic spectrum of H , in the sense of [Bur-Sar91], see Corollary 1.3 of [Har14]. Admitting the claim, we note that Hypothesis 4.6 of [Har14] is our regularity hypothesis, and Hypothesis 4.8 is true by construction. The theorem then follows from the discussion following the proof of Theorem 4.12 of [Har14]. More precisely, because $\pi_{A \quad q}$ is isolated in the automorphic spectrum, we can apply Corollary 1.3 (b) of [Har14], which is a restatement of the main result of [Bur-Sar91]. Now the isolation follows as in [Har-Li98], Thm. 7.2.1, from the existence of base change from H to $\mathrm{GL}_{n \quad 1}$, established in the generality which we need in [Shi11, KMSW14].

□

As a last conjectural ingredient, our main theorem will also make use of the following archimedean

Conjecture 4.16. *Let $\pi \quad q$ and $\pi \quad q$ be as in Theorem 4.15. Let f and f satisfy (2) of that theorem. Up to replacing $E \quad \pi \quad q$ and $E \quad \pi \quad q$ by suitable finite extensions, they can be chosen so that, for all $v \quad S$, f_v (resp. f_v) belongs to the $E \quad \pi \quad q$ - (resp. $E \quad \pi \quad q$ -) rational subspaces of the minimal $K_{H,v}$-type of $\pi \quad q \quad_v$ (resp. $K_{H \;,v}$-type of $\pi \quad q \quad_v$), with respect to the $E \quad \pi \quad q$ - (resp. $E \quad \pi \quad q$ -) deRham-rational structures defined in Corollary 3.8 of [Har13a] (and its correction in the Erratum to that paper). With such choices, we then have*

$$I_v \quad f_v, f_v \qquad E \quad \pi \quad q \qquad E \quad \pi \quad q$$

for all $v \quad S$.

5. Statement of the main theorem

5.1. **The main theorem.** For the reason of normalisation, we shall descend Π but not Π. We hence consider the L-packet $\Pi_\Phi \quad H^{\;0}, \Pi$.

Definition 5.1. Let $0 \quad q \quad n \quad 1$ be an integer. We say $\pi \quad \Pi_\Phi \quad H^{\;0}, \Pi$ is of degree q if the Harish-Chandra parameter A_{v_0} of π_{v_0} satisfies $q \quad \pi_{v_0} \quad q \quad A_{v_0} \quad q$.

Obviously, by our standing Assumption 4.7, we necessarily have $q \quad \pi_v \quad 0$ for $v \quad v_0$.

Theorem 5.2. *Let Π be a cuspidal automorphic representation of $G_n \quad \mathbb{A}_F$, such that for each $I \quad I_\iota \;_{\iota \; \Sigma} \quad 0, 1, \quad , n^{\;\Sigma}$, Π descends to a cohomological cuspidal automorphic representation of a unitary group $U_I \quad \mathbb{A}_F$ of signature $n \quad I_\iota, I_\iota$ at $v \quad \iota, \bar{\iota} \quad S$, with tempered archimedean component. We assume moreover that Π is $n \quad 1$ -regular. For $0 \quad q \quad n \quad 2$, let $\pi \quad q \quad \pi \quad q, \iota_0$ be an element in $\Pi_\Phi \quad H^{\;0}, \Pi$ of degree q and $\tilde{\pi} \quad q$ an essentially tempered, cohomological cuspidal automorphic extension of $\pi \quad q$ to $\tilde{H} \quad \mathbb{A}_\mathbb{Q}$.*

We assume the validity of

- (a) *Conjecture 3.3 (the IINH-Conjecture) for pairs of cohomological representations of totally definite unitary groups and unitary groups, which are indefinite at precisely one place (an there of real rank 1), see Ass. 4.7. Indeed, in the latter case we only need to assume Conjecture 3.3 for cohomological representations lifting to cuspidal automorphic representations.*
- (b) *Conjecture 3.7 (non-vanishing of twisted central critical values).*
- (c) *Conjecture 4.16 (rationality of archimedean integrals).*

Then, denoting by $\check{\xi}_\Pi$ the conjugate invers of the central character of Π,

$$Q \quad \pi \quad q \qquad_{E \;\pi \; q} \; p \;\; \check{\xi}_\Pi, \Sigma \quad {}_1\frac{P^{\;q \;\;1} \quad \Pi, \iota_0}{P^{\;q} \quad \Pi, \iota_0}.$$

Remark 5.3. If n is odd, for each I, we can take a unitary group U_I which is quasi-split at each finite place. Then the representation Π descends to U_I automatically. If n is even, we assume in addition that there is a finite place v_1 of F that does not split in F such that Π_{v_1} descends to the non-quasi-split unitary group over F_{v_1}. Then Π should descend by the completion of [KMSW14].

Observation 5.4. If $q \quad 0$, then by construction $Q \ \pi \ 0 \qquad {}_{E \ \pi \ 0} \ P^{\ I_0} \ \ \Pi$ where I_0 is the signature of the unitary group $V^{\ 0}$. In particular, $I_0 \ \iota \quad 0$ if $\iota \quad \iota_{v_0}$ and $I_0 \ \iota_{v_0} \quad 1$. Hence

$$
\begin{aligned}
Q \ \pi \ 0 \qquad & {}_{E \ \pi \ 0} \qquad \prod_{\iota_v \ \iota_{v_0}} P^{\ 0} \ \ \Pi, \iota_v \quad P^{\ 1} \ \ \Pi, \iota_{v_0} \\
& {}_{E \ \pi \ 0} \qquad \prod_{\iota_v \ \iota_{v_0}} p \ \ \check{\xi}_{\Pi}, \iota_v \quad {}^{1} \ P^{\ 1} \ \ \Pi, \iota_{v_0} \\
& {}_{E \ \pi \ 0} \qquad \prod_{\iota_v \ \Sigma} p \ \ \check{\xi}_{\Pi}, \iota_v \quad {}^{1} \ P^{\ 1} \ \ \Pi, \iota_{v_0} \ p \ \ \check{\xi}, \bar{\iota}_v \quad {}^{1} \\
& {}_{E \ \pi \ 0} \quad p \ \ \check{\xi}_{\Pi}, \Sigma \quad {}^{1} \frac{P^{\ 1} \ \ \Pi, \iota_{v_0}}{P^{\ 0} \ \ \Pi, \iota_{v_0}}.
\end{aligned}
\tag{5.5}
$$

In brief, Theorem 5.2 is true when $q \quad 0$. This is going to be used as the first step in our inductive argument.

Definition 5.6. Let Π and $\pi \ q$ be as in the previous theorem. We define

$$
P_q \ \ \Pi, \iota_0 \quad : \qquad
\begin{cases}
P^{\ 0} \ \ \Pi, \iota_{v_0} & \text{if} \quad q \quad 0; \\
Q \ \ \pi \ \ q \quad 1 \quad p \ \ \check{\xi}_{\Pi}, \Sigma & \text{if} \quad 1 \ \ q \ \ n \quad 1; \\
P^{\ n} \ \ \Pi, \iota_{v_0} \prod_{i \, 0}^{n \ \ 1} P_i \ \ \Pi, \iota_{v_0} \quad {}^{1} & \text{if} \quad q \quad n.
\end{cases}
$$

The above theorem immediately implies that:

Corollary 5.7. *For any* $0 \ \ i \ \ n$ *we have the following factorization:*

$$
P^{\ i} \ \ \Pi, \iota_0 \quad {}_{E \ \pi \ q} \ P_0 \ \ \Pi, \iota_0 \ \ P_1 \ \ \Pi, \iota_0 \qquad P_i \ \ \Pi, \iota_0 \ . \tag{5.8}
$$

6. Proof of the main theorem: Factorization of periods

6.1. **Beginning of the proof.** We will now show how to derive a version of the factorization of periods, conjectured in [Har97] (see Conjecture 2.8.3 and Corollary 2.8.5 of [Har97]). A proof of this conjecture when F is imaginary quadratic, under a certain regularity hypothesis and up to an unspecified product of archimedean factors, was obtained in [Har07]. The proof there was based on an elaborate argument involving the theta correspondence. The argument given here is much shorter and more efficient, but it depends on the hypotheses (or conjectures) (a), (b), and (c) of Theorem 5.2.

We return to the formulation of the main theorem: Recall that for $0 \ \ q \ \ n \quad 2$, $\pi \ \ q$ denoted an element in $\Pi_{\Phi} \ \ H^{\ 0}, \Pi$ of degree q. Their base changes are hence the same cohomological representation Π . In particular, the highest weight of the finite dimensional representation with respect to which $\pi \ \ q$ has non-trivial $\mathfrak{h} \ \ , K_{H,}$ -cohomology does not depend on q. In other words, there exists a highest weight λ as in the previous section such that $\pi \ \ q \qquad \pi_{\lambda,q}$. Then the infinity type of Π at v is $z^{a_{v,i}} \bar{z}^{\ \ a_{v,i}} \ {}_{1 i n}$ where $a_{v,i} \qquad A_{v,n \ \ 1 \ \ i}$.

Let $\pi \quad q$ and $\tilde{\pi} \quad q$ be as in theorem 4.15. The Harish-Chandra parameter of $\pi \quad q$, denoted by $A \quad q$ satisfies that $A \quad q_{\ v} \quad A_{v,n \ 1} \ \frac{1}{2} \quad A_{v,1} \ \frac{1}{2}$ and

$$A \quad q_{\ v_0} \quad A_{v_0,n \ 1} \ \frac{1}{2} \quad A_{v_0,q \ 1} \ \frac{1}{2} \quad A_{v_0,1} \ \frac{1}{2}; \ A_{v_0,q \ 1} \ \frac{1}{2} \ .$$

Let Π be the base change of the $\pi \quad q$. It is easy to see that $\pi \quad q$ is an element in $\Pi_\Phi \ H \ ^0, \Pi$ of degree $n \quad q \quad 2$. The infinity type of Π at $v \quad S$ is $z^{b_{v,j}} \bar{z} \ ^{b_{v,j}} \ _{1jn \ 1}$, with $b_{v,j} \quad A_{v,j} \quad \frac{1}{2}$ if either $v \quad v_0$, or $v \quad v_0$ and $j \quad q \quad 1$, whereas $b_{v_0,q \ 1} \quad A_{v_0,q \ 1} \quad \frac{1}{2}$.

It is easy to see that if $v \quad v_0$, then

$$sp \ \ i, \Pi; \Pi \ , \iota_v \quad \begin{matrix} 1 & \text{if} & 1 \ i \ n \quad 1 \\ 0 & \text{if} & i \quad 0 \text{ or } n \end{matrix} \quad ; sp \ \ j, \Pi \ ; \Pi, \iota_v \quad 1 \ \text{ if } \ 0 \ j \ n \quad 1 \quad .$$

At $v \quad v_0$, we have:

$$sp \ \ i, \Pi; \Pi \ , \iota_0 \quad \begin{matrix} 1 & \text{if} & 1 \ i \ n \quad 1, i \quad q, i \quad q \quad 1 \\ 0 & \text{if} & i \quad 0, q \quad 1 \text{ or } n \\ 2 & \text{if} & i \quad q \end{matrix} \quad ;$$

$$sp \ \ j, \Pi \ ; \Pi, \iota_0 \quad \begin{matrix} 1 & \text{if} & 0 \ j \ n \quad 1, j \quad n \quad q \quad 1, j \quad n \quad q \quad 2 \\ 2 & \text{if} & j \quad n \quad q \quad 2 \\ 0 & \text{if} & j \quad n \quad q \quad 1 \end{matrix} \quad .$$

By Theorem 3.12, we obtain that:

$$L^S \ \tfrac{1}{2}, \Pi \quad \Pi \qquad _{E \ \Pi \ E \ \Pi} \quad 2\pi i \ ^{dn \ n \ 1 \ 2} \ _{\iota_v \ \Sigma \quad 1in \ 1} \ P^{\ i} \ \ \Pi, \iota_v \quad _{0jn \ 1} \ P^{\ j} \ \ \Pi \ , \iota_v$$

$$\frac{P^{\ q} \ \ \Pi, \iota_0 \ \ P^{\ n \ \ q \ \ 2} \ \ \Pi \ , \iota_0}{P^{\ q \ 1} \ \ \Pi, \iota_0 \ \ P^{\ n \ \ q \ \ 1} \ \ \Pi \ , \iota_0} \tag{6.1}$$

6.2. **Analysis of the Ichino-Ikeda formula.** As in the statement of Thm. 5.2, we now assume Conjecture 3.3 applies to the pair $\ \pi, \pi \quad \pi \ q \ , \pi \quad q \ $. Here, $\pi \ q$ is as in Thm. 5.2 and $\pi \quad q$ is as in Thm. 4.15. Recall that since both Π and Π are conjugate self-dual, we have:

$$L^S \ \tfrac{1}{2}, \Pi \quad \Pi \qquad L^S \ \tfrac{1}{2}, \Pi^c \quad \Pi^{\ c} \quad L^S \ \tfrac{1}{2}, \Pi \quad \Pi \ . \tag{6.2}$$

We take factorizable $f \quad \pi, f \quad \pi$ as in Theorem 4.15. In particular $I^{can} \ \ f, f \quad 0$ and f_v (resp. f_v) in the minimal $K_{H,v}$- (resp. $K_{H \ ,v}$-type) of π_v (resp. π_v) for all $v \quad S$. We may extend f and f to $\phi \quad \tilde{\pi}$ and $\phi \quad \tilde{\pi}$ respectively. We may assume moreover that ϕ and ϕ are deRham-rational.

As in section 2.6, let $\tilde{\xi}$ be a Hecke character of $GU \ \ V_1 \quad \mathbb{A}_\mathbb{Q}$ such that $\xi_{\tilde{\pi}}^{\ 1} \quad \xi_{\tilde{\pi}} \ \tilde{\xi} \quad _{Z_{\tilde{H}} \ \mathbb{A}_\mathbb{Q}}$. Write $\xi \quad \tilde{\xi} \quad _{U \ V_1 \ \ \mathbb{A}_F}$ and $\pi \quad \pi \quad \xi$. Let ϕ_0 be a deRham-rational element of $\tilde{\xi}$. We define $\phi \quad \phi \quad \phi_0$, a deRham-rational element in $\tilde{\pi}$ and set $f \ : \ \phi_\pi$. Then by Lemma 2.8 and Lemma 2.10, the Gross-Prasad period

$$0 \quad \mathcal{P} \ \ f, f \quad \mathcal{P} \ \ f, f \qquad \frac{I^{can} \ \ f, f}{f, f \quad f \ , f} \ \frac{^2 \ I^{can} \ \ \phi, \phi}{Q \ \ \pi \ Q \ \ \pi} \ ^{2}.$$

Furthermore, by Theorem 4.12, $I^{can} \ \phi, \phi$ is in $E \ \tilde{\pi} \ E \ \tilde{\pi}$. So, Conjecture 3.3 together with (6.2) implies that

$$(6.3)\quad \begin{aligned} \frac{1}{Q \ \pi \ Q \ \pi} \ {}_{E \ \pi \ E \ \pi} \ \Delta_H \frac{L^S \ \frac{1}{2}, \Pi \ \Pi}{L^S \ 1, \pi \ q \ , Ad \ L^S \ 1, \pi \ q \ , Ad} \ {}_{v \ S} \ I_v \ f_v, f_v \\ {}_{E \ \pi \ E \ \pi} \ 2\pi i^{\ dn \ n \ 1 \ 2} \frac{L^S \ \frac{1}{2}, \Pi \ \Pi}{L^S \ 1, \pi \ q \ , Ad \ L^S \ 1, \pi \ q \ , Ad} \ {}_{v \ S} \ I_v \ f_v, f_v \ . \end{aligned}$$

Here the replacement of Δ_H by a power of $2\pi i$ is a consequence of (1.22) and (1.23) in [Gro-Lin17], and the elimination of the local factors $I_v \ f_v, f_v$ at the non-archimedean places follows from Lemma 3.6. At the archimedean places we make the following observation:

Proposition 6.4. *Under the hypotheses of Theorem 4.15, the local factors* $I_v \ f_v, f_v \quad 0$ *for* $v \quad S$.

Proof. This is an immediate consequence of the non-vanishing of the global period in (2) of Theorem 4.15. □

Hence, admitting Conj. 4.16 we obtain

$$(6.5)\quad \frac{1}{Q \ \pi \ Q \ \pi} \ {}_{E \ \pi \ E \ \pi} \ 2\pi i^{\ dn \ n \ 1 \ 2} \frac{L^S \ \frac{1}{2}, \Pi \ \Pi}{L^S \ 1, \pi \ q \ , Ad \ L^S \ 1, \pi \ q \ , Ad}.$$

6.2.1. *Review of periods of adjoint L-functions for unitary groups.* We will make use of the following proposition. A detailed proof of it (in particular providing a complete proof of Thm. 1.27 of [Gro-Lin17]) will appear in the forthcoming thesis of the first named author's student P. Lopez.

Proposition 6.6. *Under the assumptions of Thm. 5.2 one has*

$$(6.7)\quad L^S \ 1, \pi \ q \ , Ad \ {}_{E \ \Pi} \ 2\pi i^{\ dn \ n \ 1 \ 2} \prod_{\iota_v \ \Sigma 1 i n \ 1} P^{\ i} \ \Pi, \iota_v$$

which is equivariant under the action of Gal $\bar{\mathbb{Q}} \ F^{Gal}$.

By construction of $\pi \ q$ its cuspidal lift Π is $n \ 2$ -regular, whence we obtain

$$(6.8)\quad \begin{aligned} L^S \ 1, \pi \ q \ , Ad \ {}_{E \ \Pi} \ 2\pi i^{\ dn \ n \ 1 \ 2} \prod_{\iota_v \ \Sigma 1 j n \ 2} P^{\ j} \ \Pi, \iota_v \\ {}_{E \ \Pi} \ 2\pi i^{\ dn \ n \ 1 \ 2} \prod_{\iota_v \ \Sigma 0 j n \ 1} P^{\ j} \ \Pi, \iota_v \end{aligned}$$

where the last equation is due to the fact that $P^{\ 0} \ \Pi, \iota_v \ P^{\ n \ 1} \ \Pi, \iota_v \ {}_{E \ \Pi} \ 1$ (cf. Thm. 7.6.1 of [Lin15b]).

Combine equations (6.1), (6.7) and (6.8), we obtain that:

$$(6.9)\quad 2\pi i^{\ dn \ n \ 1 \ 2} \frac{L^S \ \frac{1}{2}, \Pi \ \Pi}{L^S \ 1, \pi \ q \ , Ad \ L^S \ 1, \pi \ q \ , Ad} \ {}_{E \ \Pi \ E \ \Pi} \ \frac{P^{\ q} \ \Pi, \iota_{v_0} \ P^{\ n \ q \ 2} \ \Pi, \iota_{v_0}}{P^{\ q \ 1} \ \Pi, \iota_{v_0} \ P^{\ n \ q \ 1} \ \Pi, \iota_{v_0}}.$$

6.3. **The induction step, and completion of the proof of Theorem 5.2.** Combining (6.5) and (6.9) and the fact that $Q\ \pi \quad Q\ \pi\ \ Q\ \xi$, we arrive at the following conclusion:

$$\frac{1}{Q\ \pi\ Q\ \pi\ \ Q\ \xi}\ {}_{E\ \pi\quad E\ \pi}\ \frac{P\ {}^{q}\ \ \Pi,\iota_{v_0}\ \ P\ {}^{n\ \ q\ \ 2}\ \ \Pi\ ,\iota_{v_0}}{P\ {}^{q\ \ 1}\ \ \Pi,\iota_{v_0}\ \ P\ {}^{n\ \ q\ \ 1}\ \ \Pi\ ,\iota_{v_0}} \tag{6.10}$$

Lemma 6.11.

$$Q\ \xi \quad {}_{E\ \tilde{\xi}}\ p\ \ \check{\xi}_{\Pi},\Sigma\ \ p\ \ \check{\xi}_{\Pi}\ ,\Sigma\ \ .$$

Proof. Recall that by our construction $U\ \ V_1$ is a one-dimensional unitary group of sign $1,0$ at each $\iota \quad \Sigma$. We identify $GU\ \ V_1 \quad \mathbb{R}$ with the subset of $\mathbb{C}^{\Sigma}$ defined by the same similitude. Let $h_{V_1,\Sigma}$ (resp. $h_{V_1,\bar{\Sigma}}$) be the map

$$Res_{\mathbb{C}\ \ \mathbb{R}}\ \mathbb{G}_{m,\mathbb{C}} \quad \mathbb{R} \quad GU\ \ V_1 \quad \mathbb{R}$$

sending $z \quad \mathbb{C}$ to $z\ \ {}_{\iota\ \ \Sigma}$ (resp. $\bar{z}\ \ {}_{\iota\ \ \Sigma}$).

By the same argument as in section 2.9 of [Har97], we have:

$$Q\ \xi \quad {}_{E\ \tilde{\xi}}\ p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\Sigma} \quad p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\overline{\Sigma}} \qquad {}^{1}.$$

Let T_{Σ},h_{Σ} be as in section 2.1. The inclusion of $GU\ \ V_1$ in $Res_{F\ \ \mathbb{Q}}F$ induces maps of Shimura data $GU\ \ V_1\ \ ,h_{V_1,\Sigma} \qquad T_{\Sigma},h_{\Sigma}$ and $GU\ \ V_1\ \ ,h_{V_1,\overline{\Sigma}} \qquad T_{\Sigma},h_{\overline{\Sigma}}$. We extend $\tilde{\xi}$ to an algebraic character of $\mathbb{A}_F$, still denoted by $\tilde{\xi}$. By Proposition 2.1, we know that

$$p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\Sigma} \qquad {}_{E\ \tilde{\xi}}\ p\ \ \tilde{\xi},\Sigma \quad \text{and}\ p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\bar{\Sigma}} \qquad {}_{E\ \tilde{\xi}}\ p\ \ \tilde{\xi},\bar{\Sigma}\ \ .$$

Therefore

$$\begin{aligned} Q\ \xi \quad & {}_{E\ \tilde{\xi}} \quad p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\Sigma}\ \ p\ \ \tilde{\xi},\ \ GU\ \ V_1\ \ ,h_{V_1,\overline{\Sigma}} \qquad {}^{1} \\ & {}_{E\ \tilde{\xi}} \quad p\ \ \tilde{\xi},\Sigma\ \ p\ \ \tilde{\xi},\overline{\Sigma} \quad {}^{1} \\ & {}_{E\ \tilde{\xi}} \quad p\ \ \tilde{\xi},\Sigma\ \ p\ \ \tilde{\xi}^{c,\ \ 1},\Sigma \\ & {}_{E\ \tilde{\xi}} \quad p\ \ \frac{\tilde{\xi}}{\tilde{\xi}^{c}},\Sigma \\ & {}_{E\ \tilde{\xi}} \quad p\ \ \frac{\xi}{\xi^{c}},\Sigma\ \ . \end{aligned} \tag{6.12}$$

Recall that $\xi_{\tilde{\pi}}^{\ \ 1} \quad \xi_{\tilde{\pi}}\,\tilde{\xi} \quad {}_{Z_{\tilde{H}}\ \ \mathbb{A}}$. Moreover, $\Pi \quad \Pi^{c}$ and $\Pi \qquad \Pi\ {}^{c}$ are the base changes of π and π respectively. Note that $\dfrac{\xi}{\xi^{c}}$ is the base change of ξ. One verifies easily $\xi_{\Pi}^{c}\xi_{\Pi}^{c}\ \dfrac{\xi}{\xi^{c}} \quad 1$.

We then obtain that

$$Q\ \xi \quad {}_{E\ \tilde{\xi}}\ p\ \ \check{\xi}_{\Pi},\Sigma\ \ p\ \ \check{\xi}_{\Pi}\ ,\Sigma\ \ .$$

□

The previous lemma and equation (6.10) then implies that:

$$Q\ \pi\ Q\ \pi \qquad {}_{E\ \pi\quad E\ \pi} \quad p\ \ \check{\xi}_{\Pi},\Sigma \quad {}^{1}\frac{P\ {}^{q\ \ 1}\ \ \Pi,\iota_{v_0}}{P\ {}^{q}\ \ \Pi,\iota_{v_0}} \qquad p\ \ \check{\xi}_{\Pi}\ ,\Sigma \quad {}^{1}\frac{P\ {}^{n\ \ q\ \ 1}\ \ \Pi\ ,\iota_{v_0}}{P\ {}^{n\ \ q\ \ 2}\ \ \Pi\ ,\iota_{v_0}}\ \ . \tag{6.13}$$

We now prove Theorem 5.2 by induction on n. When n 2, the integer q is necessarily 0. The theorem is clear by Observation 5.4. We assume that the theorem is true for n 1 2. Recall that our representation π is an element in Π_Φ H $^0, \Pi$ of degree n q 2. Since Π is n 1 -regular, we know that Π is automatically n 2 -regular by construction, whence Π and π satisfy the conditions of Thm. 5.2. Hence $Q\ \pi\ \ {}_{E\ \pi}\ p\ \check{\xi}_\Pi, \Sigma\ \ {}_1\dfrac{P^{\ n\ q\ 1}\ \Pi, \iota_{v_0}}{P^{\ n\ q\ 2}\ \Pi, \iota_{v_0}}$. The theorem then follows from equation (6.13) and [Gro-Lin17], Lem. 1.19. □

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